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EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS FOR INDEFINITE SEMILINEAR ELLIPTIC PROBLEMS IN $\mathbb{R}^N$

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Abstract. In this article, we study a class of indefinite semilinear elliptic problems in $\mathbb{R}^N$. By using the fibering maps and studying some properties of the Nehari manifold, we obtain the existence and multiplicity of positive solutions.

1. Introduction

In this article, we consider the existence and multiplicity of positive solutions for the semilinear elliptic problem

$$\begin{gathered} -\Delta u + u = |u|^{p-2}u + f(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \\ 0 \leq u \in H^1(\mathbb{R}^N), \end{gathered} \tag{1.1}$$

where $2 < q \leq p < 2^*$ ($2^* = 2N/(N-2)$ if $N \geq 3$, and $2^* = \infty$ if $N = 1, 2$) and f is a continuous function in $\mathbb{R}^N$.

When $q = p$ and $f > -1$, Equation (1.1) becomes to the semilinear elliptic equation with positive nonlinearity,

$$\begin{gathered} -\Delta u + u = (1 + f(x))|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{gathered} \tag{1.2}$$

It is well known that if $f \equiv 0$, then Equation (1.2) has a unique positive solution (see [24]) and infinitely many radially symmetric nodal solutions. Moreover, the existence of positive solutions has been established by several authors under various conditions. In [8, 26, 27], it was proved that if $f \geq \lim_{|x|\to\infty} f(x) = 0$, then Equation (1.2) has a positive ground state solution and if $f \leq \lim_{|x|\to\infty} f(x) = 0$, then Equation (1.2) has no any ground state solution. In [6, 7, 25], it was proved that there is at least one positive solution to Equation (1.2) when $\lim_{|x|\to\infty}(1 + f(x)) = C_0 > 0$ and $0 > f(x) \geq -C\exp(-\delta|x|)$ for some $\delta > 0$ and $0 < C < 1$. In [12], it was proved that there is at least one positive solution to Equation (1.2) when $\lim_{|x|\to\infty} f(x) = 0$ and $f(x) \geq 2^{(2-p)/2} - 1$, for $3 \leq N < 8$ and $1 < p < N/(N-2)$ if $N = 3, 4$; $1 < p < 8/N$ if $4 < N < 8$. The multiplicities of solutions of Equation (1.2) were studied in [35] as follows. Assume that $N \geq 5$, $(1 + f(x)) \geq$

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$\lim_{|x|\to\infty}(1+f(x)) = C_0 > 0$ and that there exist positive constants C, γ and R_0 such that $f(x) \geq C/|x|^\gamma$ for $|x| \geq R_0$. Then (1.2) has at least one positive solution and one nodal solution.

When $q < p$ and f is non-positive or sign-changing, Equation (1.1) is the semilinear elliptic equation with indefinite nonlinearity. In fact, if Equation (1.1) is considered in a bounded domain Ω with, say, a Dirichlet boundary condition, then there is a vast literature on existence and multiplicity results (see [3, 13] and the references cited therein). In particular, the authors of [3] seem to have been the first authors to consider such indefinite problems in bounded domains. However, little has been done for this type of problem in $\mathbb{R}^N$. We are only aware of works [14, 20, 15, 17, 21] etc. that which studied the existence of solutions for the indefinite elliptic problem

$$\begin{gathered} -\Delta u - \lambda a(x)u = b(x)h(u) \quad \text{in } \mathbb{R}^N, \\ 0 \leq u \in D^{1,2}(\mathbb{R}^N), \end{gathered}$$

where $a, b \in C(\mathbb{R}^N)$ change sign in $\mathbb{R}^N$ and h is a nonlinear function with superquadratic growth both at zero and at infinity.

Several papers have also been devoted in the past few years to the study of nonlinearities with indefinite sign. Most of them, however, deal with problems that are not directly comparable to those considered here (cf., e.g., [2, 11, 16, 19, 23, 32, 33, 34]).

Our work was motivated in part by recent papers [3, 6, 7]. The main purpose of this paper is to use the shape of the graph of the function f to prove the existence and multiplicity of positive solutions of (1.1). Here we consider the indefinite semilinear elliptic equation

$$\begin{gathered} -\Delta u + u = |u|^{p-2}u + f_\lambda(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{gathered} \tag{1.3}$$

where $2 < q < p < 2^*$ ($2^* = 2N/(N-2)$ if $N \geq 3$, and $2^* = \infty$ if $N = 1, 2$) and $\lambda \in \mathbb{R}$. We assume that $f_\lambda(x) = \lambda f_+(x) - f_-(x)$ and that the nonnegative functions f_+ and f_- satisfy the following conditions:

(D1) $f_- \in C(\mathbb{R}^N)\backslash\{0\}$ and there exists a positive number $r_- > 1$ such that
$$f_-(x) \leq \widehat{c}\exp(-r_-|x|) \quad \text{or some } \widehat{c} > 0 \text{ and for all } x \in \mathbb{R}^N;$$

(D2) $f_+ \in C(\mathbb{R}^N) \cap L^{p/(p-q)}(\mathbb{R}^N)$ and there exist positive numbers R_0 and $r_+ < \min\{r_-, q\}$ such that
$$f_+(x) \geq c_0 \exp(-r_+|x|) \quad \text{for some } c_0 > 0 \text{ and for all } x \in \mathbb{R}^N \text{ with } |x| \geq R_0.$$

The following theorem is our main result.

Theorem 1.1. *Suppose that the functions $f_\pm$ satisfy the conditions* (D1) *and* (D2). *Then we have the following statements:*

(i) *Equation* (1.3) *has a positive higher energy solution and no any ground state solution for $\lambda = 0$;*
(ii) *Equation* (1.3) *has a positive ground state solution for $\lambda \in (0, \infty)$;*
(iii) *there exists a positive number Λ_* such that Equation* (1.3) *has at least three positive solutions for $\lambda \in (0, \Lambda_*)$.*

Corollary 1.2. *If in addition to conditions* (D1) *and* (D2), *we assume*

(D3) *there exists a positive number* $1 < \overline{r}_+ \leq r_+$ *such that*

$$f_+(x) \leq \overline{c}_0 \exp(-\overline{r}_+|x|) \quad \textit{for some } \overline{c}_0 > 0 \textit{ and for all } x \in \mathbb{R}^N,$$

then we have the following statements:

(i) *Equation* (1.3) *has a positive higher energy solution and no any ground state solution for* $\lambda \in (-\infty, 0]$*;*
(ii) *Equation* (1.3) *has a positive ground state solution for* $\lambda \in (0, \infty)$*;*
(iii) *there exists a positive number* $\overline{\Lambda}_*$ *such that Equation* (1.3) *has at least three positive solutions for* $\lambda \in (0, \overline{\Lambda}_*)$.

Next we prove Theorem 1.1, by using the variational methods to find positive solutions of Equation (1.3). We consider, the energy functional J_λ in $H^1(\mathbb{R}^N)$ associated with Equation (1.3),

$$J_\lambda(u) = \frac{1}{2}\|u\|_{H^1}^2 - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx - \frac{1}{q}\int_{\mathbb{R}^N} f_\lambda |u|^q dx,$$

where

$$\|u\|_{H^1} = \Big(\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 dx\Big)^{1/2}$$

is the standard norm in $H^1(\mathbb{R}^N)$. It is well known that the solutions of Equation (1.3) are the critical points of the energy functional J_λ in $H^1(\mathbb{R}^N)$ (see Rabinowitz [29]).

This paper is organized as follows. In Section 2, we give some notations and preliminaries. In Section 3, we give some estimates of the energy. In Section 4, we establish the existence of a positive solution for all $\lambda \in \mathbb{R}$. In Section 5, we establish the existence of two positive solutions for λ sufficiently small. In Section 6, we prove Theorem 1.1.

2. Preliminaries

First, we define the Palais-Smale (or simply (PS)-) sequences, (PS)-values, and (PS)-conditions in $H^1(\mathbb{R}^N)$ for J_λ as follows.

Definition 2.1. (i) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(\mathrm{PS})_\beta$-sequence in $H^1(\mathbb{R}^N)$ for J_λ if $J_\lambda(u_n) = \beta + o(1)$ and $J'_\lambda(u_n) = o(1)$ strongly in $H^{-1}(\mathbb{R}^N)$ as $n \to \infty$. (ii) J_λ satisfies the $(\mathrm{PS})_\beta$-condition in $H^1(\mathbb{R}^N)$ if every $(\mathrm{PS})_\beta$-sequence in $H^1(\mathbb{R}^N)$ for J_λ contains a convergent subsequence.

As the energy functional J_λ is not bounded from below on $H^1(\mathbb{R}^N)$, it is useful to consider the functional on the Nehari manifold

$$\mathbf{N}_\lambda = \{u \in H^1(\mathbb{R}^N)\backslash\{0\} : \langle J'_\lambda(u), u\rangle = 0\}.$$

Thus, $u \in \mathbf{N}_\lambda$ if and only if

$$\|u\|_{H^1}^2 - \int_{\mathbb{R}^N} |u|^p dx - \int_{\mathbb{R}^N} f_\lambda |u|^q dx = 0.$$

Define

$$\psi_\lambda(u) = \|u\|_{H^1}^2 - \int_{\mathbb{R}^N} |u|^p dx - \int_{\mathbb{R}^N} f_\lambda |u|^q dx.$$

Then for $u \in \mathbf{N}_\lambda$,

$$\langle \psi'_\lambda(u), u\rangle = 2\|u\|_{H^1}^2 - p\int_{\mathbb{R}^N} |u|^p dx - q\int_{\mathbb{R}^N} f_\lambda |u|^q dx$$

$$= (2-q)\|u\|_{H^1}^2 + (q-p)\int_{\mathbb{R}^N} |u|^p dx < 0$$

Furthermore, we have the following results.

Lemma 2.2. *The energy functional J_λ is coercive and bounded from below on $\mathbf{N}_\lambda$.*

Proof. If $u \in \mathbf{N}_\lambda$, then

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\|u\|_{H^1}^2 - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx - \frac{1}{q}\int_{\mathbb{R}^N} f_\lambda |u|^q dx \\ &= \frac{q-2}{2q}\|u\|_{H^1}^2 + \frac{p-q}{pq}\int_{\mathbb{R}^N} |u|^p dx \end{aligned} \tag{2.1}$$

Thus, J_λ is coercive and bounded below on $\mathbf{N}_\lambda$. □

Lemma 2.3. *Suppose that u_0 is a local minimizer for J_λ on $\mathbf{N}_\lambda$. Then $J'_\lambda(u_0) = 0$ in $H^{-1}(\mathbb{R}^N)$.*

The proof of the above lemma is essentially the same as that in Brown and Zhang [11, Theorem 2.3] (or see Binding, Drábek and Huang [9]).

To get a better understanding of the Nehari manifold, we consider the function $m_u : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$m_u(t) = t^{2-q}\|u\|_{H^1}^2 - t^{p-q}\int_{\mathbb{R}^N} |u|^p dx \quad \text{for } t > 0.$$

Clearly, $tu \in \mathbf{N}_\lambda$ if and only if $m_u(t) - \int_{\mathbb{R}^N} f_\lambda |u|^q dx = 0$, and $m_u(\hat{t}(u)) = 0$, where

$$\hat{t}(u) = \Big(\frac{\|u\|_{H^1}^2}{\int_{\mathbb{R}^N} |u|^p dx}\Big)^{1/(p-2)} > 0. \tag{2.2}$$

Moreover,

$$m'_u(t) = t^{1-q}\Big[(2-q)\|u\|_{H^1}^2 - (p-q)t^{p-2}\int_{\mathbb{R}^N} |u|^p dx\Big].$$

Thus,

$$m'_u(t) < 0 \quad \text{for all } t > 0,$$

which implies that m_u is strictly decreasing on $(0, \infty)$ with $\lim_{t\to 0^+} m_u(t) = \infty$ and $\lim_{t\to\infty} m_u(t) = -\infty$. Moreover, we have the following lemma.

