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EXISTENCE AND MULTIPLICITY OF HOMOCLINIC SOLUTIONS FOR $p(t)$-LAPLACIAN SYSTEMS WITH SUBQUADRATIC POTENTIALS

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Abstract. By using the genus properties, we establish some criteria for the second-order $p(t)$-Laplacian system

$$\frac{d}{dt}\big(|\dot u(t)|^{p(t)-2}\dot u(t)\big)-a(t)|u(t)|^{p(t)-2}u(t)+\nabla W(t,u(t))=0$$

to have at least one, and infinitely many homoclinic orbits. where $t\in\mathbb{R}$, $u\in\mathbb{R}^N$, $p(t)\in C(\mathbb{R},\mathbb{R})$ and $p(t)>1$, $a\in C(\mathbb{R},\mathbb{R})$ and $W\in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ may not be periodic in t.

1. Introduction

Consider the second-order ordinary $p(t)$-Laplacian system

$$\frac{d}{dt}\big(|\dot u(t)|^{p(t)-2}\dot u(t)\big)-a(t)|u(t)|^{p(t)-2}u(t)+\nabla W(t,u(t))=0, \tag{1.1}$$

where $p\in C(\mathbb{R},\mathbb{R})$ and $p(t)>1$, $t\in\mathbb{R}$, $u\in\mathbb{R}^N$, $a:\mathbb{R}\to\mathbb{R}$ and $W:\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$. As usual, we say that a solution $u(t)$ of (1.1) is homoclinic (to 0) if $u(t)\to 0$ as $t\to\pm\infty$. In addition, if $u(t)\not\equiv 0$ then $u(t)$ is called a nontrivial homoclinic solution.

System (1.1) has been studied by Fan, et al. in a series of papers [10, 11, 12, 13]. The $p(t)$-Laplacian systems can be applied to describe the physical phenomena with "pointwise different properties" which first arose from the nonlinear elasticity theory (see [29]). The $p(t)$-Laplacian operator possesses more complicated nonlinearity than that of the p-Laplacian, for example, it is not homogeneous, this causes many troubles, and some classic theories and methods, such as the theory of Sobolev spaces, are not applicable.

It is well-known that homoclinic orbits play an important role in analyzing the chaos of dynamical systems. If a system has the transversely intersected homoclinic orbits, then it must be chaotic. Therefore, it is of practical importance and mathematical significance to consider the existence of homoclinic orbits of (1.1) emanating from 0.

If $p(t)\equiv p$ is a constant, system (1.1) reduces to the ordinary p-Laplacian system

$$\frac{d}{dt}\big(|\dot u(t)|^{p-2}\dot u(t)\big)-a(t)|u(t)|^{p-2}u(t)+\nabla W(t,u(t))=0. \tag{1.2}$$

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In recent years, the existence and multiplicity of homoclinic orbits for Hamiltonian systems have been investigated in many papers via variational methods and many results were obtained based on various hypotheses on the potential functions when $p = 2$, see, e.g., [2, 3, 4, 14, 15, 20, 21, 24, 25, 27, 28].

In the last decade there has been an increasing interest in the study of ordinary differential systems driven by the p-Laplacian (or the generalization of Laplacian [17]). For the existence of solutions for $p(t)$-Laplacian Dirichlet problems on a bounded domain we refer to [5, 6, 7, 8, 9, 26]. The study on the existence of solutions for $p(t)$-Laplacian equations in $\mathbb{R}$ is a new topic, which seems not to have been considered in the literature. We know that in the study of p-Laplacian equations in $\mathbb{R}$, a main difficulty arises from the lack of compactness. On the other hand, compared with the literature available for $W(t,x)$ being superquadratic as $|x| \to +\infty$, there is less literature available for the case where $W(t,x)$ is subquadratic at infinity. Motivated by papers [2, 27], we will use the genus properties to establish some existence criteria to guarantee that system (1.1) has infinitely many homoclinic solutions under more relaxed assumptions on $W(t,x)$.

For our results, we use the following assumptions:

(A1) $a \in C(\mathbb{R},(0,\infty))$ and $a(t) \to +\infty$ as $|t| \to \infty$, $b(t) = 1/a(t)$, $b(t)^{\frac{\alpha_i(t)}{p(t)}}$ belongs to $L^{r_i(t)}(\mathbb{R},\mathbb{R})$, where $r_i(t)$ satisfies

$$\frac{1}{r_i(t)} + \frac{\alpha_i(t)}{p(t)} = 1, \quad i = 1,2.$$

(P1) $1 < p^- := \inf_{t\in\mathbb{R}} p(t) \le \sup_{t\in\mathbb{R}} p(t) := p^+ < \infty$;

