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WELL-POSEDNESS OF BOUNDARY-VALUE PROBLEMS FOR PARTIAL DIFFERENTIAL EQUATIONS OF EVEN ORDER

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Abstract. In this article, we establish the well-posedness of two boundary value problems for $2k$-th order partial differential equations. It is shown that the solvability of these problems depends on the evenness and oddness of the number k.

1. Introduction and formulation of the problems

There is a huge number of theoretical and applied works devoted to the study of partial differential equations of higher order. Solutions of some classical and nonclassical problems for specific partial differential equations of higher order can be found, for example, in the monographs by Egorov and Fyodorov [10], Gazzola and Evans [11], Grunau and Sweers [12], Mizohata [18], Peetre [19], Polyanin [20], Vragov [28], and in the articles by Kirane and Qafsaoui [15] and Qafsaoui [21]. These problems are studied in various directions: qualitative properties of solutions, spectral problems, various statements of boundary value problems, and numerical investigations.

The study of well-posedness of the Cauchy problem, Goursat problem and boundary value problem for partial differential equations and mixed type partial differential equations have been studied extensively in a large number of articles; see, for example, [1, 3, 4, 7, 8, 9, 16, 17, 22, 24, 26, 27, 28, 30] and the references therein. As examples, we present some of them.

In [1, 3], in the domain $\Omega = \{(x,t) : 0 < x < p,\ 0 < t < T\}$ the boundary value problem

$$\frac{\partial^{2k} u}{\partial x^{2k}} - \frac{\partial^2 u}{\partial t^2} = f(x,t)$$

was investigated. Here $2 \leqslant k$ is a fixed integer. There its solvability was established.

In [4], p-evolution equations in (t,x) with real characteristics were studied. Sufficient conditions for the well-posedness of the Cauchy problem in Sobolev spaces, in terms of decay estimates of the coefficients as the space variable $x \to \infty$ were given.

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In [16], some correct boundary value problems for the following differential-operator equation

$$(-1)^k A\frac{d^n u(t)}{dt^n} + Bu(t) = f(t), t \in (0,T),$$

where $k = [(n-1)/2], A$ and B are linear self-adjoint operators in Hilbert space, B is a positive operator and operator A can have a spectrum of any kind, i.e. this equation can be of mixed type, were investigated. The existence of weak solutions was proved and estimates for them were obtained. Boundary conditions are satisfied in the sense of the existence of weak limits of the solution and its derivatives. The results were illustrated by three partial differential equation examples.

In [17], boundary value problems for some class of higher-order mixed type partial differential equations were considered. Note that these partial differential equations are unsolved with respect to the highest derivative. The author establishes existence and uniqueness in various settings including a so-called partially hyperbolic case. The results rely on coerciveness properties.

In [27], a partial differential operator, parabolic in the sense of Petrovskiĭ, of higher order in time, is perturbed by an unbounded Volterra-type integral operator, the maximal order of which equals the order of the leading part of the parabolic operator. As an example for such general time-varying Volterra integro-linear-partial-differential equations in sufficiently smooth bounded domains in $\mathbb{R}^n$, one may consider the equation as a model for heat conduction with memory (first order in time), while for second-order equations in t, the parabolic part can be viewed as modeling the vibrations of a structurally damped higher-order material (beams, plates, etc.) with memory effects (concerning the deformation history) included in the Volterra operator. The nonhomogeneous initial and boundary value (with homogeneous boundary conditions) problem with distributed load is solved by a general "variation of constants formula": the procedure is first to find a fundamental solution of the parabolic "reference" problem, then formally to solve the equation by considering the integral as a perturbation and (this appears to be the main difficulty) constructing thereby the fundamental solution (i.e. the integral resolvent) of the original problem.

In [29], a relation between parabolicity and coerciveness in Besov spaces for a higher-order linear evolution equation in a Banach space was investigated. It was proved that coerciveness in Besov spaces and parabolicity of the equation are in fact equivalent.

In this article, two boundary value problems for $2k$-th order partial differential equations are investigated. In the domain $\Omega = \{(x,t) : 0 < x < p, 0 < t < T\}$ we consider the partial differential equation

$$Lu \equiv \frac{\partial^{2k} u}{\partial x^{2k}} + \frac{\partial^2 u}{\partial t^2} = f(x,t), \tag{1.1}$$

where $2 \leqslant k$ is a fixed integer.

Problem 1. Obtain the solution $u(x,t)$ of equation (1.1) in the domain Ω satisfying conditions

$$\frac{\partial^{2m} u(0,t)}{\partial x^{2m}} = \frac{\partial^{2m} u(p,t)}{\partial x^{2m}} = 0, \quad m = 0,1,\dots,k-1,\ 0 \le t \le T,; \tag{1.2}$$

$$u(x,0) = 0, \quad u(x,T) = 0, \quad 0 \leqslant x \leqslant p. \tag{1.3}$$

Problem 2. Obtain the solution $u(x,t)$ of equation (1.1) in the domain Ω satisfying conditions (1.2) and

$$u(x,0)=0, \quad u_t(x,0)=0, \quad 0\leqslant x\leqslant p\,. \tag{1.4}$$

Well-posedness of boundary value problems 1 and 2 in the cases even and odd k is investigated. It is proved that solvability results of the present paper depend on the evenness or oddness of number k.

This article is organized as follows. Section 1 is an introduction where we provide the formulation of problems. In sections 2 and 3, well-posedness of problem 1 in the cases even and odd k is established. The spectrum of problem 1 is studied. In section 4, well-posedness of problem 2 in the cases even and odd k is established. The spectrum of problem 2 is studied. Finally, section 5 is a conclusion.

2. Well-posedness of problem 1 when k is odd

In this section, we present some basic definitions and preliminary facts which are used throughout the paper. We denote

$$\begin{gathered}
V_1(\Omega)=\{u(x,t): u\in C^{2k-2,0}_{x,t}(\bar\Omega)\cap C^{2k,2}_{x,t}(\Omega),\ (1.2)\text{ and }(1.3)\text{ hold}\},\\
W_1(\Omega)=\Big\{f(x,t): f\in C^{1,0}_{x,t}(\bar\Omega),\ \frac{\partial f}{\partial x}\in \operatorname{Lip}_\alpha[0,p]\text{uniformly with respect to } t,\\
0<\alpha<1,\ f(0,t)=f(p,t)=0\Big\}.
\end{gathered}$$

We define a solution of problem 1 as follows:

Definition 2.1. A function $u(x.t)\in V_1(\Omega)$ is said to be a regular solution of problem 1 with $f(x,t)\in C(\Omega)$, if it satisfies (1.1) in the domain Ω and conditions (1.2) and (1.3).

Definition 2.2. A function $u(x,t)\in L_2(\Omega)$ is said to be a strong solution of the problem 1 with $f\in L_2(\Omega)$, if there exists a sequence $\{u_n\}\subset V_1(\Omega)$, $n=1,2,\dots$, such that $\|u_n-u\|_{L_2(\Omega)}\to 0$, $\|Lu_n-f\|_{L_2(\Omega)}\to 0$ as $n\to\infty$.

