Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 11, pp. 1–19.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

A FRICTIONAL CONTACT PROBLEM WITH DAMAGE AND ADHESION FOR AN ELECTRO ELASTIC-VISCOPLASTIC BODY

ADEL AISSAOUI, NACERDINE HEMICI

Abstract. We consider a quasistatic frictional contact problem for an electro elastic-viscopalastic body with damage and adhestion. The contact is modelled with normal compliance. The adhesion of the contact surfaces is taken into account and modelled by a surface variable. We derive variational formulation for the model which is in the form of a system involving the displacement field, the electric potential field, the damage field and the adhesion field. We prove the existence of a unique weak solution to the problem. The proof is based on arguments of time-dependent variational inequalities, parabolic inequalities, differential equations and fixed point.

1. Introduction

Considerable progress has been achieved recently in modeling, mathematical analysis and numerical simulations of various contact processes and, as a result, a general mathematical theory of contact mechanics (MTCM) is currently maturing. It is concerned with the mathematical structures which underlie general contact problems with different constitutive laws (i.e., different materials), varied geometries and settings, and different contact conditions, see for instance [7, 20, 21] and the references therein. The theory's aim is to provide a sound, clear and rigorous background for the constructions of models for contact between deformable bodies; proving existence, uniqueness and regularity results; assigning precise meaning to solutions; and the necessary setting for finite element approximations of the solutions.

There is a considerable interest in frictional or frictionless contact problems involving piezoelectric materials, see for instance [3, 16, 23] and the references therein. Indeed, many actuators and sensors in engineering controls are made of piezoelectric ceramics. However, there exists virtually no mathematical results about contact problems for such materials and there is a need to expand the MTCM to include the coupling between the mechanical and electrical material properties.

The piezoelectric effect is characterized by such a coupling between the mechanical and electrical properties of the materials. This coupling, leads to the appearance of electric field in the presence of a mechanical stress, and conversely, mechanical

2000 *Mathematics Subject Classification.* 74C10, 49J40, 74M10, 74M15.
Key words and phrases. Piezoelectric; elastic-viscoplastic; friction contact; normal compliance; fixed point; variational inequality.

Submitted June 28, 2013. Published January 8, 2014.

stress is generated when electric potential is applied. The first effect is used in sensors, and the reverse effect is used in actuators.

On a nano-scale, the piezoelectric phenomenon arises from a nonuniform charge distribution within a crystal's unit cell. When such a crystal is deformed mechanically, the positive and negative charges are displaced by a different amount causing the appearance of electric polarization. So, while the overall crystal remains electrically neutral, an electric polarization is formed within the crystal. This electric polarization due to mechanical stress is called *piezoelectricity.* A deformable material which exhibits such a behavior is called a *piezoelectric material.* Piezoelectric materials for which the mechanical properties are elastic are also called *electro-elastic materials* and piezoelectric materials for which the mechanical properties are viscoelastic are also called *electro-viscoelastic materials.*

Only some materials exhibit sufficient piezoelectricity to be useful in applications. These include quartz, Rochelle salt, lead titanate zirconate ceramics, barium titanate, and polyvinylidene flouride (a polymer film), and are used extensively as switches and actuators in many engineering systems, in radioelectronics, electroacoustics and in measuring equipment. General models for electro-elastic materials can be found in [14, 15] and, more recently, in [1, 12, 18]. A static and a slip-dependent frictional contact problems for electro-elastic materials were studied in [3, 16] and in [22], respectively. A contact problem with normal compliance for electro-viscoelastic materials was investigated in [13, 23]. In the last two references [22, 23] the foundation was assumed to be insulated.

The variational formulations of the corresponding problems were derived and existence and uniqueness of weak solutions were obtained.

Here we continue this line of research and study a quasistatic frictionless contact problem for an electro-viscoelastic material, in the framework of the MTCM, when the foundation is conductive; our interest is to describe a physical process in which both contact, friction and piezoelectric effect are involved, and to show that the resulting model leads to a well-posed mathematical problem. Taking into account the conductivity of the foundation leads to new and nonstandard boundary conditions on the contact surface, which involve a coupling between the mechanical and the electrical unknowns.

The rest of the article is structured as follows. In Section 2 we describe the model of the frictional contact process between an electro-viscoplastic body and a conductive deformable foundation. In Section 3 we introduce some notation, list the assumptions on the problem's data, and derive the variational formulation of the model. It consists of a variational inequality for the displacement field coupled with a nonlinear time-dependent variational equation for the electric potential. We state our main result, the existence of a unique weak solution to the model in Theorem 3.3. The proof of the theorem is provided in Section 4, where it is carried out in several steps and is based on arguments of evolutionary inequalities with monotone operators, and a fixed point theorem.

2. The model

We describe the model for the process, we present its variational formulation. The physical setting is the following. An electro elastic-viscoplastic body occupies a bounded domain $\Omega \subset \mathbb{R}_d$ $(d = 2, 3)$ with outer Lipschitz surface Γ. The body undergoes the action of body forces of density f_0 and volume electric charges of density q_0. It also undergoes the mechanical and electric constraint on the boundary.

We consider a partition of Γ into three disjoint parts Γ_1, Γ_2 and Γ_3, on one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. We assume that $meas(\Gamma_1) > 0$ and $meas(\Gamma_a) > 0$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0, T)$, so the displacement field vanishes there. A surface traction of density f_2 act on $\Gamma_2 \times (0, T)$ and a body force of density f_0 acts in $\Omega \times (0, T)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (0, T)$ and a surface electric charge of density q_2 is prescribed on $\Gamma_b \times (0, T)$. The body is in adhesive contact with an obstacle, or foundation, over the contact surface Γ_3.

We denote by $\mathbf{u}$ the displacement field, by σ the stress tensor field and by $\varepsilon(\mathbf{u})$ the linearized strain tensor. We use an electro elastic-viscoplastic constitutive law with damage given by

$$\begin{aligned}\boldsymbol{\sigma}(t) &= \mathcal{A}(\varepsilon(\dot{\mathbf{u}}(t))) + \mathcal{B}(\varepsilon(\mathbf{u}(t))) \\ &\quad + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}(s) - \mathcal{A}(\varepsilon(\dot{\mathbf{u}}(s))) + \mathcal{E}^* E(\varphi), \varepsilon(\mathbf{u}(s)), \beta(s)\Big) ds - \mathcal{E}^* E(\varphi), \\ \mathbf{D} &= \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}) + \boldsymbol{B} E(\varphi),\end{aligned}$$

where $\mathcal{A}$ and $\mathcal{B}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, $E(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E} = (e_{ijk})$ represents the third order piezoelectric tensor $\mathcal{E}^*$ is its transpose and $\boldsymbol{B}$ denotes the electric permittivity tensor, and $\mathcal{G}$ is a nonlinear constitutive function which describes the visco-plastic behavior of the material, where β is an internal variable describing the damage of the material caused by elastic deformations. The differential inclusion used for the evolution of the damage field is

$$\dot{\beta} - k\Delta\beta + \partial\varphi_k(\beta) \ni S(\boldsymbol{\sigma} - \mathcal{A}\varepsilon(\dot{\mathbf{u}}) + \mathcal{E}^* E(\varphi), \varepsilon(\mathbf{u}), \beta)$$

where K denotes the set of admissible damage functions defined by

$$K = \{\xi \in V : 0 \le \xi(x) \le 1 \text{ a.e. } x \in \Omega\},$$

where k is a positive coefficient, $\partial\varphi_k$ denotes the subdifferential of the indicator function of the set K and S is a given constitutive function which describes the sources of the damage in the system. When $\beta = 1$ the material is undamaged, when $\beta = 0$ the material is completely damaged, and for $0 < \beta < 1$ there is partial damage. General models of mechanical damage, which were derived from thermodynamical considerations and the principle of virtual work, can be found in [8] and [9] and references therein. The models describe the evolution of the material damage which results from the excess tension or compression in the body as a result of applied forces and tractions. Mathematical analysis of one-dimensional damage models can be found in [10]. We denote by $\mathbf{x} \in \Omega \cup \Gamma$ and $t \in [0, T]$ the spatial and the time variable, respectively, and, to simplify the notation, we do not indicate in what follows the dependence of various functions on $\mathbf{x}$ and t. In this paper $i, j, k, l = 1, \dots, d$, summation over two repeated indices is implied, and the index that follows a comma represents the partial derivative with respect to the corresponding component of $\mathbf{x}$. A dot over a variable represents the time derivative. We use the notation $\mathbb{S}^d$ for the space of second order symmetric tensors on $\mathbb{R}^d$ and " $\cdot$ " and $\|\cdot\|$ represent the inner product and the Euclidean norm on

$\mathbb{S}^d$ and $\mathbb{R}^d$, respectively, that is $\mathbf{u}\cdot\mathbf{v}=u_iv_i$, $\|\mathbf{v}\|=(\mathbf{v}\cdot\mathbf{v})^{1/2}$ for $\mathbf{u},\mathbf{v}\in\mathbb{R}^d$, and $\boldsymbol{\sigma}\cdot\boldsymbol{\tau}=\sigma_{ij}\tau_{ij}$, $\|\boldsymbol{\tau}\|=(\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{1/2}$ for $\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{S}^d$. We also use the usual notation for the normal components and the tangential parts of vectors and tensors, respectively, by $u_\nu=\mathbf{u}\cdot\boldsymbol{\nu}$, $\mathbf{u}_\tau=\mathbf{u}-u_n\boldsymbol{\nu}$, $\sigma_\nu=\sigma_{ij}\nu_i\nu_j$, and $\boldsymbol{\sigma}_\tau=\boldsymbol{\sigma}\boldsymbol{\nu}-\sigma_\nu\boldsymbol{\nu}$. The classical model for the process is as follows.

