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BOUNDARY BLOW-UP SOLUTIONS TO SEMILINEAR ELLIPTIC EQUATIONS WITH NONLINEAR GRADIENT TERMS

SHUFANG LIU, YONGLIN XU

Abstract. In this article we study the blow-up rate of solutions near the boundary for the semilinear elliptic problem

$$\begin{gathered}\Delta u \pm |\nabla u|^q = b(x)f(u), \quad x \in \Omega,\\ u(x) = \infty, \quad x \in \partial\Omega,\end{gathered}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, and $b(x)$ is a nonnegative weight function which may be bounded or singular on the boundary, and f is a regularly varying function at infinity. The results in this article emphasize the central role played by the nonlinear gradient term $|\nabla u|^q$ and the singular weight $b(x)$. Our main tools are the Karamata regular variation theory and the method of explosive upper and lower solutions.

1. Introduction and statement of main results

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded domain with smooth boundary. We are interested in the asymptotic behavior of boundary blow-up solutions to the elliptic problem

$$\begin{gathered}\Delta u \pm |\nabla u|^q = b(x)f(u), \quad x \in \Omega,\\ u(x) = \infty, \quad x \in \partial\Omega.\end{gathered} \tag{1.1}$$

For the functions $f(u)$ and $b(x)$, we assume the following hypotheses:

- (F1) $f \in C^1[0,\infty), f'(s) \geq 0$ for $s \geq 0$, $f(0) = 0$ and $f(s) > 0$ for $s > 0$.
- (B1) $b \in C^\alpha(\Omega)$ for some $\alpha \in (0,1)$ and is non-negative in Ω.
- (B2) b has the property: if $x_0 \in \Omega$ and $b(x_0) = 0$, then there exists a domain Ω_0 such that $x_0 \in \Omega_0 \subset \Omega$ and $b(x) > 0$, for all $x \in \partial\Omega_0$.

The boundary condition $u(x) = \infty$, $x \in \partial\Omega$ is to be understand as $u \to \infty$ when $d(x) = \operatorname{dist}(x, \partial\Omega) \to 0+$. The solutions of problem (1.1) are called large solutions, boundary blow-up solutions or explosive solutions; that is, the boundary blow-up solutions provide uniform bounds for all other solutions to $\Delta u \pm |\nabla u|^q = b(x)f(u)$ in Ω, regardless of the boundary data by the comparison principle.

The study of boundary blow-up solutions of $\Delta u = e^u$ in Ω was initiated by Bieberbach [3], where $\Omega \subset \mathbb{R}^2$. Problems of this type arise in Riemannian geometry, more precisely: if a Riemannian metric of the form $|ds|^2 = e^{2u(x)}|dx|^2$ has constant

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Gaussian curvature $-b^2$, then $\Delta u = b^2 e^{2u}$. Rademacher [24] extended the results of Bieberbach to $\Omega \subset \mathbb{R}^3$. Later, Lazer and McKenna [22] generalized the results of [3, 24] to the case of bounded domains in $\mathbb{R}^N$ and nonlinearities $b(x)e^u$.

Recently, Cîrstea and Rădulescu[11, 12] opened a unified new approach, the Karamata regular variation theory approach, to study the uniqueness and asymptotic behavior of boundary blow-up solutions, which enables us to obtain significant information about the qualitative behavior of the boundary blow-up solutions in a general framework. Cîrstea [13] obtained the asymptotic behavior of boundary blow-up solutions to

$$\Delta u + au = b(x)f(u), \tag{1.2}$$

provided $f(x)$ and $b(x)$ satisfy

- (F2) $f \circ \mathcal{L} \in RV_\rho(\rho > 0)$ (see Definition 2.1) for some $\mathcal{L} \in C^2[A, \infty)$ satisfying $\lim_{u\to\infty} \mathcal{L}(u) = \infty$ and $\mathcal{L}' \in NRV_{-1}$,
- (B3) $\lim_{d(x)\to 0} \frac{b(x)}{k^2(d(x))} = 1$, $k(x) \in NRV_\theta(0+)$ (see Definition 2.5) for some $\theta \ge 0$, and k is nondecreasing near the origin if $\theta = 0$.

They showed that the blowup rate of boundary blow-up solutions u to problem (1.2) can be expressed by

$$\lim_{d(x)\to 0} \frac{u}{(\mathcal{L} \circ \Phi_1)(d(x))} = 1, \tag{1.3}$$

where the function Φ_1 is defined as

$$\int_{\Phi_1(t)}^{\infty} \frac{[\mathcal{L}'(y)]^{1/2}}{y^{\frac{\rho+1}{2}}[L_f(y)]^{1/2}} dy = \int_0^t k(s)dt, \quad \text{for all } x \in (0, \tau) \text{ with small } \tau > 0. \tag{1.4}$$

where L_f is a normalised slowly varying function such that

$$\lim_{u\to\infty} \frac{f(\mathcal{L}(u))}{u^\rho L_f(u)} = 1. \tag{1.5}$$

Elliptic boundary blow-up problems have been studied by a large number of authors in the last century, see [10, 5, 15, 16, 27] and references therein.

For problem (1.1), with $b \equiv 1$ in Ω, and$f(u) = u^p$, by ordinary differential equation theory and comparison principle, Bandle and Giarrusso [2] showed the following results:

(1) If $p \ge 1$ and $q < \frac{2p}{p+1} (< 2)$, then problem (1.1) possesses at least one solution. Every solution of (1.1) satisfies

$$\lim_{d(x)\to 0} \frac{u(x)}{(d(x))^{-2/(p-1)}} = \left[\sqrt{2(p+1)}/(p-1)\right]^{2/(p-1)}.$$

(2) The same statement for (1.1) is true if $\frac{2p}{p+1} < q < p$ except that in this case

$$\lim_{d(x)\to 0} u(x)\Big(\frac{p-q}{q} d(x)\Big)^{q/(p-q)} = 1.$$

(3) If $\max\{1, \frac{2p}{p+1}\} < q < 2$, then (1.1), with the minus sign, possesses a solution. Each solution of (1.1) with the minus sign satisfies

$$\lim_{d(x)\to 0} u(x)(2-q)[(q-1)d(x)]^{\frac{2-q}{q-1}} = 1.$$

(4) If $q = 2$, (1.1) with the minus sign has a solution for all $p > 0$ which satisfies
$$\lim_{d(x)\to 0} u(x)/\ln d(x) = 1.$$

Now we introduce the class of functions K_l consisting of positive monotonic functions $k \in L^1(0,\vartheta) \cap C^1(0,\vartheta)$ which satisfy
$$\lim_{t\to 0+} \frac{K(t)}{k(t)} = 0, \quad \lim_{t\to 0+} \frac{d}{dt}\Big(\frac{K(t)}{k(t)}\Big) = l, \quad \text{where } K(t) = \int_0^t k(s)ds.$$
We point out that for each $k \in \mathcal{K}_l$, $l \in [0,1]$ if k is non-decreasing and $l \in [1,\infty)$ if k is non-increasing. For more propositions of $\mathcal{K}_l$, we refer reader to [9, 10].

Some examples of functions $k \in \mathcal{K}_l$ are:

(1) $k(t) = t^q \in \mathcal{K}_l$ with $l = 1/(1+q)$;
(2) $k(t) = (-\ln t)^q \in \mathcal{K}_l$ for $q < 0$ with $l = 1$;
(3) $k(t) = -s/\ln t \in \mathcal{K}_l$ for $s > 0$ with $l = 1$;
(4) $k(t) = t^s/\ln(1+t^{-1}) \in \mathcal{K}_l$ for $s > 0$ with $l = 1/(1+s)$.

When b satisfies (B1) and (B2), Zhang [28] gave the following results: Assume f satisfies (F1), $f'(u) = u^\rho L(u)$, $\rho > 0$, $L(u)$ is slowly varying at infinity, $1 < q < \rho+1$, $b(x)$ satisfies (B1) with $b = 0$ on $\partial\Omega$,

(B4) $\lim_{d(x)\to 0} \frac{b(x)}{k^q(d(x))} = c_q$, where $k(x) \in K_l$ for some $0 < l \le 1$,

$\varphi \in C^2(0,a)$ be uniquely determined by
$$\int_{\varphi(t)}^\infty \frac{dt}{[f(y)]^{1/q}} = \int_0^t k(s)dt, \quad \text{for all } x \in (0,\tau) \text{ with small } \tau > 0.$$
(1) If $q = \frac{2(\rho+1)}{\rho+2}$ and $\lim_{u\to\infty} L(u) = (1+\rho)\gamma \in (0,+\infty)$, then every solution $u_+ \in C^2(\Omega)$ to problem (1.1), with plus sign, satisfies
$$\lim_{d(x)\to 0} \frac{u_+(x)}{\varphi(d(x))} = c_q^{-1/(\rho+1-q)},$$
where
$$\varphi(t) = \Big(\frac{2-q}{\gamma^{1/q}(q-1)}\Big)^{(2-q)/(q-1)} \Big(\int_0^t k(s)dt\Big)^{(q-2)/(q-1)}, \quad t \in (0,a),$$

(2) The same statement is true if $\frac{2(\rho+1)}{\rho+2} < q \le 2$, where $\varphi \in RVZ_{-q/l(\rho+1-q)}$ and there exists $H \in RVZ_0$ such that $\varphi(t) = H(t)t^{-q/l(\rho+1-q)}$.

Moreover, he also obtained some boundary blow-up rate of solutions to problem (1.1) if $0 \le q < 2(1+l\rho)/(2+l\rho)$. Zhang [29] considered problem (1.1) for a weight b that may be singular on the boundary.