Lemma 2.4. *Suppose that $\lambda \in \mathbb{R}$. Then for each $u \in H^1(\mathbb{R}^N)\backslash\{0\}$ we have the following.*

(i) *If $\int_{\mathbb{R}^N} f_\lambda |u|^q dx \le 0$, then there is a unique $t_\lambda(u) \ge \hat{t}(u)$ such that $t_\lambda(u)u \in \mathbf{N}_\lambda$. Furthermore,*
$$J_\lambda(t_\lambda(u)u) = \sup_{t\ge 0} J_\lambda(tu) = \sup_{t\ge \hat{t}(u)} J_\lambda(tu). \tag{2.3}$$

(ii) *If $\int_{\mathbb{R}^N} f_\lambda |u|^q dx > 0$, then there is a unique $t_\lambda(u) < \hat{t}(u)$ such that $t_\lambda(u)u \in \mathbf{N}_\lambda$. Furthermore,*
$$J_\lambda(t_\lambda(u)u) = \sup_{t\ge 0} J_\lambda(tu) = \sup_{0\le t\le \hat{t}(u)} J_\lambda(tu). \tag{2.4}$$

(iii) *$t_\lambda(u)$ is a continuous function for $u \in H^1(\mathbb{R}^N)\backslash\{0\}$.*

(iv) *$t_\lambda(u) = \frac{1}{\|u\|_{H^1}} t_\lambda(\frac{u}{\|u\|_{H^1}})$.*

(v) *$\mathbf{N}_\lambda = \{u \in H^1(\mathbb{R}^N)\backslash\{0\} : \frac{1}{\|u\|_{H^1}} t_\lambda(\frac{u}{\|u\|_{H^1}}) = 1\}$.*

Proof. Fix $u \in H^1(\mathbb{R}^N)\backslash\{0\}$. Let

$$h_u(t) = J_\lambda(tu) = \frac{t^2}{2}\|u\|_{H^1}^2 - \frac{t^p}{p}\int_{\mathbb{R}^N} |u|^p dx - \frac{t^q}{q}\int_{\mathbb{R}^N} f_\lambda |u|^q dx.$$

Then

$$\begin{aligned} h_u'(t) &= t\|u\|_{H^1}^2 - t^{p-1}\int_{\mathbb{R}^N} |u|^p dx - t^{q-1}\int_{\mathbb{R}^N} f_\lambda |u|^q dx \\ &= t^{q-1}\Big(t^{2-q}\|u\|_{H^1}^2 - t^{p-q}\int_{\mathbb{R}^N} |u|^p dx - \int_{\mathbb{R}^N} f_\lambda |u|^q dx\Big) \\ &= t^{q-1}\Big(m_u(t) - \int_{\mathbb{R}^N} f_\lambda |u|^q dx\Big). \end{aligned}$$

(i) If $\int_{\mathbb{R}^N} f_\lambda |u|^q dx \le 0$, then the equation $m_u(t) - \int_{\mathbb{R}^N} f_\lambda |u|^q dx = 0$ has a unique solution $t_\lambda(u) \ge \hat{t}(u)$, which implies that $h_u'(t_\lambda(u)) = 0$ and $t_\lambda(u)u \in \mathbf{N}_\lambda$. Moreover, h_u is strictly increasing on $(0, t_\lambda(u))$ and strictly decreasing on $(t_\lambda(u), \infty)$. Therefore, (2.3) holds.

(ii) If $\int_{\mathbb{R}^N} f_\lambda |u|^q dx > 0$, then the equation $m_u(t) - \int_{\mathbb{R}^N} f_\lambda |u|^q dx = 0$ has a unique solution $t_\lambda(u) < \hat{t}(u)$, which implies that $h_u'(t_\lambda(u)) = 0$ and $t_\lambda(u)u \in \mathbf{N}_\lambda$. Moreover, h_u is strictly increasing on $(0, t_\lambda(u))$ and strictly decreasing on $(t_\lambda(u), \infty)$. Therefore, (2.4) holds.

(iii) By the uniqueness of $t_\lambda(u)$ and the extrema property of $t_\lambda(u)$, we have $t_\lambda(u)$ is a continuous function for $u \in H^1(\mathbb{R}^N)\backslash\{0\}$.

(iv) Let $v = \frac{u}{\|u\|_{H^1}}$. Then by parts (i) and (ii), there is a unique $t_\lambda(v) > 0$ such that $t_\lambda(v)v \in \mathbf{N}_\lambda$ or $t_\lambda(\frac{u}{\|u\|_{H^1}})\frac{u}{\|u\|_{H^1}} \in \mathbf{N}_\lambda$. Thus, by the uniqueness of $t_\lambda(v)$, we can conclude that $t_\lambda(u) = \frac{1}{\|u\|_{H^1}} t_\lambda(\frac{u}{\|u\|_{H^1}})$.

(v) For $u \in \mathbf{N}_\lambda$. By parts (i)–(iii), $t_\lambda(\frac{u}{\|u\|_{H^1}})\frac{u}{\|u\|_{H^1}} \in \mathbf{N}_\lambda$. Since $u \in \mathbf{N}_\lambda$, we have $t_\lambda(\frac{u}{\|u\|_{H^1}})\frac{1}{\|u\|_{H^1}} = 1$, which implies that

$$\mathbf{N}_\lambda \subset \{u \in H^1(\mathbb{R}^N) : \frac{1}{\|u\|_{H^1}} t_\lambda(\frac{u}{\|u\|_{H^1}}) = 1\}.$$

Conversely, let $u \in H^1(\mathbb{R}^N)$ such that $\frac{1}{\|u\|_{H^1}} t_\lambda(\frac{u}{\|u\|_{H^1}}) = 1$. Then, by part (iii),

$$t_\lambda(\frac{u}{\|u\|_{H^1}})\frac{u}{\|u\|_{H^1}} \in \mathbf{N}_\lambda.$$

Thus,

$$\mathbf{N}_\lambda = \{u \in H^1(\mathbb{R}^N)\backslash\{0\} : \frac{1}{\|u\|_{H^1}} t_\lambda(\frac{u}{\|u\|_{H^1}}) = 1\}.$$

This completes the proof. □

Now we consider the elliptic problem

$$\begin{gathered} -\Delta u + u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\ \lim_{|x|\to\infty} u = 0. \end{gathered} \tag{2.5}$$

We consider the energy functional J^∞ in $H^1(\mathbb{R}^N)$ associated with (2.5),

$$J^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx.$$

Consider the minimizing problem:

$$\inf_{u \in \mathbf{N}^\infty} J^\infty(u) = \alpha^\infty,$$

where

$$\mathbf{N}^\infty = \{u \in H^1(\mathbb{R}^N) \backslash \{0\} : \langle (J^\infty)'(u), u \rangle = 0\}.$$

It is known that Equation (2.5) has a unique positive radial solution $w(x)$ such that $J^\infty(w) = \alpha^\infty$ and $w(0) = \max_{x \in \mathbb{R}^N} w(x)$ (see [24]). Then we have the following results.

Proposition 2.5. *Let $\{u_n\}$ be a (PS)$_\beta$–sequence in $H^1(\mathbb{R}^N)$ for J_λ. Then there exist a subsequence $\{u_n\}$, $m \in \mathbb{N}$, sequences $\{x_n^i\}_{n=1}^\infty$ in $\mathbb{R}^N$, and functions $v_0 \in H^1(\mathbb{R}^N)$, and $0 \neq w^i \in H^1(\mathbb{R}^N)$, for $1 \leq i \leq m$ such that:*

- (i) *$|x_n^i| \to \infty$ and $|x_n^i - x_n^j| \to \infty$ as $n \to \infty$, for $1 \leq i \neq j \leq m$;*
- (ii) *$-\Delta v_0 + v_0 = |v_0|^{p-2} v_0 + f_\lambda(x) |v_0|^{q-2} v_0$ in $\mathbb{R}^N$;*
- (iii) *$-\Delta w^i + w^i = |w^i|^{p-2} w^i$ in $\mathbb{R}^N$;*
- (iv) *$u_n = v_0 + \sum_{i=1}^m w^i(\cdot - x_n^i) + o(1)$ strongly in $H^1(\mathbb{R}^N)$;*
- (v) *$J_\lambda(u_n) = J_\lambda(v_0) + \sum_{i=1}^m J^\infty(w^i) + o(1)$.*

In addition, if $u_n \geq 0$, then $v_0 \geq 0$ and $w^i \geq 0$ for each $1 \leq i \leq m$.

The proof of the above proposition is similar to the argument in Lions [26, 27]. For $\lambda \in \mathbb{R}$, we define

$$\alpha_\lambda = \inf_{u \in \mathbf{N}_\lambda} J_\lambda(u).$$

Then, by Proposition 2.5, we have the following compactness result.

Corollary 2.6. *Suppose that $\{u_n\}$ is a (PS)$_\beta$-sequence in $H^1(\mathbb{R}^N)$ for J_λ with $0 < \beta < \alpha^\infty + \min\{\alpha_\lambda, \alpha^\infty\}$ and $\beta \neq \alpha^\infty$. Then there exists a subsequence $\{u_n\}$ and a non-zero u_0 in $H^1(\mathbb{R}^N)$ such that $u_n \to u_0$ strongly in $H^1(\mathbb{R}^N)$ and $J_\lambda(u_0) = \beta$. Furthermore, u_0 is a non-zero solution of (1.3).*

3. The estimate of energy

Let $w(x)$ be a positive radial solution of Equation (2.5) such that $J^\infty(w) = \alpha^\infty$. Then by Gidas, Ni and Nirenberg [22] and Kwong [24], for any $\varepsilon > 0$, there exist positive numbers A_ε and B_0 such that

$$A_\varepsilon \exp(-(1+\varepsilon)|x|) \leq w(x) \leq B_0 \exp(-|x|) \quad \text{for all } x \in \mathbb{R}^N. \tag{3.1}$$

Let $e \in \mathbb{S}^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$ and let $z_0 = (\delta_0, 0, \dots, 0) \in \mathbb{R}^N$, where

$$0 < \delta_0 = \frac{\min\{r_-, q, \frac{p}{2}\} - 1}{2(\min\{r_-, q, \frac{p}{2}\} + 1)} < 1.$$

Clearly,

$$1 - \delta_0 \leq |e - z_0| \leq 1 + \delta_0 \quad \text{for all } e \in \mathbb{S}^{N-1}. \tag{3.2}$$

Define

$$w_{e,l}(x) = w(x - le) \quad \text{for } l \geq 0 \text{ and } e \in \mathbb{S}^{N-1} \tag{3.3}$$

and

$$w_{z_0,l}(x) = w(x - lz_0) \quad \text{for } l \geq 0.$$

Clearly, $w_{e,l}$ and $w_{z_0,l}$ are also least energy positive solutions of (2.5) for all $l \geq 0$. Moreover, by Lemma 2.4 for each $u \in H^1(\mathbb{R}^N)\backslash\{0\}$ and $\lambda \in \mathbb{R}$ there is a unique $t_\lambda(u) > 0$ such that $t_\lambda(u)u \in \mathbf{N}_\lambda$. Let $\hat{t}$ be as in (2.2). Then we have the following results.

Lemma 3.1. *For each $s_0 \in (0,1)$ there exist $l(s_0) > 0$ and $\sigma(s_0) > 1$ such that for any $l > l(s_0)$ we have*

$$\hat{t}^{p-2}(sw_{e,l} + (1-s)w_{z_0,l}) > \frac{\sigma(s_0)}{s^{p-2} + (1-s)^{p-2}}$$

for all $e \in \mathbb{S}^{N-1}$ and for all $s \in (0,1)$ with $\min\{s, 1-s\} \geq s_0$.