Problem P. Find a displacement field $\mathbf{u}:\Omega\times[0,T]\to\mathbb{R}^d$, a stress field $\boldsymbol{\sigma}:\Omega\times[0,T]\to\mathbb{S}^d$, an electric potential $\varphi:\Omega\times[0,T]\to\mathbb{R}$, an electric displacement field $\mathbf{D}:\Omega\times[0,T]\to\mathbb{R}^d$, a damage field $\beta:\Omega\times[0,T]\to\mathbb{R}$, and a bonding field $\alpha:\Gamma_3\times[0,T]\to\mathbb{R}$ such that

$$\begin{aligned}\boldsymbol{\sigma}(t)=\mathcal{A}\varepsilon(\dot{\mathbf{u}}(t))+\mathcal{B}\varepsilon(\mathbf{u}(t))+\int_0^t\mathcal{G}\Big(\boldsymbol{\sigma}(s)-\mathcal{A}\varepsilon(\dot{\mathbf{u}}(s))\\ -\mathcal{E}^*\nabla\varphi(s),\varepsilon(\mathbf{u}(s)),\beta(s)\Big)ds+\mathcal{E}^*\nabla\varphi(t)\quad\text{in }\Omega\times(0,T),\end{aligned}\tag{2.1}$$

$$\mathbf{D}=\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u})-\boldsymbol{B}\nabla(\varphi)\quad\text{in }\Omega\times(0,T),\tag{2.2}$$

$$\dot{\beta}-k\Delta\beta+\partial\varphi_K(\beta)\ni S(\sigma-\mathcal{A}\varepsilon(\dot{u})-\mathcal{E}^*\nabla(\varphi),\varepsilon(\mathbf{u}),\beta),\tag{2.3}$$

$$\operatorname{Div}\boldsymbol{\sigma}+\mathbf{f}_0=0\quad\text{in }\Omega\times(0,T),\tag{2.4}$$

$$\operatorname{div}\mathbf{D}-q_0=0\quad\text{in }\Omega\times(0,T),\tag{2.5}$$

$$\mathbf{u}=\mathbf{0}\quad\text{on }\Gamma_1\times(0,T),\tag{2.6}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu}=\mathbf{f}_2\quad\text{on }\Gamma_2\times(0,T),\tag{2.7}$$

$$\begin{cases}-\sigma_\nu=p_\nu(u_\nu-g)\\ \|\sigma_\tau\|\le p_\tau(u_\nu-g)\end{cases}\text{on }\Gamma_3\times(0,T),\tag{2.8}$$

$$\dot{u}_\tau\neq0\Rightarrow\sigma_\tau=-p_\tau(u_\nu-g)\frac{\dot{\mathbf{u}}_\tau}{\|\dot{\mathbf{u}}_\tau\|}\quad\text{on }\Gamma_3\times(0,T),\tag{2.9}$$

$$\dot{\alpha}=-(\alpha(\gamma_\nu R_\nu(u_\nu))^2+\gamma_\tau\|\mathbf{R}_\tau(\mathbf{u}_\tau)\|^2)-\varepsilon_a)_+\quad\text{on }\Gamma_3\times(0,T),\tag{2.10}$$

$$\frac{\partial\beta}{\partial\nu}=0\quad\text{on }\Gamma\times(0,T),\tag{2.11}$$

$$\varphi=0\quad\text{on }\Gamma_a\times(0,T),\tag{2.12}$$

$$\mathbf{D}\cdot\boldsymbol{\nu}=q_2\quad\text{on }\Gamma_b\times(0,T),\tag{2.13}$$

$$\mathbf{D}\cdot\boldsymbol{\nu}=\psi(u_\nu-g)\phi_l(\varphi-\varphi_0)\quad\text{on }\Gamma_3\times(0,T),\tag{2.14}$$

$$\mathbf{u}(0)=\mathbf{u}_0,\quad\beta(0)=\beta_0\quad\text{in }\Omega,\tag{2.15}$$

$$\alpha(0)=\alpha_0\text{ on }\Gamma_3.\tag{2.16}$$

We now describe problem (2.1)-(2.16) and provide explanation of the equations and the boundary conditions.

Equations(2.1) and (2.2) represent the electro elastic-viscoplastic constitutive law with damage,the evolution of the damage field is governed by the inclusion of parabolic type given by the relation (2.3) where S is the mechanical source of the damage growth, assumed to be rather general function of the strains and damage itself, $\partial\varphi_k$ is the subdifferential of the indicator function of the admissible damage functions set K. Next equations(2.4) and (2.5) are the steady equations for the stress and electric-displacement field, respectively, in which "Div" and "div" denote the divergence operator for tensor and vector valued functions, i.e.,

$$\operatorname{Div}\boldsymbol{\sigma}=(\sigma_{ij,j}),\quad\operatorname{div}\mathbf{D}=(D_{i,i}).$$

We use these equations since the process is assumed to be mechanically quasistatic and electrically static.

Conditions (2.6) and (2.7) are the displacement and traction boundary conditions, whereas (2.12) and (2.13) represent the electric boundary conditions; the displacement field and the electrical potential vanish on Γ_1 and Γ_a, respectively, while the forces and free electric charges are prescribed on Γ_2 and Γ_b, respectively.

We turn to the boundary condition (2.8)and (2.9) which describe the mechanical conditions on the potential contact surface Γ_3. The normal compliance function p_ν in(2.8),is described below, and g represents the gap in the reference configuration between Γ_3 and the foundation, measured along the direction of ν. When positive, $u_\nu - g$ represents the interpenetration of the surface asperities into those of the foundation. This condition was first introduced in [12] and used in a large number of papers, see for instance[6, 9, 10, 16] and the references therein.

Conditions (2.9)is the associated friction law where p_τ is a given function. According to (2.9) the tangential shear cannot exceed the maximum frictional resistance $p_\tau(u_\nu - g)$, the so-called friction bound. Moreover, when sliding commences, the tangential shear reaches the friction bound and opposes the motion. Frictional contact conditions of the form (2.8), (2.9) have been used in various papers, see, e.g., [7, 8, 20] and the references therein.

Equation (2.10) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [4] see also [20, 21] for more details. Here, besides γ_ν, two new adhesion coefficients are involved, γ_τ and ϵ_a. Notice that in this model once debonding occurs bonding cannot be reestablished since, as it follows form equation (2.10), $\dot{\alpha} \leq 0$.

The contribution of the adhesive to the normal traction is represented by the term $\gamma_\nu \alpha^2 R_\nu(u_\nu)$, the adhesive traction is tensile and is proportional, with proportionality coefficient γ_ν, to the square of the intensity of adhesion and to the normal displacement, but as long as it does not exceed the bond length L. The maximal tensile traction is $\gamma_\nu L.R_\nu$ is the truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \leq s \leq 0, \\ 0 & \text{if } s > 0 \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of the operator R_ν, together with the operator $\mathbf{R}_\tau$ defined below, is motivated by mathematical arguments but it is not restrictive from the physical point of view, since no restriction on the size of the parameter L is made in what follows. Condition (2.10) represents the adhesive contact condition on the tangential plane, in which p_τ is a given function and $\mathbf{R}_\tau$ is the truncation operator given by

$$\mathbf{R}_\tau(\mathbf{v}) = \begin{cases} \mathbf{v} & \text{if } |\mathbf{v}| \leq L, \\ L\frac{\mathbf{v}}{|\mathbf{v}|} & \text{if } |\mathbf{v}| > L\,. \end{cases}$$

This condition shows that the shear on the contact surface depends on the bonding field and on the tangential displacement, but as long as it does not exceed the bond length L. The frictional tangential traction is assumed to be much smaller than the adhesive one and, therefore, omitted. The introduction of the operator R_ν, together with the operator R_τ defined above, is motivated by mathematical arguments but it is not restrictive for physical point of view, since no restriction on the size of the parameter L is made in what follows.

The relation (2.11) describes a homogeneous Neumann boundary condition where $\partial\beta/\partial\nu$ is the normal derivative of β. (2.12) and (2.13) represent the electric boundary conditions.

Next,(2.14) is the electrical contact condition on Γ_3, introduced in [13]. It may be obtained as follows. First, unlike previous papers on piezoelectric contact, we assume that the foundation is electrically conductive and its potential is maintained at φ_0. When there is no contact at a point on the surface (i.e., $u_\nu < g$), the gap is assumed to be an insulator (say, it is filled with air), there are no free electrical charges on the surface and the normal component of the electric displacement field vanishes. Thus,

$$u_\nu < g \;\Rightarrow\; \mathbf{D}\cdot\boldsymbol{\nu} = 0. \tag{2.17}$$

During the process of contact (i.e., when $u_\nu \geq g$) the normal component of the electric displacement field or the free charge is assumed to be proportional to the difference between the potential of the foundation and the body's surface potential, with k as the proportionality factor. Thus,

$$u_\nu \geq g \;\Rightarrow\; \mathbf{D}\cdot\boldsymbol{\nu} = k\,(\varphi - \varphi_0). \tag{2.18}$$

We combine (2.17), (2.18) to obtain

$$\mathbf{D}\cdot\boldsymbol{\nu} = k\,\chi_{[0,\infty)}(u_\nu - g)(\varphi - \varphi_0), \tag{2.19}$$

where $\chi_{[0,\infty)}$ is the characteristic function of the interval $[0,\infty)$; that is,

$$\chi_{[0,\infty)}(r) = \begin{cases} 0 & \text{if } r < 0, \\ 1 & \text{if } r \geq 0. \end{cases}$$

Condition (2.19) describes perfect electrical contact and is somewhat similar to the well-known Signorini contact condition. Both conditions may be over-idealizations in many applications. To make it more realistic, we regularize condition (2.19) and write it as(2.14) in which $k\,\chi_{[0,\infty)}(u_\nu - g)$ is replaced with ψ which is a regular function which will be described below, and ϕ_l is the truncation function

$$\phi_l(s) = \begin{cases} -l & \text{if } s < -l, \\ s & \text{if } -l \leq s \leq l, \\ l & \text{if } s > l, \end{cases}$$

where l is a large positive constant. We note that this truncation does not pose any practical limitations on the applicability of the model, since l may be arbitrarily large, higher than any possible peak voltage in the system, and therefore in applications $\phi_l(\varphi - \varphi_0) = \varphi - \varphi_0$. The reasons for the regularization (2.14) of (2.19) are mathematical. First, we need to avoid the discontinuity in the free electric charge when contact is established and, therefore, we regularize the function $k\chi_{[0,\infty)}$ in (2.19) with a Lipschitz continuous function ψ. A possible choice is

$$\psi(r) = \begin{cases} 0 & \text{if } r < 0, \\ k\delta r & \text{if } 0 \leq r \leq 1/\delta, \\ k & \text{if } r > \delta, \end{cases} \tag{2.20}$$

where $\delta > 0$ is a small parameter. This choice means that during the process of contact the electrical conductivity increases as the contact among the surface asperities improves, and stabilizes when the penetration $u_\nu - g$ reaches the value δ. Secondly, we need the term $\phi_l(\varphi - \varphi_0)$ to control the boundednes of $\varphi - \varphi_0$. Note that when $\psi \equiv 0$ in (2.14) then

$$\mathbf{D} \cdot \boldsymbol{\nu} = 0, \quad \text{on } \Gamma_3 \times (0, T), \tag{2.21}$$

which decouples the electrical and mechanical problems on the contact surface. Condition (2.21) models the case when the obstacle is a perfect insulator and was used in [3, 16, 22, 23]. Condition (2.14), instead of (2.21), introduces strong coupling between the mechanical and the electric boundary conditions and leads to a new and nonstandard mathematical model.