Definition 2.3. A function $u(x,t)\in W_2^{2k,2}(\Omega)$ is called an almost everywhere solution of problem 1 with $f\in L_2(\Omega)$, if it satisfies (1.1), (1.2) and (1.3) almost everywhere in the domain Ω. On $V_1(\Omega)$, we define the operator L mapping the set $V_1(\Omega)$ into $C(\Omega)$ by (1.1). The domain of definition $V_1(\Omega)$ is dense in $L_2(\Omega)$. The closure in $L_2(\Omega)$ is denoted by $\overline{L}$.

Let k be an odd number. First, we will establish a priori estimate for the regular solution of problem 1.

Lemma 2.4. *If $u\in V_1(\Omega)$ and $\frac{\partial^{m+1}u}{\partial x^m\partial t}\in C(\bar\Omega)$, $m=1,2,\dots,k$, $\frac{\partial^{2k}u}{\partial x^{2k}}\in L_2(\Omega)$, $\frac{\partial^2 u}{\partial t^2}\in L_2(\Omega)$, then there exists a constant $C_1>0$ that depends only on numbers k and T, and does not depend on the function $u(x,t)$ such that the following a priori estimate holds:*

$$\|u\|_{W_2^{2k,2}(\Omega)}\leqslant C_1\|Lu\|_{L_2(\Omega)}, \tag{2.1}$$

where

$$\|u\|_{W_2^{2k,2}(\Omega)}=\sum_{m=0}^{2k}\|\frac{\partial^m u}{\partial x^m}\|^2_{L_2(\Omega)}+\sum_{m=1}^{2}\|\frac{\partial^m u}{\partial t^m}\|^2_{L_2(\Omega)}+\sum_{m=2}^{k+1}\|\frac{\partial^m u}{\partial x^{m-1}\partial t}\|^2_{L_2(\Omega)}.$$

Proof. Multiplying by $u(x,t)$ both sides of the equation

$$\frac{\partial^{2k}u}{\partial x^{2k}}+\frac{\partial^2 u}{\partial t^2}=Lu \tag{2.2}$$

and taking the integral over the domain Ω, we obtain

$$\int_0^p\int_0^T u(\frac{\partial^{2k}u}{\partial x^{2k}}+\frac{\partial^2 u}{\partial t^2})\,dx\,dt=\int_0^p\int_0^T uLu\,dt\,dx. \tag{2.3}$$

Since k is odd number, using equality (2.3), identities

$$u\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial t}(u\frac{\partial u}{\partial t})-(\frac{\partial u}{\partial t})^2,$$

$$u\frac{\partial^{2k}u}{\partial x^{2k}}=\sum_{m=0}^{k-1}(-1)^m\frac{\partial}{\partial x}(\frac{\partial^m u}{\partial x^m}\frac{\partial^{2k-m-1}u}{\partial x^{2k-m-1}})+(-1)^k(\frac{\partial^k u}{\partial x^k})^2,$$

and conditions (1.2) and (1.3), we obain

$$\|\frac{\partial^k u}{\partial x^k}\|^2_{L_2(\Omega)}+\|\frac{\partial u}{\partial t}\|^2_{L_2(\Omega)}=\int_0^p\int_0^T(-u)Lu\,dt\,dx.$$

From this equality and the Hölder inequality it follows the estimate

$$\|\frac{\partial^k u}{\partial x^k}\|^2_{L_2(\Omega)}+\|\frac{\partial u}{\partial t}\|^2_{L_2(\Omega)}\le\|u\|_{L_2(\Omega)}\ \|Lu\|_{L_2(\Omega)}.$$

Applying the inequality

$$ab\leqslant\frac{\varepsilon}{2}a^2+\frac{1}{2\varepsilon}b^2$$

for $a>0$, $b>0$ and $\varepsilon>0$, we obtain

$$\|u_t\|^2_{L_2(\Omega)}+\|\frac{\partial^k u}{\partial x^k}\|^2_{L_2(\Omega)}\leqslant\frac{\varepsilon}{2}\|u\|^2_{L_2(\Omega)}+\frac{1}{2\varepsilon}\|Lu\|^2_{L_2(\Omega)}. \tag{2.4}$$

Now, we obtain the estimate for $\|u\|^2_{L_2(\Omega)}$. Using condition (1.3), we obtain

$$u^2(x,t)=\int_0^t\frac{\partial}{\partial t}(u^2(x,\tau))d\tau.$$

Therefore,

$$u^2(x,t)\leqslant 2\int_0^T|u(x,t)u_t(x,t)|dt.$$

Taking the integral over the domain Ω on both sides of the inequality and applying the Hölder inequality, we obtain

$$\|u\|^2_{L_2(\Omega)}\leqslant 2T\|u\|_{L_2(\Omega)}\|u_t\|_{L_2(\Omega)}.$$

From that, it follows

$$\|u\|^2_{L_2(\Omega)}\leqslant 4T^2\|u_t\|^2_{L_2(\Omega)}.$$

The last estimate and inequality (2.4) yield that

$$\|u\|^2_{L_2(\Omega)}\leqslant 4T^2(\frac{\varepsilon}{2}\|u\|^2_{L_2(\Omega)}+\frac{1}{2\varepsilon}\|Lu\|^2_{L_2(\Omega)}). \tag{2.5}$$

Adding the both sides of inequalities (2.4) and (2.5), we obtain

$$\|u\|^2_{L_2(\Omega)}+\|u_t\|^2_{L_2(\Omega)}+\|\frac{\partial^k u}{\partial x^k}\|^2_{L_2(\Omega)}\leqslant(4T^2+1)(\frac{\varepsilon}{2}\|u\|^2_{L_2(\Omega)}+\frac{1}{2\varepsilon}\|Lu\|^2_{L_2(\Omega)})$$

for any $\varepsilon > 0$. Choosing $\varepsilon = 1/(4T^2+1)$, we can write

$$\|u\|^2_{L_2(\Omega)} + \|u_t\|^2_{L_2(\Omega)} + \|\frac{\partial^k u}{\partial x^k}\|^2_{L_2(\Omega)} \leqslant (4T^2+1)^2\|Lu\|^2_{L_2(\Omega)}. \tag{2.6}$$

Taking the square of both sides of (2.2) and integrating the obtained equality over the domain Ω, we obtain

$$\|u_{tt}\|^2_{L_2(\Omega)} + \|\frac{\partial^{2k} u}{\partial x^{2k}}\|^2_{L_2(\Omega)} + 2\int_\Omega \frac{\partial^2 u}{\partial t^2}\frac{\partial^{2k} u}{\partial x^{2k}}\,dx\,dt = \|Lu\|^2_{L_2(\Omega)}. \tag{2.7}$$

We consider the integral

$$\int_\Omega \frac{\partial^2 u}{\partial t^2}\frac{\partial^{2k} u}{\partial x^{2k}}\,dx\,dt.$$

We have that

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2}\frac{\partial^{2k} u}{\partial x^{2k}} &= \sum_{m=0}^{k-1}(-1)^m\frac{\partial}{\partial x}\Big(\frac{\partial^{2+m} u}{\partial t^2\partial x^m}\frac{\partial^{2k-m-1} u}{\partial x^{2k-m-1}}\Big)\\ &\quad + (-1)^k\frac{\partial}{\partial t}\Big(\frac{\partial^{1+k} u}{\partial t\partial x^k}\frac{\partial^k u}{\partial x^k}\Big) + (-1)^{k+1}\Big(\frac{\partial^{k+1} u}{\partial t\partial x^k}\Big)^2.\end{aligned}$$