Proof. Since

$$\begin{aligned}
&\hat{t}^{p-2}(sw_{e,l} + (1-s)w_{z_0,l}) \\
&= \frac{\|sw_{e,l} + (1-s)w_{z_0,l}\|^2_{H^1}}{\int_{\mathbb{R}^N} |sw_{e,l} + (1-s)w_{z_0,l}|^p dx} \\
&= \frac{s^2\|w_{e,l}\|^2_{H^1} + (1-s)^2\|w_{z_0,l}\|^2_{H^1} + 2s(1-s)\langle w_{e,l}, w_{z_0,l}\rangle}{\int_{\mathbb{R}^N} |sw_{e,l} + (1-s)w_{z_0,l}|^p dx} \\
&= \frac{s^2\|w\|^2_{H^1} + (1-s)^2\|w\|^2_{H^1} + 2s(1-s)\langle w_{e,l}, w_{z_0,l}\rangle}{\int_{\mathbb{R}^N} |sw_{e-z_0,l} + (1-s)w|^p dx}
\end{aligned} \tag{3.4}$$

for all $s \in [0,1]$ and for all $e \in \mathbb{S}^{N-1}$. Moreover, by

$$1 - \delta_0 \leq |e - z_0| \leq 1 + \delta_0 \quad \text{for all } e \in \mathbb{S}^{N-1}, \tag{3.5}$$

and

$$\int_{\mathbb{R}^N} w_{e,l}^{p-1} w_{z_0,l} dx = \langle w_{e,l}, w_{z_0,l}\rangle = \int_{\mathbb{R}^N} w_{e,l} w_{z_0,l}^{p-1} dx. \tag{3.6}$$

we have

$$\begin{aligned}
\langle w_{e,l}, w_{z_0,l}\rangle &= \int_{\mathbb{R}^N} w^{p-1} w_{z_0-e,l} dx \\
&\leq B_0^p \int_{\mathbb{R}^N} \exp(-(p-1)|x|)\exp(-|x - l(z_0 - e)|)dx \\
&\leq B_0^p \int_{|x|<(1+\delta_0)l} \exp(-(|x| + |x - l(z_0 - e)|))dx \\
&\quad + B_0^p \int_{|x|\geq(1+\delta_0)l} \exp(-(|x| + |x - l(z_0 - e)|))dx \\
&\leq B_0^p l^N \int_{|x|<(1+\delta_0)} \exp(-l(|x| + |x - (z_0 - e)|))dx \\
&\quad + c_0 B_0^p \exp(-(1+\delta_0)l) \int_{|x|\geq(1+\delta_0)l} \exp\big(-(|x - l(z_0 - e)|)\big)dx \\
&\leq c_0 B_0^p l^N \int_{|x|<(1+\delta_0)} \exp(-(1-\delta_0)l)dx + C_0 B_0^p \exp(-(1+\delta_0)l) \\
&\leq C_0 B_0^p l^N \exp(-l(1-\delta_0)) \text{ for all } l \geq 1 \quad \text{and for all } e \in \mathbb{S}^{N-1},
\end{aligned}$$

which implies that

$$\lim_{l\to\infty} \langle w_{e,l}, w_{z_0,l}\rangle = 0 \quad \text{uniformly in } e \in \mathbb{S}^{N-1}. \tag{3.7}$$

By (3.1), (3.5) and Brézis-Lieb lemma [10], for any $s \in [0,1]$ we have

$$\begin{aligned} &\lim_{l\to\infty}\int_{\mathbb{R}^N} |sw_{e-z_0,l} + (1-s)w|^p - |sw_{e-z_0,l}|^p dx \\ &= \int_{\mathbb{R}^N} |(1-s)w|^p dx \quad \text{uniformly in } e \in \mathbb{S}^{N-1}. \end{aligned} \tag{3.8}$$

Thus, by (3.4), (3.7) and (3.8), for any $s \in [0,1]$,

$$\begin{aligned} \lim_{l\to\infty} \hat{t}^{p-2}(sw_{e,l} + (1-s)w_{z_0,l}) &= \frac{(s^2+(1-s)^2)\|w\|_{H^1}^2}{(s^p+(1-s)^p)\int_{\mathbb{R}^N}|w|^p dx} \\ &= \frac{s^2+(1-s)^2}{s^p+(1-s)^p} \quad \text{uniformly in } e \in \mathbb{S}^{N-1}. \end{aligned} \tag{3.9}$$

Since

$$\begin{aligned} \frac{(s^2+(1-s)^2)(s^{p-2}+(1-s)^{p-2})}{s^p+(1-s)^p} &= 1 + \frac{s^2(1-s)^{p-2}+(1-s)^2 s^{p-2}}{s^p+(1-s)^p} \\ &> 1 + \frac{s_0^2(1-s_0)^{p-2}+(1-s_0)^2 s_0^{p-2}}{s_0^p+(1-s_0)^p} \end{aligned} \tag{3.10}$$

for all $s \in (0,1)$ with $\min\{s, 1-s\} > s_0$, by (3.9) and (3.10), there exist $l(s_0) > 0$ and $\sigma(s_0) > 1$ such that for any $l > l(s_0)$, we have

$$\hat{t}^{p-2}(sw_{e,l} + (1-s)w_{z_0,l}) > \frac{\sigma(s_0)}{s^{p-2}+(1-s)^{p-2}}$$

for all $e \in \mathbb{S}^{N-1}$ and for all $s \in (0,1)$ with $\min\{s, 1-s\} \geq s_0$. This completes the proof. □

Proposition 3.2. (i) *For each $\lambda > 0$, there exists $\widehat{l}_1 = \widehat{l}_1(\lambda) > 0$ such that for any $l \geq \widehat{l}_1$,*

$$\sup_{t\geq 0} J_\lambda(tw_{e,l}) < \alpha^\infty \quad \textit{for all } e \in \mathbb{S}^{N-1}.$$

Furthermore, there is a unique $t_\lambda(w_{e,l}) > 0$ such that $t_\lambda(w_{e,l})w_{e,l} \in \mathbf{N}_\lambda$.

(ii) *There exists $l_1 > 0$ such that for any $l \geq l_1$*

$$\sup_{t\geq 0} J_0(t[sw_{e,l} + (1-s)w_{z_0,l}]) < 2\alpha^\infty \quad \textit{for all } 0 < s < 1 \textit{ and } e \in \mathbb{S}^{N-1},$$

where $J_0 = J_\lambda$ with $\lambda = 0$. Furthermore, there is a unique $t_\lambda(sw_{e,l}+(1-s)w_{z_0,l}) > 0$ such that

$$t_\lambda(sw_{e,l} + (1-s)w_{z_0,l})[sw_{e,l} + (1-s)w_{z_0,l}] \in \mathbf{N}_\lambda.$$

Proof. (i) We have

$$\begin{aligned} J_\lambda(tw_{e,l}) &= \frac{t^2}{2}\|w_{e,l}\|_{H^1}^2 - \frac{t^p}{p}\int_{\mathbb{R}^N}|w_{e,l}|^p dx - \frac{t^q}{q}\int_{\mathbb{R}^N} f_\lambda |w_{e,l}|^q dx \\ &= \frac{t^2}{2}\|w\|_{H^1}^2 - \frac{t^p}{p}\int_{\mathbb{R}^N} w^p dx - \frac{\lambda t^q}{q}\int_{\mathbb{R}^N} f_+ w_{e,l}^q dx + \frac{t^q}{q}\int_{\mathbb{R}^N} f_- w_{e,l}^q dx \\ &\leq \frac{t^2}{2}\|w\|_{H^1}^2 - \frac{t^p}{p}\int_{\mathbb{R}^N} w^p dx + \frac{\widehat{c}t^q}{q}\int_{\mathbb{R}^N} w^q dx. \end{aligned} \tag{3.11}$$

for all $\lambda > 0$. This implies that $J_\lambda(tw_{e,l}) \to -\infty$ as $t \to \infty$ uniformly for $e \in \mathbb{S}^{N-1}$. Thus, by $J_\lambda(0) = 0 < \alpha^\infty, J_\lambda \in C^1(H^1(\mathbb{R}^N), \mathbb{R})$ and $\|w_{e,l}\|_{H^1}^2 = \frac{2p}{p-2}\alpha^\infty$ for all $l \geq 0$, there exists $t_1, t_2 > 0$ such that

$$J_\lambda(tw_{e,l}) < \alpha^\infty \quad \text{for all } t \in [0, t_2] \cup [t_1, \infty) \quad \text{and for all } e \in \mathbb{S}^{N-1}. \tag{3.12}$$

Moreover, by Brown and Zhang [11] and Willem [31], we know that

$$J^\infty(tw) = \frac{t^2}{2}\|w\|_{H^1}^2 - \frac{t^p}{p}\int_{\mathbb{R}^N} w^p dx \leq \alpha^\infty \quad \text{for all } t > 0. \tag{3.13}$$

Thus, by (3.11),

$$J_\lambda(tw_{e,l}) \leq \alpha^\infty - \frac{\lambda t^q}{q}\int_{\mathbb{R}^N} f_+ w_{e,l}^q dx + \frac{t^q}{q}\int_{\mathbb{R}^N} f_- w_{e,l}^q dx \text{ for all } t > 0. \tag{3.14}$$

By (3.12) we only need to show that there exists $\widehat{l}_1 > 0$ such that, for any $l > \widehat{l}_1$,

$$\sup_{t_2 \leq t \leq t_1} J_\lambda(tw_{e,l}) < \alpha^\infty \quad \text{for all } e \in \mathbb{S}^{N-1}.$$

We set

$$C_0 = \min_{x \in \overline{B^N(0,1)}} w^q(x) > 0,$$

where $B^N(0,1) = \{x \in \mathbb{R}^N : |x| < 1\}$. Then, by condition (D2),

$$\begin{aligned}
\int_{\mathbb{R}^N} f_+ w_{e,l}^q dx &\geq \int_{|x| \geq R_0} f_+ w_{e,l}^q dx \\
&= \int_{|x+le| \geq R_0} f_+(x+le) w^q(x) dx \geq C_0 \int_{B^N(0,1)} f_+(x+le) dx \\
&\geq C_0 \exp(-r_+ l) \text{ for all } l \geq 2\max\{1, R_0\}.
\end{aligned}$$

Moreover, by (3.1) and condition (D1),

$$\begin{aligned}
\int_{\mathbb{R}^N} f_- w_{e,l}^q dx &\leq \widehat{c} B_0^q \int_{\mathbb{R}^N} \exp(-r_-|x|)\exp(-q|x-le|)dx \\
&\leq C_1 \exp(-\min\{r_-, q\} l)
\end{aligned} \tag{3.15}$$

Since $r_+ < \min\{r_-, q\}$ and $t_2 \leq t \leq t_1$, we can find $\widehat{l}_1 > 2\max\{1, R_0\}$ such that, for any $l > \widehat{l}_1$,

$$\frac{t^q}{q}\int_{\mathbb{R}^N} f_- w_{e,l}^q dx < \frac{\lambda t^q}{q}\int_{\mathbb{R}^N} f_+ w_{e,l}^p dx \quad \text{for all } e \in \mathbb{S} \text{ and for all } t \in [t_2, t_1]. \tag{3.16}$$

Thus, by (3.12)- (3.14) and (3.16), we obtain that for any $l > \widehat{l}_1$,

$$\sup_{t \geq 0} J_\lambda(tw_{e,l}) < \alpha^\infty \quad \text{for all } e \in \mathbb{S}^{N-1}.$$

Moreover, by Lemma 2.4, there is a unique $t_\lambda(w_{e,l}) > 0$ such that $t_\lambda(w_{e,l})w_{e,l} \in \mathbf{N}_\lambda$.

(ii) When $s = 0$ or 1, by a similar argument in part (i), there exists $\widetilde{t}_1 > 0$ such that

$$\max\{\sup_{t \geq 0} J_0(tw_{e,l}), \sup_{t \geq 0} J_0(tw_{z_0,l})\} \leq \alpha^\infty + \frac{\widetilde{t}_1 C_0}{q}\exp(-\min\{r_+, q\} l) \tag{3.17}$$

for all $e \in \mathbb{S}^{N-1}$, this implies that there exists $\widetilde{l}_1 > 0$ such that, for any $l > \widetilde{l}_1$,

$$\max\{\sup_{t \geq 0} J_0(tw_{e,l}), \sup_{t \geq 0} J_0(tw_{z_0,l})\} \leq \frac{3}{2}\alpha^\infty \quad \text{for all } e \in \mathbb{S}^{N-1}.$$

Therefore, since $J_0 \in C^2(H^1(\mathbb{R}^N),\mathbb{R})$, there exist positive constants s_0 and $\widetilde{l}$ such that, for any $l > \widetilde{l}$,

$$\sup_{t\geq 0} J_0(t[sw_{e,l} + (1-s)w_{z_0,l}]) < 2\alpha^\infty$$

for all $e \in \mathbb{S}^{N-1}$ and for all $\min\{s, 1-s\} \leq s_0$. In the following we always assume that $\min\{s, 1-s\} \geq s_0$. Since

$$\int_{\mathbb{R}^N} f_-|(sw_{e,l} + (1-s)w_{z_0,l})|^q dx \geq 0,$$

by Lemma 2.4 (i) and Lemma 3.1, we may show that there exists $l_1 \geq \widetilde{l}$ such that, for any $l > l_1$,

$$\sup_{t\geq (\frac{\sigma(s_0)}{s^{p-2}+(1-s)^{p-2}})^{1/(p-2)}} J_0(t[sw_{e,l} + (1-s)w_{z_0,l}]) < 2\alpha^\infty \text{ for all } e \in \mathbb{S}^{N-1}, \tag{3.18}$$

where $\sigma(s_0) > 1$ is as in Lemma 3.1. Since

$$\begin{aligned}
&J_0(t[sw_{e,l} + (1-s)w_{z_0,l}]) \\
&= \frac{t^2}{2}[s^2\|w\|_{H^1}^2 + (1-s)^2\|w\|_{H^1}^2 + 2s(1-s)\langle w_{e,l}, w_{z_0,l}\rangle] \\
&\quad + \frac{t^q}{q}\int_{\mathbb{R}^N} f_-[sw_{e,l} + (1-s)w_{z_0,l}]^q dx - \frac{t^p}{p}\int_{\mathbb{R}^N}[sw_{e,l} + (1-s)w_{z_0,l}]^p dx \\
&\leq \frac{t^2}{2}[s^2 + 2s(1-s) + (1-s)^2]\|w\|_{H^1}^2 \\
&\quad + \frac{C}{q}t^q[s^q + (1-s)^q]\int_{\mathbb{R}^N} w^q dx - \frac{t^p}{p}\max\{s^p, (1-s)^p\}\int_{\mathbb{R}^N} w^p dx \\
&\leq \frac{t^2}{2}\|w\|_{H^1}^2 + \frac{2C}{q}t^q\int_{\mathbb{R}^N} w^q dx - \frac{t^p}{p2^p}\int_{\mathbb{R}^N} w^p dx
\end{aligned} \tag{3.19}$$

for all $0 \leq s \leq 1$ and $e \in \mathbb{S}^{N-1}$, there exists $t_1 > 0$ such that, for any $t \geq t_1$,

$$J_0(t[sw_{e,l} + (1-s)w_{z_0,l}]) < 2\alpha^\infty \quad \text{for all } 0 \leq s \leq 1 \text{ and for all } e \in \mathbb{S}^{N-1}. \tag{3.20}$$

