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GROWTH OF SOLUTIONS TO HIGHER-ORDER LINEAR DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS

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Abstract. In this article, we discuss the order and hyper-order of the linear differential equation

$$f^{(k)} + \sum_{j=1}^{k-1} (B_j e^{b_j z} + D_j e^{d_j z}) f^{(j)} + (A_1 e^{a_1 z} + A_2 e^{a_2 z}) f = 0,$$

where $A_j(z), B_j(z), D_j(z)$ are entire functions ($\not\equiv 0$) and a_1, a_2, d_j are complex numbers ($\neq 0$), and b_j are real numbers. Under certain conditions, we prove that every solution $f \not\equiv 0$ of the above equation is of infinite order. Then, we obtain an estimate of the hyper-order. Finally, we give an estimate of the exponent of convergence for distinct zeros of the functions $f^{(j)} - \varphi$ $(j = 0, 1, 2)$, where φ is an entire function ($\not\equiv 0$) and of order $\sigma(\varphi) < 1$, while the solution f of the differential equation is of infinite order. Our results extend the previous results due to Chen, Peng and Chen and others.

1. Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory (see [13, 19]). Let $\sigma(f)$ denote the order of growth of an entire function f and the hyper-order $\sigma_2(f)$ of f is defined by (see [19])

$$\sigma_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log\log\log M(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f and

$$M(r,f) = \max_{|z|=r} |f(z)|.$$

To give some estimates of fixed points, we recall the following definition.

Definition 1.1 ([3, 15])**.** Let f be a meromorphic function. Then the exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by

$$\overline{\tau}(f) = \overline{\lambda}(f-z) = \limsup_{r\to+\infty} \frac{\log \overline{N}(r, \frac{1}{f-z})}{\log r},$$

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where $\overline{N}(r, 1/f)$ is the counting function of distinct zeros of $f(z)$ in $\{z : |z| \leq r\}$. We also define

$$\overline{\lambda}(f-\varphi) = \limsup_{r\to+\infty} \frac{\log \overline{N}(r, \frac{1}{f-\varphi})}{\log r}$$

for any meromorphic function $\varphi(z)$.

For the second-order linear differential equation

$$f'' + e^{-z} f' + B(z) f = 0, \tag{1.1}$$

where $B(z)$ is an entire function, it is well-known that each solution f of equation (1.1) is an entire function, and that if f_1 and f_2 are two linearly independent solutions of (1.1), then by [6], there is at least one of f_1, f_2 of infinite order. Hence, "most" solutions of (1.1) will have infinite order. But equation (1.1) with $B(z) = -(1 + e^{-z})$ possesses a solution $f(z) = e^z$ of finite order.

A natural question arises: What conditions on $B(z)$ will guarantee that every solution $f \not\equiv 0$ of (1.1) has infinite order? Many authors, Frei [7], Ozawa [16], Amemiya-Ozawa [1] and Gundersen [9], Langley [14] have studied this problem. They proved that when $B(z)$ is a nonconstant polynomial or $B(z)$ is a transcendental entire function with order $\sigma(B) \neq 1$, then every solution $f \not\equiv 0$ of (1.1) has infinite order.

In 2002, Chen [4] considered the question: What conditions on $B(z)$ when $\sigma(B) = 1$ will guarantee that every nontrivial solution of (1.1) has infinite order? He proved the following result, which improved results of Frei, Amemiya-Ozawa, Ozawa, Langley and Gundersen.

Theorem 1.2 ([4]). *Let $A_j(z)$ $(\not\equiv 0)$ $(j = 0, 1)$ be entire functions with $\max\{\sigma(A_j)$ $(j = 0, 1)\} < 1$. and let a, b be complex constants that satisfy $ab \neq 0$ and $a \neq b$. Then every solution $f \not\equiv 0$ of the differential equation*

$$f'' + A_1(z) e^{az} f' + A_0(z) e^{bz} f = 0$$

is of infinite order.

In [17], Peng and Chen investigated the order and hyper-order of solutions of some second order linear differential equations and have proved the following result.

Theorem 1.3 ([17]). *Let $A_j(z)$ $(\not\equiv 0)$ $(j = 1, 2)$ be entire functions with $\sigma(A_j) < 1$, a_1, a_2 be complex numbers such that $a_1 a_2 \neq 0$, $a_1 \neq a_2$ (suppose that $|a_1| \leq |a_2|$). If $\arg a_1 \neq \pi$ or $a_1 < -1$, then every solution $f (\not\equiv 0)$ of the differential equation*

$$f'' + e^{-z} f' + (A_1 e^{a_1 z} + A_2 e^{a_2 z}) f = 0$$

has infinite order and $\sigma_2(f) = 1$.

Recently in [12], the authors extend and improve the results of Theorem 1.3 to some higher order linear differential equations as follows.

Theorem 1.4 ([12]). *Let $A_j(z)$ $(\not\equiv 0)$ $(j = 1, 2)$, $B_l(z)$ $(\not\equiv 0)$ $(l = 1, \dots, k-1)$, D_m $(m = 0, \dots, k-1)$ be entire functions with $\max\{\sigma(A_j), \sigma(B_l), \sigma(D_m)\} < 1$, b_l $(l = 1, \dots, k-1)$ be complex constants such that*

- (i) *$\arg b_l = \arg a_1$ and $b_l = c_l a_1$ $(0 < c_l < 1)$ $(l \in I_1)$ and*
- (ii) *b_l is a real constant such that $b_l \leq 0$ $(l \in I_2)$, where $I_1 \neq \emptyset$, $I_2 \neq \emptyset$, $I_1 \cap I_2 = \emptyset$, $I_1 \cup I_2 = \{1, 2, \dots, k-1\}$, and a_1, a_2 are complex numbers such that $a_1 a_2 \neq 0$, $a_1 \neq a_2$ (suppose that $|a_1| \leq |a_2|$).*

If $\arg a_1 \neq \pi$ *or* a_1 *is a real number such that* $a_1 < \frac{b}{1-c}$, *where* $c = \max\{c_l : l \in I_1\}$ *and* $b = \min\{b_l : l \in I_2\}$, *then every solution* $f \not\equiv 0$ *of the differential equation*

$$f^{(k)}+(D_{k-1}+B_{k-1}e^{b_{k-1}z})f^{(k-1)}+\dots+(D_1+B_1e^{b_1z})f'+(D_0+A_1e^{a_1z}+A_2e^{a_2z})f=0$$

satisfies $\sigma(f) = +\infty$ *and* $\sigma_2(f) = 1$.

In this paper, we continue the research in this type of problems, the main purpose of this paper is to extend and improve the results of Theorems 1.2–1.4 to some higher order linear differential equations. In fact we will prove the following results.

Theorem 1.5. *Let* $k \geq 2$ *be an integer,* $A_j(z)$ $(\not\equiv 0)$ $(j = 1, 2)$ *and* $B_j(z)$ $(\not\equiv 0)$, $D_j(z)$ $(\not\equiv 0)$ $(j = 1, \dots, k-1)$ *be entire functions with*

$$\max\{\sigma(A_j)(j=1,2), \sigma(B_j)(j=1,\dots,k-1), \sigma(D_j)(j=1,\dots,k-1)\} < 1,$$

a_1, a_2 *be complex numbers such that* $a_1a_2 \neq 0, a_1 \neq a_2, d_j \neq 0$ $(j = 1, \dots, k-1)$ *be complex numbers and* b_j $(j = 1, \dots, k-1)$ *be real numbers such that* $b_j < 0$. *Suppose that there exists* α_j, β_j $(j = 1, \dots, k-1)$ *where* $0 < \alpha_j < 1$, $0 < \beta_j < 1$ *and* $d_j = \alpha_j a_1 + \beta_j a_2$. *Set* $\alpha = \max\{\alpha_j : j = 1, \dots, k-1\}$, $\beta = \max\{\beta_j : j = 1, \dots, k-1\}$ *and* $b = \min\{b_j : j = 1, \dots, k-1\}$. *If*

(1) $\arg a_1 \neq \pi$ *and* $\arg a_1 \neq \arg a_2$*; or*
(2) $\arg a_1 \neq \pi$, $\arg a_1 = \arg a_2$ *and (i)* $|a_2| > \frac{|a_1|}{1-\beta}$ *or (ii)* $|a_2| < (1-\alpha)|a_1|$*; or*
(3) $a_1 < 0$ *and* $\arg a_1 \neq \arg a_2$*; or*
(4) *(i)* $(1-\beta)a_2 - b < a_1 < 0$, $a_2 < \frac{b}{1-\beta}$ *or (ii)* $a_1 < \frac{a_2+b}{1-\alpha}$ *and* $a_2 < 0$,

then every solution $f(\not\equiv 0)$ *of the differential equation*

$$f^{(k)} + \sum_{j=1}^{k-1}(B_je^{b_jz} + D_je^{d_jz})f^{(j)} + (A_1e^{a_1z} + A_2e^{a_2z})f = 0 \tag{1.2}$$

satisfies $\sigma(f) = +\infty$ *and* $\sigma_2(f) = 1$.

