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POLYCONVOLUTION AND THE TOEPLITZ PLUS HANKEL INTEGRAL EQUATION

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Abstract. In this article we introduce a polyconvolution which related to the Hartley and Fourier cosine transforms. We prove some properties of this polyconvolution, and then solve a class of Toeplitz plus Hankel integral equations and systems of two Toeplitz plus Hankel integral equations.

1. Introduction

In studying the physical problems related to fluid dynamics, filtering theory, and wave diffraction, the following equation has been considered [4, 7, 8, 18],

$$f(x) + \int_0^\infty f(y)[k_1(x-y) + k_2(x+y)]dy = g(x), \quad x > 0, \tag{1.1}$$

here k_1, k_2, g are given, and f is unknown function. This equation is called the Toeplitz plus Hankel integral equation, with $k_1(x-y)$ is Toeplitz kernel and $k_2(x+y)$ is Hankel kernel. Up to now, solving (1.1) in the general case is still open. In recent years, there have been several results solving this equation in special cases of k_1, k_2 and the solutions are obtained in closed form by convolution tool.

In [12], the authors obtained the explicit solutions of the equation (1.1) in case the Toeplitz kernel is even function and k_1, k_2 have special forms

$$k_1(t) = \frac{1}{2\sqrt{2\pi}} \operatorname{sign}(t-1) h_1(|t-1|) - \frac{1}{2\sqrt{2\pi}} h_1(t+1) - \frac{1}{\sqrt{2\pi}} h_2(t)$$
$$k_2(t) = \frac{1}{2\sqrt{2\pi}} \operatorname{sign}(t+1) h_1(|t+1|) - \frac{1}{2\sqrt{2\pi}} h_1(t+1) + \frac{1}{\sqrt{2\pi}} h_2(|t|)$$

where $h_1(x) = (\varphi_1 \underset{2}{*} \varphi_2)(x)$, $\varphi_1, \varphi_2, h_2 \in L_1(\mathbb{R}_+)$ and $(\cdot \underset{2}{*} \cdot)$ is the generalized convolution for Fourier sine and Fourier cosine transforms [11].

Some special cases of k_1, k_2, where k_1 is still even function or special right-hand-side for arbitrary kernels are considered in [13]. In case, when k_1, k_2 are periodic functions with period 2π, the explicit solution to a class of equation (1.1) on a

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period $[0, 2\pi]$,

$$f(x) + \int_0^{2\pi} f(y)[k_1(|x-y|) + k_2(x+y)]dy = g(x), \quad x \in [0, 2\pi].$$

is introduced in [1].

In [15], the authors obtained the solution of (1.1) to the case $k_1 = k_2$ in real number

$$f(|x|) + \frac{1}{2\pi}\int_0^{\infty} f(y)[k(x-y) + k(x+y)]dy = g(x), \quad x \in \mathbb{R}.$$

In this article, we consider a modified type of the equation (1.1), with the integral real domain

$$f(x) + \lambda\int_{-\infty}^{\infty} f(y)[k_1(x-y) + k_2(x+y)]dy = p(x), \quad x \in \mathbb{R} \tag{1.2}$$

where, for $t \in \mathbb{R}$,

$$k_1(t) := \int_0^{\infty} g(v)[h(-t+v) + h(t-v) + h(-t-v) + h(t+v)]dv, \tag{1.3}$$

$$k_2(t) := \int_0^{\infty} g(v)[-h(t+v) + h(-t-v) - h(t-v) + h(-t+v)]dv, \tag{1.4}$$

and g, h, p are given functions, f is an unknown function. In this case, we see that the Toeplitz kernel k_1 is still an even function. The tool to solve this equation in closed form is a new polyconvolution related to Hartley and Fourier cosine transforms.

Convolutions have many applications [2, 11, 13, 14, 17, 18]. The concept of polyconvolution was first proposed by Kakichev in 1997 [9]. According to this definition, the polyconvolution of n, $(n \in \mathbb{N},\ n \geq 3)$ functions $f_1, f_2, \dots, f_n$ for $n+1$ arbitrary integral transforms $T, T_1, T_2, \dots, T_n$ with weight-function $\gamma(x)$ is denoted $\overset{\gamma}{*}(f_1, f_2, \dots, f_n)(x)$, for which the factorization property holds

$$T[\overset{\gamma}{*}(f_1, f_2, \dots, f_n)](y) = \gamma(y) \cdot (T_1 f_1)(y) \cdot (T_2 f_2)(y) \cdots (T_n f_n)(y).$$

Our new polyconvolution $*(f, g, h)(x)$, $x \in \mathbb{R}$ has the following factorization equalities

$$H_1[*(f, g, h)](y) = (H_1 f)(y) \cdot (F_c g)(y) \cdot (H_2 h)(y), \quad \forall y \in \mathbb{R}, \tag{1.5}$$

$$H_2[*(f, g, h)](y) = (H_2 f)(y) \cdot (F_c g)(y) \cdot (H_1 h)(y), \quad \forall y \in \mathbb{R}, \tag{1.6}$$

where H_1, H_2 are Hartley transforms and F_c is Fourier cosine transform.

The paper is organized as follows. In Section 2, we recall some known related results about convolutions. In Section 3, we define the new polyconvolution $*(f, g, h)(x)$ for Hartley and Fourier cosine integral transforms, whose factorization equalities are in the forms (1.5)-(1.6) and prove its existence on the certain function spaces. Its boundedness property on $L_p(\mathbb{R})$ is also considered. In Section 4, with the help of new polyconvolution $*(f, g, h)(x)$, we solve the equation (1.1) for the case k_1, k_2 are determined by (1.3)-(1.4). The systems of two equations are considered in this section.

2. Preliminaries

The following well-known transforms are used in this paper. We denote F by the Fourier transform, F_c by Fourier cosine transform, H_1 and H_2 by Hartley transforms, which are known in [3, 10, 11].

$$(Ff)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixy} dx, \quad y \in \mathbb{R}; \tag{2.1}$$

$$(F_c f)(y) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos(xy) dx, \quad y \in \mathbb{R}_+; \tag{2.2}$$

$$(H_1 f)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \operatorname{cas}(xy) dx, \quad y \in \mathbb{R}; \tag{2.3}$$

$$(H_2 f)(y) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x) \operatorname{cas}(-xy) dx, \quad y \in \mathbb{R}; \tag{2.4}$$

here $\operatorname{cas} u = \cos u + \sin u$.