By (3.18) and (3.20), we only need to show that there exists $l_1 \geq \widetilde{l}$ such that, for $l > l_1$,

$$\sup_{(\frac{\sigma(s_0)}{s^{p-2}+(1-s)^{p-2}})^{1/(p-2)} \leq t \leq t_1} J_0(t[sw_{e,l} + (1-s)w_{z_0,l}]) < 2\alpha^\infty \quad \text{for all } e \in \mathbb{S}^{N-1}. \tag{3.21}$$

By Bahri-Li [6, Lemma 2.1], there exists $C_p > 0$, such that, for any nonnegative real numbers c, d,

$$(c+d)^p \geq c^p + d^p + p(c^{p-1}d + cd^{p-1}) - C_p c^{p/2} d^{p/2}.$$

Then, by (3.13), (3.6), (3.19) and Lemma 3.1,

$$
\begin{aligned}
&J_0(t[sw_{e,l}+(1-s)w_{z_0,l}]) \\
&\le \frac{t^2}{2}[s^2\|w\|_{H^1}^2+(1-s)^2\|w\|_{H^1}^2+2s(1-s)\langle w_{e,l},w_{z_0,l}\rangle] \\
&\quad + \frac{t^q}{q}\int_{\mathbb{R}^N} f_-[sw_{e,l}+(1-s)w_{z_0,l}]^q dx \\
&\quad - \frac{t^p}{p}\int_{\mathbb{R}^N} (sw_{e,l})^p+[(1-s)w_{z_0,l}]^p+p(sw_{e,l})^{p-1}((1-s)w_{z_0,l}) \\
&\quad + p(sw_{e,l})[(1-s)w_{z_0,l}]^{p-1}-C_p(sw_{e,l})^{p/2}[(1-s)w_{z_0,l}]^{p/2}dx \\
&\le 2\alpha^\infty - s(1-s)t^2[t^{p-2}(s^{p-2}+(1-s)^{p-2})-1]\int_{\mathbb{R}^N} w_{e,l}^{p-1}w_{z_0,l}dx \\
&\quad + \frac{t_1^q}{q}\int_{\mathbb{R}^N} f_-[sw_{e,l}+(1-s)w_{z_0,l}]^q dx + \frac{t_1^pC_p}{p}\int_{\mathbb{R}^N} w_{e,l}^{p/2}w_{z_0,l}^{p/2}dx \\
&\le 2\alpha^\infty - C_0^2[\sigma(s_0)-1]\int_{\mathbb{R}^N} w_{e,l}^{p-1}w_{z_0,l}dx \\
&\quad + \frac{t_1^q}{q}\int_{\mathbb{R}^N} f_-[sw_{e,l}+(1-s)w_{z_0,l}]^q dx + \frac{t_1^pC_p}{p}\int_{\mathbb{R}^N} w_{e,l}^{p/2}w_{z_0,l}^{p/2}dx
\end{aligned} \tag{3.22}
$$

for all $e\in\mathbb{S}^{N-1}$.

We first estimate $\int_{\mathbb{R}^N} w_{e,l}^{p-1}w_{z_0,l}dx$. Set

$$
\overline{C}_0 = \min_{x\in\overline{B^N(0,1)}} w^{p-1}(x) > 0,
$$

then by (3.1) and (3.2), for any $\varepsilon>0$,

$$
\begin{aligned}
\int_{\mathbb{R}^N} w_{e,l}^{p-1}w_{z_0,l}dx &= \int_{\mathbb{R}^N} w^{p-1}(x)w(x-l(z_0-e))dx \\
&\ge \overline{C}_0\int_{B^N(0,1)} w(x-l(z_0-e))dx \\
&\ge \overline{C}_0A_\varepsilon\int_{B^N(0,1)} \exp(-(1+\varepsilon)|x-l(z_0-e)|)dx \\
&\ge \overline{C}_0A_\varepsilon\int_{B^N(0,1)} \exp(-(1+\varepsilon)|x|-l(1+\varepsilon)|e-z_0|)dx \\
&\ge \overline{C}_0A_\varepsilon\exp(-l(1+\varepsilon)|e-z_0|) \\
&\ge \overline{C}_0A_\varepsilon\exp(-l(1+\varepsilon)(1+\delta_0)).
\end{aligned} \tag{3.23}
$$

From (3.2) we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} w_{e,l}^{p/2}w_{z_0,l}^{p/2}dx \\
&\le B_0^p\int_{\mathbb{R}^N}\exp(-\frac{p}{2}|x|)\exp(-\frac{p}{2}|x-l(z_0-e)|)dx \\
&\le B_0^p\int_{|x|<(1+\delta_0)l}\exp(-\frac{p}{2}(|x|+|x-l(z_0-e)|))dx \\
&\quad + B_0^p\int_{|x|\ge(1+\delta_0)l}\exp(-\frac{p}{2}(|x|+|x-l(z_0-e)|))dx
\end{aligned}
$$

$$
\begin{aligned}
&\le B_0^p l^N \int_{|x|<(1+\delta_0)} \exp(-\frac{p}{2} l(|x| + |x-(z_0-e)|))dx \\
&\quad + c_0 B_0^p \exp(-\frac{(1+\delta_0)pl}{2}) \int_{|x|\ge(1+\delta_0)l} \exp(-\frac{p}{2}(|x-l(e-z_0)|))dx \\
&\le c_0 B_0^p l^N \int_{|x|<(1+\delta_0)} \exp(-\frac{pl}{2}|e-z_0|)dx + \tilde{C} B_0^p \exp(-\frac{pl}{2}|e-z_0|) \\
&\le C_0 B_0^p l^N \exp(-\frac{pl}{2}|e-z_0|) \\
&\le C_0 B_0^p l^N \exp(-\min\{r_-, q, \frac{p}{2}\}(1-\delta_0)l) \quad \text{for } l \text{ sufficiently large.}
\end{aligned}
$$

By (3.15) and conditions (D1), (D2), we also have

$$
\begin{aligned}
&\int_{\mathbb{R}^N} f_-[sw_{e,l} + (1-s)w_{z_0,l}]^q dx \\
&\le \Big(\int_{\mathbb{R}^N} f_- w_{e,l}^q dx + \int_{\mathbb{R}^N} f_- w_{z_0,l}^q dx \Big) \\
&\le C_0 B_0^q l^N \exp(-\min\{r_-, q\} l) \\
&\le C_0 B_0^q l^N \exp(-\min\{r_-, q, \frac{p}{2}\}(1-\delta_0)l) \quad \text{for } l \ge 1.
\end{aligned} \tag{3.24}
$$

Since

$$
\begin{aligned}
1+\delta_0 &= 1 + \frac{\min\{r_-, q, \frac{p}{2}\} - 1}{2(\min\{r_-, q, \frac{p}{2}\} + 1)} \\
&< \min\{r_-, q, \frac{p}{2}\}\Big(1 - \frac{\min\{r_-, q, \frac{p}{2}\} - 1}{2(\min\{r_-, q, \frac{p}{2}\} + 1)}\Big) \\
&= \min\{r_-, q, \frac{p}{2}\}(1-\delta_0),
\end{aligned}
$$

we may take $0 < \varepsilon \ll 1$ such that

$$
(1+\varepsilon)(1+\delta_0) < \min\{r_-, q, \frac{p}{2}\}(1-\delta_0).
$$

Then, by (3.22)–(3.24), there exists $l_1 \ge \max\{\widetilde{l}, 1\}$ such that (3.21) holds. Thus, we can conclude that for any $l > l_1$,

$$
\sup_{t\ge 0} J_0(t[sw_{e,l} + (1-s)w_{z_0,l}]) < 2\alpha^\infty \quad \text{for all } 0 \le s \le 1 \text{ and for all } e \in \mathbb{S}^{N-1}.
$$

Moreover, by Lemma 2.4 (i), there is a unique $t_0(sw_{e,l} + (1-s)w_{z_0,l}) > 0$ such that

$$
t_0(sw_{e,l} + (1-s)w_{z_0,l})[sw_{e,l} + (1-s)w_{z_0,l}] \in \mathbf{N}_0.
$$

This completes the proof. □

Theorem 3.3. *Suppose that $\lambda = 0$. Then we have*

$$
\alpha_0 = \inf_{u\in\mathbf{N}_0} J_0(u) = \inf_{u\in\mathbf{N}^\infty} J^\infty(u) = \alpha^\infty.
$$

where $\alpha_0 = \alpha_\lambda$ with $\lambda = 0$. Furthermore, Equation (1.3) does not admit any ground state solutions.

Proof. Let $w_{e,l}$ be as in (3.3). Then, by Lemma 2.4 (i), there is a unique $t_0(w_{e,l}) > 0$ such that $t_0(w_{e,l})w_{e,l} \in \mathbf{N}_0$ for all $e \in \mathbb{S}^{N-1}$, that is

$$\|t_0(w_{e,l})w_{e,l}\|_{H^1}^2 = \int_{\mathbb{R}^N} |t_0(w_{e,l})w_{e,l}|^p dx + \int_{\mathbb{R}^N} f_-|t_0(w_{e,l})w_{e,l}|^q dx$$

or

$$|t_0(w_{e,l})|^2\|w_{e,l}\|_{H^1}^2 = |t_0(w_{e,l})|^p \int_{\mathbb{R}^N} |w_{e,l}|^p dx + |t_0(w_{e,l})|^q \int_{\mathbb{R}^N} f_-|w_{e,l}|^q dx \quad (3.25)$$

Since

$$\int_{\mathbb{R}^N} f_-|w_{e,l}|^q dx \to 0 \quad \text{as } l \to \infty, \quad (3.26)$$

and

$$\|w_{e,l}\|_{H^1}^2 = \int_{\mathbb{R}^N} |w_{e,l}|^p dx = \frac{2p}{p-2}\alpha^\infty \quad \text{for all } l \ge 0 \text{ and for all } e \in \mathbb{S}^{N-1}, \quad (3.27)$$

by (3.25), (3.26) and (3.27) we have $t_0(w_{e,l}) \to 1$ as $l \to \infty$. Thus,

$$\lim_{l\to\infty} J_0(t_0(w_{e,l})w_{e,l}) = \lim_{l\to\infty} J^\infty(t_0(w_{e,l})w_{e,l}) = \alpha^\infty \quad \text{for all } e \in \mathbb{S}^{N-1}.$$

Then

$$\alpha_0 = \inf_{u\in\mathbf{N}_0} J_0(u) \le \inf_{u\in\mathbf{N}^\infty} J^\infty(u) = \alpha^\infty.$$

Let $u \in \mathbf{N}_0$. Then, by Lemma 2.4, $J_0(u) = \sup_{t\ge0} J_0(tu)$. Moreover, there is a unique $t^\infty > 0$ such that $t^\infty u \in \mathbf{N}^\infty$. Thus,

$$J_0(u) \ge J_0(t^\infty u) \ge J^\infty(t^\infty u) \ge \alpha^\infty$$

and so $\alpha_0 \ge \alpha^\infty$. Therefore,

$$\alpha_0 = \inf_{u\in\mathbf{N}_0} J_0(u) = \inf_{u\in\mathbf{N}^\infty} J^\infty(u) = \alpha^\infty.$$

