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PAIRS OF SIGN-CHANGING SOLUTIONS FOR SUBLINEAR ELLIPTIC EQUATIONS WITH NEUMANN BOUNDARY CONDITIONS

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Abstract. We consider the Neumann problem for a sublinear elliptic equation in a convex bounded domain of $\mathbb{R}^N$. Using an variant of Clark Theorem, we obtain the existence and multiplicity of its pairs of sign-changing solutions.

1. Introduction

Consider the Neumann problem for a semilinear elliptic equation

$$\begin{gathered} -\Delta u(x) = f(u(x)), \quad x \in \Omega, \\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geqslant 1$) is a convex and bounded domain with the smooth boundary $\partial\Omega$ and the outward normal n, $f(u) : \mathbb{R} \to \mathbb{R}$. Let $F(u) = \int_0^u f(s)ds$, the primitive of f, and assume it satisfies

$$\limsup_{|u|\to\infty} F(u)/|u|^2 \leqslant a < \infty, \tag{1.2}$$

then we say that (1.1) is sublinear (or subquadratic). If

$$\lim_{|u|\to\infty} F(u)/|u|^2 = \infty, \tag{1.3}$$

then (1.1) is superlinear (or superquadratic). For sublinear problem (1.1), there is a vast of literature. Under the assumptions of sign conditions [4, 5], or monotonicity conditions [10], or periodicity conditions[11], or Landesman-Lazer type conditions [6, 7], it has been showed that problem (1.1) possesses at least one solution. Tang [14, 15, 16] supposed that F satisfies the hypothesis

$$\lim_{u\in X_0, \|u\|_0\to\infty} \|u\|_0^{-2\alpha} \int_\Omega F(u(x))dx \to \infty, \tag{1.4}$$

where $X_0 = \{u \in H^1(\Omega) : \int_\Omega u(x)dx = 0\}$, $\|u\|_0 = (\int_\Omega |\nabla u(x)|^2 dx)^{1/2}$ for $u \in X_0$, and $0 < \alpha < 1$. He proved the existence and multiplicity results of problem (1.1) by minimax methods. Costa [1] assumed that f satisfies

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(F1) $f \in \mathbb{C}^1(\mathbb{R}, \mathbb{R})$, strictly increasing and $f(0) = 0$,

(F2') the limits $f'(\pm\infty) = \lim_{u\to\pm\infty} f'(u)$ exist, $0 < f'(\pm\infty) < \lambda_1 < f'(0)$, where λ_1 is the first positive eigenvalue of the problem

$$\begin{gathered} -\Delta u(x) = \lambda u(x), \quad x \in \Omega \\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0. \end{gathered} \tag{1.5}$$

Then Costa [1] showed that (1.1) has one nontrivial solution in $H^1(\Omega)$, which minimizes the functional

$$\varphi_\Omega(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \int_\Omega F(u(x))dx, \tag{1.6}$$

over the manifold

$$\mathcal{M} = \big\{u \in H^1(\Omega) : \int_\Omega f(u(x))dx = 0\big\}. \tag{1.7}$$

In [9], under the hypotheses that there are two sequences $\{a_j\}$,$\{b_j\} \subset \mathbb{R}$, such that $f(a_j) = 0 = f(b_j)$, $j = 1, 2, \dots$, and $f'(a_j) \geq \lambda_2$, $f'(b_j) \geq \lambda_2$, $j = 2, 4, \dots$, where λ_2 is the second positive eigenvalue of problem (1.5), Li and Li [9] proved the existences of positive, negative and sign-changing solutions for problem (1.1).

In this article, motivated by [1], a multiplicity result of pairs of sign-changing solutions for (1.1) shall be obtained, which is a generalization of [1, Theorem 3.7].

Exactly, we have the following conclusion.