If m is odd, then $2k-m-1$ is even. Therefore, from (1.2) it follows that $\frac{\partial^{2k-m-1}u}{\partial x^{2k-m-1}} = 0$ at $x=0$ and $x=p$. If m is even, then from (1.2) it follows that $\frac{\partial^{m+2}u}{\partial t^2\partial x^m} = 0$ at $x=0$ and $x=p$. Moreover, from (1.3) it follows $\frac{\partial^k u}{\partial x^k} = 0$ at $t=0$ and $t=T$. Therefore, integrating both sides of the last equality over the domain Ω, we obtain

$$2\iint_\Omega \frac{\partial^2 u}{\partial t^2}\frac{\partial^{2k} u}{\partial x^{2k}}\,dx\,dt = 2\|\frac{\partial^{k+1} u}{\partial x^k\partial t}\|^2_{L_2(\Omega)}.$$

Using this equality and (2.7), we obtain

$$\|u_{tt}\|^2_{L_2(\Omega)} + 2\|\frac{\partial^{k+1} u}{\partial x^k\partial t}\|^2_{L_2(\Omega)} + \|\frac{\partial^{2k} u}{\partial x^{2k}}\|^2_{L_2(\Omega)} = \|Lu\|^2_{L_2(\Omega)}.$$

From that it follows

$$\|u_{tt}\|^2_{L_2(\Omega)} + \|\frac{\partial^{k+1} u}{\partial x^k\partial t}\|^2_{L_2(\Omega)} + \|\frac{\partial^{2k} u}{\partial x^{2k}}\|^2_{L_2(\Omega)} \leqslant \|Lu\|^2_{L_2(\Omega)}. \tag{2.8}$$

Adding both sides of inequalities (2.6) and (2.8), we obtain

$$\begin{aligned}&\|u\|^2_{L_2(\Omega)} + \|u_t\|^2_{L_2(\Omega)} + \|u_{tt}\|^2_{L_2(\Omega)} + \|\frac{\partial^{k+1} u}{\partial x^k\partial t}\|^2_{L_2(\Omega)} + \|\frac{\partial^k u}{\partial x^k}\|^2_{L_2(\Omega)} + \|\frac{\partial^{2k} u}{\partial x^{2k}}\|^2_{L_2(\Omega)}\\ &\leqslant C_1\|Lu\|^2_{L_2(\Omega)},\end{aligned} \tag{2.9}$$

where $C_1 = (4T^2+1)^2+1$.

To obtain estimates for the norm $\|\frac{\partial^m u}{\partial x^m}\|^2_{L_2(\Omega)}$, $m = 1, 2, \dots, 2k-1$, we use the inequality

$$\|\frac{\partial^n u}{\partial x^n}\|^2_{L_2(\Omega)} \leqslant \frac{1}{2}\|\frac{\partial^{n-1} u}{\partial x^{n-1}}\|^2_{L_2(\Omega)} + \frac{1}{2}\|\frac{\partial^{n+1} u}{\partial x^{n+1}}\|^2_{L_2(\Omega)} \tag{2.10}$$

that can easily be checked.

Taking the sum of both sides of inequality (2.10) with respect to n from 1 to $2k-1$, and using inequality (2.9), we obtain

$$\|\frac{\partial u}{\partial x}\|^2_{L_2(\Omega)} + \|\frac{\partial^{2k-1} u}{\partial x^{2k-1}}\|^2_{L_2(\Omega)} \leqslant C_1\|Lu\|^2_{L_2(\Omega)}. \tag{2.11}$$

In general, taking the sum of both sides of inequality (2.10) with respect to n from m to $k-1$, and using inequality (2.9), we obtain

$$\|\frac{\partial^m u}{\partial x^m}\|^2_{L_2(\Omega)} + \|\frac{\partial^{2k-m} u}{\partial x^{2k-m}}\|^2_{L_2(\Omega)} \leqslant C_1\|Lu\|^2_{L_2(\Omega)}. \tag{2.12}$$

Taking the sum of both sides of inequalities (2.9) and (2.12) with respect to m from 1 to $k-1$, we obtain

$$\sum_{m=0}^{2}\Big\|\frac{\partial^m u}{\partial t^m}\Big\|^2_{L_2(\Omega)} + \sum_{m=1}^{2k}\Big\|\frac{\partial^m u}{\partial x^m}\Big\|^2_{L_2(\Omega)} + \|\frac{\partial^{k+1} u}{\partial x^k \partial t}\|^2_{L_2(\Omega)} \leqslant kC_1\|Lu\|^2_{L_2(\Omega)}. \tag{2.13}$$

Next, we use the inequality

$$\|\frac{\partial^m u}{\partial x^{m-1}\partial t}\|^2_{L_2(\Omega)} \leqslant \|\frac{\partial^2 u}{\partial t^2}\|_{L_2(\Omega)}\|\frac{\partial^{2m-2} u}{\partial x^{2m-2}}\|_{L_2(\Omega)}$$

that can easily be checked. Summing with respect to m from 2 to k and using inequality (2.13), we obtain

$$\sum_{m=2}^{k}\Big\|\frac{\partial^m u}{\partial x^{m-1}\partial t}\Big\|^2_{L_2(\Omega)} \leqslant k(k-1)C_1\|Lu\|^2_{L_2(\Omega)}. \tag{2.14}$$

Adding both sides of inequalities (2.13) and (2.14), we obtain estimate (2.1). The proof is complete. □

Corollary 2.5. *From estimate* (2.1) *it follows that:*

- (i) *The regular solution of problem 1 is unique and continuously depends on* $f(x,t)$.
- (ii) *The inverse operator* L^{-1} *exists and it is bounded.*
- (iii) $\|L^{-1}\| \leqslant C_2$, $C_2 = k^2[(4T^2+1)^2+1]$.
- (iv) $\ker(L) = \{0\}$.
- (v) *The adjoint problem to problem 1 is well-posed.*

Second, we will study the regular solvability of problem 1. We seek a regular solution of problem 1 in the form of a Fourier series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)X_n(x) \tag{2.15}$$

expanded along the complete orthonormal system $X_n(x) = \sqrt{\frac{2}{p}}\sin\lambda_n x$ in $L_2(0,p)$, where $\lambda_n = \frac{n\pi}{p}$, $n \in N$. It is clear that $u(x,t)$ satisfies conditions (1.2). We expand the given function $f \in W_1(\Omega)$ in the form of a Fourier series along the functions $X_n(x)$, $n \in N$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)X_n(x), \tag{2.16}$$

where

$$f_n(t) = \int_0^P f(x,t)X_n(x)dx. \tag{2.17}$$

Substituting (2.15) and (2.16) into (1.1), we obtain

$$u_n''(t) - \lambda_n^{2k}u_n(t) = f_n(t), \quad 0 < t < T. \tag{2.18}$$

The solution of equation (2.18) satisfying the conditions

$$u_n(0) = 0, \quad u_n(T) = 0 \tag{2.19}$$

has the form

$$u_n(t) = -\frac{1}{\lambda_n^k} \int_0^T K_n(t,\tau) f_n(\tau) d\tau, \tag{2.20}$$

where

$$K_n(t,\tau) = \begin{cases} \frac{sh\lambda_n^k \tau sh\lambda_n^k (T-t)}{sh\lambda_n^k T}, & 0 \leqslant \tau \leqslant t, \\ \frac{sh\lambda_n^k t sh\lambda_n^k (T-\tau)}{sh\lambda_n^k T}, & t \leqslant \tau \leqslant T. \end{cases} \tag{2.21}$$