More recently, for b satisfying (B1) and (B2), and f satisfying (F1) and (F2), Huang et al [20] obtained the following:

(1) If $0 \le q < 2$, and $b(x)$ satisfies (B3), then (1.1) has a large solution $u_\pm$, which satisfy (1.3);

(2) If $q > 2$, and $b(x)$ satisfy

(B5) $\lim_{d(x)\to 0} \frac{b(x)}{k^q(d(x))} = 1$, where $k(x) \in NRV_\theta(0+)$ for some $\theta \ge 0$, and k is nondecreasing near the origin if $\theta = 0$.

Then problem (1.1), with plus sign, has a boundary blow-up solution u_+ satisfying
$$\lim_{d(x)\to 0} \frac{u_+}{(\mathcal{L}\circ\Phi_2)(d(x))} = 1, \tag{1.6}$$

where Φ_2 is given by

$$\int_{\Phi_2(t)}^{\infty} \frac{\mathcal{L}'(y)}{y^{\rho/q}[L_f(y)]^{1/q}} dy = \int_0^t k(s)dt, \quad \text{for all } x \in (0,\tau) \text{ with small } \tau > 0. \tag{1.7}$$

(3) If $q > 2$, then $u_- = -\ln v$ is the unique solution to problem (1.1), with the minus sign, where v is the unique solution to problem $\Delta v = b(x)f(-\ln v)v$, $v > 0$, $x \in \Omega$, $v|_{\partial\Omega} = 0$.

(4) If $q = 2$, and $b(x)$ satisfies (B3), then problem (1.1), with plus sign, has a unique solution u_+ satisfying

$$u(x) \sim \frac{1}{\rho} \ln\Big(\frac{2+\rho(1+\theta)}{2}\Big) + \ln \Psi(d(x)) \quad \text{as } d(x) \to 0,$$

where $\Psi(t)$ is given by

$$\int_{\Psi(t)}^{\infty} \frac{dy}{y\sqrt{f(\ln y)}} = \int_0^t k(s)dt \quad \text{for all } t \in (0,\tau),\ \tau > 0 \text{ small enough.}$$

For more results of boundary blow-up problem with nonlinear gradient terms, see [17, 7, 21, 8, 14, 23, 6, 1, 19].

We remark at this point that $\lim_{u\to\infty} \mathcal{L}(u) = \infty$ with $\mathcal{L}' \in NRV_{-1}$ if and only if

$$\mathcal{L}(u) = C \exp\Big\{ \int_B^u \frac{s(t)}{t} dt \Big\}, \quad \forall u > B > 0,$$

where $C > 0$ is a constant and $s(t)$ is a normalised slowly varying function satisfying

$$\lim_{u\to\infty} s(u) = 0, \quad \lim_{u\to\infty} \int_B^u \frac{s(t)}{t} dt = \infty.$$

Note that $f \circ \mathcal{L} \in RV_\rho(\rho > 0)$ is equivalent to the existence of $g \in RV_\rho$ so that $f(u) = g(\mathcal{L}^{\leftarrow}(u))$ for u large, where $\mathcal{L}^{\leftarrow}$ denotes the inverse of $\mathcal{L}$, By Proposition 2.8, we know that if $\mathcal{L}' \in NRV_{-1}$, then $\mathcal{L}^{\leftarrow}$ is rapidly varying with index ∞; i.e.,

$$\lim_{u\to\infty} \frac{\mathcal{L}^{\leftarrow}(\lambda u)}{\mathcal{L}^{\leftarrow}(u)} = \begin{cases} 0, & \text{if } \lambda \in (0,1), \\ 1, & \text{if } \lambda = 1, \\ \infty, & \text{if } \lambda > 1, \end{cases}$$

Therefore, the nonlinear term $f(u)$ satisfies (F2), then it is rapidly varying at infinity with index ∞, namely $f(u) \in RV_\infty$.

The main purpose of this article is to describe the asymptotic behavior of the boundary blow-up solution to (1.1), when f satisfies

(F3) $f \circ \mathcal{L} \in RV_\rho$ $(\rho > 0)$ for some $\mathcal{L} \in C^2[A,\infty)$ satisfying $\mathcal{L}' \in NRV_{-r}$ with $0 \le r < 1$.

Our main results are the following.

thmeorem 1.1. *Let f satisfy* (F1), (F3) *with $q < \rho/(1-r)$, $b(x)$ satisfies* (B1), (B2) *and*

(B6) $\lim_{d(x)\to 0} \frac{b(x)}{k^2(d(x))} = c_0$, *where $k(x) \in K_l$ for some $0 < l < \infty$.*

(i) If

$$0 \le q < \frac{2\rho}{\rho - r + 1}, \tag{1.8}$$

then for any solution $u_\pm$ *to problem* (1.1) *satisfies*

$$\lim_{d(x)\to 0}\frac{u_\pm}{(\mathcal{L}\circ\Phi_1)(d(x))}=\xi_1, \tag{1.9}$$

where

$$\xi_1=\Big[\frac{l(\rho+r-1)+2(1-r)}{2c_0}\Big]^{\frac{1-r}{\rho+r-1}},$$

and Φ_1 *is defined by* (1.4)*, moreover,*

$$\Phi_1\in RV_{-\frac{2}{l(\rho+r-1)}}(0+).$$

(ii) The same statement is true if $q=\frac{2\rho}{\rho-r+1}$ *and* $\lim_{u\to\infty}\mathcal{L}'(u)/u^{-r}=L_0$*, moreover*

$$\lim_{d(x)\to 0}u_\pm\Big(\int_0^t k(s)dt\Big)^{\frac{2(1-r)}{\rho+r-1}}=L_0^{\frac{\rho}{\rho+r-1}}\Big(\frac{\rho+r-1}{2}\Big)^{-\frac{2(1-r)}{\rho+r-1}} \tag{1.10}$$

thmeorem 1.2. *Let* $b(x)$ *satisfy* (B1), (B2), (B4) *with* $0<l<\infty$*,* f *satisfy* (F1) *and* (F3) *with* $q<\rho/(1-r)$*. If*

$$\frac{2\rho}{\rho-r+1}<q<\frac{\rho}{1-r}. \tag{1.11}$$

Then problem (1.1)*, with plus sign, have a solution* u_+*, which satisfies*

$$\lim_{d(x)\to 0}\frac{u_+}{(\mathcal{L}\circ\Phi_2)(d(x))}=\xi_2, \tag{1.12}$$

where $\xi_2=c_q^{\frac{1-r}{q(1-r)-\rho}}$*, and* Φ_2 *is defined by* (1.6)*; moreover,*

$$\Phi_2\in RV_{-\frac{q}{l(\rho+q(r-1))}}(0+).$$

thmeorem 1.3. *Assume* f *satisfies* (F1) *and* (F3) *with* $q<\frac{\rho}{1-r}$*,* $b(x)$ *satisfies* (B1), (B2), (B6)*. If*

$$\max\Big\{\frac{2\rho}{\rho+r-1},1\Big\}<q<2, \tag{1.13}$$

then any solution u_- *of problem* (1.1)*, with the minus sign, satisfies*

$$\lim_{d(x)\to 0}\frac{u_-}{(\mathcal{L}\circ\Phi_3)(d(x))}=\Big[\frac{r-1}{q-2}\Big]^{1/(q-1)}, \tag{1.14}$$

where Φ_3 *is given by*

$$\int_{\Phi_3(t)}^\infty\frac{[\mathcal{L}'(y)]^{\frac{1-q}{2-q}}}{y^{\frac{1}{2-q}}}dy=t,\forall x\in(0,\tau)\quad\text{with small } \tau>0. \tag{1.15}$$

and $\Phi_3\in RV_{-\frac{q-2}{(q-1)(r-1)}}(0+)$.

Remark 1.4. There are many functions satisfying (F1) and (F3), for example:

(1) $f(u)=u^{\frac{\rho}{1-r}}(\ln(u+1))^\alpha$, for all $\alpha\ge 0$.
(2) $f(u)=u^{\frac{\rho}{1-r}}\exp\{(\ln u)_1^\alpha(\ln_2 u)_2^\alpha\cdots(\ln_m u)_m^\alpha\}$, where $\alpha_i\in(0,1)$ and $\ln_m(\cdot)=\ln(\ln_{m-1}(\cdot))$.
(3) $f(u)=c_0u^{\frac{\rho}{1-r}}\exp\{\int_0^u\frac{s(t)}{t}dt\}$, $u\ge 0$, $s(t)\in C[0,+\infty)$ is nonnegative such that $\lim_{t\to\infty}s(t)=0$ and $\lim_{t\to\infty}s(t)/t\in[0,+\infty)$.

Remark 1.5. Define $\phi_1(K(dx)) = \Phi_1(t)$, then ϕ_1 satisfies

$$\int_{\phi_1(t)}^{\infty} \frac{[\mathcal{L}'(y)]^{1/2}}{y^{\frac{\rho+1}{2}}[L_f(y)]^{1/2}} dy = t. \tag{1.16}$$

Define $\phi_2(K(dx)) = \Phi_2(t)$, then ϕ_2 satisfies

$$\int_{\phi_2(t)}^{\infty} \frac{\mathcal{L}'(y)}{y^{\rho/q}[L_f(y)]^{\frac{1+-}{q}}} dy = t. \tag{1.17}$$

Remark 1.6. When $k \in \mathcal{K}_l$ with $0 < l < \infty$, instead of $0 < l \le 1$, then $b(x)$ may be singular near the boundary, namely $\lim_{d(x)\to 0+} b(x) = \infty$.