Set

$$I_1 = \{2a_1, 2a_2, a_1+a_2, a_1, a_2, a_1+b_i, a_2+b_i, a_1+d_i, a_2+d_i \ (i=1,\dots,k-1)\},$$

$$I_2 = \{2a_1, 2a_2, a_1+a_2, a_1+b_1, a_2+b_1, a_1+d_1, a_2+d_1\},$$

$$\begin{aligned} I_3 = \Big\{&3a_1, 3a_2, 2a_1+a_2, a_1+2a_2, 2a_1, 2a_2, a_1+a_2, a_1+b_1, a_2+b_1, a_1+d_1,\\ &a_2+d_1, 2a_1+b_i, 2a_2+b_i, 2a_1+d_i, 2a_2+d_i, a_1+a_2+b_i, a_1+a_2+d_i,\\ &a_1+b_1+b_i, a_2+b_1+b_i, a_1+d_1+d_i, a_2+d_1+d_i, a_1+b_1+d_i,\\ &a_2+b_1+d_i(i=1,\dots,k-1), a_1+d_1+b_i, a_2+d_1+b_i \ (i=2,\dots,k-1)\Big\}. \end{aligned}$$

Theorem 1.6. *Let* $A_j(z)$ $(j = 1, 2)$, $B_j(z)$, $D_j(z)$ $(j = 1, \dots, k-1)$, a_1, a_2, b_j, d_j, α_j, β_j $(j = 1, \dots, k-1)$, α, β *and* b *satisfy the additional hypotheses of Theorem 1.5. If* $\varphi(\not\equiv 0)$ *is an entire function of order* $\sigma(\varphi) < 1$, *then every solution* $f(\not\equiv 0)$ *of equation* (1.2) *satisfies*

$$\overline{\lambda}(f - \varphi) = +\infty.$$

Furthermore, we have

(1) *If* $(2a_1) \notin I_1 \setminus \{2a_1\}$ *or* $(2a_2) \notin I_1 \setminus \{2a_2\}$, *then*

$$\overline{\lambda}(f' - \varphi) = +\infty.$$

(2) *If (i)* $(2a_1) \notin I_2 \setminus \{2a_1\}$ *or* $(2a_2) \notin I_2 \setminus \{2a_2\}$ *and (ii)* $(3a_1) \notin I_3 \setminus \{3a_1\}$ *or* $(3a_2) \notin I_3 \setminus \{3a_2\}$*, then*
$$\overline{\lambda}(f'' - \varphi) = +\infty.$$

Now set
$$J_1 = \Big\{2a_1, 2a_2, a_1 + a_2, a_1 + b_i, a_2 + b_i, a_1 + d_i, a_2 + d_i \ (i = 1, 2)\Big\},$$
$$\begin{aligned} J_2 = \Big\{&3a_1, 3a_2, 2a_1 + a_2, a_1 + 2a_2, 2a_1 + b_i, 2a_2 + b_i, 2a_1 + d_i,\\ &2a_2 + d_i, a_1 + a_2 + b_i, a_1 + a_2 + d_i, a_1 + b_1 + b_i, a_2 + b_1 + b_i, a_1\\ &+ d_1 + d_i, a_2 + d_1 + d_i, a_1 + b_1 + d_i, a_2 + b_1 + d_i \ (i = 1, 2, 3),\\ &a_1 + d_1 + b_i, a_2 + d_1 + b_i \ (i = 2, 3)\Big\}. \end{aligned}$$

From Theorem 1.6, we obtain the following corollary.

Corollary 1.7. *Let* $A_j(z)$ $(j = 1, 2)$, $B_j(z)$, $D_j(z)$ $(j = 1, \dots, k-1)$, a_1, a_2, b_j, d_j, α_j, β_j $(j = 1, \dots, k-1)$, α, β *and* b *satisfy the additional hypotheses of Theorem 1.5. If* $f(\not\equiv 0)$ *is any solution of* (1.2)*, then* f *has infinitely many fixed points and satisfies*
$$\overline{\tau}(f) = \infty.$$

Furthermore, we have

(1) *If* $(2a_1) \notin J_1 \setminus \{2a_1\}$ *or* $(2a_2) \notin J_1 \setminus \{2a_2\}$*, then* f' *has infinitely many fixed points and satisfies*
$$\overline{\tau}(f') = \infty.$$

(2) *If (i)* $(2a_1) \notin I_2 \setminus \{2a_1\}$ *or* $(2a_2) \notin I_2 \setminus \{2a_2\}$ *and (ii)* $(3a_1) \notin J_2 \setminus \{3a_1\}$ *or* $(3a_2) \notin J_2 \setminus \{3a_2\}$*, then* f'' *has infinitely many fixed points and satisfies*
$$\overline{\tau}(f'') = \infty.$$

2. Preliminary lemmas

We define the linear measure of a set $E \subset [0, +\infty)$ by $m(E) = \int_0^{+\infty} \chi_E(t)dt$ and the logarithmic measure of a set $F \subset (1, +\infty)$ by $lm(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t}dt$, where χ_H is the characteristic function of a set H.

Lemma 2.1 ([10])**.** *Let* f *be a transcendental meromorphic function with* $\sigma(f) = \sigma < +\infty$*. Let* $\varepsilon > 0$ *be a given constant, and let* k, j *be integers satisfying* $k > j \geq 0$*. Then, there exists a set* $E_1 \subset [-\frac{\pi}{2}, \frac{3\pi}{2})$ *with linear measure zero, such that, if* $\psi \in [-\frac{\pi}{2}, \frac{3\pi}{2}) \setminus E_1$*, then there is a constant* $R_0 = R_0(\psi) > 1$*, such that for all* z *satisfying* $\arg z = \psi$ *and* $|z| \geq R_0$*, we have*
$$\Big|\frac{f^{(k)}(z)}{f^{(j)}(z)}\Big| \leq |z|^{(k-j)(\sigma - 1 + \varepsilon)}. \tag{2.1}$$

Lemma 2.2 ([4])**.** *Suppose that* $P(z) = (\alpha + i\beta)z^n + \dots$ $(\alpha, \beta$ *are real numbers,* $|\alpha| + |\beta| \neq 0)$ *is a polynomial with degree* $n \geq 1$*, that* $A(z)$ $(\not\equiv 0)$ *is an entire function with* $\sigma(A) < n$*. Set* $g(z) = A(z)e^{P(z)}$*,* $z = re^{i\theta}$*,* $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$*. Then for any given* $\varepsilon > 0$*, there is a set* $E_2 \subset [0, 2\pi)$ *that has linear measure zero, such that for any* $\theta \in [0, 2\pi) \setminus (E_2 \cup E_3)$*, there is* $R > 0$*, such that for* $|z| = r > R$*, we have*

(i) *If* $\delta(P,\theta)>0$, *then*

$$\exp\{(1-\varepsilon)\delta(P,\theta)r^n\}\le|g(re^{i\theta})|\le\exp\{(1+\varepsilon)\delta(P,\theta)r^n\}. \tag{2.2}$$

(ii) *If* $\delta(P,\theta)<0$, *then*

$$\exp\{(1+\varepsilon)\delta(P,\theta)r^n\}\le|g(re^{i\theta})|\le\exp\{(1-\varepsilon)\delta(P,\theta)r^n\}, \tag{2.3}$$

where $E_3=\{\theta\in[0,2\pi):\delta(P,\theta)=0\}$ *is a finite set.*

Lemma 2.3 ([17]). *Suppose that* $n\ge 1$ *is a natural number. Let* $P_j(z)=a_{jn}z^n+\dots$ $(j=1,2)$ *be nonconstant polynomials, where* a_{jq} $(q=1,\dots,n)$ *are complex numbers and* $a_{1n}a_{2n}\ne 0$. *Set* $z=re^{i\theta}$, $a_{jn}=|a_{jn}|e^{i\theta_j}$, $\theta_j\in[-\frac{\pi}{2},\frac{3\pi}{2})$, $\delta(P_j,\theta)=|a_{jn}|\cos(\theta_j+n\theta)$, *then there is a set* $E_4\subset[-\frac{\pi}{2n},\frac{3\pi}{2n})$ *that has linear measure zero such that if* $\theta_1\ne\theta_2$, *then there exists a ray* $\arg z=\theta$, $\theta\in(-\frac{\pi}{2n},\frac{\pi}{2n})\setminus(E_4\cup E_5)$, *satisfying*

$$\delta(P_1,\theta)>0,\quad \delta(P_2,\theta)<0 \tag{2.4}$$

or

$$\delta(P_1,\theta)<0,\quad \delta(P_2,\theta)>0, \tag{2.5}$$

where $E_5=\{\theta\in[-\frac{\pi}{2n},\frac{3\pi}{2n}):\delta(P_j,\theta)=0\}$ *is a finite set, which has linear measure zero.*

Remark 2.4 ([17]). In Lemma 2.3, if $\theta\in(-\frac{\pi}{2n},\frac{\pi}{2n})\setminus(E_4\cup E_5)$ is replaced by $\theta\in(\frac{\pi}{2n},\frac{3\pi}{2n})\setminus(E_4\cup E_5)$, then we obtain the same result.

Lemma 2.5 ([5]). *Suppose that* $k\ge 2$ *and* $B_0,B_1,\dots,B_{k-1}$ *are entire functions of finite order and let* $\sigma=\max\{\sigma(B_j):j=0,\dots,k-1\}$. *Then every solution* f *of the differential equation*

$$f^{(k)}+B_{k-1}f^{(k-1)}+\dots+B_1f'+B_0f=0 \tag{2.6}$$

satisfies $\sigma_2(f)\le\sigma$.