Next, we will show that for $\lambda = 0$, Equation (1.3) does not admit any solution u_0 such that $J_0(u_0) = \alpha_0$. Suppose the contrary. Then we can assume that $u_0 \in \mathbf{N}_0$ such that $J_0(u_0) = \alpha_0$. Then, by Lemma 2.4 (i), $J_0(u_0) = \sup_{t\ge0} J_0(tu_0)$. Moreover, there is a unique $t^\infty(u_0) > 0$ such that $t^\infty(u_0)u_0 \in \mathbf{N}^\infty$. Thus,

$$\begin{aligned}
\alpha^\infty = \inf_{u\in\mathbf{N}_0} J_0(u) &= J_0(u_0) \ge J_0(t^\infty(u_0)u_0) \\
&= J^\infty(t^\infty(u_0)u_0) - \frac{[t^\infty(u_0)]^q}{q} \int_{\mathbb{R}^N} f_0|u_0|^q dx \\
&\ge \alpha^\infty - \frac{[t^\infty(u_0)]^q}{q} \int_{\mathbb{R}^N} f_0|u_0|^q dx,
\end{aligned}$$

which implies that $\int_{\mathbb{R}^N} f_-|u_0|^q dx = 0$ and so

$$u_0 \equiv 0 \quad \text{in } \{x \in \mathbb{R}^N : f_-(x) \ne 0\}, \quad (3.28)$$

form conditions (D1) and (D2). Therefore,

$$\alpha^\infty = \inf_{u\in\mathbf{N}^\infty} J^\infty(u) = J^\infty(t^\infty(u_0)u_0).$$

Since $|t^\infty(u_0)u_0| \in \mathbf{N}^\infty$ and $J^\infty(|t^\infty(u_0)u_0|) = J^\infty(t^\infty(u_0)u_0) = \alpha^\infty$, By Willem [31, Theorem 4.3] and the maximum principle, we can assume that $t^\infty(u_0)u_0$ is a positive solution of Equation (2.5). This contradicts to (3.28). This completes the proof. □

4. Existence of a positive solution

First, we establish the existence of positive ground state solutions of Equation (1.3) for $\lambda > 0$

Theorem 4.1. *For each $\lambda > 0$, Equation* (1.3) *has a positive ground state solution u_λ such that*
$$J_\lambda(u_\lambda) = \inf_{u\in \mathbf{N}_\lambda} J_\lambda(u) < \alpha^\infty.$$

Proof. By analogy with the proof of Ni and Takagi [28], one can show that by the Ekeland variational principle (see [18]), there exists a minimizing sequence $\{u_n\} \subset \mathbf{N}_\lambda$ such that
$$J_\lambda(u_n) = \inf_{u\in \mathbf{N}_\lambda} J_\lambda(u) + o(1), \quad J'_\lambda(u_n) = o(1) \text{ in } H^{-1}(\mathbb{R}^N).$$
Since $\inf_{u\in \mathbf{N}_\lambda} J_\lambda(u) < \alpha^\infty$ from Proposition 3.2 (i) and Corollary 2.6 there exists a subsequence $\{u_n\}$ and $u_\lambda \in \mathbf{N}_\lambda$, a nonzero solution of Equation (1.3), such that
$$u_n \to u_\lambda \quad \text{strongly in } H^1(\mathbb{R}^N) \text{ and } J_\lambda(u_\lambda) = \inf_{u\in \mathbf{N}_\lambda} J_\lambda(u).$$
Since $J_\lambda(u_\lambda) = J_\lambda(|u_\lambda|)$ and $|u_\lambda| \in \mathbf{N}_\lambda$, by Lemma 2.3 and the maximum principle, we obtain $u_\lambda > 0$ in $\mathbb{R}^N$. This completes the proof. □

By Theorem 3.3, for $\lambda = 0$, Equation (1.3) does not admit any solution u_0 such that $J_0(u_0) = \inf_{u\in \mathbf{N}_0} J_0(u)$ and
$$\alpha_0 = \inf_{u\in \mathbf{N}_0} J_0(u) = \inf_{u\in \mathbf{N}^\infty} J^\infty(u) = \alpha^\infty.$$
Moreover, we have the following result.

Lemma 4.2. *Assume that $\lambda = 0$ and $\{u_n\}$ is a minimizing sequence for J_0 in $\mathbf{N}_0$. Then*
$$\int_{\mathbb{R}^N} f_0|u_n|^q dx = o(1).$$
Furthermore, $\{u_n\}$ is a $(PS)_{\alpha^\infty}$-sequence for J^∞ in $H^1(\mathbb{R}^N)$.

Proof. For each n, there is a unique $t_n > 0$ such that $t_n u_n \in \mathbf{N}^\infty$; that is,
$$t_n^2\|u_n\|_{H^1}^2 = t_n^p \int_{\mathbb{R}^N} |u_n|^p dx.$$
Then, by Lemma 2.4 (i),
$$\begin{aligned} J_0(u_n) \geq J_0(t_n u_n) &= J^\infty(t_n u_n) + \frac{t_n^q}{q} \int_{\mathbb{R}^N} f_-|u_n|^q dx \\ &\geq \alpha^\infty + \frac{t_n^q}{q} \int_{\mathbb{R}^N} f_-|u_n|^q dx. \end{aligned}$$
Since $J_0(u_n) = \alpha^\infty + o(1)$ from Theorem 3.3, we have
$$\frac{t_n^q}{q} \int_{\mathbb{R}^N} f_-|u_n|^q dx = o(1).$$
We will show that there exists $c_0 > 0$ such that $t_n > c_0$ for all n. Suppose the contrary. Then we may assume $t_n \to 0$ as $n \to \infty$. Since $J_0(u_n) = \alpha^\infty + o(1)$,

by Lemma 2.2, we have $\|u_n\|$ is uniformly bounded and so $\|t_n u_n\|_{H^1} \to 0$ or $J^\infty(t_n u_n) \to 0$, and this contradicts the fact that $J^\infty(t_n u_n) \geq \alpha^\infty > 0$. Thus,

$$\int_{\mathbb{R}^N} f_-|u_n|^q dx = o(1),$$

which implies that

$$\|u_n\|_{H^1}^2 = \int_{\mathbb{R}^N} |u_n|^p dx + o(1)$$

and

$$J^\infty(u_n) = \alpha^\infty + o(1).$$

Moreover, by Wang and Wu [30, Lemma 7], we have $\{u_n\}$ is a $(\mathrm{PS})_{\alpha^\infty}$-sequence for J^∞ in $H^1(\mathbb{R}^N)$. $\square$

For $u \in H^1(\mathbb{R}^N)$, we define the center mass function from $\mathbf{N}_\lambda$ to the unit ball $B^N(0,1)$ in $\mathbb{R}^N$,

$$m(u) = \frac{1}{\|u\|_{L^p(\mathbb{R}^N)}^p} \int_{\mathbb{R}^N} \frac{x}{|x|} |u(x)|^p dx.$$

Clearly, m is continuous from $\mathbf{N}_\lambda$ to $B^N(0,1)$ and $|m(u)| < 1$. Let

$$\theta_\lambda = \inf\{J_\lambda(u) : u \in \mathbf{N}_\lambda,\ u \geq 0,\ m(u) = 0\}.$$

Note that $\theta_0 = \theta_\lambda$ with $\lambda = 0$. Then we have the following result.

Lemma 4.3. *Suppose that $\lambda = 0$. Then there exists $\xi_0 > 0$ such that $\alpha^\infty < \xi_0 \leq \theta_0$.*

Proof. Suppose the contrary. Then there exists a sequence $\{u_n\} \subset \mathbf{N}_0$ and $m(u_n) = 0$ for each n, such that $J_0(u) = \alpha^\infty + o(1)$. By Lemma 4.2, $\{u_n\}$ is a $(\mathrm{PS})_{\alpha^\infty}$-sequence in $H^1(\mathbb{R}^N)$ for J^∞. By the concentration-compactness principle (see Lions [26, 27]) and the fact that $\alpha^\infty > 0$, there exist a subsequence $\{u_n\}$, a sequence $\{x_n\} \subset \mathbb{R}^N$, and a positive solution $w \in H^1(\mathbb{R}^N)$ of Equation (2.5) such that

$$\|u_n(x) - w(x - x_n)\|_{H^1} \to 0 \quad \text{as } n \to \infty. \tag{4.1}$$

Now we will show that $|x_n| \to \infty$ as $n \to \infty$. Suppose the contrary. Then we may assume that $\{x_n\}$ is bounded and $x_n \to x_0$ for some $x_0 \in \mathbb{R}^N$. Thus, by (4.1),

$$\begin{aligned}\int_{\mathbb{R}^N} f_-|u_n|^q dx &= \int_{\mathbb{R}^N} f_-(x)|w(x - x_n)|^q dx + o(1) \\ &= \int_{\mathbb{R}^N} f_-(x + x_0)|w(x)|^q dx + o(1),\end{aligned}$$

this contradicts the result of Lemma 4.2: $\int_{\mathbb{R}^N} f_-|u_n|^q dx = o(1)$. Hence we may assume that $\frac{x_n}{|x_n|} \to e$ as $n \to \infty$, where $e \in \mathbb{S}^{N-1}$. Then, by (4.1) and the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}0 &= m(u_n) \\ &= \|u_n\|_{L^p(\mathbb{R}^N)}^{-p} \int_{\mathbb{R}^N} \frac{x}{|x|} |u_n(x)|^p dx \\ &= \|w\|_{L^p(\mathbb{R}^N)}^{-p} \int_{\mathbb{R}^N} \frac{x + x_n}{|x + x_n|} |w(x)|^p dx + o(1) \\ &= e + o(1) \quad \text{as } n \to \infty,\end{aligned}$$

which is a contradiction. Therefore, there exists $\xi_0 > 0$ such that $\alpha^\infty < \xi_0 \leq \theta_0$. $\square$

By Lemma 2.4 and Proposition 3.2, if $\lambda = 0$, for each $e \in \mathbb{S}^{N-1}$ and $l > l_1$ there exists $t_0(w_{e,l}) > 0$ such that $t_0(w_{e,l})w_{e,l} \in \mathbf{N}_0$. Moreover, we have the following result.

Lemma 4.4. *Suppose that $\lambda = 0$. Then there exists $l_0 \geq l_1$ such that, for any $l \geq l_0$*

(i) $\alpha^\infty < J_0(t_0(w_{e,l})w_{e,l}) < \xi_0$ *for all* $e \in \mathbb{S}^{N-1}$

(ii) $\langle m(t_0(w_{e,l})w_{e,l}), e\rangle > 0$, *for all* $e \in \mathbb{S}^{N-1}$.

Proof. (i) Follows from (3.13)–(3.15) and Theorem 3.3.

(ii) For $x \in \mathbb{R}^N$ with $x + le \neq 0$, we have

$$\begin{aligned}(\frac{x+le}{|x+le|}, le) &= |x+le| - \frac{1}{|x+le|}(x+le, x)\\ &\geq |x+le| - |x| \geq l|e| - 2|x| = l - 2|x|.\end{aligned}$$

Then

$$\begin{aligned}\langle m(t_0(w_{e,l})w_{e,l}), e\rangle &= \frac{1}{l\|w_{e,l}\|^p_{L^p(\mathbb{R}^N)}} \int_{\mathbb{R}^N} (\frac{x}{|x|}, le)|w_{e,l}|^p dx\\ &= \frac{1}{l\|w\|^p_{L^p(\mathbb{R}^N)}} \int_{\mathbb{R}^N} (\frac{x+le}{|x+le|}, le)|w|^p dx\\ &\geq \frac{1}{l\|w\|^p_{L^p(\mathbb{R}^N)}} \Big(l\int_{\mathbb{R}^N} |w|^p dx - 2\int_{\mathbb{R}^N} |x||w|^p dx\Big)\\ &= 1 - \frac{2c_0}{l},\end{aligned}$$

where $c_0 = \|w\|^{-p}_{L^p(\mathbb{R}^N)} \int_{\mathbb{R}^N} |x||w|^p dx$. Thus, there exists $l_0 \geq l_1$ such that

$$\langle m(t_0(w_{e,l})w_{e,l}), e\rangle \geq 1 - \frac{2c_0}{l} > 0 \quad \text{for all } l \geq l_0.$$

This completes the proof. □

In the following, we will use Bahri-Li's minimax argument [6]. Let

$$\mathbb{B} = \{u \in H^1(\mathbb{R}^N)\backslash\{0\} : u \geq 0 \text{ and } \|u\|_{H^1} = 1\}.$$