Remark 1.7. The existence of boundary blow-up solutions to (1.1) for $l \in (0, 1]$, has been shown in [28, Lemma 2.5]. The existence of boundary blow-up solutions to (1.1) for $l \in (1, \infty)$, has been shown in [29, Theorem 1.4 and Remark 1.7].

Remark 1.8. Note that, the asymptotic behavior of the boundary blow-up solutions to (1.1) is independent on $|\nabla u|$ if (1.8) holds. The asymptotic behavior of the boundary blow-up solutions to (1.1), with the minus sign, is independent on the nonlinear terms $b(x)f(u)$ if (1.13) holds.

Remark 1.9. The above Theorems 1.1 and 1.2 are independent of the choice of L_f. Indeed, if $\Phi_1(t)$ and $\Phi_1'(t)$ are defined by (1.4) corresponding to L_f and L_f', respectively, by (1.5) we infer that $\lim_{u\to+\infty} L_f/L_f' = 1$, in view of [28, Lemma 2.4], we derive that $\lim_{t\to 0} \Phi_1(t)/\Phi_1'(t) = 1$, this fact, combined with $\lim_{t\to 0} \Phi_1(t) = \lim_{t\to 0} \Phi_1'(t) = +\infty$, shows that

$$\lim_{t\to 0} \frac{(\mathcal{L} \circ \Phi_1)(t)}{(\mathcal{L} \circ \Phi_1')(t)} = 1.$$

Subject to $\Phi_2(t)$, the same conclusion is holds.

Remark 1.10. Let

$$\mathcal{J}_1(\xi) = \lim_{d(x)\to 0} b(x) \frac{f(\xi\mathcal{L}(\Phi_1(t)))}{\xi(\mathcal{L} \circ \Phi_1)''(t)}.$$

Then a direct computation shows that

$$\begin{aligned}
\mathcal{J}_1(\xi) &= \lim_{d(x)\to 0} b(x) \frac{f(\xi\mathcal{L}(\Phi_1(t)))}{\xi(\mathcal{L} \circ \Phi_1)''(t)} \\
&= \xi^{\frac{\rho}{1-r}-1} \lim_{d(x)\to 0} b(x) \frac{f(\mathcal{L}(\Phi_1(t)))}{\mathcal{L}''(\Phi_1(t)(\Phi_1'(t))^2 + \mathcal{L}'(\Phi_1(t))(\Phi_1''(t)} \\
&= \xi^{\frac{\rho}{1-r}-1} \lim_{d(x)\to 0} \frac{b(x)}{k^2(d(x))} \lim_{d(x)\to 0} \frac{f(\mathcal{L}(\Phi_1(t)))}{(\Phi_1(t))^\rho L_f(\Phi_1(t))} \\
&\times \lim_{d(x)\to 0} \frac{k^2(d(x))(\Phi_1(t))^\rho L_f(\Phi_1(t))}{\mathcal{L}''(\Phi_1(t)(\Phi_1'(t))^2 + \mathcal{L}'(\Phi_1(t))(\Phi_1''(t)} \\
&= c\xi^{\frac{\rho}{1-r}-1} \lim_{d(x)\to 0} \frac{(-\Phi_1(t))^2\mathcal{L}'(\Phi_1(t))}{\Phi_1(t)(\mathcal{L}''(\Phi_1(t)(\Phi_1'(t))^2 + \mathcal{L}'(\Phi_1(t))(\Phi_1''(t))} \\
&= c\xi^{\frac{\rho}{1-r}-1} \lim_{d(x)\to 0} \frac{1}{\frac{\Phi_1(t)\mathcal{L}''(\Phi_1(t))}{\mathcal{L}'(\Phi_1(t))} + \frac{\Phi_1''(t)\Phi_1(t)}{(-\Phi_1'(t))^2}} \\
&= \frac{2c\xi^{\frac{\rho}{1-r}-1}}{l(\rho + r - 1) + 2(r - 1)}.
\end{aligned}$$

it can be easily seen that ξ_1 satisfies $\mathcal{J}_1(\xi_1) = 1$.

In a similar way we can prove that ξ_2 satisfies $\mathcal{J}_2(\xi_2) = 1$, where

$$\mathcal{J}_2(\xi) = \lim_{d(x)\to 0} \xi^{q-1} \frac{[(\mathcal{L} \circ \Phi_2)'(t)]^q}{(\mathcal{L} \circ \Phi_2)''(t)}.$$

Remark 1.11. It is important to notice that by Proposition 2.8, $\mathcal{L}' \in NRV_{-r}$ with $0 \le r < 1$ implies that $\mathcal{L}^{\leftarrow} \in NRV_{1/(1-r)}$, then $f(u) = g(\mathcal{L}^{\leftarrow}(u)) \in RV_{\rho/(1-r)}$, instead of $f(u) \in RV_\infty$ for $r = 1$. This fact will bring a significant change in the explosion speed of the large solution of (1.1). Firstly, by [20, Lemmas 2.1 and 2.2], we know that $\Phi_1 \in NRV_{-\frac{2}{l\rho}}(0+)$, which defined by (1.4) for $r = 1$, and $\Phi_2 \in NRV_{-\frac{q}{l\rho}}(0+)$, which defined by (1.7) for $r = 1$, we conclude that $\mathcal{L}\circ\Phi_1 \in RV_0$, $\mathcal{L} \circ \Phi_2 \in RV_0$. By (1.3) and (1.6), we know that the solution regularly varying at infinity with index 0, namely the solution to problem (1.1) is slowly varying functions if $r = 1$.

While we replace $\mathcal{L}' \in NRV_{-1}$ by the hypothesis $\mathcal{L}' \in NRV_{-r}$ with $0 \le r < 1$, according to Lemma 2.9, Lemma 2.11 and Proposition 2.8 (see below), we get that

$$\mathcal{L} \circ \Phi_1 \in RV_{-\frac{2(1-r)}{l(\rho+r-1)}}, \quad \mathcal{L} \circ \Phi_2 \in RV_{-\frac{q(1-r)}{l(\rho+q(r-1))}}.$$

where Φ_1 (Φ_2) defined by (1.4)(1.7))with $0 \le r < 1$, in this case, the solution regularly varying at infinity with index

$$-\frac{2(1-r)}{l(\rho+r-1)} \quad \Big(-\frac{q(1-r)}{l(\rho+q(r-1))}\Big).$$

Secondly, for $r = 1$, we know that it is sufficient to know the bounds of

$$\lim_{d(x)\to 0+} \frac{b(x)}{k^2(x)}, \tag{1.18}$$

we can obtain that (1.3) holds, namely,

$$0 < \liminf_{d(x)\to 0+} \frac{b(x)}{k^2(x)} \quad \text{and} \quad \limsup_{d(x)\to 0+} \frac{b(x)}{k^2(x)} < \infty,$$

implies (1.3) holds.

While for $0 \le r < 1$, in order to get (1.9), the weight function $b(x)$ should satisfy (B6), that is we need to know the exact value of (1.18). Indeed, if (1.8) holds and

$$\liminf_{d(x)\to 0+} \frac{b(x)}{k^2(x)} \ge c_*,$$

we can prove that

$$\lim_{d(x)\to 0} \frac{u}{(\mathcal{L} \circ \Phi_1)(d(x))} \le \Big[\frac{l(\rho+r-1) + 2(1-r)}{2c_*}\Big]^{\frac{1-r}{\rho+r-1}},$$

and

$$\liminf_{d(x)\to 0+} \frac{b(x)}{k^2(x)} \le c_*,$$

implies

$$\lim_{d(x)\to 0} \frac{u}{(\mathcal{L} \circ \Phi_1)(d(x))} \ge \Big[\frac{l(\rho+r-1) + 2(1-r)}{2c_*}\Big]^{\frac{1-r}{\rho+r-1}}.$$

The outline of the article is as follows. Section 2 gives some notion and results from regular variation theory. The main Theorem will be proved in Section 3.

2. Preliminaries

In this section, we collect some notions and properties of regularly varying functions. For more details, we refer the reader to [4, 25, 26].

Definition 2.1. A positive measurable function f defined on $[D, \infty)$ for some $D > 0$, is called regularly varying (at infinity) with index $q \in R$ (written $f \in RV_q$) if for all $\xi > 0$

$$\lim_{u \to \infty} \frac{f(\xi u)}{f(u)} = \xi^q.$$

When the index of regular variation q is zero, we say that the function is slowly varying. We say that $f(u)$ is regularly varying (on the right) at the origin with index $q \in \mathbb{R}$ (in short $f \in RV_q(0+)$) provided $f(1/u) \in RV_{-q}$. The transformation $f(u) = u^q L(u)$ reduces regular variation to slow variation. Some typical example of slowly varying functions are given by: (1) Every measurable function on $[A, \infty)$ which has a positive limit at ∞. (2) The logarithm $\log u$, its iterates $\log_m u$ and powers of $\log_m u$. (3) $L(u) = \exp\{(\log u)^{1/3} \cos((\log u)^{1/3})\}$, exhibits infinite oscillation in the sense that

$$\lim_{u \to \infty} \inf L(u) = 0 \quad \text{and} \quad \lim_{u \to \infty} \sup L(u) = \infty.$$

This shows that the behavior at infinity for a slowly varying function cannot be predicted. Next we state a uniform convergence theorem,

Proposition 2.2. *The convergence $L(\xi u)/L(u) \to 1$ as $u \to \infty$ holds uniformly on each compact $\varepsilon-$set in $(0, \infty)$.*

Now, we have some elementary properties of slowly varying functions.