Lemma 2.6 ([10]). *Let* $f(z)$ *be a transcendental meromorphic function, and let* $\alpha>1$ *be a given constant. Then there exist a set* $E_6\subset(1,\infty)$ *with finite logarithmic measure and a constant* $B>0$ *that depends only on* α *and* i,j $(0\le i<j\le k)$, *such that for all* z *satisfying* $|z|=r\notin[0,1]\cup E_6$, *we have*

$$|\frac{f^{(j)}(z)}{f^{(i)}(z)}|\le B\{\frac{T(\alpha r,f)}{r}(\log^\alpha r)\log T(\alpha r,f)\}^{j-i}. \tag{2.7}$$

Lemma 2.7 ([11]). *Let* $\varphi:[0,+\infty)\to\mathbb{R}$ *and* $\psi:[0,+\infty)\to\mathbb{R}$ *be monotone non-decreasing functions such that* $\varphi(r)\le\psi(r)$ *for all* $r\notin E_7\cup[0,1]$, *where* $E_7\subset(1,+\infty)$ *is a set of finite logarithmic measure. Let* $\gamma>1$ *be a given constant. Then there exists an* $r_1=r_1(\gamma)>0$ *such that* $\varphi(r)\le\psi(\gamma r)$ *for all* $r>r_1$.

Lemma 2.8 ([2]). *Let* $A_0,A_1,\dots,A_{k-1}$, $F\not\equiv 0$ *be finite order meromorphic functions. If* $f(z)$ *is an infinite order meromorphic solution of the equation*

$$f^{(k)}+A_{k-1}f^{(k-1)}+\dots+A_1f'+A_0f=F, \tag{2.8}$$

then f *satisfies* $\overline{\lambda}(f)=\lambda(f)=\sigma(f)=\infty$.

The following lemma, due to Gross [8], is important in the factorization and uniqueness theory of meromorphic functions, playing an important role in this paper as well.

Lemma 2.9 ([8, 19]). *Suppose that $f_1(z), f_2(z), \dots, f_n(z) (n \geq 2)$ are meromorphic functions and $g_1(z), g_2(z), \dots, g_n(z)$ are entire functions satisfying the following conditions:*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;
(ii) $g_j(z) - g_k(z)$ *are not constants for* $1 \leq j < k \leq n$;
(iii) *For* $1 \leq j \leq n$, $1 \leq h < k \leq n$, $T(r, f_j) = o\{T(r, e^{g_h(z)-g_k(z)})\}$ $(r \to \infty$, $r \notin E_8)$, *where E_8 is a set with finite linear measure.*

Then $f_j(z) \equiv 0$ $(j = 1, \dots, n)$.

Lemma 2.10 ([18]). *Suppose that $f_1(z), f_2(z), \dots, f_n(z) (n \geq 2)$ are meromorphic functions and $g_1(z), g_2(z), \dots, g_n(z)$ are entire functions satisfying the following conditions:*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv f_{n+1}$;
(ii) *If* $1 \leq j \leq n+1, 1 \leq k \leq n$, *the order of f_j is less than the order of* $e^{g_k(z)}$. *If* $n \geq 2$, $1 \leq j \leq n+1$, $1 \leq h < k \leq n$, *and the order of f_j is less than the order of* $e^{g_h - g_k}$.

Then $f_j(z) \equiv 0$ $(j = 1, 2, \dots, n+1)$.

3. Proof of Theorem 1.5

First step. Assume that $f(\not\equiv 0)$ is a solution of equation (1.2). We prove that $\sigma(f) = +\infty$. Suppose that $\sigma(f) = \sigma < +\infty$. We rewrite (1.2) as

$$\frac{f^{(k)}}{f} + \sum_{j=1}^{k-1}(B_j e^{b_j z} + D_j e^{(\alpha_j a_1 + \beta_j a_2)z})\frac{f^{(j)}}{f} + A_1 e^{a_1 z} + A_2 e^{a_2 z} = 0. \tag{3.1}$$

Set

$$\gamma = \max\{\sigma(B_j) \ (j = 1, \dots, k-1)\} < 1.$$

Then, for any given ε $(0 < \varepsilon < 1 - \gamma)$ and for sufficiently large r, we have

$$|B_j(z)| \leq \exp\{r^{\gamma+\varepsilon}\} \quad (j = 1, \dots, k-1). \tag{3.2}$$

By Lemma 2.1, for any given ε $(0 < \varepsilon < 1 - \gamma)$, there exists a set $E_1 \subset [-\frac{\pi}{2}, \frac{3\pi}{2})$ of linear measure zero, such that if $\theta \in [-\frac{\pi}{2}, \frac{3\pi}{2}) \setminus E_1$, then there is a constant $R_0 = R_0(\theta) > 1$, such that for all z satisfying $\arg z = \theta$ and $|z| = r \geq R_0$, we have

$$|\frac{f^{(j)}(z)}{f(z)}| \leq r^{j(\sigma-1+\varepsilon)} \quad (j = 1, \dots, k). \tag{3.3}$$

Let $z = re^{i\theta}$, $a_1 = |a_1|e^{i\theta_1}$, $a_2 = |a_2|e^{i\theta_2}$, $\theta_1, \theta_2 \in [-\frac{\pi}{2}, \frac{3\pi}{2})$. We know that $\delta(\alpha_j a_1 z, \theta) = \alpha_j \delta(a_1 z, \theta)$, $\delta(\beta_j a_2 z, \theta) = \beta_j \delta(a_2 z, \theta)$ $(j = 1, \dots, k-1)$ and $\alpha < 1$, $\beta < 1$.

Case 1. Assume that $\arg a_1 \neq \pi$ and $\arg a_1 \neq \arg a_2$, which is $\theta_1 \neq \pi$ and $\theta_1 \neq \theta_2$. By Lemma 2.2 and Lemma 2.3, for any given ε,

$$0 < \varepsilon < \min\{1 - \gamma, \frac{1-\alpha}{2(1+\alpha)}, \frac{1-\beta}{2(1+\beta)}\},$$

there is a ray $\arg z = \theta$ such that $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (E_1 \cup E_4 \cup E_5)$ (where E_4 and E_5 are defined as in Lemma 2.3, $E_1 \cup E_4 \cup E_5$ is of the linear measure zero), and satisfying

$$\delta(a_1 z, \theta) > 0, \ \delta(a_2 z, \theta) < 0$$

or

$$\delta(a_1 z,\theta) < 0,\ \delta(a_2 z,\theta) > 0.$$

(a) When $\delta(a_1 z,\theta) > 0$, $\delta(a_2 z,\theta) < 0$, for sufficiently large r, we obtain by Lemma 2.2,

$$|A_1 e^{a_1 z}| \geq \exp\{(1-\varepsilon)\delta(a_1 z,\theta)r\}, \tag{3.4}$$

$$|A_2 e^{a_2 z}| \leq \exp\{(1-\varepsilon)\delta(a_2 z,\theta)r\} < 1, \tag{3.5}$$

$$\begin{aligned} |D_j e^{\alpha_j a_1 z}| &\leq \exp\{(1+\varepsilon)\alpha_j \delta(a_1 z,\theta)r\} \\ &\leq \exp\{(1+\varepsilon)\alpha\delta(a_1 z,\theta)r\} \quad (j=1,\dots,k-1), \end{aligned} \tag{3.6}$$

$$|e^{\beta_j a_2 z}| \leq \exp\{(1-\varepsilon)\beta_j \delta(a_2 z,\theta)r\} < 1 \ (j=1,\dots,k-1). \tag{3.7}$$

By (3.6) and (3.7), we obtain

$$|D_j e^{(\alpha_j a_1 + \beta_j a_2)z}| = |D_j e^{\alpha_j a_1 z}||e^{\beta_j a_2 z}| \leq \exp\{(1+\varepsilon)\alpha\delta(a_1 z,\theta)r\}, \tag{3.8}$$

where $j = 1,\dots,k-1$. For $\theta \in (-\frac{\pi}{2},\frac{\pi}{2})$, by (3.2), we have

$$|B_j e^{b_j z}| = |B_j|\,|e^{b_j z}| \leq \exp\{r^{\gamma+\varepsilon}\} e^{b_j r\cos\theta} \leq \exp\{r^{\gamma+\varepsilon}\} \tag{3.9}$$

because $b_j < 0$ and $\cos\theta > 0$ $(j = 1,\dots,k-1)$. By (3.1), we obtain

$$|A_1 e^{a_1 z}| \leq |\frac{f^{(k)}}{f}| + \sum_{j=1}^{k-1}\Big(|B_j e^{b_j z}| + |D_j e^{(\alpha_j a_1 + \beta_j a_2)z}|\Big)|\frac{f^{(j)}}{f}| + |A_2 e^{a_2 z}|. \tag{3.10}$$

Substituting (3.3) -(3.5), (3.8) and (3.9) in (3.10), we have

$$\begin{aligned} \exp\{(1-\varepsilon)\delta(a_1 z,\theta)r\} &\leq |A_1 e^{a_1 z}| \\ &\leq M_1 r^{M_2}\exp\{r^{\gamma+\varepsilon}\}\exp\{(1+\varepsilon)\alpha\delta(a_1 z,\theta)r\}, \end{aligned} \tag{3.11}$$

where $M_1 > 0$ and $M_2 > 0$ are some constants. By $0 < \varepsilon < \frac{1-\alpha}{2(1+\alpha)}$ and (3.11), we obtain

$$\exp\{\frac{1-\alpha}{2}\delta(a_1 z,\theta)r\} \leq M_1 r^{M_2}\exp\{r^{\gamma+\varepsilon}\}. \tag{3.12}$$

By $\delta(a_1 z,\theta) > 0$ and $\gamma + \varepsilon < 1$ we know that (3.12) is a contradiction.