Because of the friction condition (2.9), which is non-smooth, we do not expect the problem to have, in general, any classical solutions.

In equation (2.15) $\mathbf{u}_0$ is the initial displacement, and β_0 is the initial damage. Finally, in equation (2.16) α_0 denotes the initial bonding.

To obtain the variational formulation of the problem (2.1)-(2.16) we introduce for the bonding field the set

$$\mathcal{Z} = \left\{ \theta \in L^\infty(0, T; L^2(\Gamma_3)) : 0 \le \theta(t) \le 1, \forall t \in [0, T], \text{a.e. on } \Gamma_3 \right\}$$

For this reason, we derive in the next section a variational formulation of the problem and investigate its solvability. Moreover, variational formulations are also starting points for the construction of finite element algorithms for this type of problems.

3. Variational formulation and the main result

We use standard notation for the L^p and the Sobolev spaces associated with Ω and Γ and, for a function $\zeta \in H^1(\Omega)$ we still write ζ to denote its trace on Γ. We recall that the summation convention applies to a repeated index. For the electric displacement field we use two Hilbert spaces

$$\mathcal{W} = L^2(\Omega)^d, \quad \mathcal{W}_1 = \{ \mathbf{D} \in \mathcal{W} : \operatorname{div} \mathbf{D} \in L^2(\Omega) \},$$

endowed with the inner products

$$(\mathbf{D}, \mathbf{E})_{\mathcal{W}} = \int_\Omega D_i E_i \, dx, \quad (\mathbf{D}, \mathbf{E})_{\mathcal{W}_1} = (\mathbf{D}, \mathbf{E})_{\mathcal{W}} + (\operatorname{div} \mathbf{D}, \operatorname{div} \mathbf{E})_{L^2(\Omega)},$$

and the associated norms $\| \cdot \|_{\mathcal{W}}$ and $\| \cdot \|_{\mathcal{W}_1}$, respectively. The electric potential field is to be found in

$$W = \{ \zeta \in H^1(\Omega) : \zeta = 0 \text{ on } \Gamma_a \}.$$

Since $\operatorname{meas} \Gamma_a > 0$, the Friedrichs-Poincaré inequality holds, thus,

$$\|\nabla \zeta\|_{\mathcal{W}} \ge c_F \, \|\zeta\|_{H^1(\Omega)}, \quad \forall \, \zeta \in W, \tag{3.1}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a. On W, we use the inner product

$$(\varphi, \zeta)_W = (\nabla \varphi, \nabla \zeta)_{\mathcal{W}},$$

and let $\| \cdot \|_W$ be the associated norm. It follows from (3.1) that $\| \cdot \|_{H^1(\Omega)}$ and $\| \cdot \|_W$ are equivalent norms on W and therefore $(W, \| \cdot \|_W)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant c_0, depending only on Ω, Γ_a and Γ_3, such that

$$\|\zeta\|_{L^2(\Gamma_3)} \le c_0 \|\zeta\|_W, \quad \forall \, \zeta \in W. \tag{3.2}$$

We recall that when $\mathbf{D} \in \mathcal{W}_1$ is a sufficiently regular function, the Green type formula holds:

$$(\mathbf{D}, \nabla\zeta)_{\mathcal{W}} + (\operatorname{div}\mathbf{D}, \zeta)_{\mathcal{W}} = \int_\Gamma \mathbf{D}\cdot\boldsymbol{\nu}\zeta da, \quad \forall\zeta \in H^1(\Omega). \tag{3.3}$$

For the stress and strain variables, we use the real Hilbert spaces

$$Q = \{\boldsymbol{\tau} = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^2(\Omega)\} = L^2(\Omega)^{d\times d}_{sym},$$
$$Q_1 = \{\boldsymbol{\sigma} = (\sigma_{ij}) \in Q : \operatorname{div}\boldsymbol{\sigma} = (\sigma_{ij,j}) \in \mathcal{W}\},$$

endowed with the respective inner products

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_Q = \int_\Omega \sigma_{ij}\tau_{ij}dx, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_{Q_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_Q + (\operatorname{div}\boldsymbol{\sigma}, \operatorname{div}\boldsymbol{\tau})_{\mathcal{W}},$$

and the associated norms $\|\cdot\|_Q$ and $\|\cdot\|_{Q_1}$. For the displacement variable we use the real Hilbert space

$$H_1 = \{\mathbf{u} = (u_i) \in \mathcal{W} : \boldsymbol{\varepsilon}(\mathbf{u}) \in Q\},$$

endowed with the inner product

$$(\mathbf{u}, \mathbf{v})_{H_1} = (\mathbf{u}, \mathbf{v})_{\mathcal{W}} + (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_Q,$$

and the norm $\|\cdot\|_{H_1}$. When $\boldsymbol{\sigma}$ is a regular function, the following Green's type formula holds,

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}))_Q + (\operatorname{Div}\boldsymbol{\sigma}, \mathbf{v})_{\mathcal{W}} = \int_\Gamma \boldsymbol{\sigma}\nu\cdot\mathbf{v}da, \quad \forall\mathbf{v} \in H_1. \tag{3.4}$$

Next, we define the space

$$V = \{\mathbf{v} \in H_1 : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_1\}.$$

Since $\operatorname{meas}\Gamma_1 > 0$, Korn's inequality (e.g.,[5, pp. 16–17]) holds and

$$\|\boldsymbol{\varepsilon}(\mathbf{v})\|_Q \geq c_K\,\|\mathbf{v}\|_{H_1}, \quad \forall\,\mathbf{v} \in V, \tag{3.5}$$

where $c_K > 0$ is a constant which depends only on Ω and Γ_1. On the space V we use the inner product

$$(\mathbf{u}, \mathbf{v})_V = (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_Q, \quad \|v\|_V = \|\varepsilon(v)\|_Q, \tag{3.6}$$

and let $\|\cdot\|_V$ be the associated norm. It follows from (3.5) that the norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent on V and, therefore, the space $(V, (\cdot,\cdot)_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant $\widetilde{c}_0$, depending only on Ω, Γ_1 and Γ_3, such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \leq \widetilde{c}_0\|\mathbf{v}\|_V, \quad \forall\mathbf{v} \in V. \tag{3.7}$$

Finally, for a real Banach space $(X, \|\cdot\|_X)$ we use the usual notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$ where $1 \leq p \leq \infty$, $k = 1, 2, \dots$; we also denote by $C([0,T];X)$ and $C^1([0,T];X)$ the spaces of continuous and continuously differentiable functions on $[0,T]$ with values in X, with the respective norms

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]}\|x(t)\|_X,$$
$$\|x\|_{C^1([0,T];X)} = \max_{t\in[0,T]}\|x(t)\|_X + \max_{t\in[0,T]}\|\dot{x}(t)\|_X.$$

We complete this section with the following version of the classical theorem of Cauchy-Lipschitz (see, e.g., [21],p.48]).

Theorem 3.1. *Assume that* $(X, |\cdot|_X)$ *is a real Banach space and* $T > 0$. *Let* $F(t,.) : X \to X$ *be an operator defined a.e. on* $(0,T)$ *satisfying the following conditions:*

(1) *There exists a constant* $L_F > 0$ *such that*

$$\|F(t,x) - F(t,y)\|_X \le L_F \|x-y\|_X \quad \forall x, y \in X, \text{ a.e. } t \in (0,T).$$

(2) *There exists* $p \ge 1$ *such that* $t \mapsto F(t,x) \in L^p(0,T;X)$ *for all* $x \in X$, *then for any* $x_0 \in X$, *there exists a unique function* $x \in W^{1,p}(0,T;X)$ *such that*

$$\dot{x}(t) = F(t, x(t)) \text{ a.e. } t \in (0,T), \; x(0) = x_0.$$

Theorem 3.1 will be used in section 4 to prove the unique solvability of the intermediate problem involving the bonding field. Moreover, if X_1 and X_2 are real Hilbert spaces then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(\cdot,\cdot)_{X_1\times X_2}$. Recall that the dot represents the time derivative.

We now list the assumptions on the problem's data. The *viscosity operator* $\mathcal{A}$ and the *elasticity operator* $\mathcal{B}$ are assumed to satisfy the conditions:

(a) $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$.
(b) There exists $L_{\mathcal{A}} > 0$ such that $\|\mathcal{A}(\mathbf{x}, \boldsymbol{\xi}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\xi}_2)\| \le L_{\mathcal{A}} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|$ for all $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d$, a.e. $\mathbf{x} \in \Omega$.
(c) There exists $m_{\mathcal{A}} > 0$ such that $(\mathcal{A}(\mathbf{x}, \boldsymbol{\xi}_1) - \mathcal{A}(\mathbf{x}, \boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \ge m_{\mathcal{A}} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|^2$ for all $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d$, a.e. $\mathbf{x} \in \Omega$. (3.8)
(d) The mapping $\mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \boldsymbol{\xi})$ is Lebesgue measurable on Ω, for any $\boldsymbol{\xi} \in \mathbb{S}^d$.
(e) The mapping $\mathbf{x} \mapsto \mathcal{A}(\mathbf{x}, \mathbf{0})$ belongs to Q.