We define

$$I_0(u) = \sup_{t\geq 0} J_0(tu) : \mathbb{B} \to \mathbb{R}.$$

Then, by Lemma 2.4 (iii), for each $u \in H^1(\mathbb{R}^N)\backslash\{0\}$ there exists

$$t_0(u) = \frac{1}{\|u\|_{H^1}} t_0(\frac{u}{\|u\|_{H^1}}) > 0$$

such that $t_0(u)u \in \mathbf{N}_0$ and

$$I_0(u) = J_0(t_0(u)u) = J_0\Big(t_0(\frac{u}{\|u\|_{H^1}})\frac{u}{\|u\|_{H^1}}\Big) \tag{4.2}$$

Next, we define a map h_0 from $\mathbb{S}^{N-1}$ to $\mathbb{B}$ by

$$h_0(e) = \frac{w(x-le)}{\|w(x-le)\|_{H^1}} = \frac{w_{e,l}}{\|w_{e,l}\|_{H^1}},$$

where $e \in \mathbb{S}^{N-1}$. Then, by (3.17) and (4.2), for $l > l_0$ sufficiently large, we have

$$I_0(h_0(e)) = J_0(t_0(w_{e,l})w_{e,l}) < \theta_0 \quad \text{for all } e \in \mathbb{S}^{N-1}.$$

We define another map h^* from $\overline{B^N(0,1)}$ to $\mathbb{B}$ by

$$h^*(se+(1-s)z_0) = \frac{sw_{e,l}+(1-s)w_{z_0,l}}{\|sw_{e,l}+(1-s)w_{z_0,l}\|_{H^1}}$$

where $0 \le s \le 1$ and $e \in \mathbb{S}^{N-1}$. It is clear that $h^*|_{\mathbb{S}^{N-1}} = h_0$. It follows from Proposition 3.2 (ii) and (4.2) that

$$\begin{aligned} I_0(h^*(se+(1-s)z_0)) &= J_0(t_0(sw_{e,l}+(1-s)w_{z_0,l})[sw_{e,l}+(1-s)w_{z_0,l}]) \\ &< 2\alpha^\infty \end{aligned} \tag{4.3}$$

for all $e \in \mathbb{S}^{N-1}$. We next define a min-max value. Let

$$\beta_0 = \inf_{\gamma\in\Gamma} \max_{x\in\overline{B^N(0,1)}} I_0(\gamma(x)) \tag{4.4}$$

where

$$\Gamma = \{\gamma \in C(\overline{B^N(0,1)}, \mathbb{B}) : \gamma|_{\mathbb{S}^{N-1}} = h_0\}. \tag{4.5}$$

Note that $\mathbb{S}^{N-1} = \partial B^N(0,1)$. Then we have the following result.

Lemma 4.5. *Suppose that $\lambda = 0$. Then*

$$\alpha^\infty < \xi_0 \le \theta_0 \le \beta_0 < 2\alpha^\infty.$$

Proof. By Lemmas 4.3 and 4.4, and by (4.3) and (4.2), we only need to show that $\theta_0 \le \beta_0$. For any $\gamma \in \Gamma$, there exists $t_0(\gamma(x)) > 0$ such that $t_0(\gamma(x))\gamma(x) \in \mathbf{N}_0$ and

$$t_0(\gamma(x))\gamma(x) = t_0(w_{x,l})w_{x,l} \text{ for all } x \in \mathbb{S}^{N-1}.$$

Consider the homotopy $H(s,x) : [0,1] \times B^N(0,1) \to \mathbb{R}$ defined by

$$H(s,x) = (1-s)m(t_0(\gamma(x))\gamma(x)) + sI(x),$$

where I denotes the identity map. Note that $m(t_0(\gamma(x))\gamma(x)) = m(t_0(w_{x,l})w_{x,l})$ for all $x \in \mathbb{S}$. By Lemma 4.4 (ii), $H(s,x) \ne 0$ for $x \in \mathbb{S}^{N-1}$ and $s \in [0,1]$. Therefore,

$$\deg(m(t_0(\gamma)\gamma), B^N(0,1), 0) = \deg(I, B^N(0,1), 0) = 1.$$

There exists $x_0 \in B^N(0,1)$ such that

$$m(t_0(\gamma(x_0))\gamma(x_0)) = 0.$$

Hence, for each $\gamma \in \Gamma$, we have

$$\begin{aligned} \theta_0 &= \inf\{J_0(u) : u \in \mathbf{N}_0,\ u \ge 0,\ m(u) = 0\} \\ &\le I_0(\gamma(x_0)) \\ &\le \max_{x\in\overline{B^N(0,1)}} I_0(\gamma(x)). \end{aligned}$$

This shows that $\theta_0 \le \beta_0$. $\square$

Now, we assert that Equation (1.3) has a positive higher energy solution for $\lambda \le 0$.

Theorem 4.6. *Suppose that $\lambda = 0$. Then Equation (1.3) has a positive solution $\widetilde{u}_0$ such that $J_0(\widetilde{u}_0) = \beta_0 > \alpha^\infty$.*

Proof. By Lemma 4.5 and the minimax principle (see Ambrosetti and Rabinowitz [4]), there exists a sequence $\{u_n\} \subset \mathbb{B}$ such that

$$I_0(u_n) = \beta_0 + o(1),$$
$$\|I_0'(u_n)\|_{T^*_{u_n}\mathbb{B}} \equiv \sup\{I_0'(u_n)\phi : \phi \in T_{u_n}\mathbb{B}, \|\phi\|_{H^1} = 1\} = o(1)$$

as $n \to \infty$, where $\alpha^\infty < \beta_0 < 2\alpha^\infty$ and $T_{u_n}\mathbb{B} = \{\phi \in H^1(\mathbb{R}^N) : \langle \phi, u_n \rangle = 0\}$. By an argument similar to the proof of Adachi and Tanaka [1, Proposition 1.7], there exists $t_0(u_n) > 0$ such that $t_0(u_n)u_n \in \mathbf{N}_0$ and

$$J_0(t_0(u_n)u_n) = \beta_0 + o(1),$$
$$J_0'(t_0(u_n)u_n) = o(1) \quad \text{in } H^{-1}(\mathbb{R}^N), \text{ as } n \to \infty.$$

Thus, by Corollary 2.6, we can conclude that Equation (1.3) has a positive solution $\widetilde{u}_0$ such that $J_0(\widetilde{u}_0) = \beta_0$. □

5. Existence of two positive solutions

We need the following result.

Lemma 5.1. *Suppose that $\lambda = 0$. Then there exists $d_0 > 0$ such that if $u \in \mathbf{N}_0$ and $J_0(u) \leq \alpha^\infty + d_0$, then*

$$\int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla u|^2 + u^2)dx \neq 0,$$

where $\mathbf{N}_0 = \mathbf{N}_\lambda$ and $J_0 = J_\lambda$ with $\lambda = 0$.

Proof. Suppose the contrary. Then there exists a sequence $\{u_n\} \subset \mathbf{N}_0$ such that $J_0(u_n) = \alpha^\infty + o(1)$ and

$$\int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla u_n|^2 + u_n^2)dx = 0.$$

Moreover, by Lemma 4.2, $\{u_n\}$ is a $(\mathrm{PS})_{\alpha^\infty}$-sequence in $H^1(\mathbb{R}^N)$ for J^∞. By the concentration-compactness principle (see Lions [26, 27]) and the fact that $\alpha^\infty > 0$, there exist a subsequence $\{u_n\}$, a sequence $\{x_n\} \subset \mathbb{R}^N$, and a positive solution $w \in H^1(\mathbb{R}^N)$ of Equation (2.5) such that

$$\|u_n(x) - w(x - x_n)\|_{H^1} \to 0 \quad \text{as } n \to \infty. \tag{5.1}$$

Now we will show that $|x_n| \to \infty$ as $n \to \infty$. Suppose the contrary. Then we may assume that $\{x_n\}$ is bounded and $x_n \to x_0$ for some $x_0 \in \mathbb{R}^N$. Thus, by (5.1),

$$\begin{aligned}\int_{\mathbb{R}^N} f_-|u_n|^q dx &= \int_{\mathbb{R}^N} f_-(x)|w(x - x_n)|^q dx + o(1)\\ &= \int_{\mathbb{R}^N} f_-(x + x_0)|w(x)|^q dx + o(1),\end{aligned}$$

which contradicts the result of Lemma 4.2: $\int_{\mathbb{R}^N} f_-|u_n|^q dx = o(1)$. Hence we may assume $\frac{x_n}{|x_n|} \to e_0$ as $n \to \infty$, where $e_0 \in \mathbb{S}^{N-1}$. Then, by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}0 &= \int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla u_n|^2 + u_n^2)dx = \int_{\mathbb{R}^N} \frac{x + x_n}{|x + x_n|}(|\nabla w|^2 + w^2)dx + o(1)\\ &= \frac{2p}{p-2}\alpha^\infty e_0 + o(1),\end{aligned}$$

which is a contradiction. This completes the proof. □

For $\lambda > 0$ and $u \in \mathbf{N}_\lambda$, by Lemma 2.4, there is a unique $t_0(u) > 0$ such that $t_0(u)u \in \mathbf{N}_0$ where $\mathbf{N}_0 = \mathbf{N}_\lambda$ with $\lambda = 0$. Moreover, we have the following result.

Lemma 5.2. *There exists a continuous function $\Lambda : [0,\infty) \to [0, S_p^{p/(p-2)})$ with $\Lambda(0) = 0$ such that*

$$t_0(u) \le [1 + \lambda \|f_+\|_{L^{p/(p-q)}}^{p/(p-q)} (S_p^{p/(p-2)} - \Lambda(\lambda))^{(q-p)/p}]^{1/(p-q)}$$

for all $\lambda > 0$ and $u \in \mathbf{N}_\lambda$, where S_p be the constant for the Sobolev embedding from H^1 to L^p.

Proof. Let $u \in \mathbf{N}_\lambda$. Then we have

$$\begin{aligned} S_p\Big(\int_{\mathbb{R}^N} |u|^p dx\Big)^{2/p} &\le \|u\|_{H^1}^2 = \int_{\mathbb{R}^N} |u|^p dx + \int_{\mathbb{R}^N} f_\lambda |u|^q dx \\ &\le \int_{\mathbb{R}^N} |u|^p dx + \lambda \int_{\mathbb{R}^N} f_+ |u|^q dx \\ &\le \int_{\mathbb{R}^N} |u|^p dx + \lambda \|f_+\|_{L^{p/(p-q)}}^{p/(p-q)} \Big(\int_{\mathbb{R}^N} |u|^p dx\Big)^{q/p}, \end{aligned}$$

which implies that there exists a continuous function $\Lambda : [0,\infty) \to [0, S_p^{p/(p-2)})$ with $\Lambda(0) = 0$ such that

$$\int_{\mathbb{R}^N} |u|^p dx \ge S_p^{p/(p-2)} - \Lambda(\lambda) > 0. \tag{5.2}$$

We distinguish two cases.

Case (A): $t_0(u) < 1$. Since

$$1 + \lambda \|f_+\|_{L^{p/(p-q)}}^{p/(p-q)} (S_p^{p/(p-2)} - \Lambda(\lambda))^{(q-p)/p} \ge 1$$

for all $\lambda \ge 0$ and $p - q > 0$, we have

$$t_0(u) < 1 \le \big[1 + \lambda \|f_+\|_{L^{p/(p-q)}}^{p/(p-q)} (S_p^{p/(p-2)} - \Lambda(\lambda))^{(q-p)/p}\big]^{1/(p-q)}.$$

Case (B): $t_0(u) \ge 1$. Since

$$\begin{aligned} [t_0(u)]^p \int_{\mathbb{R}^N} |u|^p dx &= [t_0(u)]^2 \|u\|_{H^1}^2 + [t_0(u)]^q \int_{\mathbb{R}^N} f_- |u|^q dx \\ &\le [t_0(u)]^q \Big(\|u\|_{H^1}^2 + \int_{\mathbb{R}^N} f_- |u|^q dx\Big), \end{aligned}$$

by (5.2), we have

$$\begin{aligned} [t_0(u)]^{p-q} &\le \frac{\|u\|_{H^1}^2 + \int_{\mathbb{R}^N} f_- |u|^q dx}{\int_{\mathbb{R}^N} |u|^p dx} \\ &= \frac{\int_{\mathbb{R}^N} |u|^p dx + \int_{\mathbb{R}^N} f_\lambda |u|^q dx + \int_{\mathbb{R}^N} f_- |u|^q dx}{\int_{\mathbb{R}^N} |u|^p dx} \\ &= \frac{\int_{\mathbb{R}^N} |u|^p dx + \lambda \int_{\mathbb{R}^N} f_+ |u|^q dx}{\int_{\mathbb{R}^N} |u|^p dx} \\ &= 1 + \lambda \frac{\int_{\mathbb{R}^N} f_+ |u|^q dx}{\int_{\mathbb{R}^N} |u|^p dx} \end{aligned}$$

$$\leq 1+\lambda\|f_+\|_{L^{p/(p-q)}}^{p/(p-q)}\Big(\int_{\mathbb{R}^N}|u|^p dx\Big)^{(q-p)/p}$$
$$\leq 1+\lambda\|f_+\|_{L^{p/(p-q)}}^{p/(p-q)}\Big(S_p^{\frac{p}{p-2}}-\Lambda(\lambda)\Big)^{(q-p)/p}.$$

This completes the proof. □

By the proof of Proposition 3.2, there exist positive numbers $t_\lambda(w_{e,l})$ and $\widehat{l_1}$ such that $t(w_{e,l})w_{e,l}\in\mathbf{N}_\lambda$ and

$$J_\lambda(t_\lambda(w_{e,l})w_{e,l})<\alpha^\infty\quad\text{for all } l>\widehat{l_1}.$$

Let $\Lambda(\lambda)$ be as in Lemma 5.2. Then we have the following result.