Next, we recall the following convolutions, which will be used in the proof of some properties and solution of the equation (1.2).

First, the convolution for Hartley transform [5] , of two functions f and g, has the form

$$(f \underset{H}{*} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)[g(x-u)+g(-x+u)-g(-x-u)+g(x+u)]du, \quad x \in \mathbb{R},$$

with its factorization equalities

$$H_k(f \underset{H}{*} g)(y) = (H_k f)(y)(H_k g)(y), \quad \forall y \in \mathbb{R},\ k = 1, 2. \tag{2.5}$$

Second, the generalized convolution for Hartley and Fourier transforms [16], of two functions f and g, has the form

$$(f \underset{HF}{*} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)[g(x-u)+g(x+u)+ig(-x-u)-ig(x+u)]du, \quad x \in \mathbb{R}, \tag{2.6}$$

where its factorization properties are

$$H_k(f \underset{HF}{*} g)(y) = (Ff)(y)(H_k g)(y), \quad \forall y \in \mathbb{R},\ k = 1, 2. \tag{2.7}$$

Third, the generalized convolution for Hartley and Fourier cosine transforms [15], of two functions f and g has the form

$$(f \underset{HF_c}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(u)[g(x-u)+g(x+u)]du, \quad x \in \mathbb{R},$$

where its factorization properties are

$$H_k(f \underset{HF_c}{*} g)(y) = (F_c f)(y)(H_k g)(y), \quad \forall y \in \mathbb{R},\ k = 1, 2. \tag{2.8}$$

In this article, the function spaces $L_p(\mathbb{R})$ and $L_p(\mathbb{R}_+)$, $p \geq 1$, are equipped with norms,

$$\|f\|_{L_p(\mathbb{R})} = \Big(\int_{-\infty}^{\infty} |f(x)|^p dx \Big)^{1/p}, \quad \|f\|_{L_p(\mathbb{R}+)} = \Big(\int_{0}^{\infty} |f(x)|^p dx \Big)^{1/p}.$$

Also, we define the function space $L_p^{\alpha,\beta,\gamma}(\mathbb{R})$, $\alpha > -1$, $\beta > 0$, $\gamma > 0$, $p > 1$ by

$$L_p^{\alpha,\beta,\gamma}(\mathbb{R}) := \big\{ f(x) : \int_{-\infty}^{\infty} |x|^\alpha e^{-\beta|x|^\gamma} |f(x)|^p dx < \infty \big\}$$

with the norm

$$\|f\|_{L_p^{\alpha,\beta,\gamma}(\mathbb{R})} = \Big(\int_{-\infty}^{\infty} |x|^\alpha e^{-\beta|x|^\gamma} |f(x)|^p dx \Big)^{1/p}.$$

3. A polyconvolution related to Hartley and Fourier cosine transforms

In this section, we define a new polyconvolution for Hartley and Fourier cosine transforms and then prove some its properties.

Definition 3.1. The polyconvolution related to Hartley and Fourier cosine transforms of three functions f, g, h is defined by

$$[*(f,g,h)](x) := \frac{1}{4\pi} \int_{-\infty}^{\infty} f(u)[k_1(x-u) + k_2(x+u)]du, \quad x \in \mathbb{R}, \tag{3.1}$$

where k_1 and k_2 are determined by (1.3) and (1.4) respectively.

The most important feature of a new polyconvolution is its factorization property. Normally, each convolution has only one factorization equality. However, this polyconvolution is one of several convolutions or polyconvolutions, which has two factorization equalities.

Theorem 3.2. *Assume that $f, h \in L_1(\mathbb{R})$ and $g \in L_1(\mathbb{R}_+)$. Then, the polyconvolution* (3.1) *belongs to $L_1(\mathbb{R})$ and the norm inequality on $L_1(\mathbb{R})$ is of the form*

$$\| *(f,g,h)\|_{L_1(\mathbb{R})} \le \frac{2}{\pi} \|f\|_{L_1(\mathbb{R})} \|g\|_{L_1(\mathbb{R}_+)} \|h\|_{L_1(\mathbb{R})}. \tag{3.2}$$

Moreover, it satisfies the factorization identifies (1.5) *and* (1.6). *In case $h \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, the following Parseval type identity holds,*

$$*(f,g,h)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (H_{\{1,2\}} f)(y) \cdot (F_c g)(y) \cdot (H_{\{2,1\}} h)(y). \operatorname{cas}(\pm xy) dy. \tag{3.3}$$

Proof. First we prove that $*(f,g,h)(x) \in L_1(\mathbb{R})$. Indeed, using the Fubini theorem, we write

$$\begin{aligned} \int_{-\infty}^{\infty} | *(f,g,h)(x)| dx &\le \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(u)| \{ |k_1(x-u)| + |h(x+u)| \} \, dx \, du \\ &\le \frac{1}{4\pi} \int_{-\infty}^{\infty} |f(u)| du \Big[\int_{-\infty}^{\infty} |k_1(t)| dt + \int_{-\infty}^{\infty} |k_2(t)| dt \Big] \end{aligned} \tag{3.4}$$

From (1.3), we have

$$\begin{aligned} &\int_{-\infty}^{\infty} |k_1(t)| dt \\ &\le \int_{-\infty}^{\infty} \int_0^{\infty} |g(v)| [|h(-t+v)| + |h(t-v)| + |h(-t-v)| + |h(t+v)|] \, dv \, dt \\ &\le 4 \Big(\int_0^{\infty} |g(v)| dv \Big) \Big(\int_{-\infty}^{\infty} |h(t)| dt \Big). \end{aligned}$$

Similarly, with k_2, and continue with (3.4), we obtain

$$\int_{-\infty}^{\infty} |*(f,g,h)(x)|dx \le \frac{2}{\pi}\Big(\int_{-\infty}^{\infty}|f(u)|du\Big)\Big(\int_0^{\infty}|g(v)|dv\Big)\Big(\int_{-\infty}^{\infty}|h(t)|dt\Big) < \infty$$

So $*(f,g,h)(x)$ belongs to $L_1(\mathbb{R})$ and we also get inequality (3.2).