Proposition 2.3. *If L is slowly varying, then*

(1) *For any $\alpha > 0$, $u^\alpha L(u) \to \infty$, $u^{-\alpha} L(u) \to 0$ as $u \to \infty$;*
(2) *$(L(u))^\alpha$ varies slowly for every $\alpha \in \mathbb{R}$;*
(3) *If L_1 varies slowly, so do $L(u)L_1(u)$ and $L(u) + L_1(u)$.*

Proposition 2.4 (Representation Theorem)**.** *The function $L(u)$ is slowly varying if and only if it can be written in the form*

$$L(u) = M(u) \exp\Big\{ \int_B^u \frac{y(t)}{t} dt \Big\} \quad (u \geq B) \tag{2.1}$$

for some $B > 0$, where $y \in C[B, \infty)$ satisfies $\lim_{u \to \infty} y(u) = 0$ and $M(u)$ is measurable on $[B, \infty)$ such that $lim_{u \to \infty} M(u) = M \in (0, \infty)$.

If $M(u)$ is replaced by $\hat{M}$ in (2.1), we get a normalised regularly varying function.

Definition 2.5. A function $f(u)$ defined for $u > B$ is called a normalised regularly varying function of index q (in short $f \in NRV_q$) if it is C^1 and satisfies

$$\lim_{u \to \infty} \frac{u f'(u)}{f(u)} = q. \tag{2.2}$$

Note that $f \in NRV_{q+1}$ if and only if f is C^1 and $f' \in RV_q$. And $NRV_q(0+)$ (resp.,NRV_q) denote the set of all normalised regularly varying functions at 0 (resp.,∞) of index q. A typify example function $f(u) = u^{q+1} + \sin(u^{q+2})$ (defined for large u) belongs to RV_{q+1} but not NRV_{q+1}.

Next we presente[Karamata's Theorem, direct half.

Proposition 2.6. *Let $f \in RV_q$ be locally bounded in $[A,\infty)$. Then (1) For any $j \geq -(q+1)$,*

$$\lim_{u\to\infty} \frac{u^{j+1}f(u)}{\int_A^u x^j f(x)dx} = j+q+1. \tag{2.3}$$

(2) For any $j < -(q+1)$, (and for $j = -(q+1)$ if $\int^\infty x^{-(q+1)}f(x)dx < \infty$)

$$\lim_{u\to\infty} \frac{u^{j+1}f(u)}{\int_u^\infty x^j f(x)dx} = -(j+q+1). \tag{2.4}$$

Definition 2.7. A non-decreasing function f defined on (A,∞) is $\Gamma-$varying at ∞ (written $f \in \Gamma$) if $\lim_{u\to\infty} f(u) = \infty$ and there exists $\chi : (A,\infty) \to (0,\infty)$ such that

$$\lim_{u\to\infty} \frac{f(u+\lambda\chi(u))}{f(u)} = e^\lambda, \quad \text{for all } \lambda \in \mathbb{R}.$$

The function χ is called an auxiliary function and is unique up to asymptotic equivalence. The following functions f with the specified auxiliary functions χ.

(1) $f(x) = \exp(x^p)$ for $p > 0$ with

$$\chi = \begin{cases} 1, & \text{for } x \leq 0, \\ p^{-1}x^{1-p}, & \text{for } x > 0. \end{cases}$$

(2) $f(x) = \exp(x \ln_+ x)$ with

$$\chi = \begin{cases} 1, & \text{for } x \leq 1, \\ (\ln x)^{-1}, & \text{for } x > 1. \end{cases}$$

(3) $f(x) = \exp(e^x)$ with $\chi = e^{-x}$.

For a non-decreasing function H on R, we define the (left continuous) inverse of H by

$$H^{\leftarrow}(y) = \inf\{s : H(s) \geq y\}.$$

Proposition 2.8. *We have*

(i) *If $f(u) \in RV_q$, then $\lim_{u\to\infty} \ln f(u)/\ln u = q$.*
(ii) *If $f_1(u) \in RV_q$ and $f_2(u) \in RV_s$ with $\lim_{u\to\infty} f_2(u) = \infty$, then $f_1 \circ f_2 \in RV_{qs}$.*
(iii) *Suppose $f(u)$ is non-decreasing, $\lim_{u\to\infty} f(u) = \infty$ and $f(u) \in RV_q$, $0 < q < \infty$, then $f^{\leftarrow} \in RV_{1/q}$.*

Now we a Characterization of Φ_1.

Lemma 2.9. *Suppose that f satisfies* (F3). *Then*

(i) *The function Φ_1 given by* (1.3) *is well defined. Moreover, $\Phi_1 \in C^2(0,\tau)$ satisfies $\lim_{t\to 0+} \Phi(t) = \infty$;*
(ii) *$\Phi_1 \in NRV_{-\frac{2}{l(r+\rho-1)}}(0+)$ satisfies*

$$\lim_{t\to 0+} \frac{\ln_m \Phi_1(t)}{\ln_m t} = \begin{cases} -\frac{2}{l(r+\rho-1)}, & m = 1, \\ -1, & m \geq 2, \end{cases}$$

where we set $\ln_{m+1}(\cdot) = \ln(\ln_m(\cdot))$, $m \geq 1$.

(iii) $\lim_{t\to 0+} \frac{\Phi_1(t)}{\Phi_1'(t)} = \lim_{t\to 0+} \frac{\Phi_1'(t)}{\Phi_1''(t)} = \lim_{t\to 0+} \frac{\Phi_1(t)}{\Phi_1''(t)} = 0$.
(iv) $\lim_{t\to 0+} \frac{\Phi_1''(t)\Phi_1(t)}{|\Phi_1'(t)|^2} = 1 + \frac{l(r+\rho-1)}{2}$.

(v) *If* (1.8) *holds,* $\lim_{t\to 0+}\frac{(-\Phi_1'(t))^{2-q}}{(\Phi_1(t))^{r(q-1)-1}}=0$.

Proof. In a similar way as [13, Lemma 3.4], we can prove (i)-(iv). Here we only prove (v). We differentiate (1.4) to obtain

$$(-\Phi_1'(t))^2\mathcal{L}'(\Phi_1(t))=(\Phi_1(t))^{\rho+1}L_f(\Phi_1(t))k^2(t),$$

then we have

$$\begin{aligned}&\lim_{t\to 0+}\frac{(-\Phi_1'(t))^{2-q}}{(\Phi_1(t))^{r(q-1)-1}}\\&=\lim_{t\to 0+}L_f^{\frac{q-2}{2}}(\Phi_1(t))k^{q-2}(t))\left((\Phi_1(t))^r\mathcal{L}'(\Phi_1(t))\right)^{\frac{q-2}{2}}(\Phi_1(t))^{\frac{q-2}{2}(\rho+1-r)-(r-1)}.\end{aligned}$$

Recalling that $L_f\in NRV_0$, $\mathcal{L}\in NRV_{-r}$ and (1.8), by Proposition 2.3, we get that (v) holds. □

Corollary 2.10. *The function* ϕ_1 *given by* (1.16) *is well defined and satisfies*

(i) $\phi_1\in C^2(0,\tau)$ *and* $\lim_{t\to 0+}\Phi(t)=\infty$*;*

(ii) $\phi_1\in NRV_{-\frac{2}{r+\rho-1}}(0+)$ *satisfies*

$$\lim_{t\to 0+}\frac{\ln_m\phi_1(t)}{\ln_m t}=\begin{cases}-\frac{2}{r+\rho-1}, & m=1,\\ -1, & m\ge 2.\end{cases}$$

(iii) $\lim_{t\to 0+}\frac{\phi_1(t)}{\phi_1'(t)}=\lim_{t\to 0+}\frac{\phi_1'(t)}{\phi_1''(t)}=\lim_{t\to 0+}\frac{\phi_1(t)}{\phi_1''(t)}=0$.

(iv) $\lim_{t\to 0+}\frac{\phi_1''(t)\phi_1(t)}{|\phi_1'(t)|^2}=1+\frac{2}{r+\rho-1}$.

(v) $\lim_{t\to 0+}\frac{(-\phi_1'(t))^{2-q}}{(\phi_1(t))^{r(q-1)-1}}=0$.

Next we Characterize Φ_2.

Lemma 2.11. *Suppose that* f *satisfies* (F3)*. Then*

(i) *The function* Φ_2 *given by* (1.7) *is well defined. Moreover,* $\Phi_2\in C^2(0,\tau)$ *satisfies* $\lim_{t\to 0^+}\Phi_2(t)=\infty$*;*

(ii) $\Phi_2\in NRV_{-\frac{q}{l(\rho+q(r-1))}}(0+)$ *satisfies*

$$\lim_{t\to 0+}\frac{\ln_m\Phi_2(t)}{\ln_m t}=\begin{cases}-\frac{q}{l(\rho+q(r-1))}, & m=1,\\ -1, & m\ge 2,\end{cases}\tag{2.5}$$

where we set $\ln_{m+1}(\cdot)=\ln(\ln_m(\cdot)),m\ge 1$*;*

(iii) $\lim_{t\to 0+}\frac{\Phi_2(t)}{\Phi_2'(t)}=\lim_{t\to 0+}\frac{\Phi_2'(t)}{\Phi_2''(t)}=\lim_{t\to 0+}\frac{\Phi_2(t)}{\Phi_2''(t)}=0$*;*

(iv) $\lim_{t\to 0+}\frac{\Phi_2''(t)\Phi_2(t)}{|\Phi_2'(t)|^2}=1+\frac{l(\rho+q(r-1))}{q}$*;*

(v) *If* (1.10) *holds,* $\lim_{t\to 0+}\frac{(\Phi_2'(t))^{2-q}}{(\Phi_2(t))^{r(1-q)+1}}=0$.