(W1) $W \in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$, $W(t,0) = 0$ and there exist two bounded functions $a_i(t)$ ($i = 1,2.$) such that

$$|\nabla W(t,x)| \le a_1(t)\alpha_1(t)|x|^{\alpha_1(t)-1}, \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N,\ |x| \le 1,$$

and for every $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ with $|x| \ge 1$,

$$|\nabla W(t,x)| \le a_2(t)\alpha_2(t)|x|^{\alpha_2(t)-1}, \quad |W(t,x)| \le ca_2(t)(t)|x|^{\alpha_2(t)}$$

where $\alpha_i(t)$ satisfy $\alpha_i^+ < p^-$, $a_i(t) \in C(\mathbb{R},\mathbb{R}^+)$ ($i = 1,2$), and c is a constant;

(W2) There exist an open set $J \subset \mathbb{R}$ and a function $\gamma_1(t)$ such that

$$W(t,x) \ge \eta|x|^{\gamma_1(t)}, \quad \forall (t,x) \in J\times\mathbb{R}^N,\ |x| \le 1,$$

where $\gamma_1(t)$ satisfy $1 < \gamma_1^+ < p^-$, $\eta > 0$ is a constant;

(W3) $W(t,-x) = W(t,x)$ for all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$.

Our main results are the following two theorems.

Theorem 1.1. *Assume* (A1), (P1), (W1), (W2) *are satisfied. Then* (1.1) *possesses at least one nontrivial homoclinic solution.*

Theorem 1.2. *Assume* (A1), (P1), (W1), (W2), (S3) *are satisfied. Then* (1.1) *possesses infinitely many nontrivial homoclinic solutions.*

The rest of the this article is organized as follows. In Section 2, we introduce some notations, preliminary results in space $W_a^{1,p(t)}$ and establish the corresponding variational structure. In Section 3, we complete the proofs of Theorems 1.1–1.2. In Section 4, we give some examples to to illustrate our results.

2. Preliminaries

In this section, we recall some known results in critical point theory and the properties of space $W_a^{1,p(t)}$ are listed for the convenience of readers. Let Ω be an open subset of $\mathbb{R}$. Let $S = \{\text{u}|\text{u is a measurable function in } \Omega\}$, elements in S that are equal to each other almost everywhere are considered as one element. Define

$$L_a^{p(t)}(\Omega, \mathbb{R}^N) = \{u \in S(\Omega, \mathbb{R}^N) : \int_\Omega a(t)|u(t)|^{p(t)}dt < \infty\}$$

with the norm

$$|u|_{p(t),a} = \inf\{\lambda > 0 : \int_\Omega a(t)|\frac{u}{\lambda}|^{p(t)}dt \le 1\}.$$

Define

$$W_a^{1,p(t)}(\Omega, \mathbb{R}^N) = \{u \in L_a^{p(t)}(\Omega, \mathbb{R}^N) : \dot{u} \in L^{p(t)}(\Omega, \mathbb{R}^N)\}$$

with the norm

$$\|u\| = \inf\{\lambda > 0 : \int_\Omega (|\frac{\dot{u}}{\lambda}|^{p(t)} + a(t)|\frac{u}{\lambda}|^{p(t)})dt \le 1\}.$$

We call the space $L_a^{p(t)}$ a generalized Lebesgue space, it is a special kind of generalized Orlicz spaces. The space $W_a^{1,p(t)}$ is called a generalized Sobolev space, it is a special kind of generalized Orlicz-Sobolev spaces. For the general theory of generalized Orlicz spaces and generalized Orlicz-Sobolev spaces, see [1, 19]. One can find the basic theory of spaces $L_a^{p(t)}$ and $W_a^{1,p(t)}$ in [10, 11, 12, 13].

Lemma 2.1 ([11, 12]). *Let*

$$\rho(u) = \int_\Omega a(t)|u|^{p(t)}dt, \quad \forall u \in L_a^{p(t)},$$

then

(i) $|u|_{p(t),a} < 1\ (= 1; > 1)$ *if and only if* $\rho(u) < 1\ (= 1; > 1)$;

(ii) $|u|_{p(t),a} > 1$ *implies* $|u|_{p(t),a}^{p^-} \le \rho(u) \le |u|_{p(t),a}^{p^+}$,
$|u|_{p(t),a} < 1$ *implies* $|u|_{p(t),a}^{p^+} \le \rho(u) \le |u|_{p(t),a}^{p^-}$;

(iii) $|u|_{p(t),a} \to 0$ *if and only if* $\rho(u) \to 0$;
$|u|_{p(t),a} \to \infty$ *if and only if* $\rho(u) \to \infty$.

(iv) *Let* $u \in L_a^{p(t)} \setminus \{0\}$, *then* $\|u\|_{p(t),a} = \lambda$ *if and only if* $\rho(\frac{u}{\lambda}) = 1$.