Now we prove the factorization property (1.5). From (2.2) and (2.4), we write

$$\begin{aligned}&(H_1f)(y)\cdot(F_cg)(y)\cdot(H_2h)(y)\\&= \frac{1}{\pi\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} f(u)g(v)h(t)\operatorname{cas}(uy)\cos(vy)\operatorname{cas}(-ty)\,dt\,dv\,du, \quad \forall y\in\mathbb{R}.\end{aligned}$$

Using trigonometric transforms, one can easily see that

$$\begin{aligned}&(H_1f)(y)\cdot(F_cg)(y)\cdot(H_2h)(y)\\&= \frac{1}{4\pi\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} f(u)g(v)h(t)\Big\{\operatorname{cas}[(u+v-t)y] + +\operatorname{cas}[(u+v+t)y]\\&\quad -\operatorname{cas}[(-u-v+t)y]+\operatorname{cas}[(-u-v-t)y]+\operatorname{cas}[(u-v-t)y]\\&\quad +\operatorname{cas}[(u-v+t)y]-\operatorname{cas}[(-u+v+t)y]+\operatorname{cas}[(-u+v-t)y]\Big\}\,dt\,dv\,du.\end{aligned}$$

Putting the corresponding substitution with each integral term in the above expression, we obtain

$$\begin{aligned}&(H_1f)(y)\cdot(F_cg)(y)\cdot(H_2h)(y)\\&= \frac{1}{4\pi\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty} f(u)g(v)\Big[h(-x+u+v)+h(x-u-v)-h(x+u+v)\\&\quad +h(-x-u-v)+h(-x+u-v)+h(x-u+v)-h(x+u-v)\\&\quad +h(-x-u+v)\Big]\operatorname{cas}(xy)\,dx\,dv\,du.\end{aligned}$$

Using Fubini's theorem, change the order of integrating, and using (1.3), (1.4), we have

$$\begin{aligned}&(H_1f)(y)\cdot(F_cg)(y)\cdot(H_2h)(y)\\&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Big\{\frac{1}{4\pi}\int_{-\infty}^{\infty} f(u)[k_1(x-u)+k_2(x+u)]du\Big\}\operatorname{cas}(xy)dx\\&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[*(f,g,h)(x)]\operatorname{cas}(xy)dx\\&= H_1(*(f,g,h))(y), \quad \forall y\in\mathbb{R}.\end{aligned}$$

This expression implies (1.5). Since $(H_1f)(y) = (H_2f)(-y)$, replacing (y) by $(-y)$, we obtain the second factorization identify (1.6).

Now we prove the Parseval properties (3.3). Indeed, by the hypothesis $h \in L_1(\mathbb{R})\cap L_2(\mathbb{R})$ then we have $H_2[(H_2h)(y)](x) = h(x)$. By the help of Fubini's theorem and using trigonometric transforms, we write

$$\begin{aligned}&H_1[(H_1f)(y)\cdot(F_cg)(y)\cdot(H_2h)(y)](x)\\&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(H_1f)(y)\cdot(F_cg)(y)\cdot(H_2h)(y)\operatorname{cas}(xy)\,dy\\&= \frac{1}{\pi\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} f(u)g(v)(H_2h)(y)\operatorname{cas}(uy)\cos(vy)\operatorname{cas}(xy)\,dv\,du\,dy\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4\pi\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(u)g(v)(H_2h)(y)\Big\{ \operatorname{cas}[(x-u-v)y] \\
&\quad + \operatorname{cas}[(-x+u+v)y] - \operatorname{cas}[(-x-u-v)y] + \operatorname{cas}[(x+u+v)y] \\
&\quad + \operatorname{cas}[(x-u+v)y] + \operatorname{cas}[(-x+u-v)y] - \operatorname{cas}[(-x-u+v)y] \\
&\quad + \operatorname{cas}[(x+u-v)y] \Big\}\, dy\, dv\, du \\
&\quad = \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{0}^{\infty} f(u)g(v)\Big[H_2(H_2h)(-x+u+v) + H_2(H_2h)(x-u-v) \\
&\quad - H_2(H_2h)(x+u+v) + H_2(H_2h)(-x-u-v) + H_2(H_2h)(-x+u-v) \\
&\quad + H_2(H_2h)(x-u+v) - H_2(H_2h)(x+u-v) + H_2(H_2h)(-x-u+v)\Big]\, dv\, du \\
&= \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{0}^{\infty} f(u)g(v)\Big[h(-x+u+v) + h(x-u-v) - h(x+u+v) \\
&\quad + h(-x-u-v) + h(-x+u-v) + h(x-u+v) - h(x+u-v) \\
&\quad + h(-x-u+v)\Big]\, dv\, du \\
&= \frac{1}{4\pi} \int_{-\infty}^{\infty} f(u)[k_1(x-u) + k_2(x+u)] du \\
&= [*(f,g,h)](x), \quad \forall x \in \mathbb{R}.
\end{aligned}
$$

It implies the first equality of (3.3), the other follows from a similar process. The proof is complete. □

Using the factorization equalities (1.5), (1.6), we easily obtain the following corollary.

Corollary 3.3. *Let $g, l \in L_1(\mathbb{R}_+)$ and $f, k, h \in L_1(\mathbb{R})$. Then the polyconvolution* (3.1) *satisfies the following conditions:*

$$
\begin{aligned}
&*(*(f,g,h),l,k) = *(*(f,l,h),g,k) = *(*(f,g,k),l,h) = *(*(f,l,k),g,h), \\
&*(k,l,*(f,g,h)) = *(h,l,*(f,g,k)) = *(k,g,*(f,l,h)) = *(h,g,*(f,l,k))
\end{aligned}
$$

Next, we study the polyconvolution in the function space $L_s^{\alpha,\beta,\gamma}(\mathbb{R})$ and its norm estimation.