Proof. (i) Let $b>0$ such that $\mathcal{L}'(t)$, $L_f(t)$ are positive on (b,∞). Since $\mathcal{L}'\in RV_{-r}$ and $L_f\in RV_0$, by Proposition 2.3, we have

$$\lim_{t\to\infty}\frac{\mathcal{L}'(t)}{t^{\rho/q}[L_f(t)]^{1/q}}t^{1+\tau}=\lim_{t\to\infty}\frac{t^r\mathcal{L}'(t)}{[L_f(t)]^{1/q}}t^{1+\tau-\frac{\rho}{q}-r}=0,\text{ for some }\tau\in(0,\frac{\rho}{q}+r-1).$$

This shows that, for some $D>0$,

$$h(x)=\int_x^\infty\frac{\mathcal{L}'(t)}{t^{\rho/q}[L_f(t)]^{1/q}}dt<\infty,\quad\text{for all }x>D.$$

So, Φ_2 is well defined on $(0,\tau)$ for small enough τ.

We easily see that $h:(D,\infty)\to(0,h(D))$ is bijective and $\lim_{t\to 0}\int_0^t k(s)ds=0$, $\Psi=h^{-1}(\int_0^t k(s)ds)$ for $t\in(0,\tau)$, τ is small enough. Then $\lim_{t\to 0}\Phi_2(t)=\infty$. Moreover, by direct differentiating, we have $\Phi_2\in C^2$.

(ii), Note that, $k(t)\in NRV_\theta(0+)$ with $\theta=1/l-1$, then by Definition 2.5 and Proposition 2.6, it follows that

$$\lim_{t\to 0}\frac{tk'(t)}{k(t)}=\theta,\ \lim_{t\to 0}\frac{\int_0^t k(s)ds}{tk(t)}=l, \tag{2.6}$$

on the other hand, by (1.7), we have

$$\frac{-\Phi_2'(t)\mathcal{L}'(\Phi_2(t))}{\Phi_2(t)^{\rho/q}[L_f(\Phi_2(t))]^{1/q}}=k(t),\ \forall t\in(0,\tau), \tag{2.7}$$

thanks to Proposition 2.6, we obtain

$$\lim_{t\to\infty}\frac{\mathcal{L}'(t)}{t^{\frac{\rho}{q}-1}[L_f(t)]^{1/q}h(t)}=-(1-\frac{\rho}{q}-r)=\frac{\rho}{q}+r-1,$$

hence, in view of (1.7),

$$\lim_{t\to 0+}\frac{\mathcal{L}'(\Phi_2(t))}{\Phi_2(t)^{\frac{\rho}{q}-1}[L_f(\Phi_2(t))]^{1/q}\int_0^t k(s)dt}=\frac{\rho}{q}+r-1, \tag{2.8}$$

which, together with (2.7), yields,

$$\lim_{t\to 0+}\frac{\Phi_2'(t)\int_0^t k(s)ds}{\Phi_2(t)k(t)}=-\frac{q}{\rho+q(r-1)}, \tag{2.9}$$

by (2.6) and (2.9),

$$\lim_{t\to 0+}\frac{t\Phi_2'(t)}{\Phi_2(t)}=\lim_{t\to 0+}\frac{\Phi_2'(t)\int_0^t k(s)ds}{\Phi_2(t)k(t)}\times\frac{tk(t)}{\int_0^t k(s)ds}=-\frac{q}{l(\rho+q(r-1))}, \tag{2.10}$$

this implies

$$\Phi_2\in NRV_{-\frac{q}{l(\rho+q(r-1))}}(0+).$$

By (2.10) and L'Hospital's rule, we obtain

$$\lim_{t\to 0+}\frac{\ln\Phi_2(t)}{\ln t}=\lim_{t\to 0+}\frac{t\Phi_2'(t)}{\Phi_2(t)}=-\frac{q}{l(\rho+q(r-1))},$$

and

$$\lim_{t\to 0+}\frac{\ln(\ln\Phi_2(t))}{\ln(\ln t)}=\lim_{t\to 0+}\frac{t\Phi_2'(t)}{\Phi_2(t)}\cdot\frac{\ln t}{\ln\Psi}=1,$$

we now prove (2.5) by induction, Let $m=n(n\geq 2)$, we have

$$\lim_{t\to 0+}\frac{\ln_n\Phi_2(t)}{\ln_n t}=1.$$

Then, if $m=n+1$, we obtain

$$\lim_{t\to 0+}\frac{\ln_{n+1}\Phi_2(t)}{\ln_{n+1}t}=\lim_{t\to 0+}\frac{\ln(\ln_n\Phi_2(t))}{\ln(\ln_n t)}=\lim_{t\to 0+}\frac{\ln_n t}{\ln_n\Phi_2(t)}=1,$$

this prove (2.5).

(iii) Following from (ii), $\Phi_2\in NRV_{-\frac{q}{l(\rho+q(r-1))}}(0+)$, then the claim of (iii) is clear.

(iv) Differentiating (2.7), we deduce that

$$\begin{aligned} \Phi_2''(t) = -&\frac{\Phi_2'(t)k(t)\Phi_2(t)^{\frac{\rho}{q}-1}[L_f(\Phi_2(t))]^{1/q}}{\mathcal{L}'(\Phi_2(t))} \\ &\Big[\frac{\rho}{q} + \frac{k'(t)\Phi_2(t)}{k(t)\Phi_2'(t)} + \frac{L_f'(\Phi_2(t))\Phi_2(t)}{qL_f(\Phi_2(t))} - \frac{\Phi_2(t)\mathcal{L}''(\Phi_2(t))}{\mathcal{L}'(\Phi_2(t))}\Big], \end{aligned} \tag{2.11}$$

since $L_f \in NRV_0$ and $\mathcal{L}' \in NRV_{-r}$, by Definition 2.5, we have

$$\lim_{t\to 0+} \frac{\Phi_2(t)L_f'(\Phi_2(t))}{L_f(\Phi_2(t))} = 0, \quad \lim_{t\to 0+} \frac{\Phi_2(t)\mathcal{L}''(\Phi_2(t))}{\mathcal{L}'(\Phi_2(t))} = -r, \tag{2.12}$$

which combined (2.7) with (2.11), leads to

$$\lim_{t\to 0+} \frac{\Phi_2''(t)\mathcal{L}'(\Phi_2(t))}{\Phi_2'(t)k(t)\Phi_2(t)^{\frac{\rho}{q}-1}[L_f(\Phi_2(t))]^{1/q}} = -(r + \frac{l(\rho+q(r-1))}{q}),$$

then, thanks to (2.7), we have

$$\lim_{t\to 0+} \frac{\Phi_2''(t)\Phi_2(t)}{|\Phi_2'(t)|^2} = r + \frac{l(\rho+q(r-1))}{q}.$$

(v) In a similar way to Lemma 2.9 (v), we can prove that (v) holds, here we omit its proof. ☐

Corollary 2.12. *The function ϕ_2 given by* (1.17) *is well defined and*

(i) $\phi_2 \in C^2(0,\tau)$ *and* $\lim_{t\to 0^+} \phi_2(t) = \infty$*;*

(ii) $\phi_2 \in NRV_{-\frac{q}{\rho+q(r-1)}}(0+)$ *satisfies*

$$\lim_{t\to 0+} \frac{\ln_m \phi_2(t)}{\ln_m t} = \begin{cases} -\frac{q}{\rho+q(r-1)}, & m=1, \\ -1, & m \ge 2. \end{cases}$$

(iii) $\lim_{t\to 0+} \frac{\phi_2(t)}{\phi_2'(t)} = \lim_{t\to 0+} \frac{\phi_2'(t)}{\phi_2''(t)} = \lim_{t\to 0+} \frac{\phi_2(t)}{\phi_2''(t)} = 0.$

(iv) $\lim_{t\to 0+} \frac{\phi_2''(t)\phi_2(t)}{|\phi_2'(t)|^2} = 1 + \frac{\rho+q(r-1)}{q}.$

(v) $\lim_{t\to 0+} \frac{(\phi_2'(t))^{2-q}}{(\phi_2(t))^{r(1-q)+1}} = 0.$

Next we Characterize of Φ_3.

Lemma 2.13. *Suppose that f satisfies* (F3)*. Then*

(i) $\Phi_3 \in C^2(0,\tau)$ *and* $\lim_{t\to 0^+} \Phi_3(t) = \infty$*;*

(ii) $\Phi_3 \in NRV_{-\frac{q-2}{(q-1)(r-1)}}(0+)$ *satisfies*

$$\lim_{t\to 0+} \frac{\ln_m \Phi_3(t)}{\ln_m t} = \begin{cases} -\frac{q-2}{(q-1)(r-1)}, & m=1, \\ -1, & m \ge 2; \end{cases}$$

(ii) $\lim_{t\to 0+} \frac{\Phi_3(t)}{\Phi_3'(t)} = \lim_{t\to 0+} \frac{\Phi_3'(t)}{\Phi_3''(t)} = \lim_{t\to 0+} \frac{\Phi_3(t)}{\Phi_3''(t)} = 0$*;*

(iii) $\lim_{t\to 0+} \frac{\Phi_3''(t)\phi_2(t)}{|\Phi_3'(t)|^2} = 1 + \frac{(q-1)(r-1)}{q-2}$*;*

(iv) $\lim_{t\to 0+} \frac{(\Phi_3'(t))^{2-q}}{(\Phi_3(t))^{r(1-q)+1}} = 0.$

The Proof of the above Lemma is similarly to the previous lemmas, here we omit it.