Theorem 1.1. *Let d_Ω denote the diameter of Ω. Suppose that f satisfies* (F1) *and*

(F2) *the limits $f'(\pm\infty) = \lim_{u\to\pm\infty} f'(u)$ exist and $0 < f'(\pm\infty) < (\frac{\pi}{d_\Omega})^2$;*

(F3) *$F(u) = F(-u)$, for all $u \in \mathbb{R}$;*

(F4) *there exist $p \in \mathbb{N}, M > 0$ and $\rho > 0$ such that $d_\Omega > \frac{2p\pi}{\sqrt{M}}, M > 4p^2 f'(\pm\infty)$, and*

$$F(u) \geqslant \frac{1}{2}M|u|^2, \forall |u| \leqslant \rho;$$

(F4) *for Ω there are continuous functions $e_1(x), e_2(x), \dots, e_p(x) \in X_0 \setminus \{0\}$, which are orthogonal in $H^1(\Omega)$ and $L^2(\Omega)$, such that*

$$\int_\Omega |\nabla e_j(x)|^2 dx \leqslant \frac{2(j+1)\pi)}{d_\Omega}]^2 \int_\Omega |e_j(x)|^2 dx, \quad \forall 1 \leqslant j \leqslant p.$$

Then (1.1) *has p distinct pairs $(u(x), -u(x))$ of sign-changing classical solutions, and has no positive and negative solution, provided that $d_\Omega \in (\frac{2p\pi}{\sqrt{M}}, \frac{\pi}{\sqrt{f'(\pm\infty)}})$.*

Remark 1.2. If f satisfies $\lim_{|u|\to 0} F(u)/|u|^2 = \infty$, then, for all $d_\Omega > 0, p \geqslant 1$, we can find $M > 0$ and $\rho > 0$ such that (F4) holds.

Remark 1.3. Usually, in some applications, the role of $e_1(x), e_2(x), \dots, e_p(x)$ is played by the eigenfunctions $\phi_j \in X_0$ $(j \geqslant 1)$ of problem (1.5) (see (2.10) and (2.11) below).

This article is organized as follows. In Section 2, we give some Lemmas. In Section 3, we prove Theorem 1.1 by using a variant of Clark Theorem as stated next.

Theorem 1.4 ([12, 13]). *Let $\hat{X}$ be a Banach space, $\hat{\varphi} \in C^1(\hat{X}, \mathbb{R})$ be even, and $\hat{\mathcal{M}} \subset \hat{X}$ be C^1- submanifold. Suppose that $\hat{\varphi}|_{\hat{\mathcal{M}}}$ satisfies the Palais-Smale condition; $\hat{\varphi}$ is bounded from below on $\hat{\mathcal{M}}$; there exist a closed, symmetric subset $\hat{K} \subset \hat{\mathcal{M}}$ and $\hat{p} \in \mathbb{N}$ such that $\hat{K}$ is homeomorphism to $S^{\hat{p}-1} \subset \mathbb{R}^{\hat{p}}$ by an odd map, and $sup\{\hat{\varphi}(x) : x \in \hat{K}\} < \hat{\varphi}(0)$. Then $\hat{\varphi}|_{\hat{\mathcal{M}}}$ possesses at least $\hat{p}$ distinct pairs $(u, -u)$ of critical point with corresponding critical values less than $\hat{\varphi}(0)$.*

As an example, we apply Theorem 1.1 to $B_\gamma(0) \subset \mathbb{R}^2$ and yield an interesting result.

Theorem 1.5. *Let f satisfy* (F1)–(F4) *with $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < r^2\}$, then, for $r \in (\frac{2p\pi}{\sqrt{M}}, \frac{\pi}{2\sqrt{f'(\pm\infty)}})$, the problem*

$$\begin{gathered} -\Delta u(x) = f(u(x)), \quad x_1^2 + x_2^2 < r^2 \\ \frac{\partial u}{\partial n}|_{x_1^2+x_2^2=r^2} = 0, \end{gathered} \tag{1.8}$$

has p-distinct pairs $(u(x), -u(x))$ of sign-changing classical solutions.

In the proof of Theorem 1.5, by some accurate analysis about the corresponding eigenvalue problem with zero points of Bessel functions, we find that condition (F5) is naturally satisfied.