Theorem 3.4. *Let $f \in L_p(\mathbb{R}),\ g \in L_q(\mathbb{R}_+),\ h \in L_r(\mathbb{R})$, such that $p, q, r > 1$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$. Then the polyconvolution* (3.1) *is bounded in $L_s^{\alpha,\beta,\gamma}(\mathbb{R})$, where $s > 1, \alpha > -1, \beta > 0, \gamma > 0$ and the following estimation holds.*

$$
\| *(f,g,h)\|_{L_s^{\alpha,\beta,\gamma}(\mathbb{R})} \le C\|f\|_{L_p(\mathbb{R})}\|g\|_{L_q(\mathbb{R}_+)}\|h\|_{L_r(\mathbb{R})}, \tag{3.5}
$$

where

$$
C = \frac{2^{1+\frac{1}{s}}}{\pi\gamma^{\frac{1}{s}}}\beta^{-\frac{\alpha+1}{\gamma.s}}\Gamma^{\frac{1}{s}}\big(\frac{\alpha+1}{\gamma}\big)
$$

If, in addition, $f \in L_1(\mathbb{R}) \cap L_p(\mathbb{R}),\ g \in L_1(\mathbb{R}_+) \cap L_q(\mathbb{R}_+)$ and $h \in L_1(\mathbb{R}) \cap L_r(\mathbb{R})$, then the polyconvolution (3.1) *satisfies the factorization equalities* (1.5), (1.6) *and belongs to $C_0(\mathbb{R})$. Moreover, if giving more condition on h, namely $h \in L_2(\mathbb{R}) \cap L_1(\mathbb{R}) \cap L_r(\mathbb{R})$, the Parseval identities* (3.3) *hold.*

Proof. Firstly, we prove $|*(f,g,h)(x)| \leq \frac{2}{\pi}\|f\|_{L_p(\mathbb{R})}.\|g\|_{L_q(\mathbb{R}_+)}\|h\|_{L_r(\mathbb{R})}$. Indeed, from Definition 3.1 and (1.3)-(1.4), we have the estimate

$$
\begin{aligned}
&|*(f,g,h)(x)| \\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty} |f(u)|.[|k_1(x-u)| + |k_2(x+u)|]du \\
&= \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} |f(u)||g(v)|\Big\{|h(-x+u+v)| + |h(x-u-v)| \\
&\quad + |h(x+u+v)| + |h(-x-u-v)| + |h(-x+u-v)| + |h(x-u+v)| \\
&\quad + |h(x+u-v)| + |h(-x-u+v)|\Big\}\, dv\, du. \qquad (3.6)
\end{aligned}
$$

Separating the right-hand side of this expression into the sum of 8 integrals and denoting them by $I_k, k = 1, \dots, 8$, respectively, without loss of generality, we have

$$
I_1(x) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} |f(u)||g(v)||h(-x+u+v)|\, dv\, du, \quad x \in \mathbb{R}.
$$

Let p_1, q_1, r_1 be the conjugate exponentials of p, q, r and

$$
\begin{gathered}
U(u,v) = |g(v)|^{q/p_1}|h(-x+u+v)|^{r/p_1} \in L_{p_1}(\mathbb{R}\times\mathbb{R}_+) \\
V(u,v) = |h(-x+u+v)|^{\frac{r}{q_1}}|f(u)|^{\frac{p}{q_1}} \in L_{q_1}(\mathbb{R}\times\mathbb{R}_+) \\
W(u,v) = |f(u)|^{p/r_1}|g(v)|^{q/r_1} \in L_{r_1}(\mathbb{R}\times\mathbb{R}_+).
\end{gathered}
$$

We have

$$
UVW = |f(u)||g(v)||h(-x+u+v)|.
$$

Using the definition of the norm on space $L_{p_1}(\mathbb{R}\times\mathbb{R}_+)$ and the help of Fubini's Theorem, we write

$$
\begin{aligned}
\|U\|^{p_1}_{L_{p_1}(\mathbb{R}\times\mathbb{R}_+)} &= \int_{-\infty}^{\infty}\int_{0}^{\infty} \big\{|g(v)|^{q/p_1}|h(-x+u+v)|^{r/p_1}\big\}^{p_1}\, dv\, du \\
&= \int_{0}^{\infty} |g(v)|^{q}\Big(\int_{-\infty}^{\infty} |h(-x+u+v)|^{r}\, du\Big)dv \\
&= \int_{0}^{\infty} |g(v)|^{q}\|h\|^{r}_{L_r(\mathbb{R})}dv \\
&= \|g\|^{q}_{L_q(\mathbb{R}_+)}\|h\|^{r}_{L_r(\mathbb{R})}.
\end{aligned}
$$

Similarly, we obtain

$$
\|V\|^{q_1}_{L_{q_1}(\mathbb{R}\times\mathbb{R}_+)} = \|f\|^{p}_{L_p(\mathbb{R})}.\|h\|^{r}_{L_r(\mathbb{R})}; \quad \|W\|^{r_1}_{L_{r_1}(\mathbb{R}\times\mathbb{R}_+)} = \|f\|^{p}_{L_p(\mathbb{R})}\|g\|^{q}_{L_q(\mathbb{R}_+)}. \quad (3.7)
$$

From the hypothesis $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=2$, it follows $\frac{1}{p_1}+\frac{1}{q_1}+\frac{1}{r_1}=1$. Using the Hölder inequality and (3.7), we have estimate

$$
\begin{aligned}
I_1 &= \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_0^{\infty} UVW\,dv\,du \\
&\le \frac{1}{4\pi}\Big(\int_{-\infty}^{\infty}\int_0^{\infty} U^{p_1}\,du\,dv\Big)^{1/p_1}\Big(\int_{-\infty}^{\infty}\int_0^{\infty} V^{q_1} dudv\Big)^{1/p_1} \\
&\quad \times \Big(\int_{-\infty}^{\infty}\int_0^{\infty} W^{r_1} dudv\Big)^{1/r_1} \\
&= \frac{1}{4\pi}\|U\|_{L_{p_1}(\mathbb{R}\times\mathbb{R}_+)}\|V\|_{L_{q_1}(\mathbb{R}\times\mathbb{R}_+)}\|W\|_{L_{r_1}(\mathbb{R}\times\mathbb{R}_+)} \\
&= \frac{1}{4\pi}\Big(\|g\|^{\frac{q}{p_1}}_{L_q(\mathbb{R}_+)}\|h\|^{\frac{r}{p_1}}_{L_r(\mathbb{R})}\Big)\Big(\|f\|^{\frac{p}{q_1}}_{L_p(\mathbb{R})}\|h\|^{\frac{r}{q_1}}_{L_r(\mathbb{R})}\Big)\Big(\|f\|^{\frac{p}{r_1}}_{L_p(\mathbb{R})}\|g\|^{\frac{q}{r_1}}_{L_q(\mathbb{R}_+)}\Big) \\
&= \frac{1}{4\pi}\|f\|_{L_p(\mathbb{R})}\|g\|_{L_q(\mathbb{R}_+)}\|h\|_{L_r(\mathbb{R})}
\end{aligned} \tag{3.8}
$$