Proposition 2.14. *Let $\Psi(x,s,\xi)$ satisfy the following two conditions*

(i) *Ψ is non-increasing in s for each $(x,\xi)\in\Omega\times\mathbb{R}^N$.*
(ii) *Ψ is continuously differentiable with respect to the ξ variable in $\Omega\times(0,\infty)\times\mathbb{R}^N$.*

If $u,v\in C(\bar\Omega)\cap C^2(\Omega)$ satisfy $\Delta u+\Psi(x,u,\nabla u)\ge\Delta v+\Psi(x,v,\nabla v)$ in Ω and $u\le v$ on $\partial\Omega$, then $u\le v$ in Ω.

3. Proof of main results

In this section we prove Theorems 1.1-1.3. The proof of each theorem will be split in two cases according to the values of l. Given $\delta>0$, for $\forall\beta\in(0,\delta)$, denote

$$\Omega_\delta=\{x\in\Omega,0<d(x)<\delta\},\quad \partial\Omega_\delta=\{x\in\Omega,d(x)=\delta\},$$
$$\Omega_\beta^-=\Omega_{2\delta}\backslash\bar\Omega_\beta,\quad \Omega_\beta^+=\Omega_{2\delta-\beta},$$

Proof of Theorem 1.1. **Case 1**: (i) $l\in(0,1]$. Set

$$\xi^\pm=\Big(\frac{2-2r+l(\rho+r-1)}{2(c_0\pm\varepsilon)}\Big)^{\frac{1-r}{\rho+r-1}}, \tag{3.1}$$

where $\varepsilon\in(0,c_0)$ is arbitrary. We now diminish $\delta\in(0,\beta/2)$, such that

(i) $d(x)$ is a C^2-function on the set $\{x\in R^N:d(x)<2\delta\}$;
(ii) $k(x)$ is non-decreasing on $(0,2\delta)$;
(iii) $c_0k^2(d(x)-\beta)<b(x)<c_0k^2(d(x)+\beta)$; for all $x\in\{x\in R^N:d(x,\partial\Omega)<2\delta\}$.

Let $\beta\in(0,\delta)$ be arbitrary, we define

$$u_\beta^+=\xi^+\mathcal{L}(\Phi_1(d(x)+\beta)),\quad x\in\Omega_\beta^+,$$
$$u_\beta^-=\xi^-\mathcal{L}(\Phi_1(d(x)-\beta)),\quad x\in\Omega_\beta^-,$$

by the definition of $u_\beta^\pm$ we derive

$$\nabla u_\beta^\pm=\xi^\pm\mathcal{L}'(\Phi_1(d(x)\pm\beta))\Phi_1'(d(x)\pm\beta)\nabla d(x);$$

since $|\nabla d(x)|=1$, it follows that

$$\begin{aligned}\Delta u_\beta^\pm&=\xi^\pm\mathcal{L}''(\Phi_1(d(x)\pm\beta))[\Phi_1'(d(x)\pm\beta)]^2+\xi^\pm\mathcal{L}'(\Phi_1(d(x)\pm\beta))\Phi_1''(d(x)\pm\beta)\\&\quad+\xi^\pm\mathcal{L}'(\Phi_1(d(x)\pm\beta))\Phi_1'(d(x)\pm\beta)\Delta d(x).\end{aligned}$$

Then

$$\begin{aligned}&\Delta u_\beta^+\pm|\nabla u_\beta^+(x)|^q-b(x)f(u_\beta^+)\\&\ge k^2(d(x)+\beta)f(u_\beta^+)(A_1^+(d(x)+\beta)+A_2^+(d(x)+\beta)\\&\quad+A_3^+(d(x)+\beta)\pm A_4^+(d(x)+\beta)-c_0),\end{aligned}$$

and

$$\begin{aligned}&\Delta u_\beta^-\pm|\nabla u_\beta^-(x)|^q-b(x)f(u_\beta^-)\\&\le k^2(d(x)-\beta)f(u_\beta^-)(A_1^-(d(x)-\beta)+A_2^-(d(x)-\beta)\\&\quad+A_3^-(d(x)-\beta)\pm A_4^-(d(x)-\beta)-c_0),\end{aligned}$$

where we denote

$$A_1^\pm(t)=\frac{\xi^\pm\mathcal{L}''(\Phi_1(t))[\Phi_1'(t)]^2}{k^2(t)f(\xi^\pm\mathcal{L}(\Phi_1(t)))},\quad A_2^\pm(t)=\frac{\xi^\pm\mathcal{L}'(\Phi_1(t))\Phi_1''(t)}{k^2(t)f(\xi^\pm\mathcal{L}(\Phi_1(t)))},$$

$$A_3^{\pm}(t)=\frac{\xi^{\pm}\mathcal{L}'(\Phi_1(t))\Phi_1'(t)\Delta d(x)}{k^2(t)f(\xi^{\pm}\mathcal{L}(\Phi_1(t)))},\quad A_4^{\pm}(t)=\frac{(\xi^{\pm})^q(\mathcal{L}'(\Phi_1(t)))^q(-\Phi_1'(t))^q}{k^2(t)f(\xi^{\pm}\mathcal{L}(\Phi_1(t)))}.$$

According to $\mathcal{L}'\in NRV_{-r}$ and (1.5), it is clear that

$$\begin{gathered}
\lim_{t\to 0}A_1^{\pm}=(\xi^{\pm})^{1-\frac{\rho}{1-r}}\lim_{t\to 0}\frac{\mathcal{L}''(\Phi_1(t))[\Phi_1'(t)]^2}{k^2(t)f(\mathcal{L}(\Phi_1(t)))},\\
\lim_{t\to 0}A_2^{\pm}=(\xi^{\pm})^{1-\frac{\rho}{1-r}}\lim_{t\to 0}\frac{\mathcal{L}'(\Phi_1(t))\Phi_1''(t)}{k^2(t)f(\mathcal{L}(\Phi_1(t)))},\\
\lim_{t\to 0}A_3^{\pm}=(\xi^{\pm})^{1-\frac{\rho}{1-r}}\lim_{t\to 0}\frac{\mathcal{L}'(\Phi_1(t))\Phi_1'(t)\Delta d(x)}{k^2(t)f(\mathcal{L}(\Phi_1(t)))},\\
\lim_{t\to 0}A_4^{\pm}=(\xi^{\pm})^{q-\frac{\rho}{1-r}}\lim_{t\to 0}\frac{(\mathcal{L}'(\Phi_1(t)))^q(-\Phi_1'(t))^q}{k^2(t)f(\mathcal{L}(\Phi_1(t)))}.
\end{gathered}$$

Thanks to (1.4) and (1.5), we have

$$\lim_{t\to 0}\frac{(\Phi_1(t))^{\rho}L_f(\Phi_1(t))}{f(\mathcal{L}(\Phi_1(t)))}=1,\quad \lim_{t\to 0}\frac{(-\Phi_1(t))^2\mathcal{L}'(\Phi_1(t))}{k^2(t)(\Phi_1(t))^{\rho+1}L_f(\Phi_1(t))}=1.$$

By $\mathcal{L}'\in NRV_{-r}$, we obtain

$$\lim_{t\to 0}\frac{\Phi_1(t)\mathcal{L}''(\Phi_1(t))}{\mathcal{L}'(\Phi_1(t))}=-r.$$

Then

$$\begin{aligned}
&\lim_{t\to 0}\frac{\mathcal{L}''(\Phi_1(t))[\Phi_1'(t)]^2}{k^2(t)f(\mathcal{L}(\Phi_1(t)))}\\
&=\lim_{t\to 0}\frac{(\Phi_1(t))^{\rho}L_f(\Phi_1(t))}{f(\mathcal{L}(\Phi_1(t)))}\lim_{t\to 0}\frac{(-\Phi_1(t))^2\mathcal{L}'(\Phi_1(t))}{k^2(t)(\Phi_1(t))^{\rho+1}L_f(\Phi_1(t))}\lim_{t\to 0}\frac{\Phi_1(t)\mathcal{L}''(\Phi_1(t))}{\mathcal{L}'(\Phi_1(t))}=-r,
\end{aligned}$$

which implies

$$\lim_{t\to 0}A_1^{\pm}=\frac{-r}{(\xi^{\pm})^{\frac{\rho}{1-r}-1}}.$$

In view of Lemma 2.9, we have

$$\begin{aligned}
&\lim_{t\to 0}\frac{\mathcal{L}'(\Phi_1(t))\Phi_1''(t)}{k^2(t)f(\mathcal{L}(\Phi_1(t)))}\\
&=\lim_{t\to 0}\frac{(\Phi_1(t))^{\rho}L_f(\Phi_1(t))}{f(\mathcal{L}(\Phi_1(t)))}\lim_{t\to 0}\frac{(-\Phi_1(t))^2\mathcal{L}'(\Phi_1(t))}{k^2(t)(\Phi_1(t))^{\rho+1}L_f(\Phi_1(t))}\lim_{t\to 0}\frac{\Phi_1(t)\Phi_1''(t)}{(-\Phi_1(t))^2}\\
&=1+\frac{l(r+\rho-1)}{2},
\end{aligned}$$

which yields

$$\lim_{t\to 0}A_2^{\pm}=\frac{1+\frac{l(r+\rho-1)}{2}}{(\xi^{\pm})^{\frac{\rho}{1-r}-1}}.$$

We notice that

$$A_3^{\pm}=A_2^{\pm}\frac{\Phi_1'(t)}{\Phi_1''(t)},$$

and we infer that $\lim_{t\to 0}A_3^{\pm}=0$.