From (2.21) it follows $K_n(t,\tau) = K_n(\tau,t)$ and the following estimate holds:

$$K_n(t,\tau) \leqslant C_0 e^{-\lambda_n^k |t-\tau|}, C_0 = \text{const.} > 0. \tag{2.22}$$

Lemma 2.6. *If $f \in W_1(\Omega)$, then for any $t \in [0,T]$ the following estimate is valid:*

$$|u_n(t)| \leqslant \frac{C_0 C_2}{\lambda_n^{2k+1+\alpha}}. \tag{2.23}$$

Proof. Integrating by parts with respect to x in (2.17), we obtain

$$f_n(t) = \frac{1}{\lambda_n} f_n^{(1,0)}(t),$$

where

$$f_n^{(1,0)}(t) = \int_0^p \frac{\partial f}{\partial x} \sqrt{\frac{2}{p}} \cos \lambda_n x \, dx.$$

Since $\frac{\partial f}{\partial x} \in \mathrm{Lip}_\alpha[0,p]$ is uniform with respect to t, then (see [7])

$$|f_n^{(1,0)}(t)| \leqslant \frac{C_2}{\lambda_n^\alpha}.$$

So,

$$|f_n(t)| \leqslant \frac{C_2}{\lambda_n^{1+\alpha}}. \tag{2.24}$$

Now, we will estimate $|u_n(t)|$. Applying equality (2.20) and estimates (2.22) and (2.24), we obtain

$$\begin{aligned} |u_n(t)| &\leqslant \frac{C_0 C_2}{\lambda_n^{1+k+\alpha}} \Big[\int_0^t e^{-\lambda_n^k (t-\tau)} d\tau + \int_t^T e^{-\lambda_n^k (\tau - t)} d\tau \Big] \\ &\leqslant \frac{C_0 C_2}{\lambda_n^{2k+1+\alpha}} \Big(e^{-\lambda_n^k (T-t)} - e^{-\lambda_n^k t} \Big) \\ &\leqslant \frac{C_0 C_2}{\lambda_n^{2k+1+\alpha}} e^{-\lambda_n^k (T-t)} \leqslant \frac{C_0 C_2}{\lambda_n^{2k+1+\alpha}}. \end{aligned}$$

The proof is complete. □

Theorem 2.7. *If $f(x,t) \in W_1(\Omega)$, then there exists a regular solution of problem 1.*

Proof. We will prove uniform and absolute convergence of series (2.15) and

$$\frac{\partial^{2k} u}{\partial x^{2k}} = -\sum_{n=1}^{\infty} \lambda_n^{2k} u_n(t) X_n(x), \tag{2.25}$$

$$\frac{\partial^2 u}{\partial t^2} = \sum_{n=1}^{\infty} f_n(t)X_n(x) + \sum_{n=1}^{\infty} \lambda_n^{2k} u_n(t)X_n(x). \tag{2.26}$$

From (2.24) it follows that the series (2.15) and the first series in (2.26) uniformly and absolutely converge. In the same manner, from (2.23) it follows that the second series in (2.26) and the series (2.25) uniformly and absolutely converge. Adding equality (2.25) and (2.26), we note that solution (2.15) satisfies equation (1.1). Solution (2.15) satisfies boundary conditions (1.2) owing to properties of the function $X_n(x)$ and conditions (1.3) owing to properties of the kernel (2.21). The proof is complete. □

Third, we will study an existence and uniqueness of the strong solution of problem 1.

Theorem 2.8. *For any $f \in L_2(\Omega)$ there exists a unique strong solution of problem 1 and it satisfies estimate (2.1), it continuously depends on $f(x,t)$ and it can be represented in the form of Fourier series*

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)X_n(x)$$

expanded in full orthonormal system $X_n(x) = \sqrt{\frac{2}{p}} \sin \lambda_n x$ in $L_2(0,p)$, where $\lambda_n = \frac{n\pi}{p}$, $n \in N$.

Proof. It is possible to represent solution (2.15) in the form

$$u(x,t) = \int_0^T \int_0^p K(x,t;\xi,\tau) f(\xi,\tau) d\xi \partial\tau, \tag{2.27}$$

where

$$K(x,t;\xi,\tau) = -\sum_{n=1}^{\infty} \frac{X_n(x)X_n(\xi)}{\lambda_n^k} K_n(t,\tau). \tag{2.28}$$

From estimate (2.22) and $k \geqslant 2$ it follows that series (2.28) converges uniformly and absolutely. Consequently

$$|K(x,t;\xi,\tau)| \leqslant C_3,\ C_3 = \text{const} > 0. \tag{2.29}$$

From relations $C_0^\infty(\Omega) \subset W_1(\Omega) \subset L_2(\Omega)$ it follows that $W_1(\Omega)$ is dense in $L_2(\Omega)$. Then for any $f \in L_2(\Omega)$ there exists a sequence $\{f_m\} \subset W_1(\Omega)$, $m = 1, 2, \dots$ such that $\|f_m - f\|_{L_2(\Omega)} \to 0$ as $m \to \infty$. Consequently, $\{f_m\}$ is a Cauchy sequence in $L_2(\Omega)$. We denote by $u_m(x,t) \in V_1(\Omega)$ the solution of (1.1) with the right side term $f_m(x,t)$. According to Theorem 2.7 and Corollary 2.5, there exists a unique solution of the form

$$u_m(x,t) = \int_0^T \int_0^p K(x,t;\xi,\tau) f_m(\xi,\tau)\, d\xi\, d\tau, \quad m = 1, 2, \dots \tag{2.30}$$

of the equation

$$Lu_m(x,t) = f_m(x.t), \quad m = 1, 2, \dots. \tag{2.31}$$

Since $u_m(x,t) \in V_1(\Omega)$, according to (2.1) we have

$$\|u_m - u_l\|_{W_2^{2k,2}} \leqslant C_1 \|Lu_m - Lu_l\|_{L_2(\Omega)} = C_1 \|f_m - f_l\|_{L_2(\Omega)} \to 0$$

as $m \to \infty$, $l \to \infty$, that $\{u_m\}$ is a Cauchy sequence in $W_2^{2k,2}(\Omega)$. According to completeness of the $W_2^{2k,2}(\Omega)$, there exists a unique limit $u(x,t) = \lim_{m\to\infty} u_m(x,t) \in W_2^{2k,2}(\Omega)$. Passing to limit in (2.31) as $m \to \infty$, we obtain

$$\overline{L}u = \lim_{m\to\infty} Lu_m(x,t) = \lim_{m\to\infty} f_m(x,t) = f, \quad f \in L_2(\Omega); \tag{2.32}$$