(b) When $\delta(a_1 z,\theta) < 0$, $\delta(a_2 z,\theta) > 0$, for sufficiently large r, we obtain

$$|A_2 e^{a_2 z}| \geq \exp\{(1-\varepsilon)\delta(a_2 z,\theta)r\}, \tag{3.13}$$

$$|A_1 e^{a_1 z}| \leq \exp\{(1-\varepsilon)\delta(a_1 z,\theta)r\} < 1, \tag{3.14}$$

$$|D_j e^{\alpha_j a_1 z}| \leq \exp\{(1-\varepsilon)\alpha_j\delta(a_1 z,\theta)r\} < 1 \quad (j=1,\dots,k-1), \tag{3.15}$$

$$\begin{aligned} |e^{\beta_j a_2 z}| &\leq \exp\{(1+\varepsilon)\beta_j\delta(a_2 z,\theta)r\} \\ &\leq \exp\{(1+\varepsilon)\beta\delta(a_2 z,\theta)r\} \quad (j=1,\dots,k-1). \end{aligned} \tag{3.16}$$

By (3.15) and (3.16), we have

$$|D_j e^{(\alpha_j a_1 + \beta_j a_2)z}| = |D_j e^{\alpha_j a_1 z}|\,|e^{\beta_j a_2 z}| \leq \exp\{(1+\varepsilon)\beta\delta(a_2 z,\theta)r\}, \tag{3.17}$$

where $j = 1,\dots,k-1$. By (3.1), we obtain

$$|A_2 e^{a_2 z}| \leq |\frac{f^{(k)}}{f}| + \sum_{j=1}^{k-1}\Big(|B_j e^{b_j z}| + |D_j e^{(\alpha_j a_1 + \beta_j a_2)z}|\Big)|\frac{f^{(j)}}{f}| + |A_1 e^{a_1 z}|. \tag{3.18}$$

Substituting (3.3), (3.9), (3.13), (3.14) and (3.17) in (3.18), we have

$$\begin{aligned}\exp\{(1-\varepsilon)\delta(a_2z,\theta)r\} &\le |A_2e^{a_2z}| \\ &\le M_1r^{M_2}\exp\{r^{\gamma+\varepsilon}\}\exp\{(1+\varepsilon)\beta\delta(a_2z,\theta)r\}.\end{aligned} \tag{3.19}$$

By $0<\varepsilon<\frac{1-\beta}{2(1+\beta)}$ and (3.19), we obtain

$$\exp\Big\{\frac{1-\beta}{2}\delta(a_2z,\theta)r\Big\} \le M_1r^{M_2}\exp\{r^{\gamma+\varepsilon}\}. \tag{3.20}$$

By $\delta(a_2z,\theta)>0$ and $\gamma+\varepsilon<1$ we know that (3.20) is a contradiction.

Case 2. Assume that $\arg a_1\neq\pi$, $\arg a_1=\arg a_2$, which is $\theta_1\neq\pi$, $\theta_1=\theta_2$. By Lemma 2.3, for any given ε

$$0<\varepsilon<\min\Big\{1-\gamma,\frac{(1-\alpha)|a_1|-|a_2|}{2[(1+\alpha)|a_1|+|a_2|]},\frac{(1-\beta)|a_2|-|a_1|}{2[(1+\beta)|a_2|+|a_1|]}\Big\},$$

there is a ray $\arg z=\theta$ such that $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})\setminus(E_1\cup E_4\cup E_5)$ and $\delta(a_1z,\theta)>0$. Since $\theta_1=\theta_2$, then $\delta(a_2z,\theta)>0$.

(i) $|a_2|>\frac{|a_1|}{1-\beta}$. For sufficiently large r, we have (3.6), (3.13) , (3.16) hold and

$$|A_1e^{a_1z}|\le\exp\{(1+\varepsilon)\delta(a_1z,\theta)r\}. \tag{3.21}$$

By (3.6) and (3.16), we obtain

$$|D_je^{(\alpha_ja_1+\beta_ja_2)z}|\le\exp\{(1+\varepsilon)\alpha\delta(a_1z,\theta)r\}\exp\{(1+\varepsilon)\beta\delta(a_2z,\theta)r\}, \tag{3.22}$$

where $j=1,\dots,k-1$. Substituting (3.3), (3.9), (3.13), (3.21) and (3.22) in (3.18), we have

$$\begin{aligned}&\exp\{(1-\varepsilon)\delta(a_2z,\theta)r\}\\ &\le|A_2e^{a_2z}|\\ &\le k\exp\{r^{\gamma+\varepsilon}\}\exp\{(1+\varepsilon)\alpha\delta(a_1z,\theta)r\}\exp\{(1+\varepsilon)\beta\delta(a_2z,\theta)r\}r^{k(\sigma-1+\varepsilon)}\\ &\quad+\exp\{(1+\varepsilon)\delta(a_1z,\theta)r\}\\ &\le M_1r^{M_2}\exp\{r^{\gamma+\varepsilon}\}\exp\{(1+\varepsilon)\delta(a_1z,\theta)r\}\exp\{(1+\varepsilon)\beta\delta(a_2z,\theta)r\}.\end{aligned} \tag{3.23}$$

From (3.23), we obtain

$$\exp\{\eta_1r\}\le M_1r^{M_2}\exp\{r^{\gamma+\varepsilon}\}, \tag{3.24}$$

where

$$\eta_1=(1-\varepsilon)\delta(a_2z,\theta)-(1+\varepsilon)\delta(a_1z,\theta)-(1+\varepsilon)\beta\delta(a_2z,\theta).$$

Since

$$0<\varepsilon<\frac{(1-\beta)|a_2|-|a_1|}{2[(1+\beta)|a_2|+|a_1|]},$$

$\theta_1=\theta_2$ and $\cos(\theta_1+\theta)>0$, we have

$$\begin{aligned}\eta_1&=[1-\beta-\varepsilon(1+\beta)]\delta(a_2z,\theta)-(1+\varepsilon)\delta(a_1z,\theta)\\ &=[1-\beta-\varepsilon(1+\beta)]|a_2|\cos(\theta_1+\theta)-(1+\varepsilon)|a_1|\cos(\theta_1+\theta)\\ &=\{[1-\beta-\varepsilon(1+\beta)]|a_2|-(1+\varepsilon)|a_1|\}\cos(\theta_1+\theta)\\ &=\{(1-\beta)|a_2|-|a_1|-\varepsilon[(1+\beta)|a_2|+|a_1|]\}\cos(\theta_1+\theta)\\ &>\frac{(1-\beta)|a_2|-|a_1|}{2}\cos(\theta_1+\theta)>0.\end{aligned}$$

Since $\eta_1>0$ and $\gamma+\varepsilon<1$, we know that (3.24) is a contradiction.

(ii) $|a_2| < (1-\alpha)|a_1|$. For sufficiently large r, we have (3.4), (3.6), (3.16) and (3.22) hold; then we obtain

$$|A_2 e^{a_2 z}| \le \exp\{(1+\varepsilon)\delta(a_2 z,\theta) r\}. \tag{3.25}$$

Substituting (3.3), (3.4), (3.9), (3.22) and (3.25) in (3.10), we have

$$\begin{aligned} &\exp\{(1-\varepsilon)\delta(a_1 z,\theta) r\} \\ &\le |A_1 e^{a_1 z}| \\ &\le k\exp\{r^{\gamma+\varepsilon}\}\exp\{(1+\varepsilon)\alpha\delta(a_1 z,\theta) r\}\exp\{(1+\varepsilon)\beta\delta(a_2 z,\theta) r\} r^{k(\sigma-1+\varepsilon)} \\ &\quad + \exp\{(1+\varepsilon)\delta(a_2 z,\theta) r\} \\ &\le M_1 r^{M_2}\exp\{r^{\gamma+\varepsilon}\}\exp\{(1+\varepsilon)\alpha\delta(a_1 z,\theta) r\}\exp\{(1+\varepsilon)\delta(a_2 z,\theta) r\}. \end{aligned} \tag{3.26}$$

From the above inequality we obtain

$$\exp\{\eta_2 r\} \le M_1 r^{M_2}\exp\{r^{\gamma+\varepsilon}\}, \tag{3.27}$$

where

$$\eta_2 = (1-\varepsilon)\delta(a_1 z,\theta) - (1+\varepsilon)\alpha\delta(a_1 z,\theta) - (1+\varepsilon)\delta(a_2 z,\theta).$$

Since $0 < \varepsilon < \frac{(1-\alpha)|a_1|-|a_2|}{2[(1+\alpha)|a_1|+|a_2|]}$, $\theta_1 = \theta_2$ and $\cos(\theta_1+\theta) > 0$, then we obtain

$$\begin{aligned} \eta_2 &= \{(1-\alpha)|a_1| - |a_2| - \varepsilon[(1+\alpha)|a_1| + |a_2|]\}\cos(\theta_1+\theta) \\ &> \frac{(1-\alpha)|a_1| - |a_2|}{2}\cos(\theta_1+\theta) > 0. \end{aligned}$$

By $\eta_2 > 0$ and $\gamma+\varepsilon < 1$ we know that (3.27) is a contradiction.