The same way, we obtain the estimates for I_k, $k=2,3\dots,8$:

$$
I_k \le \frac{1}{4\pi}\|f\|_{L_p(\mathbb{R})}\|g\|_{L_q(\mathbb{R}_+)}\|h\|_{L_r(\mathbb{R})}, \quad \text{for } k=2,3,\dots,8 \tag{3.9}
$$

From (3.6)–(3.9), it follows that

$$
|*(f,g,h)(x)| \le \frac{2}{\pi}\|f\|_{L_p(\mathbb{R})}\|g\|_{L_q(\mathbb{R}_+)}\|h\|_{L_r(\mathbb{R})}. \tag{3.10}
$$

Now, using the [6, formula 3.381.10], we have

$$
\int_{-\infty}^{\infty}|x|^{\alpha}e^{-\beta|x|^{\gamma}}dx = \frac{2}{\gamma}\beta^{-\frac{\alpha+1}{\gamma}}\Gamma\big(\frac{\alpha+1}{\gamma}\big). \tag{3.11}
$$

From (3.10) and (3.11), we have

$$
\begin{aligned}
\|*(f,g,h)\|^s_{L_s^{\alpha,\beta,\gamma}(\mathbb{R})} &= \int_{-\infty}^{\infty}|x|^{\alpha}e^{-\beta|x|^{\gamma}}|*(f,g,h)(x)|^s dx \\
&\le \int_{-\infty}^{\infty}|x|^{\alpha}e^{-\beta|x|^{\gamma}}\big(\frac{2}{\pi}\big)^s\|f\|^s_{L_p(\mathbb{R})}\|g\|^s_{L_q(\mathbb{R}_+)}\|h\|^s_{L_r(\mathbb{R})}dx \\
&= C^s\|f\|^s_{L_p(\mathbb{R})}\|g\|^s_{L_q(\mathbb{R}_+)}\|h\|^s_{L_r(\mathbb{R})}.
\end{aligned}
$$

where

$$
C = \frac{2^{1+\frac{1}{s}}}{\pi\gamma^{\frac{1}{s}}}\beta^{-\frac{\alpha+1}{\gamma.s}}\Gamma^{\frac{1}{s}}\big(\frac{\alpha+1}{\gamma}\big),
$$

which gives (3.5).

Since $f\in L_1(\mathbb{R})\cap L_p(\mathbb{R})$, $g\in L_1(\mathbb{R}_+)\cap L_q(\mathbb{R}_+)$ and $h\in L_1(\mathbb{R})\cap L_r(\mathbb{R})$, three functions f,g and h satisfy the hypothesis of Theorem 3.2, it implies that $*(f,g,h)\in C_0(\mathbb{R})\cap L_1(\mathbb{R})$, then the factorization identities (1.5), (1.6) hold. Moreover, if $h\in L_2(\mathbb{R})\cap L_1(\mathbb{R})\cap L_r(\mathbb{R})$, it also satisfies the hypothesis of Theorem 3.2 to get Parseval equalities (3.3). The proof is complete. □

Next, we have a Titchmarch type theorem.

Theorem 3.5. *Let $f,h\in L_1(\mathbb{R},e^{|x|})$ and $g\in L_1(\mathbb{R}_+,e^{x})$. If $*(f,g,h)(x)=0$, $\forall x\in\mathbb{R}$, then either $f(x)=0,\forall x\in\mathbb{R}$, or $g(x)=0,\forall x\in\mathbb{R}_+$ or $h(x)=0,\forall x\in\mathbb{R}$.*

Proof. The hypothesis $*(f,g,h)(x)=0$, for all $x\in\mathbb{R}$ implies $H_1(*(f,g,h))(y)=0$, for all $y\in\mathbb{R}$. Due to factorization equality (1.5) we have

$$(H_1f)(y)(F_cg)(y)(H_2h)(y)=0,\quad \forall y\in\mathbb{R}. \tag{3.12}$$

Now we show that $(H_1f)(y)$, $(F_cg)(y)$, $(H_2h)(y)$ are real analytic. Without loss of generality, we prove that $(H_1f)(y)$ can be expanded into convergent Taylor series in $\mathbb{R}$. Indeed, by using the Lebesgue Dominated Convergence Theorem, we can exchange the orders of integration and differentiation, we have

$$\begin{aligned}
\Big|\frac{d^n}{dy^n}(H_1f)(y)\Big| &\le \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Big|\frac{d^n}{dy^n}[f(x)\operatorname{cas}(xy))]\Big|dx\\
&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\big|f(x)x^n\big[\cos\big(xy+n\frac{\pi}{2})\big)+\sin\big(xy+n\frac{\pi}{2})\big)\big]\big|dx\\
&\le \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|2f(x)x^n|dx\\
&\le \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}e^{-|x|}\frac{|x|^n}{n!}n!|f(x)|e^{|x|}dx\\
&\le \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}n!|f(x)|e^{|x|}dx=\sqrt{\frac{2}{\pi}}n!\|f\|_{L_1(\mathbb{R},e^{|x|})}=n!M,
\end{aligned}$$

here, $M=\sqrt{\frac{2}{\pi}}\|f\|_{L_1(\mathbb{R},e^{|x|})}$.