Lemma 2.2 ([11, 12]). *Let*

$$\varphi(u) = \int_\Omega (|\dot{u}|^{p(t)} + a(t)|u|^{p(t)})dt, \quad \forall u \in W_a^{1,p(t)},$$

(i) $\|u\| < 1\ (= 1; > 1)$ *if and only if* $\varphi(u) < 1\ (= 1; > 1)$;

(ii) $\|u\| > 1$ *implies* $\|u\|^{p^-} \le \varphi(u) \le \|u\|^{p^+}$,
$\|u\| < 1$ *implies* $\|u\|^{p^+} \le \varphi(u) \le \|u\|^{p^-}$;

(iii) $\|u\| \to 0$ *if and only if* $\varphi(u) \to 0$;
$\|u\| \to \infty$ *if and only if* $\varphi(u) \to \infty$.

Lemma 2.3 ([11]). *Let* $\rho(u) = \int_\Omega a(t)|u|^{p(t)}dt$ *for* $u, u_n \in L_a^{p(t)}$ $(n = 1, 2, \cdots)$, *then the following statements are equivalent to each other*

(i) $\lim_{n\to\infty} |u_n - u|_{p(t),a} = 0$;

(ii) $\lim_{n\to\infty} \rho(u_n - u) = 0$;

(iii) $u_n \to u$ *a.e.* $t \in \Omega$ *and* $\lim_{n\to\infty} \rho(u_n) = \rho(u)$.

Lemma 2.4 ([11]). *If* $\frac{1}{p(t)} + \frac{1}{q(t)} = 1$, *then*

(i) $(L^{p(t)})^* = L^{q(t)}$, *where* $(L^{p(t)})^*$ *is the conjugate space of* $L^{p(t)}$;
(ii) *for all* $u \in L^{p(t)}$ *and all* $v \in L^{q(t)}$, *we have*

$$\Big| \int_\Omega u(t)v(t)dt\Big| \le 2|u|_{p(t)}|v|_{q(t)}.$$

Lemma 2.5 ([10]). *If* $\frac{1}{p(t)} + \frac{1}{q(t)} + \frac{1}{r(t)} = 1$ *and for any* $u \in L^{p(t)}(\mathbb{R},\mathbb{R})$, $v \in L^{q(t)}(\mathbb{R},\mathbb{R})$ *and* $w \in L^{r(t)}(\mathbb{R},\mathbb{R})$, *we have*

$$\int_{\mathbb{R}} |uvw|dt \le 3|u|_{p(t)}|v|_{q(t)}|w|_{r(t)}.$$

Lemma 2.6 ([10]). *If* $|u|^{q(x)} \in L^{s(x)/q(x)}$, *where* $q, s \in L_+^\infty(\Omega), q(x) \le s(x)$, *then* $u \in L^{s(x)}(\Omega)$ *and there is a number* $\overline{q} \in [q^-, q^+]$ *such that* $|u|^{q(x)}|_{s(x)/q(x)} = (|u|_{s(x)})^{\overline{q}}$.

Lemma 2.7 ([16]). *If* $a^{\alpha(t)/p(t)}|u|^{\alpha(t)} \in L^{p(t)/\alpha(t)}$, *then* $u \in L_a^{p(t)}(\mathbb{R},\mathbb{R})$ *and* $|a^{\alpha(t)/p(t)}|u|^{\alpha(t)}|_{p(t)/\alpha(t)} = |u|_{p(t),a}^{\widetilde{\alpha}}$, *where* α, p *satisfy the condition* (P1) *and* $\alpha(t) < p(t)$ *for all* $t \in \mathbb{R}$, $\widetilde{\alpha} \in [\alpha^-, \alpha^+]$ *is a constant.*

Now, we establish the variational structure of system (1.1). Define

$$E = W_a^{1,p(t)}(\mathbb{R},\mathbb{R}^N) = \{u \in L_a^{p(t)}(\mathbb{R},\mathbb{R}^N) | \dot{u} \in L^{p(t)}(\mathbb{R},\mathbb{R}^N)\}.$$

Let $I : E \to \mathbb{R}$ be defined by

$$I(u) = \int_{\mathbb{R}} \frac{1}{p(t)}(|\dot{u}|^{p(t)} + a(t)|u|^{p(t)})dt - \int_{\mathbb{R}} W(t, u(t))dt. \tag{2.1}$$

For convenience, we denote

$$J(u) = \int_{\mathbb{R}} \frac{1}{p(t)}(|\dot{u}|^{p(t)} + a(t)|u|^{p(t)})dt, \quad F(u) = \int_{\mathbb{R}} W(t, u(t))dt. \tag{2.2}$$

Lemma 2.8 ([11]). (i) $J \in C^1(E,\mathbb{R})$, *and*

$$\langle J'(u), v\rangle = \int_{\mathbb{R}} \Big(|\dot{u}(t)|^{p(t)-2}(\dot{u}(t), \dot{v}(t)) + a(t)|u(t)|^{p(t)-2}(u(t), v(t))\Big)dt,$$

for all $u, v \in E$;

(ii) $J' : E \to E^*$ *is a mapping of type* (S_+), *i.e., if* $u_n \rightharpoonup u$ *and*

$$\limsup_{n\to\infty}(J'(u_n), u_n - u) \le 0,$$

then u_n *has a convergent subsequence in* E.