2. Preliminaries

In this article, for simplicity, for $u \in L^2(\Omega)$, we denote by $\|u\|_{L^2}$ its L^2-norm. Clearly, problem (1.1) has the trivial solution $u(x) = 0$. In order to find its nontrivial solutions, we consider the functional

$$\begin{aligned} \varphi_\Omega(u) &= \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \int_\Omega F(u(x))dx \\ &= \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \psi_\Omega(u), u \in X, \end{aligned} \tag{2.1}$$

where $\psi_\Omega(u) = \int_\Omega F(u(x))dx$, $X = H^1(\Omega)$ is the usual Sobolev space with the inner product

$$(u, w) = \int_\Omega [\dot{u}(x)\dot{w}(x) + u(x)w(x)]dx \tag{2.2}$$

and the corresponding norm

$$\|u\| = (u, u)^{1/2} = (\int_\Omega [|\nabla u(x)|^2 + |u(x)|^2]dx)^{1/2}. \tag{2.3}$$

Under the assumptions (F1) and (F2), we know that $\varphi_\Omega \in \mathbb{C}^2(X, \mathbb{R})$, $\psi_\Omega(u)$ is weakly continuous in X, and $\psi'_\Omega(u) : X \to X^*$ is completely continuous. Moreover, critical points of φ_Ω in X are classical solutions of problem (1.1).

Next, we decompose the Sobolev space $X = H^1(\Omega)$ as

$$X = X_0 \oplus X_1, \quad X_0 = \{u \in X : \int_\Omega u(x)dx = 0\}, \quad X_1 = \mathbb{R}. \tag{2.4}$$

Let us recall the problem (1.5) has eigenvalues

$$0 = \lambda_0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \to \infty, \tag{2.5}$$

and the corresponding eigenfunctions

$$\phi_0(x) \equiv 1, \phi_1(x), \phi_2(x), \phi_3(x), \quad \dots . \tag{2.6}$$

In particular, for the first positive eigenvalue λ_1, one has the Poincare type inequality

$$\int_\Omega |u(x)|^2 dx \leqslant \frac{1}{\lambda_1} \int_\Omega |\nabla u(x)|^2 dx, \forall u \in X_0. \tag{2.7}$$

Using the estimate of lower bound for λ_1, [8],

$$\lambda_1 \geqslant (\frac{\pi}{d_\Omega})^2, \tag{2.8}$$

we have

$$\int_\Omega |u(x)|^2 dx \leqslant (\frac{d_\Omega}{\pi})^2 \int_\Omega |\nabla u(x)|^2 dx, \forall u \in X_0. \tag{2.9}$$

In addition, it is a well-known that

$$\int_\Omega \phi_j(x) dx = 0, \quad \forall j \geqslant 1, \tag{2.10}$$

which implies

$$\phi_j \in X_0, \quad \forall j \geqslant 1. \tag{2.11}$$

Under assumptions of (F1) and (F2), Costa [1] proved that

(i) $\mathcal{M} = \{u \in X = H^1(\Omega) : \int_\Omega f(u(x))dx = 0\} \subset X$ is a $\mathbb{C}^1-$ manifold of codimension 1;

(ii) $u \in X$ is a critical point of φ_Ω in X if and only if $u \in \mathcal{M}$ and it is a critical point of $\varphi_\Omega|_\mathcal{M}$.

Also for $u \in X$, writing $u = \nu + c$, $\nu \in X_0$, $c \in \mathbb{R}$, he also obtained that

(iii) $\int_\Omega F(\nu + c)dx \leqslant \int_\Omega F(\nu)dx$;

(iv) $\|\nu_n\| \to \infty$ as $\|\nu_n + c_n\| \to \infty, \nu_n + c_n \in \mathcal{M}$.

Lemma 2.1. *If f satisfies* (F1) *and* (F2)*, then*

(v) *for each $\nu \in X_0$, there exists a unique $c(\nu) \in \mathbb{R}$ such that $\nu + c(\nu) \in \mathcal{M}$;*

(vi) *$c(-\nu) = -c(\nu)$ for all $\nu \in X_0$ if f is also odd.*

Proof. (v) For any fixed $\nu \in X_0$, define

$$g_\nu(c) = \int_\Omega f(\nu + c)dx, \quad \forall c \in \mathbb{R}. \tag{2.12}$$

If $\nu \in \mathbb{C}^1(\bar\Omega)$, we easily know that $f(\nu(x) + c_1) > 0$ for all $x \in \bar\Omega$ and $c_1 > \max_{\bar\Omega} |\nu(x)|$, while $f(\nu(x)+c_2) < 0$ for all $x \in \bar\Omega$ and $c_2 < -\max_{\bar\Omega} |\nu(x)|$. Therefore, by the continuity of $g_\nu(\cdot)$, there exists $c = c(\nu) \in \mathbb{R}$ such that $\int_\Omega f(\nu + c(\nu))dx = 0$.