i.e.,

$$\|Lu_m - f\|_{L_2(\Omega)} \to 0 \quad \text{as } m \to \infty.$$

Consequently, according to Definition 2.2, the function $u(x,t) \in W_2^{2k,2}(\Omega)$ is the strong solution of problem 1. Hence, it follows that the domain $D(\overline{L})$ of definition of operator $\overline{L}$ consists of all strong solutions of problem 1 and $R(\overline{L}) = L_2(\Omega)$. So, $u_m(x,t) \in V_1(\Omega)$ from (2.1) we have that

$$(u_m, u_m)_{W_2^{2k,2}(\Omega)} \leqslant C_1^2 (Lu_m, Lu_m)_{L_2(\Omega)}.$$

Passing to the limit in the above inequality as $m \to \infty$, we obtain

$$\|u\|^2_{W_2^{2k,2}} \leqslant C_1^2 \|\overline{L}u\|^2_{L_2(\Omega)}, \tag{2.33}$$

where $u \in W_2^{2k,2}(\Omega)$, $\overline{L}u = f \in L_2(\Omega)$. We conclude that estimate (2.1) is also true for the strong solution $u(x,t)$. Passing to limit in (2.31) as $m \to \infty$, we obtain

$$u(x,t) = \int_0^T \int_0^P K(x,t;\xi,\tau) f(\xi,\tau) d\xi \partial\tau,$$

where $u(x,t) \in W_2^{2k,2}(\Omega)$, $f \in L_2(\Omega)$. From estimate (2.33) it follows that the strong solution of problem 1 is unique and it continuously depends on $f(x,t)$. The proof of Theorem 2.8 is complete. $\square$

Corollary 2.9. *The strong solution $u(x,t)$ is almost everywhere solution in Ω. This follows from* (2.32) *and $u(x,t) \in W_2^{2k,2}(\Omega)$.*

Fourth, we will study the spectrum of problem 1.

Definition 2.10. *The spectrum of a problem is the set of eigenvalues of the operator of the problem.*

According to Definition 2.4, the spectrum of problem 1 is the set of eigenvalues of operator $\overline{L}$.

Theorem 2.11. *The spectrum of problem 1 consists of real eigenvalues of finite multiplicity of the operator $\overline{L}$.*

Proof. From (2.1) and (2.27) we conclude that it is defined a bounded symmetric operator L^{-1} on $W_1(\Omega)$ which is the inverse of the operator L and acts from $W_1(\Omega)$ to $V_1(\Omega)$ by the rule

$$(L^{-1}f)(x,t) = \int_0^T \int_0^p K(x,t;\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau.$$

It can be extended to whole space $L_2(\Omega)$. This extension will be denoted by $\overline{L^{-1}}$, the closure of L^{-1}, $D(\overline{L^{-1}}) = L_2(\Omega)$. The operator $\overline{L^{-1}}$ is symmetric, bounded and defined on the whole space $L_2(\Omega)$, so it is self-adjoint. It follows from (2.29) that $K(x,t;\xi,\tau)) \in L_2(\Omega \times \Omega)$; therefore $\overline{L^{-1}}$ is a compact operator in $L_2(\Omega)$. Then the spectrum of the operator $\overline{L^{-1}}$ is discrete and consists of real eigenvalues of finite multiplicity. The relation between eigenvalues of the operator $\overline{L^{-1}}$ and $\overline{L}$ is as follows of monograph [8]: if $\mu_n \neq 0$ is an eigenvalue of the operator $\overline{L^{-1}}$, then μ_n^{-1} is eigenvalue of the operator $\overline{L}$. The proof is complete. □

3. Well-posedness of problem 1 when k is even

Let k be even number. In the same manner, we seek a regular solution of problem 1 in the form of Fourier series (2.15) and we obtain the equation

$$u_n'' + \lambda_n^{2k} u_n(t) = f_n(t), \quad 0 < t < T$$

for the function $u_n(t)$. The solution of this equation satisfying the condition

$$u_n(0) = 0, \quad u_n(T) = 0$$

which has the form

$$u_n(t) = -\frac{1}{\lambda_n^k} \int_0^T K_n(t,\tau) f_n(\tau) d\tau,$$

where

$$K_n(t,\tau) = \begin{cases} \frac{\sin\lambda_n^k \tau \sin\lambda_n^k (T-t)}{\sin\lambda_n^k T}, & 0 \leqslant \tau \leqslant t, \\ \frac{\sin\lambda_n^k t \sin\lambda_n^k (T-\tau)}{\sin\lambda_n^k T}, & t \leqslant \tau \leqslant T. \end{cases}$$

Therefore, the solution of problem 1 exists, if the following condition holds

$$\left|\sin\lambda_n \left(\frac{\pi n}{p}\right)^k T\right| \geqslant \delta > 0.$$

It is difficult to show the numbers p and T satisfying the last condition. Therefore, we give an other variant of solution of problem 1 in the case of even k. In the case of even k, solvability of problem 1 depends on domain geometry. We consider the following spectral problem.

Problem 3. Find the solution $u(x,t)$ of the equation

$$\frac{\partial^{2k} u}{\partial x^{2k}} + \frac{\partial^2 u}{\partial t^2} = \lambda u \tag{3.1}$$

satisfying (1.2) and (1.3), where λ is spectral parameter. This problem has the following eigenvalues

$$\lambda_{nm} = \left(\frac{n\pi}{p}\right)^{2k} - \left(\frac{n\pi}{T}\right)^2$$

and eigenfunctions

$$u_{nm}(x,t) = \frac{2}{\sqrt{pT}} \sin\frac{n\pi}{p} x \sin\frac{n\pi}{T} t, \quad m,n = 1,2,\dots$$

and they form a complete orthonormal system in $L_2(\Omega)$.

We expand the functions $u(x,t)$ and $f(x,t)$ into the Fourier series in functions $u_{nm}(x,t)$,

$$u(x,t) = \sum_{n,m=1}^{\infty} \alpha_{nm} u_{nm}(x,t), \tag{3.2}$$

$$f(x,t) = \sum_{n,m}^{\infty} f_{nm} u_{nm}(x,t), \tag{3.3}$$

where

$$f_{nm} = \int_0^P \int_0^T f(x,t) u_{nm}(x,t)\, dt\, dx \tag{3.4}$$

and α_{nm} are unknown Fourier coefficients of the function $u(x,t)$. Substituting (3.2) and (3.3) into (1.1), we obtain the solution of problem 1 in the form

$$u(x,t)=\sum_{n,m=1}^{\infty}\frac{f_{nm}}{\lambda_{nm}}u_{nm}(x,t). \tag{3.5}$$

Suppose that $m=n^k$, $T=p^k/\pi^{k-1}$. Then $\lambda_{nm}=\lambda_{nn^k}=0$ and problem 1 has infinitely many of linearly independent solutions in the form

$$u_{nn^k}(x,t)=\sin\big(\frac{n\pi}{p}x\big)\sin\big(\frac{n^k\pi^k}{p^k}t\big),\quad n=1,2,\dots$$

at $f(x,t)=0$. Number $\lambda_{nn^k}=0$ is an eigenvalue of infinite multiplicity of the spectral problem (3.1), (1.2), (1.3).