Lemma 5.3. *There exists a positive number λ_0 such that for every $\lambda\in(0,\lambda_0)$, we have*

$$\int_{\mathbb{R}^N}\frac{x}{|x|}\big(|\nabla u|^2+u^2\big)dx\neq 0$$

for all $u\in\mathbf{N}_\lambda$ with $J_\lambda(u)<\alpha^\infty$.

Proof. (i) Let $u\in\mathbf{N}_\lambda$ with $J_\lambda(u)<\alpha^\infty$. Then, by Lemma 2.4, there exists $t_0(u)>0$ such that $t_0(u)u\in\mathbf{N}_0$. Moreover,

$$\begin{aligned}J_\lambda(u)&=\sup_{t\geq 0}J_\lambda(tu)\geq J_\lambda(t_0(u)u)\\&=J_0(t_0(u)u)-\lambda[t_0(u)]^q\int_{\mathbb{R}^N}f_+|u|^q dx.\end{aligned}$$

Thus, by Lemma 5.2 and the Hölder inequality,

$$\begin{aligned}&J_0(t_0(u)u)\\&\leq J_\lambda(u)+\lambda[t_0(u)]^q\int_{\mathbb{R}^N}f_+|u|^q dx\\&<\alpha^\infty+\lambda c_0[1+\lambda\|f_+\|_{L^{p/(p-q)}}^{p/(p-q)}(S_p^{p/(p-2)}-\Lambda(\lambda))^{(q-p)/p}]^{q/(p-q)}\|u\|_{H^1}^q\end{aligned}\tag{5.3}$$

for some $c_0>0$. Moreover, by (2.1),

$$\alpha^\infty>J_\lambda(u)\geq\frac{q-2}{2q}\|u\|_{H^1}^2,$$

which implies

$$\|u\|_{H^1}<(\frac{2q\alpha^\infty}{q-2})^{1/2}\tag{5.4}$$

for all $u\in\mathbf{N}_\lambda$ with $J_\lambda(u)<\alpha^\infty$. Therefore, by (5.3) and (5.4),

$$\begin{aligned}&J_0(t_0(u)u)\\&<\alpha^\infty\lambda c_0[1+\lambda\|f_+\|_{L^{p/(p-q)}}^{p/(p-q)}(S_p^{p/(p-2)}-\Lambda(\lambda))^{(q-p)/p}]^{q/(p-q)}(\frac{2q\alpha^\infty}{q-2})^{q/2}.\end{aligned}$$

Let $d_0>0$ be as in Lemma 5.1. Then there exists a positive number λ_0 such that for $\lambda\in(0,\lambda_0)$,

$$J_0(t_0(u)u)<\alpha^\infty+d_0.\tag{5.5}$$

Since $t_0(u)u\in\mathbf{N}_0$ and $t_0(u)>0$, by Lemma 5.1 and (5.5),

$$\int_{\mathbb{R}^N}\frac{x}{|x|}(|\nabla(t_0(u)u)|^2+(t_0(u)u)^2)dx\neq 0,$$

which implies that there exists a positive number λ_0 such that for every $\lambda \in (0, \lambda_0)$,

$$\int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla u|^2 + u^2)dx \neq 0$$

for all $u \in \mathbf{N}_\lambda$ with $J_\lambda(u) < \alpha^\infty$. □

In the following, we use an idea by Adachi and Tanaka [1]. For $c \in \mathbb{R}^+$, we define

$$[J_\lambda \leq c] = \{u \in \mathbf{N}_\lambda : u \geq 0, J_\lambda(u) \leq c\}.$$

We then try to show that for a sufficiently small $\sigma > 0$,

$$\operatorname{cat}([J_\lambda \leq \alpha^\infty - \sigma]) \geq 2. \tag{5.6}$$

To prove (5.6), we need some preliminaries. Recall the definition of the Lusternik-Schnirelman category.

Definition 5.4. (i) For a topological space X, we say that a non-empty, closed subset $Y \subset X$ is contractible to a point in X if and only if there exists a continuous mapping $\xi : [0,1] \times Y \to X$ such that, for some $x_0 \in X$

$$\xi(0,x) = x \quad \text{for all } x \in Y,$$
$$\xi(1,x) = x_0 \quad \text{for all } x \in Y.$$

(ii) We define

$$\operatorname{cat}(X) = \min\big\{k \in \mathbb{N} : \text{there exist closed subsets } Y_1, \dots, Y_k \subset X \text{ such that } Y_j \text{ is contractible to a point in } X \text{ for all } j \text{ and } \cup_{j=1}^k Y_j = X\big\}.$$

When there do not exist finitely many closed subsets $Y_1, \dots, Y_k \subset X$ such that Y_j is contractible to a point in X for all j and $\cup_{j=1}^k Y_j = X$, we say that $\operatorname{cat}(X) = \infty$.

We need the following two lemmas.

Lemma 5.5. *Suppose that X is a Hilbert manifold and $F \in C^1(X, \mathbb{R})$. Assume that there exist $c_0 \in \mathbb{R}$ and $k \in \mathbb{N}$ such that*

(i) *F satisfies the Palais-Smale condition for energy levels $c \leq c_0$;*
(ii) $\operatorname{cat}(\{x \in X : F(x) \leq c_0\}) \geq k$

Then F has at least k critical points in $\{x \in X : F(x) \leq c_0\}$.

For a proof of the above lemma see Ambrosetti [5, Theorem 2.3]. We have the following results.

Lemma 5.6. *Let X be a topological space. Suppose that there are two continuous maps*

$$\Phi : \mathbb{S}^{N-1} \to X, \quad \Psi : X \to \mathbb{S}^{N-1}$$

such that $\Psi \circ \Phi$ is homotopic to the identity map of $\mathbb{S}^{N-1}$; that is, there exists a continuous map $\zeta : [0,1] \times \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}$ such that

$$\zeta(0,x) = (\Psi \circ \Phi)(x) \quad \text{for each } x \in \mathbb{S}^{N-1},$$
$$\zeta(1,x) = x \quad \text{for each } x \in \mathbb{S}^{N-1}.$$

Then $\operatorname{cat}(X) \geq 2$.

For a proof of the above lemma see Adachi and Tanaka [1, Lemma 2.5].

For $l > \widehat{l}_1$, we define a map $\Phi_{\lambda,l} : \mathbb{S}^{N-1} \to H^1(\mathbb{R}^N)$ by

$$\Phi_{\lambda,l}(e) = t_\lambda(w_{e,l})(w_{e,l}) \quad \text{for } e \in \mathbb{S}^{N-1},$$

where $t_\lambda(w_{e,l})(w_{e,l})$ is as in the proof of Proposition 3.2. Then we have the following result.

Lemma 5.7. *There exists a sequence $\{\sigma_l\} \subset \mathbb{R}^+$ with $\sigma_l \to 0$ as $l \to \infty$ such that*

$$\Phi_{\lambda,l}(\mathbb{S}^{(N-1)}) \subset [J_\lambda \le \alpha^\infty - \sigma_l].$$

Proof. By Proposition 3.2, for each $l > \widehat{l}_1$ we have $t_\lambda(w_{e,l})(w_{e,l}) \in \mathbf{N}_\lambda$ and

$$\sup_{l > \widehat{l}_1} J_\lambda(t_\lambda(w_{e,l})(w_{e,l})) < \alpha^\infty \quad \text{for all } e \in \mathbb{S}^{N-1}.$$

Since $\Phi_{\lambda,l}(\mathbb{S}^{N-1})$ is compact,

$$J_\lambda(t_\lambda(w_{e,l})(w_{e,l})) \le \alpha^\infty - \sigma_l,$$

so the conclusion follows. □

From Lemma 5.3, for $\lambda \in (0, \lambda_0)$, we define $\Psi_\lambda : [J_\lambda < \alpha^\infty] \to \mathbb{S}^{N-1}$ by

$$\Psi_\lambda(u) = \frac{\int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla u|^2 + u^2)dx}{|\int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla u|^2 + u^2)dx|}.$$

Then we have the following results.

Lemma 5.8. *Let $\lambda_0 > 0$ be as in Lemma 5.3. Then for each $\lambda \in (0, \lambda_0)$ there exists $\widehat{l}_0 \ge \widehat{l}_1$ such that for $l > \widehat{l}_0$, the map*

$$\Psi_\lambda \circ \Phi_{\lambda,l} : \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}$$

is homotopic to the identity.

Proof. Let $\Sigma = \{u \in H^1(\mathbb{R}^N)\backslash\{0\} : \int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla u|^2 + u^2)dx \ne 0\}$. We define $\overline{\Psi}_\lambda : \Sigma \to \mathbb{S}^{N-1}$ by

$$\overline{\Psi}_\lambda(u) = \frac{\int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla u|^2 + u^2)dx}{|\int_{\mathbb{R}^N} \frac{x}{|x|}(|\nabla u|^2 + u^2)dx|},$$

an extension of Ψ_λ. Since $w_{e,l} \in \Sigma$ for all $e \in \mathbb{S}^{N-1}$ and for l sufficiently large, we let $\gamma : [s_1, s_2] \to \mathbb{S}^{N-1}$ be a regular geodesic between $\overline{\Psi}_\lambda(w_{e,l})$ and $\overline{\Psi}_\lambda(\Phi_{\lambda,l}(e))$ such that $\gamma(s_1) = \overline{\Psi}_\lambda(w_{e,l}), \gamma(s_2) = \overline{\Psi}_\lambda(\Phi_{\lambda,l}(e))$. By an argument similar to that in Lemma 5.1, there exists a positive number $\widehat{l}_0 \ge \widehat{l}_1$ such that, for $l > \widehat{l}_0$,

$$w_{\frac{e}{2(1-\theta)},l} \in \Sigma \quad \text{for all } e \in \mathbb{S}^{N-1} \text{ and } \theta \in [1/2, 1).$$

We define $\zeta_l(\theta, e) : [0,1] \times \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}$ by

$$\zeta_l(\theta, e) = \begin{cases} \gamma(2\theta(s_1 - s_2) + s_2) & \text{for } \theta \in [0, 1/2); \\ \overline{\Psi}_\lambda(w_{\frac{e}{2(1-\theta)},l}) & \text{for } \theta \in [1/2, 1); \\ e & \text{for } \theta = 1. \end{cases}$$

Then $\zeta_l(0, e) = \overline{\Psi}_\lambda(\Phi_{\lambda,l}(e)) = \Psi_\lambda(\Phi_{\lambda,l}(e))$ and $\zeta_l(1, e) = e$. First, we claim that $\lim_{\theta \to 1^-} \zeta_l(\theta, e) = e$ and $\lim_{\theta \to \frac{1}{2}^-} \zeta_l(\theta, e) = \overline{\Psi}_\lambda(w_{e,l})$.

(a) $\lim_{\theta\to 1^-}\zeta_l(\theta,e)=e$: since

$$\begin{aligned}&\int_{\mathbb{R}^N}\frac{x}{|x|}(|\nabla[w_{\frac{e}{2(1-\theta)},l}]|^2+[w_{\frac{e}{2(1-\theta)},l}]^2)dx\\&=\int_{\mathbb{R}^N}\frac{x+\frac{le}{2(1-\theta)}}{|x+\frac{le}{2(1-\theta)}|}(|\nabla[w(x)]|^2+[w(x)]^2)dx\\&=(\frac{2p}{p-2})\alpha^\infty e+o(1)\quad\text{as }\theta\to 1^-,\end{aligned}$$

it follows that $\lim_t \theta\to 1^-\zeta_l(\theta,e)=e$.