For the general case $\nu \in X_0$, one can take $\nu_k \in \mathbb{C}^1(\bar\Omega) \cap X_0$, $\nu_k \to \nu$ in X. There are $c(\nu_k) \in \mathbb{R}$ such that $\int_\Omega f(\nu_k + c(\nu_k))dx = 0$. We claim that $\{c(\nu_k)\}$ is bounded. Otherwise, $|c(\nu_k)| \to \infty$, then $\|\nu_k + c(\nu_k)\| \to \infty$. Since $\nu_k + c(\nu_k) \in \mathcal{M}$, by (iv), we have $\|\nu_n\| \to \infty$, a contraction. Therefore, we may assume that $c(\nu_k) \to c(\nu) \in \mathbb{R}$. By (F2), there are constants $\eta > 0$ such that $0 \leqslant f'(u) \leqslant \eta$, for all $u \in \mathbb{R}$. Thus, we have

$$\begin{aligned} |\int_\Omega f(\nu + c(\nu))dx| &= |\int_\Omega f(\nu + c(\nu))dx - \int_\Omega f(\nu_k + c(\nu_k))dx| \\ &\leqslant \eta \int_\Omega [|\nu - \nu_k| + |c(\nu_k) - c(\nu)|]dx \to 0; \end{aligned} \tag{2.13}$$

that is, $\nu + c(\nu) \in \mathcal{M}$. The uniqueness of $c(\nu)$ can be obtained from the monotonicity of $f(u)$.

(vi) for all $\nu \in X_0$ naturally $-\nu \in X_0$, by (v), there is $c(-\nu) \in \mathbb{R}$ such that

$$\int_\Omega f(-\nu + c(-\nu))dx = 0. \tag{2.14}$$

Since $f(u)$ is odd, we have

$$\int_\Omega f(\nu - c(-\nu))dx = 0. \tag{2.15}$$

By the uniqueness of $c(\nu)$, we obtain $c(\nu) = -c(-\nu)$, namely, $c(-\nu) = -c(\nu)$. $\square$

Lemma 2.2. *Suppose f satisfies* (F1) *and* (F2). *Then the functional $\varphi_\Omega(u)$ is bounded from below on $\mathcal{M}$ and satisfies the Palais-Smale condition on $\mathcal{M}$.*

Proof. By (F2), there exist m and b,

$$0 < m < (\frac{\pi}{d_\Omega})^2 \leqslant \lambda_1, \quad b > 0, \tag{2.16}$$

such that

$$F(s) \leqslant b + \frac{1}{2}m|s|^2, \quad \forall s \in \mathbb{R}. \tag{2.17}$$

For $u \in \mathcal{M}$, writing $u = \nu + c \in X_0 \oplus X_1$, we have

$$\begin{aligned} \varphi_\Omega(u) &= \frac{1}{2}\int_\Omega |\nabla\nu(x)|^2 dx - \int_\Omega F(\nu + c)dx \\ &\geqslant \frac{1}{2}\int_\Omega |\nabla\nu(x)|^2 dx - \int_\Omega F(\nu)dx \\ &\geqslant \frac{1}{2}\|\nabla\nu\|_{L^2}^2 - \frac{1}{2}m\|\nu\|_{L^2}^2 - b|\Omega|\,. \end{aligned} \tag{2.18}$$

This inequality and (2.9) implies

$$\varphi_\Omega(u) \geqslant \frac{1}{2}[1 - m(\frac{d_\Omega}{\pi})^2]\|\nabla\nu\|_{L^2}^2 - b|\Omega| = \frac{1}{2}D\|\nabla\nu\|_{L^2}^2 - b|\Omega| \geqslant -b|\Omega| \tag{2.19}$$

with $D = 1 - m(\frac{d_\Omega}{\pi})^2 > 0$. Thus, $\varphi_\Omega(u)$ is bounded from below on $\mathcal{M}$.