In this case, the solution of problem 1 exists, if the following orthogonality conditions hold

$$\int_0^p\int_0^T f(x,t)\sin\big(\frac{n\pi}{p}x\big)\sin\big(\frac{n^k\pi^k}{p^k}t\big)\,dt\,dx=0,\quad n=1,2,\dots.$$

Now, suppose that p and T such that $\frac{P^k}{T\pi^{k-1}}$ is irrational algebraic number of second degree (see [25]). Then according to Liouville's theorem [25] there exists number $\varepsilon_0>0$ such that

$$\big|\frac{p^k}{T\pi^{k-1}}-\frac{n^k}{m}\big|\geqslant\frac{\varepsilon_0}{m^2}.$$

In this case

$$\lambda_{nm}\geqslant\frac{\pi^{k+1}}{Tp^k}\varepsilon_0. \tag{3.6}$$

Theorem 3.1. *Let $f\in L_2(\Omega)$ and number $\frac{P^k}{T\pi^{k-1}}$ be irrational algebraic number of second degree. Then there exists the solution of problem 1, it belongs to $L_2(\Omega)$, and satisfies estimate*

$$\|u\|_{L_2(\Omega)}\leqslant C\|f\|_{L_2(\Omega)}, \tag{3.7}$$

where $C=\frac{Tp^k}{\pi^{k+1}\varepsilon_0}$ and it continuously depends on $f(x,t)$.

Proof. Since $f\in L_2(\Omega)$. Then for f, the Parseval equality is true

$$\sum_{n,m=1}^{\infty}|f_{nm}|^2=\|f\|^2_{L_2(\Omega)}.$$

Owing to it, for any $S>0$ and natural number N, we have

$$\begin{aligned}\|u_{N+S,N+S}-u_{NN}\|^2_{L_2(\Omega)}&=\sum_{n=N+1}^{N+S}\sum_{m=N+1}^{N+S}\frac{|f_{nm}|^2}{\lambda^2_{nm}}\\&\leqslant C\sum_{n=N+1}^{N+S}\sum_{m=N+1}^{N+S}|f_{nm}|^2\to 0\quad\text{as } N\to\infty.\end{aligned}$$

Consequently, the series (3.5) converges in $L_2(\Omega)$ and $u(x,t)\in L_2(\Omega)$. It is easy to show that $\|u\|_{L_2(\Omega)}\leqslant C\|f\|_{L_2(\Omega)}$. From that it follows continuously dependence of the solution of problem 1 on $f(x,t)$. The proof is complete. $\square$

Now, we study the regular solvability of the problem 1 for k even. We denote

$$\begin{aligned} W_2(\Omega)=\Big\{f: f\in C^{2k,2}_{x,t}(\overline{\Omega}),\ \frac{\partial^{2k+2}f}{\partial x^{k+1}\partial t}\in L_2(\Omega),\ \frac{\partial^4 f}{\partial x\partial t^3}\in L_2(\Omega),\\ \frac{\partial^{2l}f(0,t)}{\partial x^{2l}}=\frac{\partial^{2l}f(p,t)}{\partial x^{2l}}=0,\ l=0,1,2,\dots,k;\\ \frac{\partial^{2s}f(x,0)}{\partial t^{2s}}=\frac{\partial^{2s}f(x,T)}{\partial t^{2s}}=0,\ s=0,1\Big\}. \end{aligned}$$

Theorem 3.2. *If $P^k/(T\pi^{k-1})$ is irrational algebraic number of second degree and $f\in W_2(\Omega)$, then there exists regular solution of problem 1 and satisfies estimate (3.7).*

Proof. We will prove uniform and absolute convergence of series (3.2) and that

$$\frac{\partial^{2k}u}{\partial x^{2k}}=\sum_{n,m=1}^{\infty}\frac{f_{nm}}{\lambda_{nm}}\big(\frac{n\pi}{p}\big)^{2k}u_{nm}(x,t), \tag{3.8}$$

$$\frac{\partial^{2}u}{\partial t^{2}}=-\sum_{n,m=1}^{\infty}\frac{f_{nm}}{\lambda_{nm}}\big(\frac{m\pi}{T}\big)^{2}u_{nm}(x,t). \tag{3.9}$$

The series

$$\sum_{n,m=1}^{\infty}n^{2k}|f_{nm}| \tag{3.10}$$

is majorant for series (3.8) and the series

$$\sum_{n,m=1}^{\infty}m^{2}|f_{nm}| \tag{3.11}$$

is majorant for series (3.9) .

Integrating (3.4) by parts $2k+1$ times with respect to x and one time with respect to t, we obtain

$$|f_{nm}|=\frac{Tp^{2k+1}}{\pi^{2k+2}}\frac{1}{mn^{2k+1}}|f^{(2k+1,1)}_{nm}|, \tag{3.12}$$

$$f_{nm}=-\frac{pT^3}{\pi^4 nm^3}f^{(1,3)}_{nm}, \tag{3.13}$$

where

$$f^{(2k+1,1)}_{nm}=\int_0^p\int_0^T\frac{\partial^{2k+2}f}{\partial x^{2k+1}\partial t}\frac{2}{\sqrt{pT}}\cos\big(\frac{n\pi}{p}x\big)\cos\big(\frac{m\pi}{T}t\big)\,dt\,dx,$$

$$f^{(1,3)}_{nm}=\int_0^P\int_0^T\frac{\partial^{4}f}{\partial x\partial t^3}\frac{2}{\sqrt{PT}}\cos\big(\frac{n\pi}{P}x\big)\cos\big(\frac{m\pi}{T}t\big)\,dt\,dx.$$

Using (3.12), the Hölder inequality and the Bessel inequality from (3.10), we obtain

$$\sum_{n,m=1}^{\infty}n^{2k}|f_{nm}|\leqslant\frac{Tp^{2k+1}}{6\pi^{2k}}\|\frac{\partial^{2k+2}f}{\partial x^{2k+1}\partial t}\|_{L_2(\Omega)}.$$

Consequently, series (3.8) uniformly and absolutely converges. In 3.11 using (3.13), the Hölder inequality and the Bessel inequality, we obtain

$$\sum_{n,m=1}^{\infty}m^{2}|f_{nm}|\leqslant\frac{pT^3}{6\pi^2}\|\frac{\partial^{4}f}{\partial x\partial t^3}\|_{L_2(\Omega)}.$$

Hence, absolutely and uniformly convergence of series (3.9) follows.

Adding equality (3.8) and (3.9), we note that solution (3.2) satisfies equation (1.1). Solution (3.2) satisfies boundary conditions (1.2) and (1.3) owing to properties of functions $u_{nm}(x,t)$. From $W_2(\Omega) \subset L_2(\Omega)$ for $f \in W_2(\Omega)$ it follows estimate (42). The proof is complete. □

Corollary 3.3. *In the case even k, solvability of problem 1 depends on domain geometry. A minor change of numbers p and T can lead to an ill-posed problem.*

4. Well-posedness of problem 2

First, we will study the regular solvability of problem 2. Let k be odd number, then problem 2 is not correct. Indeed, if we seek the solution of problem 2 in the form (2.15), we obtain

$$u_n(t) = \frac{1}{\lambda_n^k} \int_0^t f_n(\tau) sh\lambda_n^k (t-\tau) d\tau$$

and $|u_n(t)| \to \infty$ as $n \to \infty$. Therefore, we will consider the case when k is even. We denote

$$V_2(\Omega) = \{U : U \in C_{x,t}^{2k-2,1}(\overline{\Omega}) \cap C_{x,t}^{2k,2}(\Omega);\ (1.2) \text{ and } (1.4) \text{ are satisfied}\}$$

$$W_3(\Omega) = \Big\{ f : f \in C_{x,t}^{k,0}(\overline{\Omega}),\ \frac{\partial^{k+1} f}{\partial x^{k+1}} \in L_2(\Omega);\ \frac{\partial^{2m} f}{\partial x^{2m}} = 0 \\ \text{at } x = 0 \text{ and } x = p,\ m = 0, 1, \dots, \frac{k-2}{2} \Big\}.$$

We define the operator A mapping $V_2(\Omega)$ into $C(\Omega)$ by the rule $Au = Lu$. Let $\overline{A}$ be the closure of A in $L_2(\Omega)$.