Case 3. Assume that $a_1 < 0$ and $\arg a_1 \ne \arg a_2$, which is $\theta_1 = \pi$ and $\theta_2 \ne \pi$. By Lemma 2.2, for the above ε, there is a ray $\arg z = \theta$ such that $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (E_1 \cup E_4 \cup E_5)$ and $\delta(a_2 z,\theta) > 0$. Because $\cos\theta > 0$, $\delta(a_1 z,\theta) = |a_1|\cos(\theta_1+\theta) = -|a_1|\cos\theta < 0$. Using the same reasoning as in Case 1 (b), we can get a contradiction.

Case 4. Assume that (i) $(1-\beta)a_2 - b < a_1 < 0$ and $a_2 < \frac{b}{1-\beta}$ or (ii) $a_1 < \frac{a_2+b}{1-\alpha}$ and $a_2 < 0$, which is $\theta_1 = \theta_2 = \pi$. By Lemma 2.2, for any given ε satisfying

$$0 < \varepsilon < \min\big\{1-\gamma, \frac{(1-\alpha)|a_1| - |a_2| + b}{2[(1+\alpha)|a_1| + |a_2|]}, \frac{(1-\beta)|a_2| - |a_1| + b}{2[(1+\beta)|a_2| + |a_1|]}\big\},$$

there is a ray $\arg z = \theta$ such that $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2}) \setminus (E_1 \cup E_4 \cup E_5)$, then $\cos\theta < 0$, $\delta(a_1 z,\theta) = |a_1|\cos(\theta_1+\theta) = -|a_1|\cos\theta > 0$ and

$$\delta(a_2 z,\theta) = |a_2|\cos(\theta_2+\theta) = -|a_2|\cos\theta > 0.$$

(i) $(1-\beta)a_2 - b < a_1 < 0$ and $a_2 < \frac{b}{1-\beta}$. For sufficiently large r, we obtain (3.6), (3.13), (3.16), (3.21) and (3.22) hold. For $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2})$, by (3.2) we have

$$|B_j e^{b_j z}| = |B_j||e^{b_j z}| \le \exp\{r^{\gamma+\varepsilon}\}e^{b_j r\cos\theta} \le \exp\{r^{\gamma+\varepsilon}\}e^{br\cos\theta} \tag{3.28}$$

because $b \le b_j < 0$ and $\cos\theta < 0$ ($j = 1,\dots,k-1$). Substituting (3.3), (3.13), (3.21), (3.22) and (3.28) in (3.18), we obtain

$$\begin{aligned} &\exp\{(1-\varepsilon)\delta(a_2 z,\theta) r\} \\ &\le |A_2 e^{a_2 z}| \\ &\le M_1 r^{M_2} e^{br\cos\theta}\exp\{r^{\gamma+\varepsilon}\}\exp\{(1+\varepsilon)\delta(a_1 z,\theta) r\}\exp\{(1+\varepsilon)\beta\delta(a_2 z,\theta) r\}. \end{aligned} \tag{3.29}$$

From (3.29) we have

$$\exp\{\eta_3 r\} \le M_1 r^{M_2}\exp\{r^{\gamma+\varepsilon}\}, \tag{3.30}$$

where

$$\eta_3 = (1-\varepsilon)\delta(a_2z,\theta) - (1+\varepsilon)\delta(a_1z,\theta) - (1+\varepsilon)\beta\delta(a_2z,\theta) - b\cos\theta.$$

Since $(1-\beta)a_2 - b < a_1$, $a_2 = -|a_2|$ and $a_1 = -|a_1|$, then we obtain $(1-\beta)|a_2| - |a_1| + b > 0$. We can see that $0 < (1-\beta)|a_2| - |a_1| + b < (1-\beta)|a_2| - |a_1| < 2[(1+\beta)|a_2| + |a_1|]$. Therefore,

$$0 < \frac{(1-\beta)|a_2| - |a_1| + b}{2[(1+\beta)|a_2| + |a_1|]} < 1.$$

From $0 < \varepsilon < \frac{(1-\beta)|a_2|-|a_1|+b}{2[(1+\beta)|a_2|+|a_1|]}$, $\theta_1 = \theta_2 = \pi$ and $\cos\theta < 0$, we obtain

$$\begin{aligned}
\eta_3 &= [1-\beta-\varepsilon(1+\beta)]\delta(a_2z,\theta) - (1+\varepsilon)\delta(a_1z,\theta) - b\cos\theta \\
&= -[1-\beta-\varepsilon(1+\beta)]|a_2|\cos\theta + (1+\varepsilon)|a_1|\cos\theta - b\cos\theta \\
&= (-\cos\theta)\{[1-\beta-\varepsilon(1+\beta)]|a_2| - (1+\varepsilon)|a_1| + b\} \\
&= (-\cos\theta)\{(1-\beta)|a_2| - |a_1| + b - \varepsilon[(1+\beta)|a_2| + |a_1|]\} \\
&> \frac{-1}{2}[(1-\beta)|a_2| - |a_1| + b]\cos\theta > 0.
\end{aligned}$$

From $\eta_3 > 0$ and $\gamma + \varepsilon < 1$ we know that (3.30) is a contradiction.

(ii) $a_1 < \frac{a_2+b}{1-\alpha}$ and $a_2 < 0$. For sufficiently large r, we obtain (3.4), (3.6), (3.16), (3.22), and (3.25) hold. Substituting (3.3), (3.4), (3.22), (3.25) and (3.28) in (3.10), we obtain

$$\begin{aligned}
\exp\{(1-\varepsilon)\delta(a_1z,\theta)r\} &\le |A_1e^{a_1z}| \\
&\le M_1r^{M_2}e^{br\cos\theta}\exp\{r^{\gamma+\varepsilon}\}\exp\{(1+\varepsilon)\alpha\delta(a_1z,\theta)r\} \\
&\quad \times\exp\{(1+\varepsilon)\delta(a_2z,\theta)r\}.
\end{aligned} \tag{3.31}$$

From this inequality we have

$$\exp\{\eta_4r\} \le M_1r^{M_2}\exp\{r^{\gamma+\varepsilon}\}, \tag{3.32}$$

where

$$\eta_4 = (1-\varepsilon)\delta(a_1z,\theta) - (1+\varepsilon)\alpha\delta(a_1z,\theta) - (1+\varepsilon)\delta(a_2z,\theta) - b\cos\theta.$$

Since $a_1 < \frac{a_2+b}{1-\alpha}$, $a_2 = -|a_2|$ and $a_1 = -|a_1|$, then we obtain $(1-\alpha)|a_1| - |a_2| + b > 0$. We can see that $0 < (1-\alpha)|a_1| - |a_2| + b < (1-\alpha)|a_1| - |a_2| < 2[(1+\alpha)|a_1| + |a_2|]$. Therefore,

$$0 < \frac{(1-\alpha)|a_1| - |a_2| + b}{2[(1+\alpha)|a_1| + |a_2|]} < 1.$$

From

$$0 < \varepsilon < \frac{(1-\alpha)|a_1| - |a_2| + b}{2[(1+\alpha)|a_1| + |a_2|]},$$

$\theta_1 = \theta_2 = \pi$ and $\cos\theta < 0$, we obtain

$$\begin{aligned}
\eta_4 &= (-\cos\theta)\{(1-\alpha)|a_1| - |a_2| + b - \varepsilon[(1+\alpha)|a_1| + |a_2|]\} \\
&> \frac{-1}{2}[(1-\alpha)|a_1| - |a_2| + b]\cos\theta > 0.
\end{aligned}$$

By $\eta_4 > 0$ and $\gamma + \varepsilon < 1$ we know that (3.32) is a contradiction. Concluding the above proof, we obtain $\sigma(f) = +\infty$.