(a) $\mathcal{B} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$.
(b) There exists $L_{\mathcal{B}} > 0$ such that $\|\mathcal{B}(\mathbf{x}, \boldsymbol{\xi}_1) - \mathcal{B}(\mathbf{x}, \boldsymbol{\xi}_2)\| \le L_{\mathcal{B}} \|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|$ for all $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{S}^d$, a.e. $\mathbf{x} \in \Omega$.
(c) The mapping $\mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \boldsymbol{\xi})$ is Lebesque measurable on Ω, for any $\boldsymbol{\xi} \in \mathbb{S}^d$. (3.9)
(d) The mapping $\mathbf{x} \mapsto \mathcal{B}(\mathbf{x}, \mathbf{0})$ belongs to Q.

The *plasticity operator* $\mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{S}^d$ satisfies

(a) There exists a constant $L_{\mathcal{G}} > 0$ such that $\|\mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \theta_1, \varsigma_1) - \mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \theta_2, \varsigma_2)\| \le L_{\mathcal{G}}(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\theta_1 - \theta_2\| + \|\varsigma_1 - \varsigma_2\|)$ for all $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in \mathbb{S}^d$, for all $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d$ for all $\theta_1, \theta_2 \in \mathbb{R}$, for all $\varsigma_1, \varsigma_2 \in \mathbb{R}$ a.e. $\mathbf{x} \in \Omega$; (3.10)
(b) The mapping $\mathbf{x} \to \mathcal{G}(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \theta, \varsigma)$ is Lebesgue measurable on Ω for all $\boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d$, for all $\theta, \varsigma \in \mathbb{R}$,
(c) The mapping $\mathbf{x} \to \mathcal{G}(x, 0, 0, 0, 0) \in Q$.

The *electric permittivity operator* $\boldsymbol{B} = (\boldsymbol{B}_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

(a) $\boldsymbol{B}(x, E) = (\boldsymbol{B}_{ij}(x) E_j)$ for all $E = (E_i) \in \mathbb{R}^d$, a. e. $x \in \Omega$.
(b) $\boldsymbol{B}_{ij} = \boldsymbol{B}_{ji} \in L^\infty(\Omega)$, $1 \le i, j \le d$. (3.11)
(c) There exists a constant $M_{\boldsymbol{B}} > 0$ such that $\boldsymbol{B}E.E \ge M_{\boldsymbol{B}} |E|^2$ for all $E = (E_i) \in \mathbb{R}^d$, a. e. in Ω.

The *piezoelectric operator* $\mathcal{E} : \Omega \times S^d \to \mathbb{R}^d$ satisfies

$$\begin{aligned}&\text{(a) } \mathcal{E} = (e_{ijk}),\ 4e_{ijk} \in L^\infty(\Omega),\ 1 \le i,j,k \le d.\\ &\text{(b) } \mathcal{E}(\mathbf{x})\boldsymbol{\sigma}.\boldsymbol{\tau} = \boldsymbol{\sigma}.\mathcal{E}^*\boldsymbol{\tau} \text{ for all } \boldsymbol{\sigma} \in S^d \text{ and all } \boldsymbol{\tau} \in \mathbb{R}^d.\end{aligned} \tag{3.12}$$

The *damage source function* $S : \Omega \times S^d \times \times S^d \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\begin{aligned}&\text{(a) There exists a constant } M_S > 0 \text{ such that } \|S(\mathbf{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1,\beta_1) - \\ &S(\mathbf{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2,\beta_2)\| \le M_S(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\beta_1 - \beta_2\|) \text{ for all}\\ &\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in S^d, \text{ for all } \beta_1,\beta_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Omega.\\ &\text{(b) for all } \boldsymbol{\sigma},\boldsymbol{\varepsilon} \in S^d, \beta \in \mathbb{R},\ S(\mathbf{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon},\beta) \text{ is Lebesgue measurable on } \Omega.\\ &\text{(c) The mapping } \mathbf{x} \mapsto S(\mathbf{x},0,0,0) \text{ belongs to } L^2(\Omega).\end{aligned} \tag{3.13}$$

The *normal compliance functions* $p_r : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$, $(r = \nu, \tau)$ satisfy

$$\begin{aligned}&\text{(a) There exists } L_r > 0 \text{ such that } \|p_r(\mathbf{x},u_1) - p_r(\mathbf{x},u_2)\| \le L_r\|u_1 - u_2\|\\ &\text{for all } u_1,u_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3.\\ &\text{(b) } \mathbf{x} \mapsto p_r(\mathbf{x},u) \text{ is measurable on } \Gamma_3 \text{ for all } u \in \mathbb{R}.\\ &\text{(c) } \mathbf{x} \mapsto p_r(\mathbf{x},u) = 0 \text{ for all } u \le 0.\end{aligned} \tag{3.14}$$

An example of a normal compliance function p_ν which satisfies conditions (3.14) is $p_\nu(u) = c_\nu u_+$ where $c_\nu \in L^\infty(\Gamma_3)$ is a positive surface stiffness coefficient, and $u_+ = \max\{0,u\}$. The choices $p_\tau = \mu p_\nu$ and $p_\tau = \mu p_\nu(1 - \delta p_\nu)_+$ in (2.9), where $\mu \in L^\infty(\Gamma_3)$ and $\delta \in L^\infty(\Gamma_3)$ are positive functions, lead to the usual or to a modified Coulomb's law of dry friction, respectively, see [7, 8, 24] for details. Here, μ represents the coefficient of friction and δ is a small positive material constant related to the wear and hardness of the surface. We note that if p_ν satisfies condition (3.14) then p_τ satisfies it too, in both examples. Therefore, we conclude that the results below are valid for the corresponding piezoelectric frictional contact models.

The *surface electrical conductivity function* $\psi : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies:

$$\begin{aligned}&\text{(a) There exists } L_\psi > 0 \text{ such that } \|\psi(\mathbf{x},u_1) - \psi(\mathbf{x},u_2)\| \le L_\psi\|u_1 - u_2\|\\ &\text{for all } u_1,u_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3.\\ &\text{(b) There exists } M_\psi > 0 \text{ such that } \|\psi(\mathbf{x},u)\| \le M_\psi \text{ for all } u \in \mathbb{R}, \text{ a.e.}\\ &\mathbf{x} \in \Gamma_3.\\ &\text{(c) } \mathbf{x} \mapsto \psi(\mathbf{x},u) \text{ is measurable on } \Gamma_3, \text{ for all } u \in \mathbb{R}.\\ &\text{(d) } \mathbf{x} \mapsto \psi(\mathbf{x},u) = \mathbf{0} \text{ for all } u \le 0.\end{aligned} \tag{3.15}$$

An example of a conductivity function which satisfies condition (3.15) is given by (2.20) in which case $M_\psi = k$. Another example is provided by $\psi \equiv 0$, which models the contact with an insulated foundation, as noted in Section 2. We conclude that our results below are valid for the corresponding piezoelectric contact models.

The adhesion coefficients and the limit bound satisfy

$$\gamma_\nu, \gamma_\tau \in L^\infty(\Gamma_3), \quad \boldsymbol{\varepsilon}_a \in L^2(\Gamma_3), \quad \gamma_\nu, \gamma_\tau, \boldsymbol{\varepsilon}_a \ge \mathbf{0}, \quad \text{a.e. on } \Gamma_3. \tag{3.16}$$

the initial bonding field satisfies

$$\alpha_0 \in L^2(\Gamma_3), \quad \mathbf{0} \le \alpha_0 \le \mathbf{1}, \quad \text{a.e. on } \Gamma_3. \tag{3.17}$$

and the initial damage field satisfies

$$\beta_0 \in K. \tag{3.18}$$

Finally, we assume that the gap function, the given potential and the initial displacement satisfy

$$g \in L^2(\Gamma_3), \quad g \ge \mathbf{0}, \quad \text{a.e. on } \Gamma_3, \tag{3.19}$$

$$\varphi_0 \in L^2(\Gamma_3), \tag{3.20}$$
$$\mathbf{u}_0 \in V. \tag{3.21}$$

The forces, tractions, volume and surface free charge densities satisfy

$$\mathbf{f}_0 \in W^{1,p}(0,T;\mathcal{W}), \quad \mathbf{f}_2 \in W^{1,p}(0,T;L^2(\Gamma_2)^d), \tag{3.22}$$
$$q_0 \in W^{1,p}(0,T;L^2(\Omega)), \quad q_2 \in W^{1,p}(0,T;L^2(\Gamma_b)). \tag{3.23}$$

Here, $1 \le p \le \infty$. We define the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$

$$a(\xi,\varphi) = k\int_\Omega \nabla\xi.\nabla\varphi dx. \tag{3.24}$$

Next, we define the four mappings $j : V \times V \to \mathbb{R}$, $h : V \times W \to W$, $\mathbf{f} : [0,T] \to V$ and $q : [0,T] \to W$, respectively, by

$$j(\mathbf{u},\mathbf{v}) = \int_{\Gamma_3} p_\nu(u_\nu - g)v_\nu\, da + \int_{\Gamma_3} p_\tau(u_\nu - g)\|\mathbf{v}_\tau\|\, da, \tag{3.25}$$
$$(h(\mathbf{u},\varphi),\zeta)_W = \int_{\Gamma_3} \psi(u_\nu - g)\phi_l(\varphi - \varphi_0)\zeta\, da, \tag{3.26}$$
$$(\mathbf{f}(t),\mathbf{v})_V = \int_\Omega \mathbf{f}_0(t)\cdot\mathbf{v}\, dx + \int_{\Gamma_2} \mathbf{f}_2(t)\cdot\mathbf{v}\, da, \tag{3.27}$$
$$(q(t),\zeta)_W = \int_\Omega q_0(t)\zeta\, dx - \int_{\Gamma_b} q_2(t)\zeta\, da, \tag{3.28}$$

for all $\mathbf{u},\mathbf{v} \in V$, $\varphi,\zeta \in W$ and $t \in [0,T]$. We note that the definitions of h, $\mathbf{f}$ and q are based on the Riesz representation theorem, moreover, it follows from assumptions (3.13)-(3.21) that the integrals in (3.25)-(3.28) are well-defined. Using Green's formulas (3.3) and (3.4),it is easy to see that if $(\mathbf{u},\boldsymbol{\sigma},\varphi,\mathbf{D})$ are sufficiently regular functions which satisfy (2.4)–(2.9)and(2.12)–(2.14) then

$$(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t))_Q + j(\mathbf{u}(t),\mathbf{v}) - j(\mathbf{u}(t),\dot{\mathbf{u}}(t)) \ge (\mathbf{f}(t),\mathbf{v} - \dot{\mathbf{u}}(t))_V, \tag{3.29}$$
$$(\mathbf{D}(t),\nabla\zeta)_{\mathcal{W}} + (q(t),\zeta)_W = (h(\mathbf{u}(t),\varphi(t)),\zeta)_W, \tag{3.30}$$

for all $\mathbf{v} \in V$, $\zeta \in W$ and $t \in [0,T]$. We substitute (2.1) in (3.29), (2.2) in (3.30), we use the initial condition (2.15) and derive a variational formulation of problem $\mathcal{P}$.