Considering $A_4^{\pm}$, since

$$\frac{(\mathcal{L}'(\Phi_1(t)))^q(-\Phi_1'(t))^q}{k^2(t)f(\mathcal{L}(\Phi_1(t)))}$$

$$
\begin{aligned}
&= \frac{(\Phi_1(t))^{\rho} L_f(\Phi_1(t))}{f(\mathcal{L}(\Phi_1(t)))} \frac{(-\Phi_1(t))^2 \mathcal{L}'(\Phi_1(t))}{k^2(t)(\Phi_1(t))^{\rho+1} L_f(\Phi_1(t))} ((\Phi_1(t))^r \mathcal{L}'(\Phi_1(t)))^{q-1} \\
&\quad \times \frac{(-\Phi_1'(t))^{2-q}}{(\Phi_1(t))^{r(q-1)-1}}
\end{aligned}
$$

we use Lemma 2.9 (v) to obtain $\lim_{t\to 0} A_4^{\pm} = 0$. Then we have

$$
\begin{aligned}
&\lim_{d(x)+\beta\to 0} \big(A_1^+(d(x)+\beta) + A_2^+(d(x)+\beta) + A_3^+(d(x)+\beta) \pm A_4^+(d(x)+\beta) - c_0\big) \\
&= +\varepsilon, \\
&\lim_{d(x)\to \beta} \big(A_1^-(d(x)-\beta) + A_2^-(d(x)-\beta) + A_3^-(d(x)-\beta) \pm A_4^-(d(x)-\beta) - c_0\big) \\
&= -\varepsilon,
\end{aligned}
$$

Now we can choose $\delta > 0$ small enough so that

$$
\begin{gathered}
\Delta u_\beta^+ \pm |\nabla d(x)|^q - b(x) f(u_\beta^+) \le 0, \quad x \in \Omega_\beta^+, \\
\Delta u_\beta^- \pm |\nabla d(x)|^q - b(x) f(u_\beta^-) \ge 0, \quad x \in \Omega_\beta^-,
\end{gathered}
$$

Let $u(x)$ be a non-negative solution of (1.1) and $M(2\delta) = \max_{d(x)\ge 2\delta} u(x)$, $N(2\delta) = \mathcal{L}(\xi^- B(2\delta))$, it follows that

$$
\begin{gathered}
u(x) \le M(2\delta) + u_\beta^-, \quad x \in \partial\Omega_\beta^-, \\
u_\beta^+ \le N(2\delta) + u(x), \quad x \in \partial\Omega_\beta^+,
\end{gathered}
$$

This, combined with Proposition 2.14, yields

$$
\begin{gathered}
u(x) \le M(2\delta) + u_\beta^-, \quad x \in \Omega_\beta^-, \\
u_\beta^+ \le N(2\delta) + u(x),\ x \in \Omega_\beta^+,
\end{gathered}
$$

for each $x \in \Omega_\beta^- \cap \Omega_\beta^+$, we have

$$
u_\beta^+ - N(2\delta) \le u \le M(2\delta) + u_\beta^-,
$$

we arrive at

$$
\begin{aligned}
\frac{u_\beta^+}{(\mathcal{L}\circ\Phi_1)(d(x))} - \frac{N(2\delta)}{(\mathcal{L}\circ\Phi_1)(d(x))} &\le \frac{u(x)}{(\mathcal{L}\circ\Phi_1)(d(x))} \\
&\le \frac{u_\beta^-}{(\mathcal{L}\circ\Phi_1)(d(x))} + \frac{M(2\delta)}{(\mathcal{L}\circ\Phi_1)(d(x))},
\end{aligned}
$$

we note that (1.9) and Proposition 2.8 leads to

$$
\mathcal{L}\circ\Phi_1 \in RV_{\frac{2(r-1)}{l(\rho+r-1)}}(0+)
$$

thus, we deduce that $\lim_{d(x)\to 0} \mathcal{L}\circ\Phi_1(d(x) = \infty$. Then letting $d(x) \to 0$, we conclude (1.3).

Case 2: $l \in (1, +\infty)$. We now diminish $\delta \in (0, \beta/2)$, such that

(i) $d(x)$ is a C^2-function for all $x \in \Omega_{2\delta}$;
(ii) $k(x)$ is non-increasing on $(0, 2\delta)$;
(iii) $c_0 k^2(d(x)) < b(x) < c_0 k^2(d(x))$ for all $x \in \Omega_{2\delta}$.

Let $\beta \in (0, \delta)$ be arbitrary, we define

$$
\begin{gathered}
u_\beta^+ = \xi^+ \mathcal{L}(\phi_1(d(x)+\beta)), \quad x \in \Omega_\beta^+, \\
u_\beta^- = \xi^- \mathcal{L}(\phi_1(d(x)-\beta)), \quad x \in \Omega_\beta^-,
\end{gathered}
$$

where $\xi^{\pm}$ defined by (3.1) and ϕ_1 defined by (1.16). We infer that

$$\begin{aligned}\Delta u_\beta^{\pm} &= \xi^{\pm}\mathcal{L}''(\xi^{\pm}\phi_1(d(x)\pm\beta))[\phi_1'(d(x)\mp\beta)]^2k^2(d(x))\\ &\quad+\xi^{\pm}\mathcal{L}'(\xi^{\pm}\phi_1(d(x)\pm\beta))\phi_1''(d(x)\pm\beta)k^2(d(x))\\ &\quad+\xi^{\pm}\mathcal{L}'(\xi^{\pm}\phi_1(d(x)\pm\beta))\phi_1'(d(x)\pm\beta)k'(d(x))\\ &\quad+\xi^{\pm}\mathcal{L}'(\xi^{\pm}\phi_1(d(x)\pm\beta))\phi_1'(d(x)\pm\beta)k(d(x))\Delta d(x).\end{aligned}$$

Then we obtain

$$\begin{aligned}&\Delta u_\beta^{+}\pm|\nabla u_\beta^{+}(x)|^q-b(x)f(u_\beta^{+})\\ &\geq k^2(d(x)+\beta)f(u_\beta^{+})(A_1^{+}(d(x)+\beta)+A_2^{+}(d(x)+\beta)\\ &\quad+A_3^{+}(d(x)+\beta)+A_4^{+}(d(x)+\beta)\pm A_5^{+}(d(x)+\beta)-c_0),\end{aligned}$$

and

$$\begin{aligned}&\Delta u_\beta^{-}\pm|\nabla u_\beta^{-}(x)|^q-b(x)f(u_\beta^{-})\\ &\leq k^2(d(x)-\beta)f(u_\beta^{-})(A_1^{-}(d(x)-\beta)+A_2^{-}(d(x)-\beta)\\ &\quad+A_3^{-}(d(x)-\beta)+A_4^{-}(d(x)-\beta)\pm A_5^{-}(d(x)-\beta)-c_0),\end{aligned}$$

where we denote

$$A_1^{\pm}(t)=\frac{(\xi^{\pm})^2\mathcal{L}''(\xi^{\pm}\phi_1(t))[\phi_1'(t)]^2}{f(\mathcal{L}(\xi^{\pm}\phi_1(t)))},\quad A_2^{\pm}(t)=\frac{\xi^{\pm}\mathcal{L}'(\xi^{\pm}\phi_1(t))\phi_1''(t)}{f(\mathcal{L}(\xi^{\pm}\phi_1(t)))},$$

$$A_3^{\pm}(t)=\frac{\xi^{\pm}\mathcal{L}'(\xi^{\pm}\phi_1(t))\phi_1'(t)k'(d(x))}{k^2(d(x))f(\mathcal{L}(\xi^{\pm}\phi_1(t)))},$$

$$A_4^{\pm}(t)=\frac{\xi^{\pm}\mathcal{L}'(\xi^{\pm}\phi_1(t))\Phi_1'(t)k'(d(x))\Delta d(x)}{k^2(d(x))f(\mathcal{L}(\xi^{\pm}\phi_1(t)))},$$

$$A_5^{\pm}(t)=\frac{(\xi^{\pm})^q(\mathcal{L}'(\xi^{\pm}\phi_1(t)))^q(-\phi_1'(t))^q}{k^{2-q}(d(x))f(\mathcal{L}(\xi^{\pm}\phi_1(t)))}.$$

Following the same arguments as above we obtain

$$\lim_{t\to0}A_1^{\pm}(t)=\frac{-r}{(\xi^{\pm})^{\rho+r-1}},\quad \lim_{t\to0}A_2^{\pm}(t)=\frac{1+\frac{r+\rho-1}{2}}{(\xi^{\pm})^{\rho+r-1}},$$

$$\lim_{t\to0}A_3^{\pm}(t)=\frac{\frac{r+\rho-1}{2}(l-1)}{(\xi^{\pm})^{\rho+r-1}},\quad \lim_{t\to0}A_4^{\pm}(t)=0,\quad \lim_{t\to0}A_5^{\pm}(t)=0.$$

Then we obtain

$$\begin{aligned}&\lim_{d(x)+\beta\to0}(A_1^{+}(d(x)+\beta)+A_2^{+}(d(x)+\beta)\\ &+A_3^{+}(d(x)+\beta)+A_4^{+}(d(x)+\beta)\pm A_5^{+}(d(x)+\beta)-c_0)=+\varepsilon,\\ &\lim_{d(x)\to\beta}(A_1^{-}(d(x)-\beta)+A_2^{-}(d(x)-\beta)\\ &+A_3^{-}(d(x)-\beta)+A_4^{-}(d(x)-\beta)\pm A_5^{-}(d(x)-\beta)-c_0)=-\varepsilon,\end{aligned}$$

The remaining arguments in case 1 also apply here, so that case 2 is proved.