Second step. We prove that $\sigma_2(f) = 1$. By

$$\max\{\sigma(B_je^{b_jz} + D_je^{d_jz})\ (j = 1,\dots,k-1), \sigma(A_1e^{a_1z} + A_2e^{a_2z})\} = 1$$

and Lemma 2.5, we obtain $\sigma_2(f) \le 1$. By Lemma 2.6, we know that there exists a set $E_6 \subset (1, +\infty)$ with finite logarithmic measure and a constant $C > 0$, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_6$, we obtain

$$|\frac{f^{(j)}(z)}{f(z)}| \le C[T(2r, f)]^{j+1} \quad (j = 1, \dots, k). \tag{3.33}$$

Case 1. $\arg a_1 \ne \pi$ and $\arg a_1 \ne \arg a_2$. In first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (E_1 \cup E_4 \cup E_5)$, satisfying

$$\delta(a_1 z, \theta) > 0, \quad \delta(a_2 z, \theta) < 0 \quad \text{or} \quad \delta(a_1 z, \theta) < 0, \quad \delta(a_2 z, \theta) > 0.$$

(a) When $\delta(a_1 z, \theta) > 0$, $\delta(a_2 z, \theta) < 0$, for sufficiently large r, we obtain (3.4)–(3.8) hold. Substituting (3.4), (3.5), (3.8), (3.9) and (3.33) in (3.10), we obtain that for all $z = re^{i\theta}$ satisfying $|z| = r \notin [0, 1] \cup E_6$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (E_1 \cup E_4 \cup E_5)$,

$$\begin{aligned} \exp\{(1-\varepsilon)\delta(a_1 z, \theta) r\} &\le |A_1 e^{a_1 z}| \\ &\le M \exp\{r^{\gamma+\varepsilon}\} \exp\{(1+\varepsilon)\alpha\delta(a_1 z, \theta) r\}[T(2r, f)]^{k+1}, \end{aligned} \tag{3.34}$$

where $M > 0$ is a constant. From (3.34) and $0 < \varepsilon < \frac{1-\alpha}{2(1+\alpha)}$, we obtain

$$\exp\big\{\frac{1-\alpha}{2}\delta(a_1 z, \theta) r\big\} \le M \exp\{r^{\gamma+\varepsilon}\}[T(2r, f)]^{k+1}. \tag{3.35}$$

Since $\delta(a_1 z, \theta) > 0$ and $\gamma + \varepsilon < 1$, then by using Lemma 2.7 and (3.35), we obtain $\sigma_2(f) \ge 1$. Hence $\sigma_2(f) = 1$.

(b) When $\delta(a_1 z, \theta) < 0, \delta(a_2 z, \theta) > 0$, for sufficiently large r, we obtain (3.13)–(3.17) hold. By using the a same reasoning as above, we can get $\sigma_2(f) = 1$.

Case 2. $\arg a_1 \ne \pi$, $\arg a_1 = \arg a_2$. In the first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (E_1 \cup E_4 \cup E_5)$, satisfying $\delta(a_1 z, \theta) > 0$ and $\delta(a_2 z, \theta) > 0$.

(i) $|a_2| > \frac{|a_1|}{1-\beta}$. For sufficiently large r, we have (3.6), (3.13), (3.16), (3.21) and (3.22) hold. Substituting (3.9), (3.13), (3.21), (3.22) and (3.33) in (3.18), we obtain that for all $z = re^{i\theta}$ satisfying $|z| = r \notin [0, 1] \cup E_6$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (E_1 \cup E_4 \cup E_5)$,

$$\begin{aligned} \exp\{(1-\varepsilon)\delta(a_2 z, \theta) r\} &\le |A_2 e^{a_2 z}| \\ &\le M \exp\{r^{\gamma+\varepsilon}\} \exp\{(1+\varepsilon)\delta(a_1 z, \theta) r\} \\ &\quad \times \exp\{(1+\varepsilon)\beta\delta(a_2 z, \theta) r\}[T(2r, f)]^{k+1}. \end{aligned} \tag{3.36}$$

From this inequality, we obtain

$$\exp\{\eta_1 r\} \le M \exp\{r^{\gamma+\varepsilon}\}[T(2r, f)]^{k+1}, \tag{3.37}$$

where

$$\eta_1 = (1-\varepsilon)\delta(a_2 z, \theta) - (1+\varepsilon)\delta(a_1 z, \theta) - (1+\varepsilon)\beta\delta(a_2 z, \theta).$$

Since $\eta_1 > 0$ and $\gamma + \varepsilon < 1$, then by using Lemma 2.7 and (3.37), we obtain $\sigma_2(f) \ge 1$. Hence $\sigma_2(f) = 1$.

(ii) $|a_2| < (1-\alpha)|a_1|$. For sufficiently large r, we have (3.4), (3.6), (3.16), (3.22) and (3.25) hold. By using the same reasoning as above, we can get $\sigma_2(f) = 1$.

Case 3. $a_1 < 0$ and $\arg a_1 \ne \arg a_2$. In the first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (E_1 \cup E_4 \cup E_5)$, satisfying $\delta(a_2 z, \theta) > 0$ and $\delta(a_1 z, \theta) < 0$. Using the same reasoning as in second step (Case 1 (b)), we can get $\sigma_2(f) = 1$.

Case 4. (i) $(1-\beta)a_2 - b < a_1 < 0$ and $a_2 < \frac{b}{1-\beta}$ or (ii) $a_1 < \frac{a_2+b}{1-\alpha}$ and $a_2 < 0$. In the first step, we have proved that there is a ray $\arg z = \theta$, where $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2}) \setminus (E_1 \cup E_4 \cup E_5)$, satisfying $\delta(a_2 z, \theta) > 0$ and $\delta(a_1 z, \theta) > 0$.

(i) $(1-\beta)a_2 - b < a_1 < 0$ and $a_2 < \frac{b}{1-\beta}$. For sufficiently large r, we obtain (3.6), (3.13), (3.16), (3.21) and (3.22) hold. Substituting (3.13), (3.21), (3.22), (3.28) and (3.33) in (3.18), we obtain that for all $z = re^{i\theta}$ satisfying $|z| = r \notin [0,1] \cup E_6$, $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2}) \setminus (E_1 \cup E_4 \cup E_5)$,

$$\begin{aligned} \exp\{(1-\varepsilon)\delta(a_2 z,\theta)r\} &\leq |A_2 e^{a_2 z}| \\ &\leq M e^{br\cos\theta} \exp\{r^{\gamma+\varepsilon}\} \exp\{(1+\varepsilon)\delta(a_1 z,\theta)r\} \\ &\quad \times \exp\{(1+\varepsilon)\beta\delta(a_2 z,\theta)r\}[T(2r,f)]^{k+1}. \end{aligned} \tag{3.38}$$

From this inequality we obtain

$$\exp\{\eta_3 r\} \leq M \exp\{r^{\gamma+\varepsilon}\}[T(2r,f)]^{k+1}, \tag{3.39}$$

where

$$\eta_3 = (1-\varepsilon)\delta(a_2 z,\theta) - (1+\varepsilon)\delta(a_1 z,\theta) - (1+\varepsilon)\beta\delta(a_2 z,\theta) - b\cos\theta.$$

Since $\eta_3 > 0$ and $\gamma + \varepsilon < 1$, then by using Lemma 2.7 and (3.39), we obtain $\sigma_2(f) \geq 1$. Hence $\sigma_2(f) = 1$.

(ii) $a_1 < \frac{a_2+b}{1-\alpha}$ and $a_2 < 0$. For sufficiently large r, we obtain (3.4), (3.6), (3.16), (3.22) and (3.25) hold. By using the same reasoning as above, we can get $\sigma_2(f) = 1$. Concluding the above proof, we obtain that every solution $f(\not\equiv 0)$ of (1.2) satisfies $\sigma_2(f) = 1$. The proof of Theorem 1.5 is complete.

4. Proof of Theorem 1.6

Set $R_0(z) = A_1 e^{a_1 z} + A_2 e^{a_2 z}$ and $R_i(z) = B_i e^{b_i z} + D_i e^{d_i z}$ $(i = 1, \dots, k-1)$. Assume $f(\not\equiv 0)$ is a solution of (1.2). Then $\sigma(f) = +\infty$ by Theorem 1.5. Set $g_0(z) = f(z) - \varphi(z)$. Then we have $\sigma(g_0) = \sigma(f) = \infty$. Substituting $f = g_0 + \varphi$ into (1.2), we obtain

$$\begin{aligned} &g_0^{(k)} + R_{k-1} g_0^{(k-1)} + \dots + R_2 g_0'' + R_1 g_0' + R_0 g_0 \\ &= -[\varphi^{(k)} + R_{k-1}\varphi^{(k-1)} + \dots + R_2\varphi'' + R_1\varphi' + R_0\varphi]. \end{aligned} \tag{4.1}$$

We can rewrite (4.1) in the form

$$g_0^{(k)} + h_{0,k-1} g_0^{(k-1)} + \dots + h_{0,2} g_0'' + h_{0,1} g_0' + h_{0,0} g_0 = h_0, \tag{4.2}$$

where

$$h_0 = -[\varphi^{(k)} + R_{k-1}\varphi^{(k-1)} + \dots + R_2\varphi'' + R_1\varphi' + R_0\varphi].$$

We prove that $h_0 \not\equiv 0$. In fact, if $h_0 \equiv 0$, then

$$\varphi^{(k)} + R_{k-1}\varphi^{(k-1)} + \dots + R_2\varphi'' + R_1\varphi' + R_0\varphi = 0.$$

Hence, $\varphi \not\equiv 0$ is a solution of (1.2) with $\sigma(\varphi) = +\infty$ by Theorem 1.5, which is a contradiction. Hence, $h_0 \not\equiv 0$ is proved. By Lemma 2.8 and (4.2) we know that $\overline{\lambda}(g_0) = \overline{\lambda}(f - \varphi) = \sigma(g_0) = \sigma(f) = \infty$.