If (A1), (W1) *or* (W2) *hold, then* $I \in C^1(E,\mathbb{R})$ *and one can easily check that*

$$\begin{aligned}\langle I'(u), v\rangle = \int_{\mathbb{R}} \big[&|\dot{u}(t)|^{p(t)-2}(\dot{u}(t), \dot{v}(t)) + a(t)|u(t)|^{p(t)-2}(u(t), v(t)) \\ &- (\nabla W(t, u(t)), v(t))\big]dt.\end{aligned} \tag{2.3}$$

Furthermore, the critical points of I *in* E *are classical solutions of* (1.1) *with* $u(\pm\infty) = 0$.

Lemma 2.9 ([16]). *For* $u \in E$, *then* $u \in C(\mathbb{R},\mathbb{R}^N)$, *and* $u(t) \to 0$, $|t| \to \infty$. *Furthermore, the embedding* $E \hookrightarrow L^\infty(\mathbb{R},\mathbb{R}^N)$ *is continuous and compact.*

Remark 2.10. By Lemma 2.9, there exists a constant $C > 0$ such that

$$\|u\|_{L^\infty} \le C\|u\|_E. \tag{2.4}$$

Lemma 2.11 ([18]). *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfy the (PS)-condition. If I is bounded from below, then $c = \inf_E I$ is a critical value of I.*

To find nontrivial critical points of I, we will use the "genus" properties, so we recall the following definitions and results (see [18]). Let E be a Banach space, $f \in C^1(E, \mathbb{R})$ and $c \in \mathbb{R}$. We set

$$\Sigma = \{A \subset E - \{0\} : A \text{ is closed in } E \text{ and symmetric with respect to } 0\},$$

$$K_c = \{u \in E : f(u) = c,\ f'(u) = 0\}, \quad f^c = \{u \in E : f(u) \le c\}.$$

Definition 2.12 ([18]). *For $A \in \Sigma$, we say genus of A is n (denoted by $\gamma(A) = n$) if there is an odd map $\phi \in C(A, \mathbb{R}^n \setminus \{0\})$ and n is the smallest integer with this property.*

Lemma 2.13 ([18]). *Let f be an even C^1 functional on E and satisfy the (PS)-condition. For any $n \in \mathbb{N}$, set*

$$\Sigma_n = \{A \in \Sigma : \gamma(A) \ge n\}, \quad c_n = \inf_{A \in \Sigma_n} \sup_{u \in A} f(u).$$

(i) If $\Sigma_n \ne \emptyset$ and $c_n \in \mathbb{R}$, then c_n is a critical value of f;

(ii) If there exists $r \in \mathbb{N}$ such that

$$c_n = c_{n+1} = \dots = c_{n+r} = c \in \mathbb{R},$$

and $c \ne f(0)$, then $\gamma(K_c) \ge r + 1$.

3. Proof of main results

Proof of Theorem 1.1. In view of Lemma 2.8 and (W1), $I \in C^1(E, \mathbb{R})$. In what follows, we first show that I is coercive. By (W1), we have