Note that the definition of regular solution $u(x,t) \in V_2(\Omega)$ of problem 2 is same as Definition 2.1. Moreover, the definition of strong solution of problem 2 is same as Definition 2.2, but in this case $\{u_n\} \subset V_2(\Omega)$.

We seek a regular solution of problem 2 in the form (2.15). Then, for unknown functions $u_n(t)$ we have

$$u_n'' + \lambda_n^{2k} u_n(t) = f_n(t), 0 < t < T.$$

The solution of this equation satisfying following two conditions $u_n(0) = 0$, $u_n'(0) = 0$ has the form

$$u_n(t) = \frac{1}{\lambda_n^k} \int_0^t f_n(\tau) \sin \lambda_n^k (t-\tau) d\tau. \tag{4.1}$$

Lemma 4.1. *If $f \in W_3(\Omega)$, then for any $t \in [0,T]$ the following inequalities hold:*

$$|u_n(t)| \leqslant \frac{\sqrt{T}}{\lambda_n^k} \|f_n\|_{L_2(0,T)}, \tag{4.2}$$

$$|u_n'(t)| \leqslant \frac{\sqrt{T}}{\lambda_n} \|f_n^{(1,0)}\|_{L_2(0,T)}, \tag{4.3}$$

$$|u_n''(t)| \leqslant \frac{1}{\lambda_n^2} |f_n^{(2,0)}(t)| + \frac{\sqrt{T}}{\lambda_n} \|f_n^{(k+1,0)}\|_{L_2(0,T)}. \tag{4.4}$$

Proof. Integrating (2.17) by parts $k+1$ times with respect to x, we obtain

$$|f_n(t)| = \frac{1}{\lambda_n^{k+1}} |f_n^{(k+1,0)}(t)|, \tag{4.5}$$

where

$$f_n^{(k+1,0)}(t) = \int_0^p \frac{\partial^{k+1} f}{\partial x^{k+1}} \sqrt{\frac{2}{p}} \cos \lambda_n x dx.$$

Differentiating (4.1) twice with respect to t, we obtain

$$u_n'(t) = \int_0^t f_n(\tau) \cos \lambda_n^k (t-\tau) d\tau, \tag{4.6}$$

$$u_n''(t) = f_n(t) - \lambda_n^{2k} u_n(t). \tag{4.7}$$

Applying the Hölder inequality to (4.1), we obtain inequality (4.2). Using (4.5), when $k = 0$ and applying Hölder inequality to (4.6), we obtain inequality (4.3). Taking into account (4.2), we obtain

$$\lambda_n^{2k} |u_n(t)| \leqslant \sqrt{T} \lambda_n^k \|f_n\|_{L_2(0,T)}.$$

Furthermore, using (4.5), we obtain

$$\lambda_n^{2k} |u_n(t)| \le \frac{\sqrt{T}}{\lambda_n} \|f_n^{(k+1,0)}\|_{L_2(0,T)}.$$

Using (4.5) for $k = 1$ and owing to (4.7), we obtain inequality (4.4). The proof is complete. □

Theorem 4.2. *If $f \in W_3(\Omega)$, then there exists the regular solution of problem 2.*

Proof. We will prove uniform and absolute convergence of series (2.15) and

$$\frac{\partial^{2k} u}{\partial x^{2k}} = \sum_{n=1}^{\infty} \lambda_n^{2k} u_n(t) X_n(x), \tag{4.8}$$

$$\frac{\partial^{2k} u}{\partial x^{2k}} = \sum_{n=1}^{\infty} f_n(t) X_n(x) - \sum_{n=1}^{\infty} \lambda_n^{2k} u_n(t) X_n(x). \tag{4.9}$$

From Lemma 4.2 it follows that series (2.15), (4.8), and (4.9) uniformly and absolutely converges. Adding equalities (4.8) and (4.9), we note that solution (2.15) satisfies equation (1.1). Solution (2.15) satisfies boundary conditions (1.2) owing to properties of the function $X_n(x)$. From (4.1) and (4.6) it follows that solution (2.15) satisfies conditions (1.4). The proof is complete. □

Second, we will establish a priori estimate for a solution of problem 2.

Lemma 4.3. *If $u(x,t) \in V_2(\Omega)$, $\frac{\partial^2 u}{\partial t^2} \in L_2(\Omega)$, $\frac{\partial^{2k} u}{\partial x^{2k}} \in L_2(\Omega)$, then there exists a constant $C > 0$ that depends only on numbers T and k, and does not depend on the function $u(x,t)$ such that*

$$\|u\|_{W_2^{(k,1)}(\Omega)} \leqslant C \|Lu\|_{L_2(\Omega)}, \tag{4.10}$$

where

$$\|u\|^2_{W_2^{k,1}(\Omega)} = \sum_{m=0}^{k} \Big\| \frac{\partial^m u}{\partial x^m} \Big\|^2_{L_2(\Omega)} + \Big\| \frac{\partial u}{\partial t} \Big\|^2_{L_2(\Omega)}.$$

Proof. Multiplying by $\frac{\partial u}{\partial t}$ both sides of the identity (2.2) and integrating it over the domain $\Omega_\tau = \{(x,t) : 0 < x < p, 0 < t < \tau; \tau < T\}$, we obtain

$$\int_0^P \int_0^\tau \frac{\partial u}{\partial t} \Big(\frac{\partial^{2k} u}{\partial x^{2k}} + \frac{\partial^2 u}{\partial t^2} \Big) \, dt \, dx = \int_0^P \int_0^\tau \frac{\partial u}{\partial t} Lu \, dt \, dx. \tag{4.11}$$

Using the identities

$$\frac{\partial u}{\partial t}\frac{\partial^{2k}u}{\partial x^{2k}} = \sum_{m=0}^{k-1}(-1)^m\frac{\partial}{\partial x}\Big(\frac{\partial^{m+1}u}{\partial t\partial x^m}\frac{\partial^{2k-1-m}u}{\partial x^{2k-1-m}}\Big) + (-1)^k\frac{1}{2}\frac{\partial}{\partial t}\Big(\frac{\partial^k u}{\partial x^k}\Big)^2,$$

$$\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} = \frac{1}{2}\frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial t}\Big)^2$$

and conditions (1.2), (1.4) and equality (4.11), we obtain

$$\frac{1}{2}\int_0^p\Big[\frac{\partial^k u(x,\tau)}{\partial x^k}\Big]^2dx + \frac{1}{2}\int_0^p\Big[\frac{\partial u(x,\tau)}{\partial \tau}\Big]^2dx = \int_0^p\int_0^\tau u_t Lu\,dt\,dx. \tag{4.12}$$