Now we prove that $\overline{\lambda}(f'-\varphi)=\infty$. Set $g_1(z)=f'(z)-\varphi(z)$. Then we have $\sigma(g_1)=\sigma(f')=\sigma(f)=\infty$. Differentiating both sides of equation (1.2), we obtain

$$\begin{aligned} &f^{(k+1)}+R_{k-1}f^{(k)}+(R'_{k-1}+R_{k-2})f^{(k-1)}+(R'_{k-2}+R_{k-3})f^{(k-2)}\\ &\quad+\dots+(R'_3+R_2)f'''+(R'_2+R_1)f''+(R'_1+R_0)f'+R'_0f=0. \end{aligned} \tag{4.3}$$

By (1.2), we have

$$f=-\frac{1}{R_0}[f^{(k)}+R_{k-1}f^{(k-1)}+\dots+R_2f''+R_1f']. \tag{4.4}$$

Substituting (4.4) into (4.3), we have

$$\begin{aligned} &f^{(k+1)}+\Big(R_{k-1}-\frac{R'_0}{R_0}\Big)f^{(k)}+\Big(R'_{k-1}+R_{k-2}-R_{k-1}\frac{R'_0}{R_0}\Big)f^{(k-1)}\\ &+\Big(R'_{k-2}+R_{k-3}-R_{k-2}\frac{R'_0}{R_0}\Big)f^{(k-2)}+\dots+\Big(R'_3+R_2-R_3\frac{R'_0}{R_0}\Big)f'''\\ &+\Big(R'_2+R_1-R_2\frac{R'_0}{R_0}\Big)f''+\Big(R'_1+R_0-R_1\frac{R'_0}{R_0}\Big)f'=0. \end{aligned} \tag{4.5}$$

We can write equation (4.5) in the form

$$f^{(k+1)}+h_{1,k-1}f^{(k)}+h_{1,k-2}f^{(k-1)}+\dots+h_{1,2}f'''+h_{1,1}f''+h_{1,0}f'=0, \tag{4.6}$$

where

$$h_{1,i}=R'_{i+1}+R_i-R_{i+1}\frac{R'_0}{R_0}\quad(i=0,1,\dots,k-2),$$
$$h_{1,k-1}=R_{k-1}-\frac{R'_0}{R_0}.$$

Substituting $f^{(j+1)}=g_1^{(j)}+\varphi^{(j)}$ $(j=0,\dots,k)$ into (4.6), we obtain

$$g_1^{(k)}+h_{1,k-1}g_1^{(k-1)}+h_{1,k-2}g_1^{(k-2)}+\dots+h_{1,2}g''_1+h_{1,1}g'_1+h_{1,0}g_1=h_1, \tag{4.7}$$

where

$$h_1=-[\varphi^{(k)}+h_{1,k-1}\varphi^{(k-1)}+h_{1,k-2}\varphi^{(k-2)}+\dots+h_{1,2}\varphi''+h_{1,1}\varphi'+h_{1,0}\varphi].$$

We can get

$$h_{1,i}(z)=\frac{N_i(z)}{R_0(z)}\quad(i=0,1,\dots,k-1), \tag{4.8}$$

where

$$N_0=R'_1R_0+R_0^2-R_1R'_0, \tag{4.9}$$
$$N_i=R'_{i+1}R_0+R_iR_0-R_{i+1}R'_0\quad(i=1,2,\dots,k-2), \tag{4.10}$$
$$N_{k-1}=R_{k-1}R_0-R'_0. \tag{4.11}$$

Now we prove that $h_1\not\equiv 0$. In fact, if $h_1\equiv 0$, then $\frac{h_1}{\varphi}\equiv 0$. Hence, by (4.8) we obtain

$$\frac{\varphi^{(k)}}{\varphi}R_0+\frac{\varphi^{(k-1)}}{\varphi}N_{k-1}+\frac{\varphi^{(k-2)}}{\varphi}N_{k-2}+\dots+\frac{\varphi''}{\varphi}N_2+\frac{\varphi'}{\varphi}N_1+N_0=0. \tag{4.12}$$

Obviously, $\frac{\varphi^{(j)}}{\varphi}$ $(j = 1, \dots, k)$ are meromorphic functions with $\sigma(\frac{\varphi^{(j)}}{\varphi}) < 1$. By (4.9)–(4.11) we can rewrite (4.12) in the form

$$A_1^2 e^{2a_1 z} + A_2^2 e^{2a_2 z} + \sum_{\lambda \in I_1'} f_\lambda e^{\lambda z} = 0, \tag{4.13}$$

where $I_1' = I_1 \setminus \{2a_1, 2a_2\}$ and f_λ $(\lambda \in I_1')$ are meromorphic functions with order less than 1.

(1) If $(2a_1) \notin I_1 \setminus \{2a_1\}$, then we write (4.13) in the form

$$A_1^2 e^{2a_1 z} + \sum_{\lambda \in \Gamma_1} g_{1,\lambda} e^{\lambda z} = 0,$$

where $\Gamma_1 \subseteq I_1 \setminus \{2a_1\}$, $g_{1,\lambda}$ $(\lambda \in \Gamma_1)$ are meromorphic functions with order less than 1 and $2a_1$, λ $(\lambda \in \Gamma_1)$ are distinct numbers. By Lemmas 2.9 and 2.10, we obtain $A_1 \equiv 0$, which is a contradiction.

(2) If $(2a_2) \notin I_1 \setminus \{2a_2\}$, then we write (4.13) in the form

$$A_2^2 e^{2a_2 z} + \sum_{\lambda \in \Gamma_2} g_{2,\lambda} e^{\lambda z} = 0,$$

where $\Gamma_2 \subseteq I_1 \setminus \{2a_2\}$, $g_{2,\lambda}$ $(\lambda \in \Gamma_2)$ are meromorphic functions with order less than 1 and $2a_2$, λ $(\lambda \in \Gamma_2)$ are distinct numbers. By Lemmas 2.9 and 2.10, we obtain $A_2 \equiv 0$, which is a contradiction. Hence, $h_1 \not\equiv 0$ is proved. By Lemma 2.8 and (4.7) we know that $\overline{\lambda}(g_1) = \overline{\lambda}(f' - \varphi) = \sigma(g_1) = \sigma(f) = \infty$.

Now we prove that $\overline{\lambda}(f'' - \varphi) = \infty$. Set $g_2(z) = f''(z) - \varphi(z)$. Then we have $\sigma(g_2) = \sigma(f'') = \sigma(f) = \infty$. Differentiating both sides of equation (4.3), we have

$$\begin{aligned} &f^{(k+2)} + R_{k-1} f^{(k+1)} + (2R_{k-1}' + R_{k-2}) f^{(k)} + (R_{k-1}'' + 2R_{k-2}' + R_{k-3}) f^{(k-1)} \\ &+ (R_{k-2}'' + 2R_{k-3}' + R_{k-4}) f^{(k-2)} + \dots + (R_3'' + 2R_2' + R_1) f''' \\ &+ (R_2'' + 2R_1' + R_0) f'' + (R_1'' + 2R_0') f' + R_0'' f = 0. \end{aligned} \tag{4.14}$$

By (4.4) and (4.14), we have

$$\begin{aligned} &f^{(k+2)} + R_{k-1} f^{(k+1)} + \Big(2R_{k-1}' + R_{k-2} - \frac{R_0''}{R_0}\Big) f^{(k)} \\ &+ \Big(R_{k-1}'' + 2R_{k-2}' + R_{k-3} - R_{k-1}\frac{R_0''}{R_0}\Big) f^{(k-1)} + \dots \\ &+ \Big(R_4'' + 2R_3' + R_2 - R_4\frac{R_0''}{R_0}\Big) f^{(4)} + \Big(R_3'' + 2R_2' + R_1 - R_3\frac{R_0''}{R_0}\Big) f''' \\ &+ \Big(R_2'' + 2R_1' + R_0 - R_2\frac{R_0''}{R_0}\Big) f'' + \Big(R_1'' + 2R_0' - R_1\frac{R_0''}{R_0}\Big) f' = 0. \end{aligned} \tag{4.15}$$

Now we prove that $R_1' + R_0 - R_1\frac{R_0'}{R_0} \not\equiv 0$. Suppose that $R_1' + R_0 - R_1\frac{R_0'}{R_0} \equiv 0$, then we have

$$A_1^2 e^{2a_1 z} + A_2^2 e^{2a_2 z} + \sum_{\lambda \in I_2'} f_\lambda e^{\lambda z} = 0, \tag{4.16}$$

where $I_2' = I_2 \setminus \{2a_1, 2a_2\}$ and f_λ $(\lambda \in I_2')$ are meromorphic functions with order less than 1. By using the same reasoning as above, we can get a contradiction. Hence, $R_1' + R_0 - R_1\frac{R_0'}{R_0} \not\equiv 0$ is proved. Set

$$\psi(z) = R_1' R_0 + R_0^2 - R_1 R_0' \quad \text{and} \quad \phi(z) = R_1'' R_0 + 2R_0' R_0 - R_1 R_0''. \tag{4.17}$$

By (4.5) and (4.17), we obtain

$$
\begin{aligned}
f' = \frac{-R_0}{\psi(z)}\Big\{ f^{(k+1)} + \Big(R_{k-1} - \frac{R_0'}{R_0}\Big) f^{(k)} + \Big(R_{k-1}' + R_{k-2} - R_{k-1}\frac{R_0'}{R_0}\Big) f^{(k-1)} \\
+ \Big(R_{k-2}' + R_{k-3} - R_{k-2}\frac{R_0'}{R_0}\Big) f^{(k-2)} + \dots + \Big(R_2' + R_1 - R_2\frac{R_0'}{R_0}\Big) f'' \Big\}.
\end{aligned} \tag{4.18}
$$