Problem $\mathcal{P}_V$. Find a displacement field $\mathbf{u} : [0,T] \to V$, $\sigma : [0,T] \to \mathcal{Q}$ and an electric potential $\varphi : [0,T] \to W$,a damage field $\beta : [0,T] \to H^1(\Omega)$ and a bonding field $\alpha : [0,T] \to L^\infty(\Gamma_3)$ such that

$$\begin{aligned}\sigma(t) = \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) + \mathcal{B}\varepsilon(\mathbf{u}(t)) + \mathcal{E}^*\nabla\varphi(t) + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}(s) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}(s))\\ - \mathcal{E}^*\nabla\varphi(s),\varepsilon(\mathbf{u}(s)),\beta(s)\Big)ds \quad \text{in } \Omega \times (0,T),\end{aligned} \tag{3.31}$$
$$(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t))_Q + j(\mathbf{u}(t),\mathbf{v}) - j(\mathbf{u}(t),\dot{\mathbf{u}}(t)) \ge (\mathbf{f}(t),\mathbf{v} - \dot{\mathbf{u}}(t))_V, \tag{3.32}$$

for all $\mathbf{v} \in V$ and $t \in [0,T]$,

$$(\boldsymbol{B}\nabla\varphi(t),\nabla\zeta)_{\mathcal{W}} - (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}(t)),\nabla\zeta)_{\mathcal{W}} + (h(\mathbf{u}(t),\varphi(t)),\zeta)_W = (q(t),\zeta)_W, \tag{3.33}$$

for all $\zeta \in W$ and $t \in [0,T]$, we have $\beta(t) \in K$, and

$$\begin{aligned}&(\dot{\beta}(t), \xi - \beta(t))_{L^2(\Omega)} + a(\beta(t), \xi - \beta(t)) \\ &\geq (S(\sigma(t) - \mathcal{A}\varepsilon(\dot{\mathbf{u}}(t)) - \mathcal{E}^* \nabla\varphi(t), \varepsilon(\mathbf{u}(t)), \beta(t)), \xi - \beta(t))_{L^2(\Omega)},\end{aligned} \tag{3.34}$$

for all $\xi \in K$ and $t \in [0,T]$,

$$\dot{\alpha} = -(\alpha(\gamma_\nu R_\nu(u_\nu))^2 + \gamma_\tau \|\mathbf{R}_\tau(\mathbf{u}_\tau)\|^2) - \varepsilon_a)_+ \quad \text{on } \Gamma_3 \times (0,T), \tag{3.35}$$

and

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \beta(0) = \beta_0, \quad \alpha(0) = \alpha_0. \tag{3.36}$$

To study problem $\mathcal{P}_V$ we use the smallness assumption

$$M_\psi < \frac{m_B}{c_0^2}\,, \tag{3.37}$$

where M_ψ, c_0 and m_β are given in (3.15), (3.2) and (3.11), respectively. We note that only the trace constant, the coercivity constant of $\boldsymbol{B}$ and the bound of ψ are involved in (3.37); therefore, this smallness assumption involves only the geometry and the electrical part, and does not depend on the mechanical data of the problem. Moreover, it is satisfied when the obstacle is insulated, since then $\psi \equiv 0$ and so $M_\psi = 0$.

Removing this assumption remains a task for future research, since it is made for mathematical reasons, and does not seem to relate to any inherent physical constraints of the problem.

Remark 3.2. We note that, in Problem $\mathbf{P}$ and in Problem $\mathcal{PV}$, we do not need to impose explicitly the restriction $0 \leq \alpha \leq 1$. Indeed, (3.35) guarantees that $\alpha(x,t) \leq \alpha_0(x)$ and, therefore, assumption (3.31) shows that $\alpha(x,t) \leq 1$ for $t \geq 0$, a.e. $x \in \Gamma_3$. On the other hand, if $\alpha(x,t_0) = 0$ at time t_0, then it follows from (3.35) that $\dot{\alpha}(x,t) = 0$ for all $t \geq t_0$ and therefore, $\alpha(x,t) = 0$ for all $t \geq t_0$, a.e. $x \in \Gamma_3$. We conclude that $0 \leq \alpha(x,t) \leq 1$ for all $t \in [0;T]$, a.e. $x \in \Gamma_3$.

Now, we propose our existence and uniqueness result.

Theorem 3.3. *Assume that* (3.7)-(3.23) *hold. Then there exists a unique solution* $\mathbf{u}, \varphi, \beta, \alpha$ *to problem* $\mathcal{PV}$. *Moreover, the solution satisfies*

$$\mathbf{u} \in W^{2,p}(0,T;V) \cap C^1(0,T;V), \tag{3.38}$$

$$\varphi \in W^{1,p}(0,T;W), \tag{3.39}$$

$$\boldsymbol{\sigma} \in W^{1,p}(0,T;Q), \quad Div\sigma \in W^{1,p}(0,T;\mathcal{W}), \tag{3.40}$$

$$\beta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \tag{3.41}$$

$$\alpha \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Z}. \tag{3.42}$$

The functions $\mathbf{u}, \varphi, \boldsymbol{\sigma}, \boldsymbol{D}, \beta$ and α which satisfy (3.31)-(3.36)are called a weak solution of the contact problem $\mathbf{P}$. We conclude that, under the assumptions(3.8)-(3.23)and (3.37), the mechanical problem (2.1)-(2.16) has a unique weak solution satisfying (3.38)-(3.42).

The regularity of the weak solution is given by(3.38)-(3.42) and, in term of electric displacement

$$\boldsymbol{D} \in W^{1,p}(0,T;\mathcal{W}). \tag{3.43}$$

It follows from (3.33) that $\operatorname{div}\mathbf{D}(t) - q_0(t) = 0$ for all $t \in [0,T]$, and therefore the regularity (3.39) of φ, combined with (3.11),(3.12), and (3.23) implies (3.43). In this section we suppose that assumptions of Theorem3.3 hold, and we consider that C is a generic positive constant which depends on $\Omega, \Gamma_1, \Gamma_2, \Gamma_3, p_\nu, p_\tau, \gamma_\nu, \gamma_\tau$ and L and may change from place to place. Let $\boldsymbol{\eta} \in C(0, T; V)$ be given. In the first step we consider the following variational problem Our main existence and uniqueness result that we state now and prove in the next section is the following.

4. Existence and uniqueness of a solution

Let $\boldsymbol{\eta} \in C([0, T], Q)$ be given, and in the first step consider the following intermediate mechanical problem.

Problem $\mathcal{PV}^1_\eta$. Find a displacement field $\mathbf{u}_\eta : [0, T] \to V$ such that

$$\begin{aligned} &(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)))_Q + (\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)))_Q \\ &+ (\boldsymbol{\eta}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)))_Q + j(\mathbf{u}_\eta(t), \mathbf{v}) - j(\mathbf{u}_\eta(t), \dot{\mathbf{u}}_\eta(t)) \\ &\geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}_\eta(t))_V, \quad \forall \mathbf{v} \in V,\ t \in [0, T], \end{aligned} \tag{4.1}$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0. \tag{4.2}$$

We have the following result for $\mathcal{PV}^1_\eta$.

Lemma 4.1. *(1) There exists a unique solution $\mathbf{u}_\eta \in C^1([0, T]; V)$ to the problem* (4.1) *and* (4.2).

(2) If $\mathbf{u}_1$ and $\mathbf{u}_2$ are two solutions of (4.1) *and* (4.2) *corresponding to the data $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in C([0, T]; Q)$, then there exists $c > 0$ such that*

$$\|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)\|_V \leq c\,(\|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_Q + \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V), \quad \forall\, t \in [0, T]. \tag{4.3}$$

(3) If, moreover, $\boldsymbol{\eta} \in W^{1,p}(0, T; Q)$ for some $p \in [1, \infty]$, then the solution satisfies $\mathbf{u}_\eta \in W^{2,p}(0, T; V)$.

The proof of the above lemma, we use an abstract existence and unique result which may be found in [13, pp. 12]. In the next step we use the solution $\mathbf{u}_{\boldsymbol{\eta}} \in C^1([0, T], V)$, obtained in Lemma 4.1, to construct the following variational problem for the electrical potential.

Problem $\mathcal{PV}^2_{\boldsymbol{\eta}}$. Find an electrical potential $\varphi_\eta : [0, T] \to W$ such that

$$(\boldsymbol{B}\nabla\varphi_\eta(t), \nabla\zeta)_{\mathcal{W}} - (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)), \nabla\zeta)_{\mathcal{W}} + (h(\mathbf{u}_\eta(t), \varphi_\eta(t)), \zeta)_W = (q(t), \zeta)_W \tag{4.4}$$

for all $\zeta \in W$, $t \in [0, T]$.

The well-posedness of problem $\mathcal{PV}^2_\eta$ follows.