(b) $\lim_{\theta\to\frac12^-}\zeta_l(\theta,e)=\overline{\Psi}_\lambda(w_{e,l})$: since $\overline{\Psi}_\lambda\in C(\Sigma,\mathbb{S}^{N-1})$, we obtain that $\lim_{\theta\to\frac12^-}\zeta_l(\theta,e)=\overline{\Psi}_\lambda(w_{e,l})$. Thus, $\zeta_l(\theta,e)\in C([0,1]\times\mathbb{S}^{N-1},\mathbb{S}^{N-1})$ and

$$\begin{gathered}\zeta_l(0,e)=\Psi_\lambda(\Phi_{\lambda,l}(e))\quad\text{for all } e\in\mathbb{S}^{N-1},\\ \zeta_l(1,e)=e\quad\text{for all } e\in\mathbb{S}^{N-1},\end{gathered}$$

provided $l>\widehat{l}_0$. This completes the proof. □

Theorem 5.9. *For each $\lambda\in(0,\lambda_0)$, the functional J_λ has at least two critical points in $[J_\lambda<\alpha^\infty]$. In particular, Equation (E_λ) has two positive solutions $u_0^{(1)}$ and $u_0^{(2)}$ such that $u_0^{(i)}\in\mathbf{N}_\lambda$ for $i=1,2$.*

Proof. Applying Lemmas 5.6, 5.8, for $\lambda\in(0,\lambda_0)$, we have

$$\operatorname{cat}([J_\lambda\le\alpha^\infty-\sigma_l])\ge 2.$$

By Proposition 2.6 and Lemma 5.5, $J_\lambda(u)$ has at least two critical points in $[J_\lambda<\alpha^\infty]$. This implies that Equation (1.3) has two positive solutions $u_\lambda^{(1)}$ and $u_\lambda^{(2)}$ such that $u_\lambda^{(i)}\in\mathbf{N}_\lambda$ for $i=1,2$. □

6. Proof of Theorem 1.1

Given a positive real number $r_0>\frac{q}{p-q}$. Let

$$\Lambda_0=\min\{(\frac{r_0p}{q(r_0+1)}-1),\lambda_0\}>0,$$

where $\lambda_0>0$ is as in Lemma 5.3. Then we have the following results.

Lemma 6.1. *We have*

$$\begin{gathered}\frac12(1+\lambda)^{r_0}-\frac1p(1+\lambda)^{r_0+1}-\frac{p-2}{2p}>0,\\ \frac1q(1+\lambda)^{r_0}-\frac1p(1+\lambda)^{r_0+1}-\frac{p-q}{pq}>0\end{gathered}$$

for all $\lambda\in(0,\Lambda_0)$.

Proof. Let

$$k(\lambda)=\frac1q(1+\lambda)^{r_0}-\frac1p(1+\lambda)^{r_0+1}-\frac{p-q}{pq}.$$

Then $k(0)=0$ and

$$k'(\lambda)=\frac{r_0}{q}(1+\lambda)^{r_0-1}-\frac{r_0+1}{p}(1+\lambda)^{r_0}$$

$$= (1+\lambda)^{r_0-1}(\frac{r_0}{q} - \frac{r_0+1}{p}(1+\lambda)) > 0$$

for all $\lambda \in (0, \Lambda_0)$. This implies that $k(\lambda) > 0$ or

$$\frac{1}{2}(1+\lambda)^{l_0} - \frac{1}{p}(1+\lambda)^{r_0+1} - \frac{p-q}{pq} > 0 \quad \text{for all } \lambda \in (0, \Lambda_0).$$

By a similar argument, we have

$$\frac{1}{2}(1+\lambda)^{r_0} - \frac{1}{p}(1+\lambda)^{r_0+1} - \frac{p-2}{2p} > 0 \quad \text{for all } \lambda \in (0, \Lambda_0).$$

This completes the proof. □

We define

$$I_\lambda(u) = \sup_{t \geq 0} J_\lambda(tu) : \mathbb{B} \to \mathbb{R}.$$

Then we have the following result.

Lemma 6.2. *For each $\lambda \in (0, \Lambda_0)$ and $u \in \mathbb{B}$ we have*

$$(1+\lambda)^{-r_0} I_0(u) - \frac{\lambda(p-q)}{pq} \|f_+\|_{L^{p/(p-q)}}^{p/(p-q)} \leq I_\lambda(u) \leq I_0(u),$$

where $I_0 = I_\lambda$ with $\lambda = 0$.

Proof. Let $u \in \mathbb{B}$. Then by Lemmas 2.4, 6.1 and (4.2),

$$\begin{aligned}
I_\lambda(u) &= \sup_{t\geq 0} J_\lambda(tu) \geq J_\lambda(t_0(u)u) \\
&= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla t_0(u)u|^2 + (t_0(u)u)^2 dx + \frac{1}{q}\int_{\mathbb{R}^N} f_-|t_0(u)u|^q dx \\
&\quad - \frac{\lambda}{q}\int_{\mathbb{R}^N} f_+|t_0(u)u|^q dx - \frac{1}{p}\int_{\mathbb{R}^N} |t_0(u)u|^p dx \\
&\geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla t_0(u)u|^2 + (t_0(u)u)^2 dx + \frac{1}{q}\int_{\mathbb{R}^N} f_-|t_0(u)u|^q dx \\
&\quad - \frac{1+\lambda}{p}\int_{\mathbb{R}^N} |t_0(u)u|^p dx - \frac{\lambda(p-q)}{pq}\|f_+\|_{L^{p/(p-q)}}^{p/(p-q)} \\
&= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla t_0(u)u|^2 + (t_0(u)u)^2 dx + \frac{1}{q}\int_{\mathbb{R}^N} f_-|t_0(u)u|^q dx \\
&\quad - \frac{1+\lambda}{p}[\int_{\mathbb{R}^N} |\nabla t_0(u)u|^2 + (t_0(u)u)^2 dx + \int_{\mathbb{R}^N} f_-|t_0(u)u|^q dx] \\
&\quad - \frac{\lambda(p-q)}{pq}\|f_+\|_{L^{p/(p-q)}}^{p/(p-q)} \\
&= (\frac{1}{2} - \frac{1+\lambda}{p})\int_{\mathbb{R}^N} |\nabla t_0(u)u|^2 + (t_0(u)u)^2 dx \\
&\quad + (\frac{1}{q} - \frac{1+\lambda}{p})\int_{\mathbb{R}^N} f_-|t_0(u)u|^q dx - \frac{\lambda(p-q)}{pq}\|f_+\|_{L^{p/(p-q)}}^{p/(p-q)} \\
&\geq \frac{(p-2)(1+\lambda)^{-r_0}}{2p}\int_{\mathbb{R}^N} |\nabla t_0(u)u|^2 + (t_0(u)u)^2 dx \\
&\quad + \frac{(p-q)(1+\lambda)^{-r_0}}{pq}\int_{\mathbb{R}^N} f_-|t_0(u)u|^q dx - \frac{\lambda(p-q)}{pq}\|f_+\|_{L^{p/(p-q)}}^{p/(p-q)}
\end{aligned}$$

$$
\begin{aligned}
&\geq (1+\lambda)^{-r_0}(\frac{1}{2}-\frac{1}{p})\int_{\mathbb{R}^N}|\nabla t_0(u)u|^2+(t_0(u)u)^2dx\\
&\quad +(1-\lambda)^{-r_0}(\frac{1}{q}-\frac{1}{p})\int_{\mathbb{R}^N}f_-|t_0(u)u|^q dx-\frac{\lambda(p-q)}{pq}\|f_+\|^{p/(p-q)}_{L^{p/(p-q)}}\\
&\geq (1+\lambda)^{-r_0}\Big[\frac{1}{2}\int_{\mathbb{R}^N}|\nabla t_0(u)u|^2+(t_0(u)u)^2dx+\frac{1}{q}\int_{\mathbb{R}^N}f_-|t_0(u)u|^q dx\\
&\quad -\frac{1}{p}\Big(\int_{\mathbb{R}^N}|\nabla t_0(u)u|^2+(t_0(u)u)^2dx+\int_{\mathbb{R}^N}f_-|t_0(u)u|^q dx\Big)\Big]\\
&\quad -\frac{\lambda(p-q)}{pq}\|f_+\|^{p/(p-q)}_{L^{p/(p-q)}}\\
&=(1+\lambda)^{-r_0}J_0(t_0(u)u)-\frac{\lambda(p-q)}{pq}\|f_+\|^{p/(p-q)}_{L^{p/(p-q)}}\\
&=(1+\lambda)^{-r_0}I_0(u)-\frac{\lambda(p-q)}{pq}\|f_+\|^{p/(p-q)}_{L^{p/(p-q)}}.
\end{aligned}
$$

Moreover,

$$J_\lambda(tu)\leq J_0(tu)\leq I_0(u) \text{ for all } t>0.$$

Then $I_\lambda(u)\leq I_0(u)$. This completes the proof. □

We observe that if λ is sufficiently small, the minimax argument in Section 4 also works for J_λ. Let $l>\max\{l_0,\widehat{l_0}\}$ be very large and let

$$\beta_\lambda=\inf_{\gamma\in\Gamma}\max_{y\in\overline{B^N(0,1)}}I_\lambda(\gamma(y)),$$

where Γ is as in (4.5). Then by (4.4) and Lemma 6.2, for $\lambda\in(0,\Lambda_0)$, we have

$$(1+\lambda)^{-r_0}\beta_0-\frac{\lambda(p-q)}{pq}\|f_+\|^{p/(p-q)}_{L^{p/(p-q)}}\leq\beta_\lambda\leq\beta_0. \tag{6.1}$$

Moreover, we have the following result.

Theorem 6.3. *There exists a positive number $\Lambda_*\leq\Lambda_0$ such that for $\lambda\in(0,\Lambda_*)$,*

$$\alpha^\infty<\beta_\lambda<2\alpha^\infty.$$

Furthermore, Equation (1.3) has a positive solution $u_0^{(3)}$ such that $J_\lambda(u_0^{(3)})=\beta_\lambda$.

Proof. By Theorems 3.3 and 4.1, and Lemma 6.2, we also have that

$$(1+\lambda)^{-r_0}\alpha^\infty-\frac{\lambda(p-q)}{pq}\|f_+\|^{p/(p-q)}_{L^{p/(p-q)}}\leq\alpha_\lambda<\alpha^\infty.$$

For any $\varepsilon>0$ there exists a positive number $\overline{\lambda}_1\leq\Lambda_0$ such that for $\lambda\in(0,\overline{\lambda}_1)$,

$$\alpha^\infty-\varepsilon<\alpha_\lambda<\alpha^\infty.$$

Thus,

$$2\alpha^\infty-\varepsilon<\alpha^\infty+\alpha_\lambda<2\alpha^\infty.$$

Applying (6.1) for any $\delta>0$ there exists a positive number $\overline{\lambda}_2\leq\Lambda_0$ such that for $\lambda\in(0,\overline{\lambda}_2)$,

$$\beta_0-\delta<\beta_\lambda\leq\beta_0.$$

Moreover, by Theorem 4.6,

$$\alpha^\infty<\beta_0<2\alpha^\infty.$$

Fix a small $0 < \varepsilon < 2\alpha^{\infty} - \beta_0$, choosing a $\delta > 0$ such that for $\lambda \in (0, \lambda_*)$ we obtain

$$\alpha^{\infty} < \beta_\lambda < 2\alpha^{\infty} - \varepsilon < \alpha^{\infty} + \alpha_\lambda < 2\alpha^{\infty},$$

where $\Lambda_* = \min\{\overline{\lambda}_1, \overline{\lambda}_2\}$. Similar to the argument in the proof of Theorem 4.6, we can conclude that the Equation (1.3) has a positive solution $u_0^{(3)}$ such that $J_\lambda(u_0^{(3)}) = \beta_\lambda$. This completes the proof. □

We can now complete the proof of Theorem 1.1: By Theorems 3.3, 4.1 and 4.6, the results (i) and (ii) hold. (iii) By Theorems 5.9 and 6.3, there exists a positive number Λ_* such that for $\lambda \in (0, \Lambda_*)$, Equation (1.3) has three positive solutions $u_0^{(1)}, u_0^{(2)}$ and $u_0^{(3)}$ with

$$0 < J_\lambda(u_0^{(i)}) < \alpha^{\infty} < J_\lambda(u_0^{(3)}) < 2\alpha^{\infty} \text{ for } i = 1, 2.$$

This completes the proof of Theorem 1.1.

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