Substituting (4.17) and (4.18) into (4.15), we obtain

$$
\begin{aligned}
& f^{(k+2)} + [R_{k-1} - \frac{\phi}{\psi}] f^{(k+1)} + \Big[2R_{k-1}' + R_{k-2} - \frac{R_0''}{R_0} - \frac{\phi}{\psi}\big(R_{k-1} - \frac{R_0'}{R_0}\big)\Big] f^{(k)} \\
& + \Big[R_{k-1}'' + 2R_{k-2}' + R_{k-3} - R_{k-1}\frac{R_0''}{R_0} - \frac{\phi}{\psi}\big(R_{k-1}' + R_{k-2} - R_{k-1}\frac{R_0'}{R_0}\big)\Big] f^{(k-1)} \\
& + \dots + \Big[R_3'' + 2R_2' + R_1 - R_3\frac{R_0''}{R_0} - \frac{\phi}{\psi}\big(R_3' + R_2 - R_3\frac{R_0'}{R_0}\big)\Big] f''' \\
& + \Big[R_2'' + 2R_1' + R_0 - R_2\frac{R_0''}{R_0} - \frac{\phi}{\psi}\big(R_2' + R_1 - R_2\frac{R_0'}{R_0}\big)\Big] f'' = 0.
\end{aligned} \tag{4.19}
$$

We can write (4.19) in the form

$$
f^{(k+2)} + h_{2,k-1} f^{(k+1)} + h_{2,k-2} f^{(k)} + \dots + h_{2,2} f^{(4)} + h_{2,1} f''' + h_{2,0} f'' = 0, \tag{4.20}
$$

where

$$
\begin{gathered}
h_{2,i} = R_{i+2}'' + 2R_{i+1}' + R_i - R_{i+2}\frac{R_0''}{R_0} \\
- \frac{\phi(z)}{\psi(z)}\Big(R_{i+2}' + R_{i+1} - R_{i+2}\frac{R_0'}{R_0}\Big) \quad (i = 0, 1, \dots, k-3), \\
h_{2,k-2} = 2R_{k-1}' + R_{k-2} - \frac{R_0''}{R_0} - \frac{\phi(z)}{\psi(z)}\Big(R_{k-1} - \frac{R_0'}{R_0}\Big), \\
h_{2,k-1} = R_{k-1} - \frac{\phi(z)}{\psi(z)}.
\end{gathered}
$$

Substituting $f^{(j+2)} = g_2^{(j)} + \varphi^{(j)}$ $(j = 0, \dots, k)$ in (4.20) we have

$$
g_2^{(k)} + h_{2,k-1} g_2^{(k-1)} + h_{2,k-2} g_2^{(k-2)} + \dots + h_{2,1} g_2' + h_{2,0} g_2 = h_2, \tag{4.21}
$$

where

$$
h_2 = -[\varphi^{(k)} + h_{2,k-1}\varphi^{(k-1)} + h_{2,k-2}\varphi^{(k-2)} + \dots + h_{2,2}\varphi'' + h_{2,1}\varphi' + h_{2,0}\varphi].
$$

We obtain

$$
h_{2,i} = \frac{L_i(z)}{\psi(z)} \quad (i = 0, 1, \dots, k-1), \tag{4.22}
$$

where

$$
\begin{aligned}
L_0(z) = {} & R_2'' R_1' R_0 + R_2'' R_0^2 - R_2'' R_1 R_0' + 2{R_1'}^2 R_0 + 3R_1' R_0^2 - 2R_1' R_1 R_0' + R_0^3 \\
& - 3R_1 R_0' R_0 - R_2 R_1' R_0'' - R_2 R_0'' R_0 - R_2' R_1'' R_0 - 2R_2' R_0' R_0 + R_2' R_1 R_0'' \\
& - R_1'' R_1 R_0 + R_1^2 R_0'' + R_2 R_1'' R_0' + 2R_2 {R_0'}^2,
\end{aligned} \tag{4.23}
$$

$$\begin{aligned} L_i &= R''_{i+2}R'_1R_0 + R''_{i+2}R_0^2 - R''_{i+2}R_1R'_0 + 2R'_{i+1}R'_1R_0 + 2R'_{i+1}R_0^2 - 2R'_{i+1}R_1R'_0 \\ &\quad + R_iR'_1R_0 + R_iR_0^2 - R_iR_1R'_0 - R_{i+2}R'_1R''_0 - R_{i+2}R''_0R_0 - R'_{i+2}R''_1R_0 \\ &\quad - 2R'_{i+2}R'_0R_0 + R'_{i+2}R_1R''_0 - R_{i+1}R''_1R_0 - 2R_{i+1}R'_0R_0 + R_{i+1}R_1R''_0 \\ &\quad + R_{i+2}R''_1R'_0 + 2R_{i+2}{R'_0}^2 \quad (i = 1, 2, \dots, k-3), \end{aligned} \tag{4.24}$$

$$\begin{aligned} L_{k-2} &= 2R'_{k-1}R'_1R_0 + 2R'_{k-1}R_0^2 - 2R'_{k-1}R_1R'_0 + R_{k-2}R'_1R_0 + R_{k-2}R_0^2 \\ &\quad - R_{k-2}R_1R'_0 - R'_1R''_0 - R''_0R_0 - R_{k-1}R''_1R_0 - 2R_{k-1}R'_0R_0 \\ &\quad + R_{k-1}R_1R''_0 + R''_1R'_0 + 2{R'_0}^2, \end{aligned} \tag{4.25}$$

$$L_{k-1} = R_{k-1}R'_1R_0 + R_{k-1}R_0^2 - R_{k-1}R_1R'_0 - R''_1R_0 - 2R'_0R_0 + R_1R''_0. \tag{4.26}$$

Therefore,

$$\frac{-h_2}{\varphi} = \frac{1}{\psi}[\frac{\varphi^{(k)}}{\varphi}\psi + \frac{\varphi^{(k-1)}}{\varphi}L_{k-1} + \dots + \frac{\varphi''}{\varphi}L_2 + \frac{\varphi'}{\varphi}L_1 + L_0]. \tag{4.27}$$

Now we prove that $h_2 \not\equiv 0$. In fact, if $h_2 \equiv 0$, then $\frac{-h_2}{\varphi} \equiv 0$. Hence, by (4.27) we obtain

$$\frac{\varphi^{(k)}}{\varphi}\psi + \frac{\varphi^{(k-1)}}{\varphi}L_{k-1} + \dots + \frac{\varphi''}{\varphi}L_2 + \frac{\varphi'}{\varphi}L_1 + L_0 = 0. \tag{4.28}$$

Obviously, $\frac{\varphi^{(j)}}{\varphi}$ $(j = 1, \dots, k)$ are meromorphic functions with $\sigma(\frac{\varphi^{(j)}}{\varphi}) < 1$. By (4.17) and (4.23)–(4.26), we can rewrite (4.28) in the form

$$A_1^3e^{3a_1z} + A_2^3e^{3a_2z} + \sum_{\lambda\in I'_3} f_\lambda e^{\lambda z} = 0, \tag{4.29}$$

where $I'_3 = I_3 \setminus \{3a_1, 3a_2\}$ and f_λ $(\lambda \in I'_3)$ are meromorphic functions with order less than 1.

(1) If $(3a_1) \notin I_3 \setminus \{3a_1\}$, then we write (4.29) in the form

$$A_1^3e^{3a_1z} + \sum_{\lambda\in\Gamma_1} g_{1,\lambda}e^{\lambda z} = 0,$$

where $\Gamma_1 \subseteq I_3 \setminus \{3a_1\}$, $g_{1,\lambda}$ $(\lambda \in \Gamma_1)$ are meromorphic functions with order less than 1 and $3a_1$, λ $(\lambda \in \Gamma_1)$ are distinct numbers. By Lemmas 2.9 and 2.10, we obtain $A_1 \equiv 0$, which is a contradiction.

(2) If $(3a_2) \notin I_3 \setminus \{3a_2\}$, then we write (4.29) in the form

$$A_2^3e^{3a_2z} + \sum_{\lambda\in\Gamma_2} g_{2,\lambda}e^{\lambda z} = 0,$$

where $\Gamma_2 \subseteq I_3 \setminus \{3a_2\}$, $g_{2,\lambda}$ $(\lambda \in \Gamma_2)$ are meromorphic functions with order less than 1 and $3a_2$, λ $(\lambda \in \Gamma_2)$ are distinct numbers. By Lemmas 2.9 and 2.10, we obtain $A_2 \equiv 0$, which is a contradiction. Hence, $h_2 \not\equiv 0$ is proved. By Lemma 2.8 and (4.21), we have $\overline{\lambda}(g_2) = \overline{\lambda}(f'' - \varphi) = \sigma(g_2) = \sigma(f) = \infty$. The proof of Theorem 1.6 is complete.

Proof of Corollary 1.7. Setting $\varphi(z) = z$ and using the same reasoning as in the proof of Theorem 1.6, we obtain Corollary 1.7. □

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