$$|W(t, x)| \le a_1(t)|x|^{\alpha_1(t)}, \quad |x| \le 1, \tag{3.1}$$

$$|W(t, x)| \le ca_2(t)|x|^{\alpha_2(t)}, \quad |x| > 1. \tag{3.2}$$

Assume that $\|u\| \ge 1$, by (W1), Lemma 2.2 and Lemma 2.7, we have

$$\begin{aligned}
I(u) &= \int_{\mathbb{R}} \frac{1}{p(t)}(|\dot u|^{p(t)} + a(t)|u|^{p(t)})dt - \int_{\mathbb{R}} W(t, u(t))dt \\
&\ge \frac{1}{p^+}\|u\|^{p^-} - \int_{\{t:|u(t)|\le 1\}} W(t, u(t))dt - \int_{\{t:|u(t)|>1\}} W(t, u(t))dt \\
&\ge \frac{1}{p^+}\|u\|^{p^-} - \int_{\{t:|u(t)|\le 1\}} a_1(t)|u(t)|^{\alpha_1(t)}dt - \int_{\{t:|u(t)|>1\}} a_2(t)|u(t)|^{\alpha_2(t)}dt \\
&\ge \frac{1}{p^+}\|u\|^{p^-} - C_1 \int_{\{t:|u(t)|\le 1\}} b^{\alpha_1(t)/p(t)} a^{\alpha_1(t)/p(t)} |u(t)|^{\alpha_1(t)}dt \\
&\quad - cC_2 \int_{\{t:|u(t)|>1\}} b^{\frac{\alpha_2(t)}{p(t)}} a^{\frac{\alpha_2(t)}{p(t)}} |u(t)|^{\alpha_2(t)}dt \\
&\ge \frac{1}{p^+}\|u\|^{p^-} - 2C_1 |b^{\frac{\alpha_1(t)}{p(t)}}|_{L^{r_1(t)}} |u|^{\widetilde{\alpha_1}}_{p(t),a} - 2cC_2 |b^{\frac{\alpha_2(t)}{p(t)}}|_{L^{r_2(t)}} |u|^{\widetilde{\alpha_2}}_{p(t),a} \\
&\ge \frac{1}{p^+}\|u\|^{p^-} - 2C_1 |b^{\frac{\alpha_1(t)}{p(t)}}|_{L^{r_1(t)}} \|u\|^{\widetilde{\alpha_1}} - 2cC_2 |b^{\frac{\alpha_2(t)}{p(t)}}|_{L^{r_2(t)}} \|u\|^{\widetilde{\alpha_2}}.
\end{aligned} \tag{3.3}$$

Where $C_i = \sup_{t\in\mathbb{R}} a_i(t), \alpha_i(t), r_i(t)$ satisfy $\frac{1}{r_i(t)} + \frac{\alpha_i(t)}{p(t)} = 1, \widetilde{\alpha_i} \in [\alpha_i^-, \alpha_i^+]$ is a constant, $(i = 1, 2)$. By (W1), $\alpha_i^- < \alpha_i^+ < p^-$, this implies that $\widetilde{\alpha_i} < p^-$. By (A), we have $I(u) \to +\infty$ as $\|u\| \to +\infty$. Consequently, I is bounded from below.

Next, we prove that I satisfies the (PS)-condition. Assume that $\{u_k\}_{k\in\mathbb{N}} \subset E$ is a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$. Then by (2.1) and (3.3), there exists a constant $A > 0$ such that

$$\|u_k\| \le A, \quad k \in \mathbb{N}. \tag{3.4}$$

So passing to a subsequence if necessary, it can be assumed that $u_k \rightharpoonup u$ in E.

By (A), $b^{\alpha_i(t)/p(t)} \in L^{r_i(t)}$ $(i = 1, 2)$, for any $\varepsilon > 0$, there exists $R > 0$ such that

$$|b(t)^{\alpha_i(t)/p(t)}|_{L^{r_i(t)}(\Omega_2)} < \varepsilon, \tag{3.5}$$

where $\Omega_1 = \{t \in \mathbb{R} : |t| \le R\}$, $\Omega_2 = \mathbb{R}\setminus\Omega_1$, by Lemma 2.9, if $u_k \rightharpoonup u_0$, then $u_k \to u$ in L^∞, hence, we have

$$\int_{\Omega_1} |W(t, u_k) - W(t, u)|dt < \varepsilon, \quad k \to \infty, \tag{3.6}$$

$$\int_{\Omega_1} |\nabla W(t, u_k) - \nabla W(t, u)|dt < \varepsilon, \quad k \to \infty. \tag{3.7}$$

Without loss of generality, suppose that $\max\{\|u_k\|, \|u\|\} \le 1$, it follows from (A1), (3.5)-(3.7), Lemma 2.4 and Lemma 2.7 that