Thus, the remainder of Taylor expansion for $(H_1f)(y)$ at neighbourhood of an arbitrary $y_0\in\mathbb{R}$ is

$$\Big|\frac{1}{n!}\frac{d^n(H_1f)(c)}{dy^n}(y-y_0)^n\Big|\le\frac{1}{n!}n!M|y-y_0|^n=M|y-y_0|^n,$$

From analytic property and (3.12), it implies

$$(H_1f)(y)=0,\forall y\in\mathbb{R},\quad\text{or}\quad(F_cg)(y)=0,\forall y\in\mathbb{R}_+,\quad\text{or}\quad(H_2h)(y)=0,\forall y\in\mathbb{R}.$$

Since the uniqueness of Hartley and Fourier cosine transforms in $L_1(\mathbb{R})$, then it follows

$$f(x)=0,\ \forall y\in\mathbb{R},\quad\text{or}\quad g(x)=0,\ \forall x\in\mathbb{R}_+,\quad\text{or}\quad h(x)=0,\ \forall x\in\mathbb{R}.$$

The proof is complete. □

4. Applications

4.1. **Integral equations.** In this subsection we apply the obtained result in solving the modified equation (1.2) of the Toeplitz plus Hankel integral equation (1.1).

Theorem 4.1. *Let $g\in L_1(\mathbb{R}_+)$ and $h,p\in L_1(\mathbb{R})$ be given functions, λ is given constant. The sufficient and necessary condition for the integral equation* (1.2) *to have a unique solution in space $L_1(\mathbb{R})$ is*

$$1+\lambda(F_cg)(y)(H_2h)(y)\neq 0,\quad\forall y\in\mathbb{R}, \tag{4.1}$$

and the solution has the form

$$f(x)=p(x)-(l\underset{HF}{*}p)(x),\quad\forall x\in\mathbb{R},$$

where $l \in L_1(\mathbb{R})$ is defined by

$$(Fl)(y) = \frac{\lambda(F_c g)(y)(H_2 h)(y)}{1 + \lambda(F_c g)(y)(H_2 h)(y)}, \quad \forall y \in \mathbb{R}. \tag{4.2}$$

Proof. Using Definition 3.1, the equation (1.2) can be rewritten in the form

$$f(x) + \lambda[*(f, g, h)](x) = p(x).$$

Applying the Hartley transform H_1 on both sides of the equation, using the factorization property (1.5) and (2.8), we obtain

$$\begin{aligned}(H_1 f)(y) + \lambda(H_1 f(y) \cdot (F_c g)(y) \cdot (H_2 h)(y) &= (H_1 p)(y) \\ (H_1 f)(y)[1 + \lambda(F_c g)(y) \cdot (H_2 h)(y)] &= (H_1 p)(y).\end{aligned} \tag{4.3}$$

Due to (4.1), the equation (4.3) has a unique solution

$$\begin{aligned}(H_1 f)(y) &= (H_1 p)(y) \frac{1}{[1 + \lambda(F_c g)(y) \cdot (H_2 h)(y)]} \\ &= (H_1 p)(y)\Big[1 - \frac{\lambda(F_c g)(y) \cdot n(H_2 h)(y)}{1 + \lambda(F_c g)(y) \cdot (H_2 h)(y)}\Big].\end{aligned} \tag{4.4}$$

By the Wiener-Levy theorem [10], if q is the Fourier transform of a some function in $L_1(\mathbb{R})$, $\varphi(z)$ is analytic, $\varphi(0) = 0$ and defined at area of q values, then $\varphi(q)$ also is a Fourier transform of a some function in $L_1(\mathbb{R})$. Note that we can write $(F_c g)(y) \cdot (H_2 h)(y) = H_2\big(g \underset{HF_c}{*} h\big)(y)$ and we have the relationship between Hartley transform and Fourier transform

$$(H_2 q)(y) = \frac{1+i}{2}(Fq)(-y) + \frac{1-i}{2}(Fq)(y), \quad \forall y \in \mathbb{R},$$

then the expression

$$\frac{\lambda(F_c g)(y).(H_2 h)(y)}{1 + \lambda(F_c g)(y).(H_2 h)(y)}$$

defines the Fourier transform of a some function l in $L_1(\mathbb{R})$. It means that, there exists a function $l \in L_1(\mathbb{R})$ such that

$$(Fl)(y) = \frac{\lambda(F_c g)(y).(H_2 h)(y)}{1 + \lambda(F_c g)(y).(H_2 h)(y)}. \tag{4.5}$$

So, from (4.4), (4.5) and using the factorization equality (2.7), we obtain

$$\begin{aligned}(H_1 f)(y) &= (H_1 p)(y)[1 - (Fl)(y)] = (H_1 p)(y) - (Fl)(y)(H_1 p)(y) \\ &= (H_1 p)(y) - H_1(l \underset{HF}{*} p)(y) = H_1[p - (l \underset{HF}{*} p)](y), \quad \forall y \in \mathbb{R}.\end{aligned}$$

It follows that $f(x) = p(x) - (l \underset{HF}{*} p)(x) \in L_1(\mathbb{R})$. The proof is complete. □

We see that the condition (4.1) is still true for $g, h \in L_1(\mathbb{R})$. However, determining $l(x)$ from (4.2) depends on g, h. Below, we will show an example to find $l(x)$ with given functions g, h.