Let $\{u_j\} \subset \mathcal{M}$ be such that $\{\varphi_\Omega(u_j)\}$ is bounded and $(\varphi_\Omega|_{\mathcal{M}})'(u_j) \to 0$. Let $u_j = \nu_j + c_j \in X_0 \bigoplus X_1$. Then (2.19) implies $\|\nabla\nu_j\|_{L^2}^2 \leqslant \frac{2}{D}(\varphi_\Omega(u_j) + b|\Omega|)$, thus $\{\nu_j\}$ is bounded in X. The fact $u_j = \nu_j + c_j \in \mathcal{M}$ and (iv) derives $\{u_j\}$ is also bounded in X, so we may assume that, by passing to a subsequence if necessary,

$$u_j \rightharpoonup u \in X \quad \text{weakly in } X. \tag{2.20}$$

$$u_j \to u \in X \quad \text{strongly in } L^1(\Omega) \text{ and in } L^2(\Omega). \tag{2.21}$$

Thus, for $j \geqslant 1$, noticing

$$\int_\Omega f(u(x))dx = \int_\Omega [f(u(x)) - f(u_j(x))]dx = \int_\Omega f'(\zeta)(u(x) - u_j(x))dx, \tag{2.22}$$

with ζ between $u(x)$ and $u_j(x)$, it follows that

$$|\int_\Omega f(u(x))dx| \leqslant \int_\Omega |f'(\zeta)||u(x) - u_j(x)|dx \leqslant \eta \int_\Omega |u(x) - u_j(x)|dx \to 0;$$

consequently, $u \in \mathcal{M}$.

Let us denote by $\nabla\varphi_\Omega, \nabla J_\Omega : X \to X$ the gradient of φ_Ω, J_Ω, respectively, which are defined by the Riesz-Frechet representation theorem, namely, $\nabla\varphi_\Omega, \nabla J_\Omega \in X$ are unique elements such that

$$\varphi'_\Omega(w)h = \langle \nabla\varphi_\Omega(w), h\rangle, J'_\Omega(w)h = \langle \nabla J_\Omega(w), h\rangle, \forall w, h \in X. \tag{2.23}$$

Then, from the boundness of $\{u_j\}$, we easily know that $\nabla\varphi_\Omega(u_j), \nabla J_\Omega(u_j)$ are bounded. Moreover,

$$\begin{aligned}
&[(\varphi_\Omega|_{\mathcal{M}})'(u_j) - (\varphi_\Omega|_{\mathcal{M}})'(u)](u_j - u) \\
&= (\varphi'_\Omega(u_j) - \varphi'_\Omega(u))(u_j - u) - \frac{\langle \nabla\varphi_\Omega(u_j), \nabla J_\Omega(u_j)\rangle}{\|\nabla J_\Omega(u_j)\|^2} J'_\Omega(u_j)(u_j - u) \\
&\quad + \frac{\langle \nabla\varphi_\Omega(u), \nabla J_\Omega(u)\rangle}{\|\nabla J_\Omega(u)\|^2} J'_\Omega(u)(u_j - u) \\
&= \|\nabla u_j - \nabla u\|^2_{L^2} - \int_\Omega (f(u_j) - f(u))(u_j - u)dx - C_j J'_\Omega(u_j)(u_j - u) \\
&\quad + C_0 J'_\Omega(u)(u_j - u),
\end{aligned} \tag{2.24}$$

where $C_0 = \frac{\langle \nabla\varphi_\Omega(u), \nabla J_\Omega(u)\rangle}{\|\nabla J_\Omega(u)\|^2}$ is a constant, $C_j = \frac{\langle \nabla\varphi_\Omega(u_j), \nabla J_\Omega(u_j)\rangle}{\|\nabla J_\Omega(u_j)\|^2}$ is bounded since $\nabla J_\Omega(u_j) \to \nabla J_\Omega(u) \neq 0$.