$$\begin{aligned}
&|F(u_k) - F(u)| \\
&= \Big|\int_{\mathbb{R}} (W(t, u_k(t)) - W(t, u(t)))dt\Big| \\
&\le \int_{\Omega_1} |W(t, u_k(t)) - W(t, u(t))|dt + \int_{\Omega_2} |W(t, u_k(t)) - W(t, u(t))|dt \\
&\le \varepsilon + \int_{\mathbb{R}} [|W(t, u_k(t))| + |W(t, u(t))|]dt \\
&\le \varepsilon + \varepsilon \int_{\mathbb{R}} a_1(t)(|u_k|^{\alpha_1(t)} + |u|^{\alpha_1(t)})dt + c\varepsilon \int_{\mathbb{R}} a_2(t)(|u_k|^{\alpha_2(t)} + |u|^{\alpha_2(t)})dt \\
&\le \varepsilon + 2C_1|b(t)^{\alpha_1(t)/p(t)}|_{L^{r_1(t)}} |a^{\alpha_1(t)/p(t)}|u_k(t)|^{\alpha_1(t)}|_{L^{p(t)/alpha_1(t)}} \\
&\quad + 2C_1|b(t)^{\alpha_1(t)/p(t)}|_{L^{r_1(t)}} |a^{\alpha_1(t)/p(t)}|u(t)|^{\alpha_1(t)}|_{L^{p(t)/alpha_1(t)}} \\
&\quad + 2cC_2|b(t)^{\frac{\alpha_2(t)}{p(t)}}|_{L^{r_2(t)}} |a^{\frac{\alpha_2(t)}{p(t)}}|u_k(t)|^{\alpha_2(t)}|_{L^{p(t)/\alpha_2(t)}} \\
&\quad + 2cC_2|b(t)^{\frac{\alpha_2(t)}{p(t)}}|_{L^{r_2(t)}} |a^{\frac{\alpha_2(t)}{p(t)}}|u(t)|^{\alpha_2(t)}|_{L^{p(t)/\alpha_2(t)}} \\
&\le \varepsilon + 2C_1\varepsilon(|u_k|^{\widetilde{\alpha_{1,1}}}_{L_a^{p(t)}} + |u|^{\widetilde{\alpha_{1,2}}}_{L_a^{p(t)}}) + 2cC_2\varepsilon|u_k|^{\widetilde{\alpha_{2,1}}}_{L_a^{p(t)}} + |u|^{\widetilde{\alpha_{2,2}}}_{L_a^{p(t)}}) \\
&\le \varepsilon + 4C_1\varepsilon + 4cC_2\varepsilon,
\end{aligned} \tag{3.8}$$

where $C_i = \sup_{t\in\mathbb{R}} a_i(t) (i = 1, 2)$, $\widetilde{\alpha_{1,1}}, \widetilde{\alpha_{1,2}} \in [\alpha_1^-, \alpha_1^+], \widetilde{\alpha_{2,1}}, \widetilde{\alpha_{2,2}} \in [\alpha_2^-, \alpha_2^+]$. Hence, there exists a constant C' such that $|F(u_k) - F(u)| < C'\varepsilon$, this implies that $F(u_k) \to F(u)$, $k \to \infty$.

On the other hand, for any $v \in E$ with $\|v\| = 1$, by (W1), Lemmas 2.5 and 2.7, we have

$$|(F'(u_k) - F'(u), v)|$$

$$
\begin{aligned}
&\le \int_{\Omega_1} |\nabla W(t,u_k(t)) - \nabla W(t,u(t)))||v|dt \\
&\quad + \int_{\Omega_2} |\nabla W(t,u_k(t)) - \nabla W(t,u(t)))||v|dt \\
&\le \varepsilon\|v\|_{L^\infty} + \int_{\mathbb{R}} (|\nabla W(t,u_k(t))| + |\nabla W(t,u(t))|)|v|dt \\
&\le C\varepsilon + \int_{\mathbb{R}} \alpha_1(t)a_1(t)(|u_k|^{\alpha_1(t)-1} + |u|^{\alpha_1(t)-1})|v|dt \\
&\quad + \int_{\mathbb{R}} \alpha_2(t)a_2(t)(|u_k|^{\alpha_2(t)-1} + |u|^{\alpha_2(t)-1})|v|dt \\
&\le C\varepsilon + C_3\int_{\mathbb{R}} (|u_k|^{\alpha_1(t)-1} + |u|^{\alpha_1(t)-1})|v|dt + C_4\int_{\mathbb{R}} (|u_k|^{\alpha_2(t)-1} + |u|^{\alpha_2(t)-1})|v|dt \\
&= C\varepsilon + C_3\int_{\mathbb{R}} b^{\alpha_1(t)/p(t)} a^{\frac{\alpha_1(t)-1}{p(t)}} |u_k|^{\alpha_1(t)-1} a^{\frac{1}{p(t)}}|v|dt \\
&\quad + C_3\int_{\mathbb{R}} b^{\alpha_1(t)/p(t)} a^{\frac{\alpha_1(t)-1}{p(t)}} |u|^{\alpha_1(t)-1} a^{\frac{1}{p(t)}}|v|dt \\
&\quad + C_4\int_{\mathbb{R}} b^{\frac{\alpha_2(t)}{p(t)}} a^{\frac{\alpha_2(t)-1}{p(t)}} |u_k|^{\alpha_2(t)-1} a^{\frac{1}{p(t)}}|v|dt \\
&\quad + C_4\int_{\mathbb{R}} b^{\frac{\alpha_2(t)}{p(t)}} a^{\frac{\alpha_2(t)-1}{p(t)}} |u|^{\alpha_2(t)-1} a^{\frac{1}{p(t)}}|v|dt \\
&\le C\varepsilon + 3C_3|b(t)|^{\alpha_1(t)/p(t)}_{L^{r_1(t)}} |a^{\frac{\alpha_1(t)-1}{p(t)}}|u_k|^{\alpha_1(t)-1}|_{L^{\frac{p(t)}{\alpha_1(t)-1}}} |a^{\frac{1}{p(t)}}v|_{L^{p(t)}} \\
&\quad + 3C_3|b(t)|^{\alpha_1(t)/p(t)}_{L^{r_1(t)}} |a^{\frac{\alpha_1(t)-1}{p(t)}}|u|^{\alpha_1(t)-1}|_{L^{\frac{p(t)}{\alpha_1(t)-1}}} |a^{\frac{1}{p(t)}}v|_{L^{p(t)}} \\
&\quad + 3C_4|b(t)|^{\frac{\alpha_2(t)}{p(t)}}_{L^{r_2(t)}} |a^{\frac{\alpha_2(t)-1}{p(t)}}|u_k|^{\alpha_2(t)-1}|_{L^{\frac{p(t)}{\alpha_2(t)-1}}} |a^{\frac{1}{p(t)}}v|_{L^{p(t)}} \\
&\quad + 3C_4|b(t)|^{\frac{\alpha_2(t)}{p(t)}}_{L^{r_2(t)}} |a^{\frac{\alpha_2(t)-1}{p(t)}}|u|^{\alpha_2(t)-1}|_{L^{\frac{p(t)}{\alpha_2(t)-1}}} |a^{\frac{1}{p(t)}}v|_{L^{p(t)}} \\
&\le C\varepsilon + 3C_3\varepsilon(|u_k|^{\widetilde{\alpha_{1,3}}-1}_{L^{p(t)}_a} + |u|^{\widetilde{\alpha_{1,4}}-1}_{L^{p(t)}_a})|v|_{L^{p(t)}_a} + 3C_4\varepsilon(|u_k|^{\widetilde{\alpha_{2,3}}-1}_{L^{p(t)}_a} + |u|^{\widetilde{\alpha_{2,4}}-1}_{L^{p(t)}_a})|v|_{L^{p(t)}_a} \\
&\le C\varepsilon + 6C_3\varepsilon + 6C_4\varepsilon.
\end{aligned}
$$