(ii) The proof this case follows from [28, Lemma 2.4]. □

Proof of Theorem 1.2. **Case 1:** $l\in(0,1]$. Set

$$\xi^{\pm}=\Big(\frac{1}{c_q\pm\varepsilon}\Big)^{\frac{1}{\rho-q(1-r)}},$$

where $\varepsilon \in (0, c_q)$ is arbitrary. We now diminish $\delta > 0$, such that

(i) $d(x)$ is a C^2-function on the set $\{x \in R^N : d(x) < 2\delta\}$;
(ii) $k(x)$ is non-decreasing on $(0, 2\delta)$;
(iii) $c_q k^q(d(x) - \beta) < b(x) < c_q k^q(d(x) + \beta)$, for all $x \in \{x \in R^N : d(x, \partial\Omega) < 2\delta\}$.

Let $\beta \in (0, \delta)$ be arbitrary, we define

$$\begin{aligned} u_\beta^+ &= \mathcal{L}(\xi^+ \Phi_2(d(x) + \beta)), \quad x \in \Omega_\beta^+, \\ u_\beta^- &= \mathcal{L}(\xi^- \Phi_2(d(x) - \beta)), \quad x \in \Omega_\beta^-. \end{aligned}$$

By the definition of $u_\beta^\pm$ we have

$$\nabla u_\beta^\pm = \xi^\pm \mathcal{L}'(\xi^\pm \Phi_2(d(x) \pm \beta))\Phi_2'(d(x) \pm \beta)\nabla d(x).$$

Since $|\nabla d(x)| = 1$ it follows that

$$\begin{aligned} \Delta u_\beta^\pm &= (\xi^\pm)^2 \mathcal{L}''(\xi^\pm(d(x) \pm \beta))[\Phi_2'(d(x) \pm \beta)]^2 \\ &\quad + \xi^\pm \mathcal{L}'(\xi^\pm \Phi_2(d(x) \pm \beta))\Phi_2''(d(x) \pm \beta) \\ &\quad + \xi^\pm \mathcal{L}'(\xi^\pm \Phi_2(d(x) \pm \beta))\Phi_2'(d(x) \pm \beta)\Delta d(x)\,. \end{aligned}$$

Then

$$\begin{aligned} &\Delta u_\beta^+ + |\nabla u_\beta^+(x)|^q - b(x) f(u_\beta^+) \\ &\geq k^q(d(x) + \beta) f(u_\beta^+)[B_1^+(d(x) + \beta) + B_2^+(d(x) + \beta) \\ &\quad + B_3^+(d(x) + \beta) + B_4^+(d(x) + \beta) - c_q], \end{aligned}$$

and

$$\begin{aligned} &\Delta u_\beta^- + |\nabla u_\beta^-(x)|^q - b(x) f(u_\beta^-) \\ &\leq k^q(d(x) - \beta) f(u_\beta^-)[B_1^-(d(x) - \beta) + B_2^-(d(x) - \beta) \\ &\quad + B_3^-(d(x) - \beta) + B_4^-(d(x) - c_q], \end{aligned}$$

where we denote

$$B_1^\pm(t) = \frac{(\xi^\pm)^2 \mathcal{L}''(\xi^\pm \Phi_2(t))[\Phi_2'(t)]^2}{k^q(t) f(\mathcal{L}(\xi^\pm \Phi_2(t)))}, \quad B_2^\pm(t) = \frac{\xi^\pm \mathcal{L}'(\xi^\pm \Phi_2(t))\Phi_2''(t)}{k^q(t) f(\mathcal{L}(\xi^\pm \Phi_2(t)))},$$
$$B_3^\pm(t) = \frac{\xi^\pm \mathcal{L}'(\xi^\pm \Phi_2(t))\Phi_2'(t)\Delta d(x)}{k^q(t) f(\mathcal{L}(\xi^\pm \Phi_2(t)))}, \quad B_4^\pm(t) = \frac{(\xi^\pm)^q (\mathcal{L}'(\xi^\pm \Phi_2(t)))^q (-\Phi_2'(t))^q}{k^q(t) f(\mathcal{L}(\xi^\pm \Phi_2(t)))}.$$

A direct computation shows that

$$\lim_{t\to 0} B_1^\pm(t) = 0, \quad \lim_{t\to 0} B_2^\pm(t) = 0,\ \lim_{t\to 0} B_3^\pm(t) = 0, \quad \lim_{t\to 0} B_4^\pm(t) = \frac{1}{(\xi^\pm)^{q(r-1)+\rho}}.$$

Thus

$$\begin{aligned} &\lim_{d(x)+\beta\to 0} \big(B_1^+(d(x) + \beta) + B_2^+(d(x) + \beta) + B_3^+(d(x) + \beta) + B_4^+(d(x) + \beta) - c_q\big) \\ &= +\varepsilon, \\ &\lim_{d(x)\to \beta} \big(B_1^-(d(x) - \beta) + B_2^-(d(x) - \beta) + B_3^-(d(x) - \beta) + B_4^-(d(x) - \beta) - c_q\big) \\ &= -\varepsilon, \end{aligned}$$

Similar arguments show that (1.12) holds.

Case 2: $l \in (1, \infty)$. This case is similarly, here we omit it. □

Proof of Theorem 1.3. The main idea is the same as in the proof of Theorems 1.1 and 1.2. We consider two cases, and give the proof of the case $l \in (0,1]$, the other case is omitted. Set

$$\xi^{\pm} = \Big(\frac{r-1}{(q-2)(1\pm\varepsilon)}\Big)^{\frac{1}{(q-1)(1-r)}}.$$

Define

$$u_\beta^{\pm} = \xi^{\pm}\mathcal{L}(\Phi_3(d(x)\pm\beta)), \quad x \in \Omega_\beta^{\pm}.$$

We infer that

$$\begin{aligned}
&\Delta u_\beta^+ + |\nabla u_\beta^+(x)|^q - b(x)f(u_\beta^+)\\
&\geq (\xi^{\pm})^q(\mathcal{L}'(\Phi_3(d(x)+\beta)))^q(\Phi_3'(d(x)+\beta))^q[C_1^+(d(x)+\beta)+C_2^+(d(x)+\beta)\\
&\quad + C_3^+(d(x)+\beta) - 1 - C_4^+(d(x)+\beta)],
\end{aligned}$$

and

$$\begin{aligned}
&\Delta u_\beta^- + |\nabla u_\beta^-(x)|^q - b(x)f(u_\beta^-)\\
&\leq (\xi^{\pm})^q(\mathcal{L}'(\Phi_3(d(x)-\beta)))^q(\Phi_3'(d(x)-\beta))^q[C_1^-(d(x)-\beta)+C_2^-(d(x)-\beta)\\
&\quad + C_3^-(d(x)-\beta) - 1 + C_4^-(d(x)],
\end{aligned}$$

where

$$\begin{gathered}
C_1^{\pm}(t) = \frac{\xi^{\pm}\mathcal{L}''(\Phi_1(t))[\Phi_1'(t)]^2}{(\xi^{\pm})^q(\mathcal{L}'(\Phi_3(t)))^q(\Phi_3'(t))^q}, \quad C_2^{\pm}(t) = \frac{\xi^{\pm}\mathcal{L}'(\Phi_1(t))\Phi_1''(t)}{(\xi^{\pm})^q(\mathcal{L}'(\Phi_3(t)))^q(\Phi_3'(t))^q},\\
C_3^{\pm}(t) = \frac{\xi^{\pm}\mathcal{L}'(\Phi_1(t))\Phi_1'(t)\Delta d(x)}{(\xi^{\pm})^q(\mathcal{L}'(\Phi_3(t)))^q(\Phi_3'(t))^q}, \quad C_4^{\pm}(t) = \frac{b(x)f(u_\beta^{\pm})}{(\xi^{\pm})^q(\mathcal{L}'(\Phi_3(t)))^q(\Phi_3'(t))^q}.
\end{gathered}$$

A direct computation shows that

$$\begin{gathered}
\lim_{t\to 0} C_1^{\pm}(t) = \frac{-r}{(\xi^{\pm})^{(q-1)(1-r)}}, \quad \lim_{t\to 0} C_2^{\pm}(t) = \frac{1+\frac{(q-1)(r-1)}{q-2}}{(\xi^{\pm})^{(q-1)(1-r)}},\\
\lim_{t\to 0} C_3^{\pm}(t) = \lim_{t\to 0} C_4^{\pm}(t) = 0.
\end{gathered}$$

then we can choose $\delta > 0$ small enough so that

$$\begin{gathered}
\Delta u_\beta^+ \pm |\nabla d(x)|^q - b(x)f(u_\beta^+) \leq 0, \quad x \in \Omega_\beta^+,\\
\Delta u_\beta^- \pm |\nabla d(x)|^q - b(x)f(u_\beta^-) \geq 0, \quad x \in \Omega_\beta^-,
\end{gathered}$$

In a similar way we can prove that that (1.14) holds. □

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Shufang Liu
Department of Mathematics, Gansu Normal University for Nationalities, Hezuo, Gansu 747000, China
E-mail address: shuxueliushufang@163.com

Yonglin Xu (Corresponding author)
School of Mathematics and Computer Science Institute, Northwest University for Nationalities, Lanzhou, Gansu 730030, China
E-mail address: xuyonglin000@163.com