Example 4.2. Choose $g(x) = \sqrt{\frac{2}{\pi}} K_0(x)$ and $h(x) = \sqrt{\frac{2}{\pi}} K_0(|x|)$, where $K_0(x)$ is Bessel function. By property of Bessel $K_0(x)$ (see the [6, formula 6.511.12]), we see that $\int_0^\infty |K_0(x)| dx = \pi/2$. This implies $K_0(x)$ is a function in $L_1(\mathbb{R}_+)$. Thus,

$g(x) \in L_1(\mathbb{R}_+)$, $h(x) \in L_1(\mathbb{R})$ and from the [3, formula 1.2.17] (or [6, formula 3.754.2]) we have

$$(F_c g)(y) = (H_2 h)(y) = \frac{1}{\sqrt{1+y^2}}.$$

Next, we choose $\lambda = 1$, then

$$(Fl)(y) = \frac{(F_c g)(y)(H_2 h)(y)}{1 + (F_c g)(y)(H_2 h)(y)} = \frac{1}{y^2+2} \in L_1(\mathbb{R}).$$

Based on [6, formula 17.23.14], we have

$$l(x) = F^{-1}\big(\frac{1}{y^2+2}\big)(x) = \sqrt{\pi}\frac{e^{-\sqrt{2}|x|}}{2} \in L_1(\mathbb{R}).$$

Now, choosing $p(x) = e^{-x^2} \in L_1(\mathbb{R})$, and using (2.6), we have

$$\begin{aligned}
(l \underset{HF}{*} p)(x) &= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} l(u)[p(x-u) + p(x+u) + ip(-x-u) - ip(x+u)]du \\
&= \frac{1}{4\sqrt{2}} \int_{-\infty}^{\infty} e^{-\sqrt{2}|u|}\big[e^{-(x-u)^2} + e^{-(x+u)^2}\big]du \\
&= \frac{1}{4\sqrt{2}}.e^{\frac{1}{2}-\sqrt{2}x}\sqrt{\pi}\Big(\operatorname{Erfc}\big[\frac{1}{\sqrt{2}} - x\big] + e^{2\sqrt{2}x}\operatorname{Erfc}\big[\frac{1}{\sqrt{2}} + x\big]\Big).
\end{aligned}$$

where $\operatorname{Erfc}(x)$ is the complementary error function. By using the Mathematica software program, we have

$$\int_{-\infty}^{\infty}\Big|\frac{1}{4\sqrt{2}}e^{\frac{1}{2}-\sqrt{2}x}\sqrt{\pi}\Big(\operatorname{Erfc}[\frac{1}{\sqrt{2}} - x] + e^{2\sqrt{2}x}\operatorname{Erfc}[\frac{1}{\sqrt{2}} + x]\Big)\Big|dx = \frac{\pi}{2}.$$

This implies $(l \underset{HF}{*} p)(x) \in L_1(\mathbb{R})$. So, the solution of the equation (1.2) is

$$\begin{aligned}
f(x) &= p(x) - (l \underset{HF}{*} p)(x) \\
&= e^{-x^2} - \frac{1}{4\sqrt{2}}e^{\frac{1}{2}-\sqrt{2}x}\sqrt{\pi}\Big(\operatorname{Erfc}[\frac{1}{\sqrt{2}} - x] + e^{2\sqrt{2}x}\operatorname{Erfc}[\frac{1}{\sqrt{2}} + x]\Big) \in L_1(\mathbb{R}).
\end{aligned}$$

4.2. **System of two integral equations.** Consider a system of two Toeplitz plus Hankel integral equations of the form:

$$\begin{gathered}
f(x) + \lambda_1 \int_{-\infty}^{\infty} g(u)[k_1(x-u) + k_2(x+u)]du = p(x) \\
\lambda_2 \int_{-\infty}^{\infty} f(u)[k_3(x-u) + k_4(x+u)]du + g(x) = q(x), \quad \forall x \in \mathbb{R},
\end{gathered} \tag{4.6}$$

where, for $t \in \mathbb{R}$,

$$\begin{aligned}
k_1(t) &:= \int_0^{\infty} \varphi_1(v)[\psi_1(-t+v) + \psi_1(t-v) + \psi_1(-t-v) + \psi_1(t+v)]dv, \\
k_2(t) &:= \int_0^{\infty} \varphi_1(v)(v)[-\psi_1(t+v) + \psi_1(-t-v) - \psi_1(t-v) + \psi_1(-t+v)]dv, \\
k_3(t) &:= \int_0^{\infty} \varphi_2(v)[\psi_2(-t+v) + \psi_2(t-v) + \psi_2(-t-v) + \psi_2(t+v)]dv, \\
k_4(t) &:= \int_0^{\infty} \varphi_2(v)(v)[-\psi_2(t+v) + \psi_2(-t-v) - \psi_2(t-v) + \psi_2(-t+v)]dv,
\end{aligned}$$

and λ_1, λ_2 are complex constants; φ_1, φ_2 are functions in $L_1(\mathbb{R}_+)$; $\psi_1, \psi_2, p(x), q(x)$ are functions in $L_1(\mathbb{R})$; and f, g are unknown functions.

Theorem 4.3. *If the condition*

$$1 - \lambda_1\lambda_2(F_c\varphi_1)(y)(F_c\varphi_2)(y)(H_2\psi_1)(y)(H_2\psi_2)(y) \neq 0, \quad \forall y \in \mathbb{R}.$$

holds, then there exists a unique solution in $L_1(\mathbb{R}) \times L_1(\mathbb{R})$ *of system* (4.6) *defined by*

$$f(x) = p(x) - \lambda_1(*(q, \varphi_1, \psi_1))(x) + \big\{l \underset{HF}{*} [p - \lambda_1(*(q, \varphi_1, \psi_1))]\big\}(x),$$

$$g(x) = q(x) - \lambda_2(*(p, \varphi_2, \psi_2))(x) + \big\{l \underset{HF}{*} [q - \lambda_2(*(p, \varphi_2, \psi_2))]\big\}(x),$$

here $l \in L_1(\mathbb{R})$ *is given by*

$$(Fl)(y) = \frac{\lambda_1\lambda_2(F_c\varphi_1)(y)(F_c\varphi_2)(y)(H_2\psi_1)(y)(H_2\psi_2)(y)}{1 - \lambda_1\lambda_2(F_c\varphi_1)(y)(F_c\varphi_2)(y)(H_2\psi_1)(y)(H_2\psi_2)(y)}, \quad \forall y \in \mathbb{R}.$$

Proof. Using Definition 3.1, the system of equations (4.6) can be rewritten in the form

$$\begin{gathered} f(x) + \lambda_1[*(g, \varphi_1, \psi_1)](x) = p(x), \\ \lambda_2[*(f, \varphi_2, \psi_2)](x) + g(x) = q(x), \quad x \in \mathbb{R}. \end{gathered} \tag{4.7}$$