Where C is defined in (2.4), $C_3 = \sup_{t\in\mathbb{R}} \alpha_1(t)a_1(t)$, $C_4 = \sup_{t\in\mathbb{R}} \alpha_2(t)a_2(t)$, $\widetilde{\alpha_{1,3}}, \widetilde{\alpha_{1,4}} \in [\alpha_1^-, \alpha_1^+]$, $\widetilde{\alpha_{2,3}}, \widetilde{\alpha_{2,4}} \in [\alpha_2^-, \alpha_2^+]$. Hence, there exists a constant C'' such that $|F'(u_k) - F'(u)| < C''\varepsilon$, this implies that $F'(u_k) \to F'(u)$, $k \to \infty$. This implies that $(J'(u_k), u_k - u) \to 0$. By Lemma 2.8, J' is a mapping type (S_+), hence, $u_k \to u$. So, I satisfies (PS) condition.

By Lemma 2.13, $c = \inf_E I(u)$ is a critical value of I, that is there exists a critical point $u^* \in E$ such that $I(u^*) = c$.

Finally, we show that $u^* \neq 0$. Let $u_0 \in (W_0^{1,p(t)}(J) \bigcap E) \setminus \{0\}$ and $\|u_0\| = 1$, then by (W2) and Lemma 2.2, we have

$$
I(su_0) = \int_{\mathbb{R}} \frac{1}{p(t)}((|s\dot{u}|^{p(t)} + a(t)|su|^{p(t)}))dt - \int_{\mathbb{R}} W(t,u(t))dt nonumber \tag{3.9}
$$

$$
\le \frac{s^{p^-}}{p^-} - \int_J W(t,su_0(t))dt nonumber \tag{3.10}
$$

$$\le \frac{s^{p^-}}{p^-} - \eta s^{\gamma_1^+} \int_J |u_0(t)|^{\gamma_1(t)} dt, \quad 0 < s < 1. \tag{3.11}$$

Since $1 < \gamma_1^+ < p^-$, it follows from (3.11) that $I(su_0) < 0$ for $s > 0$ small enough. Hence $I(u^*) = c < 0$, therefore u^* is nontrivial critical point of I, and so $u^* = u^*(t)$ is a nontrivial homoclinic solution of (1.1). The proof is complete. □

Proof of Theorem 1.2. By (W3), I is an even functional. Denote by $\gamma(A)$ the genus of A. Set

$$\Sigma = \{A \subset E - \{0\} : A \text{ is closed in } E \text{ and symmetric with respect to } 0\},$$
$$\Sigma_k = \{A \in \Sigma : \gamma(A) \ge k\}, \quad k = 1, 2, \dots,$$
$$c_k = \inf_{A \in \Sigma_k} \sup_{u \in A} I(u), \quad k = 1, 2, \dots,$$

we have

$$-\infty < c_1 \le c_2 \le \dots \le c_k \le c_{k+1} \le \dots .$$

Now let us prove that $c_k < 0$ for every k.