Lemma 4.2. *There exists a unique solution $\varphi_\eta \in W^{1,p}(0, T; W)$ which satisfies* (4.4). *Moreover, if $\varphi_{\boldsymbol{\eta}_1}$ and $\varphi_{\boldsymbol{\eta}_2}$ are the solutions of* (4.4) *corresponding to $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2 \in C([0, T]; Q)$ then, there exists $c > 0$, such that*

$$\|\varphi_{\eta_1}(t) - \varphi_{\eta_2}(t)\|_W \leq c\,\|\mathbf{u}_{\eta_1}(t) - \mathbf{u}_{\eta_2}(t)\|_V \quad \forall t \in [0, T]. \tag{4.5}$$

To prove the above lemma, we use an abstract existence and unique result which may be found in [13, pp. 13].

In the third step we let $\theta \in L^2(0, T; L^2(\Omega))$ be given and consider the following variational problem for the damage filed.

Problem $\mathcal{PV}_\theta$. Find the damage field $\beta_\theta : [0,T] \to H^1(\Omega)$ such that $\beta_\theta(t) \in K$ and

$$\begin{aligned}&(\dot\beta_\theta(t), \xi - \beta_\theta)_{L^2(\Omega)} + a(\beta_\theta(t), \xi - \beta_\theta(t))\\&\geq (\theta(t), \xi - \beta_\theta(t))_{L^2(\Omega)} \quad \forall \xi \in K, \text{ a.e. } t \in (0,T),\end{aligned} \tag{4.6}$$

$$\beta_\theta(0) = \beta_0. \tag{4.7}$$

For the study of problem $\mathcal{PV}_\theta$, we have the following result.

Lemma 4.3. *There exists a unique solution β_θ to the auxiliary problem $\mathcal{PV}_\theta$ satisfying* (3.41).

The above lemma follows from a standard result for parabolic variational inequalities, see [21, p.47].

Problem $\mathcal{PV}_\alpha$. Find the adhesion field $\alpha_{\boldsymbol{\eta}} : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot{\alpha_{\boldsymbol{\eta}}} = -(\alpha_{\boldsymbol{\eta}}(\gamma_\nu R_\nu(u_{\boldsymbol{\eta}\nu})^2 + \gamma_\tau \|\mathbf{R}_\tau(\mathbf{u}_{\eta\tau})\|^2) - \varepsilon_a)_+ \tag{4.8}$$

$$\alpha_{\boldsymbol{\eta}}(0) = \alpha_0 \in \Omega. \tag{4.9}$$

We have the following result.

Lemma 4.4. *There exists a unique solution $\alpha_{\boldsymbol{\eta}} \in W^{1,\infty}(0,T; L^2(\Gamma_3)) \cap \mathcal{Z}$ to Problem $\mathcal{PV}_\alpha$.*

The above lemma follows from a standard result for differential equations, see [19, pp.108-109].

In the fifth step, we use $\mathbf{u}_\eta, \varphi_\eta, \beta_\theta$ and α_η obtained in Lemmas 4.1, 4.2, 4.3 and 4.4, respectively to construct the following Cauchy problem for the stress field.

Problem$\mathcal{PV}_{\eta,\theta}$. Find the stress field $\boldsymbol{\sigma}_{\eta,\theta} : [0,T] \to Q$ which is a solution of the problem

$$\boldsymbol{\sigma}_{\eta,\theta}(t) = \mathcal{B}(\varepsilon(\mathbf{u}_\eta(t))) + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}_{\eta,\theta}(s), \varepsilon(\mathbf{u}_\eta(s)), \beta_\theta(s)\Big) ds, \quad \text{a.e. } t \in (0,T). \tag{4.10}$$

Lemma 4.5. *There exists a unique solution of Problem $\mathcal{PV}_{\boldsymbol{\eta},\theta}$ and it satisfies* (3.40)*. Moreover, if $\mathbf{u}_{\eta_i}, \beta_{\theta_i}$ and $\boldsymbol{\sigma}_{\eta_i,\theta_i}$ represent the solutions of problems $\mathcal{PV}^1_{\eta_i}$, $\mathcal{PV}_{\theta_i}$ and $\mathcal{PV}_{\eta_i,\theta_i}$, respectively, for $(\eta_i, \theta_i) \in W^{1,p}(0,T; Q \times L^2(\Omega))$ $i = 1,2$, then there exists $C > 0$ such that*

$$\begin{aligned}\|\boldsymbol{\sigma}_{\eta_1,\theta_1}(t) - \boldsymbol{\sigma}_{\eta_2,\theta_2}(t)\|_Q^2 &\leq C\Big(\|\mathbf{u}_{\eta_1}(t) - \mathbf{u}_{\eta_2}(t)\|_V^2 + \int_0^t \|\mathbf{u}_{\eta_1}(s) - \mathbf{u}_{\eta_2}(s)\|_V^2\\&\quad + \int_0^t \|\beta_{\theta_1}(s) - \beta_{\theta_2}(s)\|_{L^2(\Omega)}^2 ds\Big).\end{aligned} \tag{4.11}$$

Proof. Let $\boldsymbol{\Sigma}_{\eta,\theta} : W^{1,p}(0,T;Q) \to W^{1,p}(0,T;Q)$ be the mapping given by

$$\boldsymbol{\Sigma}_{\eta,\theta}\boldsymbol{\sigma}(t) = \mathcal{B}(\varepsilon(\mathbf{u}_\eta(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \boldsymbol{\varepsilon}(\mathbf{u}_\eta(s)), \beta_\theta(s)) ds, \tag{4.12}$$

for all $\boldsymbol{\sigma} \in W^{1,p}(0,T;Q)$ and $t \in (0,T)$. For $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2 \in W^{1,p}(0,T;Q)$ we use (3.10) and (4.12) to obtain for all $t \in (0,T)$,

$$\|\boldsymbol{\Sigma}_{\eta,\theta}\boldsymbol{\sigma}_1(t) - \boldsymbol{\Sigma}_{\eta,\theta}\boldsymbol{\sigma}_2(t)\|_Q \leq L_{\mathcal{G}} \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_Q ds. \tag{4.13}$$

It follows from this inequality that for n large enough, a power $\boldsymbol{\Sigma}_{\eta,\theta}^n$ of the mapping $\boldsymbol{\Sigma}_{\eta,\theta}$ is a contraction on the Banach space $W^{1,p}(0,T;Q)$ and, therefore, exists a unique element $\boldsymbol{\sigma} \in W^{1,p}(0,T;Q)$ such that $\boldsymbol{\Sigma}_{\eta,\theta}\boldsymbol{\sigma} = \boldsymbol{\sigma}$. Moreover, which $\boldsymbol{\sigma}$ is the unique solution to problem $\mathcal{PV}_{\eta,\theta}$. And using (4.10), the regularity of $\mathbf{u}_\eta, \beta_\theta$ and the properties of the operators $\mathcal{B}$ and $\mathcal{G}$, it follows that $\boldsymbol{\sigma}_i \in W^{1,p}(0,T;Q)$. Consider now $(\boldsymbol{\eta}_1,\theta_1)$, $(\boldsymbol{\eta}_2,\theta_2) \in W^{1,p}(0,T;Q\times L^2(\Omega))$ and for $i=1,2$, denote $\mathbf{u}_{\eta_i} = \mathbf{u}_i$, $\boldsymbol{\sigma}_{\eta_i,\theta_i} = \boldsymbol{\sigma}_i$, $\beta_{\theta_i} = \beta_i$, $\varphi_{\theta_i} = \varphi_i$. We have

$$\boldsymbol{\sigma}_i(t) = \mathcal{B}(\varepsilon(\mathbf{u}_i(t)) + \int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}_i(s), \boldsymbol{\varepsilon}(\mathbf{u}_i(s)), \beta_i(s)\Big) ds, \quad \text{a.e. } t \in (0,T), \tag{4.14}$$

and using the properties (3.9) and (3.11) of $\mathcal{B}$ and $\mathcal{G}$, we find

$$\begin{aligned} &\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_Q^2 \\ &\le C\Big(\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 + \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_Q^2 ds \\ &\quad + \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 ds + \int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Omega)}^2 ds\Big), \quad \forall t \in [0,T]. \end{aligned} \tag{4.15}$$

We use a Gronwall argument in the obtained inequality to deduce (4.11), which concludes the proof of Lemma 4.5 □

Finally, as a consequence of these results and using the properties of the operator $\mathcal{G}$ the operator $\mathcal{E}$, the function S for $t \in (0,T)$, we consider the element

$$\Lambda(\boldsymbol{\eta},\theta)(t) = (\Lambda^1(\boldsymbol{\eta},\theta)(t), \Lambda^2(\boldsymbol{\eta},\theta)(t)) \in Q \times L^2(\Omega), \tag{4.16}$$

defined by

$$\begin{aligned} &(\Lambda^1(\boldsymbol{\eta},\theta)(t), \mathbf{v})_{Q\times V} \\ &= (\mathcal{E}^*\nabla\varphi_\eta(t), \boldsymbol{\varepsilon}(\mathbf{v}))_Q + \Big(\int_0^t \mathcal{G}\Big(\boldsymbol{\sigma}_{\eta,\theta}(s), \boldsymbol{\varepsilon}(\mathbf{u}_\eta(s)), \beta_\theta(s)\Big) ds, \boldsymbol{\varepsilon}(\mathbf{v})\Big)_Q, \quad \forall \mathbf{v} \in V. \end{aligned} \tag{4.17}$$

$$\Lambda^2(\boldsymbol{\eta},\theta)(t) = S\big(\boldsymbol{\sigma}_{\eta,\theta}(t), \boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)), \beta_\theta(t)\big). \tag{4.18}$$

Let us consider the mapping

$$\Lambda : W^{1,p}(0,T;Q\times L^2(\Omega)) \to W^{1,p}(0,T;Q\times L^2(\Omega)),$$

which maps every element $(\boldsymbol{\eta},\theta) \in W^{1,p}(0,T;Q\times L^2(\Omega))$ to the element $\Lambda(\boldsymbol{\eta},\theta) \in W^{1,p}(0,T;Q\times L^2(\Omega))$. $\mathbf{u}_\eta$, φ_η, β_θ, α_η and $\boldsymbol{\sigma}_{\eta,\theta}$, represent the displacement field, the potential field, the damage field and the stress field obtained in Lemmas (4.1), (4.2), (4.3), (4.4) and (4.5).