So $\|\nabla u_j - \nabla u\|_{L^2} \to 0$. Thus, with the aid of (2.21), we conclude that $u_j \to u \in \mathcal{M}$ in X. $\square$

3. Proof of Theorem 1.1

For $p \in \mathbb{N}$, $\rho > 0$, and $e_1(x), e_2(x), \dots, e_p(x)$ in (F4) and (F5), we define the subset $K \subset \mathcal{M}$ as follows

$$K = \{\nu + c(\nu) \in \mathcal{M} : \nu = \sum_{j=1}^p \mu_j e_j(x), \mu_j \in \mathbb{R}(1 \leqslant j \leqslant p), \sum_{j=1}^p \mu_j^2 = \hat\rho^2\}, \tag{3.1}$$

where $\hat\rho = \rho/(2\sqrt{p})$. Then, by Lemma 2.1, the map

$$\nu + c(\nu) \mapsto \big(-\frac{\mu_1}{\hat\rho}, -\frac{\mu_2}{\hat\rho}, \dots, -\frac{\mu_p}{\hat\rho}\big) \tag{3.2}$$

is an odd homeomorphism from K to $S^{p-1} \subset \mathbb{R}^p$.

Proof of Theorem 1.1. We consider the subset $K \subset \mathcal{M}$ in (31). Without loss of generality, we may assume that $|e_j(x)| \leqslant 1$, for all $1 \leqslant j \leqslant p, x \in \Omega$. Thus, for any $u(x) = \nu(x) + c(\nu) = \sum_{j=1}^p \mu_j e_j(x) + c(\nu) \in K$, we have

$$|\nu(x)|^2 \leqslant \sum_{j=1}^p \mu_j^2 \sum_{j=1}^p |e_j(x)|^2 \leqslant p\hat\rho^2, \quad \forall x \in \Omega. \tag{3.3}$$

From $\int_\Omega f(\nu(x) + c(\nu))dx = 0$, we know that there exists $\hat x \in \Omega$ such that $f(\nu(\hat x) + c(\nu)) = 0$. By (f_1), we obtain $\nu(\hat x) + c(\nu) = 0$, namely, $c(\nu) = -\nu(\hat x)$. Thus

$$|c(\nu)| = |\nu(\hat x)| \leqslant \sqrt{p}\hat\rho, \tag{3.4}$$

$$|u(x)| \leqslant |\nu(x)| + |c(\nu)| \leqslant 2\sqrt{p}\hat\rho = \rho, \quad \forall x \in \Omega, \tag{3.5}$$

so, combining (3.5) with (F4)–(F5) shows that

$$\varphi_\Omega(u) = \frac{1}{2}\int_\Omega |\nabla u(x)|^2 dx - \int_\Omega F(u(x))dx$$

$$
\begin{aligned}
&= \frac{1}{2}\int_\Omega |\nabla\nu(x)|^2dx - \int_\Omega F(\nu(x)+c(\nu))dx \\
&\leqslant \frac{1}{2}\int_\Omega |\nabla\nu(x)|^2dx - \frac{M}{2}\int_\Omega (\nu(x)+c(\nu))^2dx \\
&= \frac{1}{2}\sum_{j=1}^{p}\mu_j^2\|\nabla e_j(x)\|_{L^2}^2 - \frac{1}{2}M\sum_{j=1}^{p}\mu_j^2\|e_j(x)\|_{L^2}^2 - Mc(\nu)\sum_{j=1}^{p}\mu_j\int_\Omega e_j(x)dx \\
&\quad - \frac{1}{2}Mc^2(\nu)|\Omega| \\
&\leqslant \frac{1}{2}\sum_{j=1}^{p}\mu_j^2\|\nabla e_j(x)\|_{L^2}^2 - \frac{1}{2}M\sum_{j=1}^{p}\mu_j^2\|e_j(x)\|_{L^2}^2 \\
&\leqslant \frac{1}{2}\sum_{j=1}^{p}\mu_j^2[(\frac{2(j+1)\pi}{d_\Omega})^2 - M]\|e_j(x)\|_{L^2}^2 < 0,
\end{aligned}
$$