From equality (4.12) it follows the inequality

$$\int_0^p\Big[\frac{\partial^k u}{\partial x^k}\Big]^2dx + \int_0^p\Big[\frac{\partial u}{\partial \tau}\Big]^2dx \leqslant \int_0^p\int_0^T |u_t Lu|\,dt\,dx.$$

Integrating it with respect to τ from 0 to T, we obtain

$$\|\frac{\partial^k u}{\partial x^k}\|^2_{L_2(\Omega)} + \|\frac{\partial u}{\partial t}\|^2_{L_2(\Omega)} \leqslant 2T\int_0^p\int_0^T |u_t Lu|\,dt\,dx.$$

Then, in the same manner as the proof of Lemma 2.7, we complete the proof of Lemma 4.5. ☐

Corollary 4.4. *Applying estimate* (4.10)*, we can obtain the following facts:*

- (i) *The regular solution of problem 2 is unique and continuously depends on $f(x,t)$.*
- (ii) *The inverse operator A^{-1} exists and it is bounded.*
- (iii) $\|A^{-1}\| \leqslant C$.
- (iv) $Ker(A) = \{0\}$.
- (v) *The adjoint problem to problem 2 is well-posed.*

Third, we will study an existence and uniqueness of the strong solution of the problem 1. It is possible to represent the regular solution (2.15) in the form

$$u(x,t) = \int_0^t\int_0^p K(x,t;\xi,\tau)f(\xi,\tau)\,d\xi\,d\tau, \tag{4.13}$$

where

$$K(x,t;\xi,\tau) = \sum_{n=1}^{\infty}\frac{X_n(x)X_n(\xi)}{\lambda_n^k}\sin\big(\lambda_n^k(t-\tau)\big). \tag{4.14}$$

Obviously

$$|K(x,t;\xi,\tau)| \leqslant C, \quad C = \text{const.} > 0. \tag{4.15}$$

The method of proof of Theorem 2.8 enables us to establish the following fact.

Theorem 4.5. *For any $f \in L_2(\Omega)$ there exists unique strong solution of problem 2 and it satisfies estimate* (4.10)*, it continuously depends on $f(x,t)$ and it can be represented in the form* (4.13)*.*

Fourth, we will study the spectrum of problem 2.

Definition 4.6. We say that a problem has the Volterra property (see [14]) if the inverse operator of the problem has the Volterra property.

Theorem 4.7. *If the number k is even, then the spectrum of the problem 2 is empty.*

Proof. From (4.10), (4.13) and (4.15) we conclude that it is defined operator A^{-1} on $W_3(\Omega)$ which is inverse of the operator A and acts from $W_3(\Omega)$ to $V_2(\Omega)$ by the rule

$$(A^{-1}f)(x,t) = \int_0^p \int_0^t K(x,t;\xi,\tau)f(\xi,\tau)\,d\tau\,d\xi. \tag{4.16}$$

It follows from (4.15) that $K(x,t;\xi,\tau) \in L_2(\Omega\times\Omega)$, therefore A^{-1} is a compact operator in $L_2(\Omega)$. As $D(A^{-1}) \equiv W_3(\Omega)$ is dense in $L_2(\Omega)$, the operator A^{-1} can be extended to whole space $L_2(\Omega)$. This extension, we denote it by $\overline{A^{-1}}$, the closure of A^{-1}, $D(\overline{A^{-1}}) = L_2(\Omega)$. $\overline{A^{-1}}$ is a compact operator in $L_2(\Omega)$. Now we show that $\sigma(\overline{A^{-1}}) = \{0\}$, where $\sigma(\overline{A^{-1}})$ is the spectrum of the operator $\overline{A^{-1}}$ (see [23]). For this purpose we calculate the spectral radius of the operator $\overline{A^{-1}}$ (see [23]).

$$r(\overline{A^{-1}}) = \lim_{n\to\infty}(\|\overline{A^{-n}}\|)^{1/n}.$$

It is easy to show that

$$(\overline{A^{-n}}f)(x,t) = \int_0^p \int_0^t K_n(x,t;\xi,\tau)f(\xi,\tau)\,d\tau\,d\xi,$$

where

$$K_n(x,t;\xi,\tau) = \int_0^p \int_0^T K(x,t;\xi',\tau')K_{n-1}(\xi',\tau';\xi,\tau)d\tau' d\xi', \quad n = 2,3,\dots,$$

$$K_1(x,t;\xi,\tau) = K(x,t;\xi,\tau),$$

$$|K_n(x,t;\xi,\tau)| \leqslant \frac{C^n p^{n-1}(t-\tau)^{n-1}}{(n-1)!}.$$

By direct computation, we obtain

$$\|\overline{A^{-n}}\| \leqslant \frac{(CpT)^n}{n!}.$$

Using Stirling formula (see [13]) for $n!$ we convince that $r(\overline{A^{-1}}) = \{0\}$. Since the spectrum of $\overline{A^{-1}}$ lays in the circle $|\lambda| \leqslant r(\overline{A^{-1}})$, the spectrum consists of one point a zero. Consequently, operator $\overline{A^{-1}}$ has the Volterra property. The relation between eigenvalues of the operator $\overline{A^{-1}}$ and $\overline{A}$ is as follows [6]:

If $\mu_n \neq 0$ is eigenvalue of the operator $\overline{A^{-1}}$, then μ_n^{-1} is eigenvalue of operator $\overline{A}$. Since zero is not eigenvalue of the operator $\overline{A^{-1}}$, then the spectrum of the problem 2 is empty set. The proof of Theorem 4.7 is complete. □

Corollary 4.8. *For even k the problem 2 has the Volterra property.*

Note that problem 2 is not self-adjoint.

Problem 2^***.** Obtain the solution $v(x,t)$ of (1.1) in the domain Ω satisfying conditions (1.2) and $v(x,T) = 0$, $v_t(x,T) = 0$, $0 < x < p$.

The method of investigation of problem 2 is applicable also for problem 2^*. Therefore, all statements concerning to problem 2 are true and for the problem 2^*.

Corollary 4.9. *If k is odd, then from estimate* (2.33) *follows that the solution of* (1.1) *belongs to $C^\infty(\Omega)$. Therefore, for k odd,* (1.1) *is hypoelliptic. If k is even, then* (1.1) *is not hypoelliptic.*

5. Conclusion

In this article we investigated two boundary value problems for the equation of the even order in a rectangular domain. Problem 1 for k odd has a unique solution. For the solution of problem 1, a priori estimate in the norm of space $W_2^{2k,2}(\Omega)$ is obtained. For k even, the solvability of problem 1 depends on geometry of the domain. In this case, there is a condition for the sizes of the domain. If this condition is satisfied, then problem 1 is well-posed. Problem 2 for k odd is not correct. For k even, this problem has the unique solution. For the solution of this problem, a priori estimate in the norm of space $W_2^{k,1}(\Omega)$ is obtained. The spectrum of problems 1 and 2 is studied. These statements without proof are formulated in [2]. Moreover, applying the result in [5] the two-step difference schemes of a high order of accuracy schemes for the numerical solution of well-posed problems 1 and 2 can be presented. Of course, the stability inequalities for the solution of these difference schemes have been established without any assumptions about the grid steps τ in t and h in the space variable x.

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