Lemma 4.6. *The mapping Λ has a fixed point $(\boldsymbol{\eta}^*,\theta^*) \in W^{1,p}(0,T;Q\times L^2(\Omega))$. Such that $\Lambda(\boldsymbol{\eta}^*,\theta^*) = (\boldsymbol{\eta}^*,\theta^*)$*

Proof. Let $t \in (0,T)$ and $(\boldsymbol{\eta}_1,\theta_1)$, $(\boldsymbol{\eta}_2,\theta_2) \in W^{1,p}(0,T;Q\times L^2(\Omega))$. We use the notation $\mathbf{u}_{\eta_i} = \mathbf{u}_i$, $\dot{\mathbf{u}}_{\eta_i} = \dot{\mathbf{u}}_i$, $\alpha_{\eta_i} = \alpha_i$, $\varphi_{\eta_i} = \varphi_i$ and $\boldsymbol{\sigma}_{\eta_i,\theta_i} = \boldsymbol{\sigma}_i$, for $i=1,2$. Let us start by using (3.7) hypotheses (3.10), (3.12) and (3.14), we have.

$$\begin{aligned}
&\|\Lambda^1(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda^1(\boldsymbol{\eta}_2,\theta_2)(t)\|_Q^2\\
&\le \|\mathcal{E}^*\nabla\varphi_1(t)-\mathcal{E}^*\nabla\varphi_2(t)\|_Q^2\\
&\quad+\int_0^t \|\mathcal{G}(\boldsymbol{\sigma}_1(s),\varepsilon(\mathbf{u}_1(s)),\beta_1(s))-\mathcal{G}(\boldsymbol{\sigma}_2(s),\boldsymbol{\varepsilon}(\mathbf{u}_2(s)),\beta_2(s))\|_Q^2 ds\\
&\le C\Big(\|\varphi_1(t)-\varphi_2(t)\|_Q^2+\int_0^t\|\boldsymbol{\sigma}_1(s)-\boldsymbol{\sigma}_2(s)\|_Q^2ds\\
&\quad+\int_0^t\|\beta_1(s)-\beta_2(s)\|_{L^2(\Omega)}^2+\int_0^t\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V^2ds\Big).
\end{aligned} \tag{4.19}$$

We use estimates (4.5), (4.11) to obtain

$$\begin{aligned}
&\|\Lambda^1(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda^1(\boldsymbol{\eta}_2,\theta_2)(t)\|_Q^2\\
&\le C\Big(\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V^2+\int_0^t\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V^2ds\\
&\quad+\|\alpha_1(s)-\alpha_2(s)\|_{L^2(\Gamma_3)}^2+\int_0^t\|\beta_1(s)-\beta_2(s)\|_{L^2(\Omega)}^2ds\Big).
\end{aligned} \tag{4.20}$$

By similar arguments, from (4.18), (4.11) and (3.13) we obtain

$$\begin{aligned}
&\|\Lambda^2(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda^2(\boldsymbol{\eta}_2,\theta_2)(t)\|_Q^2\\
&\le C\Big(\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V^2+\int_0^t\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V^2\\
&\quad+\|\beta_1(t)-\beta_2(t)\|_{L^2(\Omega)}^2+\int_0^t\|\beta_1(s)-\beta_2(s)\|_{L^2(\Omega)}^2ds\Big)\quad \text{a.e. } t\in(0,T).
\end{aligned} \tag{4.21}$$

Therefore,

$$\begin{aligned}
&\|\Lambda(\boldsymbol{\eta}_1,\theta_1)(t)-\Lambda(\boldsymbol{\eta}_2,\theta_2)(t)\|_{Q\times L^2(\Omega)}^2\\
&\le C\Big(\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V^2+\int_0^t\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V^2ds+\|\beta_1(t)-\beta_2(t)\|_{L^2(\Omega)}^2\\
&+\int_0^t\|\beta_1(s)-\beta_2(s)\|_{L^2(\Omega)}^2ds+\|\alpha_1(t)-\alpha_2(t)\|_{L^2(\Gamma_3)}^2\Big)\quad \text{a.e. } t\in(0,T).
\end{aligned} \tag{4.22}$$

On the other hand, since $\mathbf{u}_i(t)=\mathbf{u}_0+\int_0^t\dot{\mathbf{u}}_i(s)ds$, we know that for a.e. $t\in(0,T)$,

$$\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_V\le\int_0^t\|\dot{\mathbf{u}}_1(s)-\dot{\mathbf{u}}_2(s)\|_Vds, \tag{4.23}$$

and using this inequality in (4.3) yields

$$\|\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t)\|_V\le c\Big(\|\boldsymbol{\eta}_1(t)-\boldsymbol{\eta}_2(t)\|_Q+\int_0^t|\dot{\mathbf{u}}_1(s)-\dot{\mathbf{u}}_2(s)|_Vds\Big), \tag{4.24}$$

for all $t\in[0,T]$. It follows now from a Gronwall-type argument that

$$\int_0^t\|\dot{\mathbf{u}}_1(s)-\dot{\mathbf{u}}_2(s)\|_Vds\le c\int_0^t\|\boldsymbol{\eta}_1(t)-\boldsymbol{\eta}_2(t)\|_Qds, \tag{4.25}$$

which also implies, using a variant of (4.23), that

$$\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V^2\le C\int_0^t\|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_Q^2ds,\quad \text{a.e. } t\in(0,T). \tag{4.26}$$

On the other hand, from the Cauchy problem (4.8)-(4.9) we can write

$$\alpha_i(t) = \alpha_0 - \int_0^t (\alpha_i(s)(\gamma_\nu R_\nu(u_{i\nu}(s))^2 + \gamma_\tau \|\mathbf{R}_\tau(\mathbf{u}_{i\tau}(s))\|^2) - \varepsilon_a)_+ ds,$$

and then

$$\begin{aligned}
&\|\alpha_1(t) - \alpha_2(t)\|_{L^2(\Gamma_3)} \\
&\le C \int_0^t \|\alpha_1(s)(R_\nu(u_{1\nu}(s))^2 - \alpha_2(s)(R_\nu(u_{2\nu}(s))^2\|_{L^2(\Gamma_3)} ds \\
&\quad + C \int_0^t \|\alpha_1(s)\|\mathbf{R}_\tau(\mathbf{u}_{1\tau}(s))\|^2 - \alpha_2(s)\|\mathbf{R}_\tau(\mathbf{u}_{2\tau}(s))\|^2\|_{L^2(\Gamma_3)} ds,
\end{aligned}$$

Using the definition of R_ν and $\mathbf{R}_\tau$ and writing $\alpha_1 = \alpha_1 - \alpha_2 + \alpha_2$, we obtain

$$\begin{aligned}
&\|\alpha_1(t) - \alpha_2(t)\|_{L^2(\Gamma_3)} \\
&\le C\Big(\int_0^t \|\alpha_1(s) - \alpha_2(s)\|_{L^2(\Gamma_3)} ds + \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_{L^2(\Gamma_3)^d}\Big) ds.
\end{aligned} \tag{4.27}$$

Next, we apply Gronwall's inequality to deduce

$$\|\alpha_1(t) - \alpha_2(t)\|_{L^2(\Gamma_3)} \le C \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_{L^2(\Gamma_3)^d} ds,$$

and from (3.7), we obtain

$$\|\alpha_1(t) - \alpha_2(t)\|^2_{L^2(\Gamma_3)} \le C \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|^2_V ds. \tag{4.28}$$

Form (4.6), we deduce that

$$(\dot\beta_1 - \dot\beta_2, \beta_1 - \beta_2)_{L^2(\Omega)} + a(\beta_1 - \beta_2, \beta_1 - \beta_2) \le (\theta_1 - \theta_2, \beta_1 - \beta_2)_{L^2(\Omega)}, \quad \forall\, t \in [0, T].$$

Integrating the previous inequality with respect to time, using the initial conditions $\beta_1(0) = \beta_2(0) = \beta_0$ and inequality $a(\beta_1 - \beta_2, \beta_1 - \beta_2) \ge 0$ to find

$$\frac{1}{2}\|\beta_1(t) - \beta_2(t)\|^2_{L^2(\Omega)} \le \int_0^t (\theta_1(s) - \theta_2(s), \beta_1(s) - \beta_2(s))_{L^2(\Omega)} ds, \tag{4.29}$$

which implies

$$\|\beta_1(t) - \beta_2(t)\|^2_{L^2(\Omega)} \le \int_0^t \|\theta_1(s) - \theta_2(s)\|^2_{L^2(\Omega)} ds + \int_0^t \|\beta_1(s) - \beta_2(s)\|^2_{L^2(\Omega)} ds.$$

This inequality, combined with Gronwall's inequality, leads to

$$\|\beta_1(t) - \beta_2(t)\|^2_{L^2(\Omega)} \le C \int_0^t \|\theta_1(s) - \theta_2(s)\|^2_{L^2(\Omega)} ds, \quad \forall t \in [0, T]. \tag{4.30}$$

Form the previous inequality and estimates (4.22), (4.26), (4.28) and (4.30) it follows now that

$$\|\Lambda(\boldsymbol{\eta}_1, \theta_1)(t) - \Lambda(\boldsymbol{\eta}_2, \theta_2)(t)\|^2_{Q\times L^2(\Omega)} \le C \int_0^t \|(\boldsymbol{\eta}_1, \theta_1)(s) - (\boldsymbol{\eta}_2, \theta_2)(s)\|^2_{Q\times L^2(\Omega)} ds.$$

Reiterating this inequality n times leads to

$$\begin{aligned}
&\|\Lambda^n(\boldsymbol{\eta}_1, \theta_1) - \Lambda^n(\boldsymbol{\eta}_2, \theta_2)\|^2_{W^{1,p}(0,T;Q\times L^2(\Omega))} \\
&\le \frac{C^n T^n}{n!}\|(\boldsymbol{\eta}_1, \theta_1) - (\boldsymbol{\eta}_2, \theta_2)\|^2_{W^{1,p}(0,T;Q\times L^2(\Omega))}.
\end{aligned}$$

Thus, for n sufficiently large, Λ^n is a contraction on $W^{1,p}(0,T;Q\times L^2(\Omega))$, and so Λ has a unique fixed point in this Banach space. □

Now, we have all the ingredients to prove Theorem 3.3.