using $\int_\Omega e_j(x)dx = 0$. Thus $sup\{\varphi_\Omega(u) : u \in K\} < 0 = \varphi_\Omega(0)$. Hence, by Lemma 2.2 and Theorem 1.4, $\varphi_\Omega|_{\mathcal{M}}$ possesses at least p distinct pairs $(u_j, -u_j)$ of critical points on $\mathcal{M}$ such that $\varphi_\Omega(u_j) < 0$ with $u_j \neq 0 (1 \leqslant j \leqslant p)$. Since $u_j \in \mathcal{M}\setminus\{0\}$; that is, $\int_\Omega f(u_j(x))dx = 0$, however, the continuous function $f(s)$ satisfies $f(s) > 0$ if $s > 0$, and $f(s) < 0$ if $s < 0$, thus, we conclude that u_j must change its sign. In addition, from (ii), we also know that there is no positive and negative critical point of φ_Ω. In other words, problem (1.1) possesses p distinct pairs $(u_j(x), -u_j(x))$ of sign-changing classical solutions $(1 \leqslant j \leqslant p)$, and has no positive and negative solution. □

Remark 3.1. Costa [1, Theorem 3.7], by minimizing method, shows that there exists $u_0 = u_0(x) \in \mathcal{M}\setminus\{0\}$ such that

$$\varphi_\Omega(u_0) = \inf_{u\in\mathcal{M}} \varphi_\Omega(u) < 0. \tag{3.6}$$

In fact, by our previous arguments in Theorem 1.1, we know that $u_0(x)$ is a sign-changing classical solution of (1.1).

As an application and illustration of Theorem 1.1, Theorem 1.5 is applied to the Neumann problem:

$$
\begin{gathered}
-\Delta u(x) = f(u(x)), \quad x_1^2 + x_2^2 < r^2 \\
\frac{\partial u}{\partial n}\,|_{x_1^2+x_2^2=r^2} = 0,
\end{gathered} \tag{3.7}
$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $r > 0$. To prove Theorem 1.5, we shall use some properties of Bessel functions.

Proof of Theorem 1.5. First consider the eigenvalue problem

$$
\begin{gathered}
-\Delta u(x) = \lambda u(x), \quad x_1^2 + x_2^2 < r^2 \\
\frac{\partial u}{\partial n}|_{x_1^2+x_2^2=r^2} = 0.
\end{gathered} \tag{3.8}
$$

By [3, Chpater 5,Section 5], positive eigenvalues λ_k^j of (3.8) satisfy

$$J_j'(r\sqrt{\lambda_k^j}) = 0, \quad j = 0,1,2\dots,\ k = 1,2\dots, \tag{3.9}$$

where $J_j(\cdot)$ is the j-order Bessel function, and the corresponding eigenfunctions are

$$u_k^j(x) = u_k^j(x_1, x_2) = J_j(\sqrt{\lambda_k^j}\tau)(\cos j\theta + \sin j\theta)$$

with $x_1 = \tau\cos\theta$, $x_2 = \tau\sin\theta$, $0 \leqslant \tau \leqslant r$, $0 \leqslant \theta \leqslant 2\pi$. Now we choose

$$e_j(x) = u_j^0(x) = J_0(\sqrt{\lambda_j^0}\tau), \quad j = 1, 2, \dots p. \tag{3.10}$$

From the integral expression

$$J_0(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(t\sin\theta)d\theta, \tag{3.11}$$

we know that $|e_j(x)| \leqslant 1$. And since $-\triangle e_j(x) = \lambda_j^0 e_j(x)$, by Green's formula, we obtain

$$\int_\Omega |\nabla e_j(x)|^2 dx = \lambda_j^0 \int_\Omega |e_j(x)|^2 dx. \tag{3.12}$$

Since $J_0'(t) = -J_1(t)$ for all $t \in \mathbb{R}$, by (3.9), we have

$$J_1(r\sqrt{\lambda_j^0}) = 0, \quad j = 1, 2, \dots. \tag{3.13}$$

Let a_j^0 be the jth positive zero point of $J_0(t)$. Then according to Schafheitlin's investigation of the zero points of $J_0(t)$ [17, Section 15.32, P.489], a_j^0 satisfies the estimate

$$(j-1)\pi + \frac{3}{4}\pi < a_j^0 < (j-1)\pi + \frac{7}{8}\pi, \quad j = 1, 2, \dots, \tag{3.14}$$

thus by (3.14) and the property of positive zero points of $J_1(t)$, we obtain

$$(j-1)\pi + \frac{3}{4}\pi < r\sqrt{\lambda_j^0} < j\pi + \frac{7}{8}\pi, \quad j = 1, 2, \dots; \tag{3.15}$$

therefore,

$$\lambda_j^0 < \frac{(j+1)^2\pi^2}{r^2}, \quad j = 1, 2, \dots p. \tag{3.16}$$

From (3.12) and (3.16), for $1 \leqslant j \leqslant p$, we derive that

$$\int_\Omega |\nabla e_j(x)|^2 dx = \lambda_j^0 \int_\Omega |e_j(x)|^2 dx \leqslant \frac{(j+1)^2\pi^2}{r^2}\int_\Omega |e_j(x)|^2 dx. \tag{3.17}$$