By (W2), there exists a bounded open set $J \subset \mathbb{R}$ such that $W(t, x) \ge \eta |x|^{\gamma_1(t)}$, for all $t \in J$. Since $W_0^{1,p(t)}(J) \subset E$, For any k, we can choose a k-dimensional linear subspace $E_k \subset W_0^{1,p(t)}(J)$. Since all norms of a finite dimensional normed space are equivalent, there exists $\rho_k \in (0, 1)$ such that $u \in E_k$ with $\|u\| \le \rho_k$ implies $|u|_{L^\infty} \le 1$. Set

$$S_{\rho_k}^{(k)} = \{u \in E_k : \|u\| = \rho_k\},$$

for any $u \in S_{\rho_k}^{(k)}$, $s \in (0, 1)$, we have

$$\begin{aligned}
I(su) &= \int_J \frac{1}{p(t)} [|s\dot{u}|^{p(t)} + a(t)|su|^{p(t)}] dt - \int_J W(t, su(t)) dt \\
&\le \frac{s^{p^-}}{p^-} \rho_k^{p^-} - s^{\gamma_1^+} \int_J |u|^{\gamma_1(t)} dt \\
&\le \frac{s^{p^-}}{p^-} \rho_k^{p^-} - d_k s^{\gamma_1^+}.
\end{aligned}$$

Where $d_k = \int_J |u|^{\gamma_1(t)} dt$, since $\gamma_1^+ < p^-$, there exist $s_k \in (0, 1), \varepsilon_k > 0$ such that

$$I(s_k u) \le -\varepsilon_k < 0, \quad \forall u \in S_{\rho_k}^{(k)}.$$

We know that $\gamma(S_{s_k \rho_k}^{(k)}) = k$, so $c_k \le -\varepsilon_k < 0$.

By genus theory [22] and Lemma 2.13, each c_k is a critical value of I, hence there is a sequence of solutions $\{\pm u_k : k = 1, 2, \dots,\}$ of system (1.1) such that $I(\pm u_k) = c_k < 0$. By the arbitraries of k, we can conclude that system (1.1) have infinitely many homoclinic solutions. The proof is complete. □

4. An example

In this section, we give an example to illustrate our results. Consider the second-order ordinary $p(t)$-Laplacian system

$$\frac{d}{dt}\Big(|\dot{u}(t)|^{8+10|\sin t|} \dot{u}(t)\Big) - a(t)|u(t)|^{8+10|\sin t|} u(t) + \nabla W(t, u(t)) = 0, \tag{4.1}$$

where $p(t) = 10 + 10|\sin t|, a(t) = \big(1 + t^2\big)^4$, let

$$W(t,x) = \frac{|x|^{4|\sin t|+4}}{1+t^2} + \frac{|x|^{2|\sin t|+2}}{1+t^2},$$

then

$$\nabla W(t,x) = \frac{(4|\sin t|+4)|x|^{4|\sin t|+2}x}{1+t^2} + \frac{(2|\sin t|+2)|x|^{2|\sin t|}x}{1+t^2},$$

$$|\nabla W(t,x)| \le \frac{3}{1+t^2}(2+2|\sin t|)|x|^{2|\sin t|+1}, \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N,\ |x| \le 1,$$

$$|\nabla W(t,x)| \le \frac{3}{2(1+t^2)}(4+4|\sin t|)|x|^{4|\sin t|+3}, \quad \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N,\ |x| \ge 1,$$

Let $J = (-2,2)$, $\gamma_1(t) = 2|\sin t| + 2$ and

$$W(t,x) \ge \frac{1}{5}|x|^{2|\sin t|+2}, \quad \forall (t,x) \in J\times\mathbb{R}^N,\ |x| \le 1.$$

These inequalitires show that all conditions of Theorem 1.2 are satisfied, where

$$\alpha_1(t) = 2 + 2|\sin t|, \quad \alpha_2(t) = 4 + 4|\sin t|,$$

$$a_1(t) = \frac{3}{1+t^2}, \quad a_2(t) = \frac{3}{2(1+t^2)}, \quad c = \frac{4}{3},$$

$$r_1(t) = \frac{5}{4}, \quad r_2(t) = \frac{5}{3}.$$

By Theorem 1.2, system (1.1) has infinitely many nontrivial homoclinic solutions.

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