Due to the factorization property of the polyconvolution (1.5), we obtain the linear system of algebraic equations with respect to $(H_1f)(y)$ and $(H_1g)(y)$

$$\begin{gathered} (H_1f)(y) + \lambda_1(H_1g)(y)(F_c\varphi_1)(y)(H_2\psi_1)(y) = (H_1p)(y), \\ \lambda_2(H_1f)(y)(F_c\varphi_2)(y)(H_2\psi_2)(y) + (H_1g)(y) = (H_1q)(y), \quad \forall y \in \mathbb{R}. \end{gathered} \tag{4.8}$$

Let Δ be the determinant of the system,

$$\begin{aligned} \Delta &= \begin{vmatrix} 1 & \lambda_1(F_c\varphi_1)(y)(H_2\psi_1)(y) \\ \lambda_2(F_c\varphi_2)(y)(H_2\psi_2)(y) & 1 \end{vmatrix} \\ &= 1 - \lambda_1\lambda_2(F_c\varphi_1)(y)(F_c\varphi_2)(y)(H_2\psi_1)(y)(H_2\psi_2)(y). \end{aligned}$$

Due to the hypothesis, $\Delta \neq 0$, the system (4.7) has a unique solution. By using (2.8) and (2.5), we present $1/\Delta$ as below

$$\begin{aligned} \frac{1}{\Delta} &= \frac{1}{1 - \lambda_1\lambda_2(F_c\varphi_1)(y)(F_c\varphi_2)(y)(H_2\psi_1)(y)(H_2\psi_2)(y)} \\ &= 1 + \frac{\lambda_1\lambda_2(F_c\varphi_1)(y)(F_c\varphi_2)(y)(H_2\psi_1)(y)(H_2\psi_2)(y)}{1 - \lambda_1\lambda_2(F_c\varphi_1)(y)(F_c\varphi_2)(y)(H_2\psi_1)(y)(H_2\psi_2)(y)} \\ &= 1 + \frac{\lambda_1\lambda_2 H_2[(\varphi_1 \underset{HF_c}{*} \psi_1) \underset{H}{*} (\varphi_2 \underset{HF_c}{*} \psi_2)](y)}{1 - \lambda_1\lambda_2 H_2[(\varphi_1 \underset{HF_c}{*} \psi_1) \underset{H}{*} (\varphi_2 \underset{HF_c}{*} \psi_2)](y)}. \end{aligned}$$

Furthermore, according to Wiener-Levy theorem [10] and the relationship between Hartley transform and Fourier transform, it exists a function $l \in L_1(\mathbb{R})$ such that

$$(Fl)(y) = \frac{\lambda_1\lambda_2 H_2[(\varphi_1 \underset{HF_c}{*} \psi_1) \underset{H}{*} (\varphi_2 \underset{HF_c}{*} \psi_2)](y)}{1 - \lambda_1\lambda_2 H_2[(\varphi_1 \underset{HF_c}{*} \psi_1) \underset{H}{*} (\varphi_2 \underset{HF_c}{*} \psi_2)](y)}, \ \forall y \in \mathbb{R}.$$

So, we can write $\frac{1}{\Delta} = 1 + (Fl)(y)$. To find the solution of the system (4.7), we need to determine the two following determinants

$$\Delta_1 = \begin{vmatrix} (H_1p)(y) & \lambda_1(F_c\varphi_1)(y)(H_2\psi_1)(y) \\ (H_1q)(y) & 1 \end{vmatrix}$$

$$= (H_1p)(y) - \lambda_1 H_1[*(q,\varphi_1,\psi_1)](y) = H_1[p - \lambda_1(*(q,\varphi_1,\psi_1))](y), \quad y \in \mathbb{R}.$$

Based on (2.7), we have

$$\begin{aligned}(H_1f)(y) = \frac{\Delta_1}{\Delta} &= H_1[p - \lambda_1(*(q,\varphi_1,\psi_1))](y)[1 + (Fl)(y)] \\ &= H_1[p - \lambda_1(*(q,\varphi_1,\psi_1))](y) + H_1\{l \underset{HF}{*} [p - \lambda_1(*(q,\varphi_1,\psi_1))]\}(y) \\ &= H_1\{p - \lambda_1(*(q,\varphi_1,\psi_1)) + l \underset{HF}{*} [p - \lambda_1(*(q,\varphi_1,\psi_1))]\}(y), \quad \forall y \in \mathbb{R}.\end{aligned}$$

It follows that

$$f(x) = p(x) - \lambda_1(*(q,\varphi_1,\psi_1))(x) + \{l \underset{HF}{*} [p - \lambda_1(*(q,\varphi_1,\psi_1))]\}(x) \in L_1(\mathbb{R}).$$

Similarly, we compute the second component determinant of system (4.7),

$$\begin{aligned}\Delta_2 &= \begin{vmatrix} 1 & (H_1p)(y) \\ \lambda_2(F_c\varphi_2)(y)(H_2\psi_2)(y) & (H_1q)(y)\end{vmatrix} \\ &= (H_1q)(y) - \lambda_2 H_1[*(p,\varphi_2,\psi_2)](y) \\ &= H_1[q - \lambda_2(*(p,\varphi_2,\psi_2))](y), \quad y \in \mathbb{R}.\end{aligned}$$

Based on (2.7) we obtain

$$\begin{aligned}(H_1g)(y) = \frac{\Delta_2}{\Delta} &= H_1[q - \lambda_2(*(p,\varphi_2,\psi_2))](y).[1 + (Fl)(y)] \\ &= H_1[q - \lambda_2(*(p,\varphi_2,\psi_2))](y) + H_1\{l \underset{HF}{*} [q - \lambda_2(*(p,\varphi_2,\psi_2))]\}(y) \\ &= H_1\big\{q - \lambda_2(*(p,\varphi_2,\psi_2)) + l \underset{HF}{*} [q - \lambda_2(*(p,\varphi_2,\psi_2))]\big\}(y), \quad \forall y \in \mathbb{R}.\end{aligned}$$

It follows that

$$g(x) = q(x) - \lambda_2(*(p,\varphi_2,\psi_2))(x) + \{l \underset{HF}{*} [q - \lambda_2(*(p,\varphi_2,\psi_2))]\}(x) \in L_1(\mathbb{R}).$$

The proof is complete. □

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