So, by Theorem 1.1 with $d_\Omega = 2r$, φ_Ω possesses at least p distinct pairs $(u_j, -u_j)$ of critical points such that $\varphi_\Omega(u_j) < 0$, which are p distinct sign-changing classical solutions of (1.8). $\square$

Theorem 3.2. *Suppose f satisfies* (F1)–(F3) *and* $\lim_{|u|\to 0} F(u)/|u|^2 = \infty$. *Then, for all* $r \in (0, \frac{\pi}{2\sqrt{f'(\pm\infty)}})$, (1.8) *has infinitely many distinct pairs* $(u(x), -u(x))$ *of sign-changing classical solutions.*

The above theorem is a corollary of Theorem 1.5 with Remark 1.2.

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References

[1] D. G. Costa; *An invitation to variational methods in differential equations*, Birkhauser, 2007.
[2] D. C. Clark; *A variant of the Lusternic-Schnirelmann theory*, Indiana Univ. Math. J. 22 (1972), 65-74.
[3] R. Courant, D. Hilbert; *Methods of mathematical physics (Vol. I)*, Interscience Publishers, Inc., New York, 1953.
[4] C. P. Gupta; *Perturbations of second order linear elliptic problems by unbounded nonlinearities*, Nonlinear Analysis: TMA, 6 (1982), 919-933.
[5] R. Iannacci, M. N. Nkashama; *Nonlinear two point boundary value problem at resonance without Landesman-Lazer condition*, Proc. AMS, 106(1989), 943-952.
[6] R. Iannacci, M. Nkashama; *Nonlinear boundary value problems at resonance*, Nonlinear Analysis: TMA, 11(1987), 455-473.
[7] Chung-cheng Kuo; *On the solvability of a nonlinear second order elliptic equations at resonance*, Proc. AMS, 124(1996), 83-87.
[8] P. Li; *Poincare inequality on Reimann manifolds*, Seminar on Differential Geometry (Yau edited), Princeton Press (1982), 73-83.
[9] C. Li, S. J. Li; *Multiple solutions and sign-changing solutions of a class of nonlinear elliptic equations with Neumann boundary condition*, J. Math. Anal. Appl. 298(2004), 14-32.
[10] J. Mawhin; *Semi-coercive monotone variational problems*, Acad Roy Belg Bull Cl Sciences, 73(1987), 118-130.
[11] P. H. Rabinowitz; *On a class of functionals invariant under a Zn action*, Trans Amer Math Soc, 310(1988), 303-311.
[12] P. H. Rabinowitz; *Minimax methods in critical point theory with applications to differential equations*, CBMS, Regional Conf. Ser. in Math., Vol.65. AMS, Providence, Rhode Island (1986).
[13] A. Szulkin; *Ljusternik-Schnirelmann theory on c1-manifolds*, Ann. Inst. H. Poincare, Analyse non lineaire 5(1988), 119-139.
[14] Chun-Lei Tang; *some existence theorems for sublinear Neumann boundary value problem*, Nonlinear Analysis:TMA, 48(2002), 1003-1011.
[15] Chun-Lei Tang, Xing-Ping Wu; *Existence and multiplicity for solutions of Neumann problem for semilinear elliptic equations*, J. Math. Anal. Appl, 288 (2003), 660-670.
[16] Chun-Lei Tang, Xing-Ping Wu; *Multiple solutions of a class of Neumann problem for semilinear elliptic equations*, Nonlinear Analysis: TMA, 62(2005), 455-465.
[17] G. N. Watson; *A treatise on the theory of Bessel functions*, Cambridge, The University Press, 1922.

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