Existence. Let $(\boldsymbol{\eta}^*,\theta^*)\in W^{1,p}(0,T;Q\times L^2(\Omega))$ be the fixed point of Λ defined by (4.16)-(4.18) and denote

$$\mathbf{u}=\mathbf{u}_{\boldsymbol{\eta}^*},\quad \varphi_{\boldsymbol{\eta}^*}=\varphi,\quad \boldsymbol{\sigma}=\mathcal{A}\varepsilon(\dot{\mathbf{u}})+\mathcal{E}^*\nabla(\varphi)+\boldsymbol{\sigma}_{\boldsymbol{\eta}^*\theta^*}, \tag{4.31}$$

$$\beta=\beta_{\theta^*},\quad \alpha=\alpha_{\boldsymbol{\eta}^*}. \tag{4.32}$$

We prove that $(\mathbf{u},\varphi,\boldsymbol{\sigma},\beta,\alpha)$ satisfies (3.31)–(3.36) and (3.38), (3.42). Indeed, we write (4.10) for $\boldsymbol{\eta}^*=\boldsymbol{\eta}$, $\theta^*=\theta$ and use (4.31), (4.32) to obtain that (3.31) is satisfied. Now we consider (4.1) for $\boldsymbol{\eta}^*=\boldsymbol{\eta}$ and use the first equality in (4.31) to find

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)),\boldsymbol{\varepsilon}(\mathbf{v})-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q+(\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}(t)),\boldsymbol{\varepsilon}(\mathbf{v})-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q\\&+(\boldsymbol{\eta}^*(t),\boldsymbol{\varepsilon}(\mathbf{v})-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_Q+j(\mathbf{u}(t),\mathbf{v})-j(\mathbf{u}(t),\dot{\mathbf{u}}(t))\\&\geq\big(\mathbf{f}(t),\mathbf{v}-\dot{\mathbf{u}}(t)\big)_V,\quad\forall\mathbf{v}\in V,\ t\in[0,T].\end{aligned} \tag{4.33}$$

The equalities $\Lambda^1(\boldsymbol{\eta}^*,\theta^*)=\boldsymbol{\eta}^*$ and $\Lambda^2(\boldsymbol{\eta}^*,\theta^*)=\theta^*$ combined with (4.17), (4.18), (4.31) and (4.32) show that for all $\mathbf{v}\in V$,

$$\begin{aligned}(\boldsymbol{\eta}^*(t),\mathbf{v})_{Q\times V}=(\mathcal{E}^*\nabla\varphi(t),\boldsymbol{\varepsilon}(\mathbf{v}))_Q+\Big(\int_0^t\mathcal{G}\Big(\sigma(s)-\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(s))\\-\mathcal{E}^*\nabla\varphi(s),\boldsymbol{\varepsilon}(\mathbf{u}(s)),\beta(s)\Big)ds,\boldsymbol{\varepsilon}(\mathbf{v})\Big)_Q,\end{aligned} \tag{4.34}$$

$$\theta^*(t)=S\big(\boldsymbol{\sigma}(t)-\mathcal{A}(\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))-\mathcal{E}^*\nabla\varphi(t),\boldsymbol{\varepsilon}(\mathbf{u}(t)),\beta(t)\big). \tag{4.35}$$

We now substitute (4.34) in (4.35) and use (3.31) to see that (3.32) is satisfied. We write (4.6) for $\theta^*=\theta$ and use (4.32) and (4.35) to find that (3.34) is satisfied. We consider now (4.8) for $\boldsymbol{\eta}^*=\boldsymbol{\eta}$ and use (4.31), (4.32) to obtain that (3.35) is satisfied. Next (3.36), and regularities (3.38), (3.39), (3.41) and (3.42) follow Lemmas 4.1, 4.4 and 4.5. The regularity $\boldsymbol{\sigma}\in W^{1,p}(0,T;Q)$ follows from Lemmas 4.1 and 4.6, the second equality in 4.31 and 3.8. Finally (3.32) implies that

$$\operatorname{Div}\boldsymbol{\sigma}(t)+\mathbf{f}_0(t)=\mathbf{0}\quad\text{in }\mathcal{W},\ \text{a.e. }t\in(0,T),$$

and therefore by (3.22) we obtain that $\operatorname{Div}\boldsymbol{\sigma}\in W^{1,p}(0,T;\mathcal{W})$. We deduce that the regularity (3.40) holds which concludes the existence part of the theorem.

Uniqueness. The uniqueness part of theorem (3.3) is a consequence of the uniqueness of the fixed point of the operator Λ defined by (4.16), (4.18) and the unique solvability of the Problem $\mathcal{PV}^1_\eta$, $\mathcal{PV}_\theta$ and $\mathcal{PV}_{\eta,\theta}$ which completes the proof.

References

[1] R. C. Batra, J. S. Yang ; *Saint Venant's principle in linear piezoelectricity*, Journal of Elasticity, 38(1995), 209-218.

[2] S. Barna; *Existence results for hemivariational inequality involving $p(x)$-Laplacian*, Opuscula Math. 32 (2012), no. 3, 439-454.

[3] P. Bisenga, F. Lebon, F. Maceri; *The unilateral frictional contact of a piezoelectric body with a rigid support, in Contact Mechanics*, J. A. C. Martins and Manuel D. P. Monteiro Marques (Eds), Kluwer, Dordrecht, 2002, 347-354.

[4] O. Chau, J. R. Fernández, M. Shillor, M. Sofonea; *Variational and numerical analysis of a quasistatic viscoelastic contact problem with adhesion*, J. Comput. Appl. Math, 159, pp. 431-465, 2003.

[5] C. Eck, J. Jarušek, M. Krbeč; *Unilateral Contact Problems: Variational Methods and Existence Theorems*, Pure and Applied Mathematics **270**, Chapman/CRC Press, New York, 2005.
[6] W. Han, M. Sofonea; *Evolutionary Variational inequalities arising in viscoelastic contact problems*, SIAM Journal of Numerical Analysis **38** (2000), 556–579.
[7] W. Han, M. Sofonea; *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, Studies in Advanced Mathematics **30**, American Mathematical Society, Providence, RI - Intl. Press, Sommerville, MA, 2002.
[8] M. Rochdi, M. Shillor, M. Sofonea; Quasistatic viscoelastic contact with normal compliance and friction, *Journal of Elasticity* **51** (1998), 105–126.
[9] M. Frémond and B. Nedjar; *Damage in concrete: the unilateral phenomenon*, Nuclear Engng. Design, 156, (1995), 323-335.
[10] M. Frémond and B. Nedjar; *Damage, Gradient of Damage and Principle of Virtual Work*, Int. J. Solids Structures, 33 (8), 1083-1103. (1996).
[11] J. R. Fernández-García, M. Sofonea, J. M. Viaño; *A Frictionless Contact Problem for Elastic-Viscoplastic Materials with Normal Compliance*, Numerische Mathemätik 90 (2002), 689–719.
[12] T. Ikeda; *Fundamentals of Piezoelectricity*, Oxford University Press, Oxford, 1990.
[13] Z. Lerguet, M. shillor, M. Sofonea; *A frictional contact problem for an electro-viscoelastic body*, Electronic Journal of Diferential Equations, Vol. 2007(2007), No. 170, pp. 1-16.
[14] R. D. Mindlin; *Polarisation gradient in elastic dielectrics*, Int. J. Solids Structures 4 (1968),637-663.
[15] R. D. Mindlin; *Elasticity piezoelasticity and crystal lattice dynamics*, Journal of Elasticity 4 (1972), 217-280.
[16] F. Maceri, P. Bisegna; *The unilateral frictionless contact of a piezoelectric body with a rigid support*, Math. Comp. Modelling 28 (1998), 19-28.
[17] S. Ntouyas' *Boundary value problems for nonlinear fractional differential equations and inclusions with nonlocal and fractional integral boundary conditions*, Opuscula Math. 33 (2013), no. 1, 117-138.
[18] V. Z. Patron, B. A. Kudryavtsev; *Electromagnetoelasticity, Piezoelectrics and Electrically Conductive Solids*, Gordon Breach, London, 1988.
[19] M. Selmani, L. Selmani; *Analysis of frictionless Contact problem for elastic-viscoplastic materials*, Nonlinear Analysis, Modelling and control, 2012, Vol. 17, No. 1, 99-77.
[20] M. Shillor, M. Sofonea, J. J. Telega; *Models and Analysis of Quasistatic Contact*, Lecture Notes in Physics 655, Springer, Berlin, 2004.
[21] M. Sofonea, W. Han, M. Shillor; *Analysis and Approximation of Contact Problems with Adhesion or Damage*, Pure and Applied Mathematics, Vol. 276, Chapman, Hall/CRC Press, New york, 2006.
[22] M. Sofonea, El H. Essoufi; *A Piezoelectric contact problem with slip dependent coeficient of friction*, Mathematical Modelling and Analysis 9 (2004), 229-242.
[23] M. Sofonea, El H. Essoufi; *Quasistatic frictional contact of a viscoelastic piezoelectric body*, Adv. Math. Sci. Appl. 14 (2004), 613-631.
[24] N. Strömberg, L. Johansson, A. Klarbring; *Derivation and analysis of a generalized standard model for contact friction and wear*, Int. J. Solids Structures, **33** (1996), 1817–1836.

Adel Aissaoui
Department of Mathematics, University of Ouargla, Ouargla 30000, Algeria
E-mail address: `aissaouiadel@gmail.com`

Nacerdine Hemici
Department of Mathematics, University of Setif, Setif 19000, Algeria
E-mail address: `na_hemici@yahoo.fr`