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MULTIPLICITY AND MINIMALITY OF PERIODIC SOLUTIONS TO DELAY DIFFERENTIAL SYSTEM

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Abstract. In this article, we study periodic solutions of a class of delay differential equations. By restricting our discussion on generalized Nehari Manifold, some sufficient conditions are obtained to guarantee the existence of infinitely many pairs of periodic solutions. Also, there exists at least one periodic solution with prescribed minimal period.

1. Introduction

The existence of periodic solutions to delay differential equations have been investigated since 1962. Various methods, such as fixed point theory, Kaplan-Yorke method, coincidence degree theory, the Hopf bifurcation theorem, the Poicaré-Bendixson theorem and critical point theory, have been used to study such a problem. We refer to[4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 26, 27].

As far as the authors know, there are only a few results concerning with periodic solutions with prescribed minimal period to delay differential equations. In 1974, Kaplan and Yorke [10] translated the problem of existence of periodic solutions to delay differential equations to that of ordinary differential equations. Then, they showed the existence of solutions with minimal period to delay differential equations with one or two delays. In 1978, by using a completely different approach, Nussbaum [22] extended Kaplan and Yorke's result to differential equation with arbitrary delays. In 2012, by making use of rigorous analysis techniques, Yu [26] proved the existence, nonexistence, multiplicity and minimality of periodic solutions to differential equation with single delay. In 2013, Yu and the author [27] made use of Maslov-type index and Fourier series showing the existence of multiplicity periodic solutions with the same minimal period to a class of nonautonomous differential equation with single delay.

Because of lacking effective tools, no more results on solutions with prescribed minimal period to delay differential system have been obtained so far. As we know, critical point theory is a useful tool to prove the minimality of periodic solutions to ordinary differential systems and difference systems. A natural question is weather or not critical point theory can be used to study the minimality of period for periodic solutions to delay differential system.

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Nehari manifold method, introduced by Nehari [20, 21] and generalized by Szukin and Weth [24], has been widely used to study the existence of ground state solutions to partial differential equations. We refer to [2, 23]. Recall that a ground state solution is a solution which minimizes the variational functional on the set of nontrivial solutions. When a variational functional is restricted on a Nehari Manifold, it possesses some minimal characteristic. Such a characteristic can be used to prove the minimality of periodic solutions. Thus Nehari manifold had been used to study solutions with minimal period to Hamiltonian systems, see for example [1, 16]. In this article, we devote to making use of critical point theory combining with generalized Nehari manifold to study the existence of solutions with prescribed minimal period to delay differential system.

Before going too far, let us introduce some notation. Denote by $\mathbb{N}, \mathbb{Z}, \mathbb{R}^*, \mathbb{R}$ the set of all natural numbers, integers, nonnegative real numbers and real numbers, respectively. For $N \in \mathbb{N}$, denote by $\mathbb{R}^N$ the $N-$deminsional Hilbert space with the usual inner product $(\cdot,\cdot)$ and the usual norm $|\cdot|$. Let $S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ and $\varphi \in C^1(X, \mathbb{R})$. A sequence $\{x_n\} \subset X$ is called $(PS)_c$ sequence (resp. (PS) sequence) of φ if it satisfies $\varphi(x_n) \to c$ and $\varphi'(x_n) \to 0$ (resp. $\varphi(x_n)$ is bounded and $\varphi'(x_n) \to 0$) as $n \to \infty$. We say that φ satisfies $(PS)_c$ condition (resp. (PS) condition) if every $(PS)_c$ sequence (resp. (PS) sequence) has a convergent subsequence. Obviously, φ satisfying (PS) condition implies φ satisfying $(PS)_c$ condition for each $c \in \mathbb{R}$.

Consider the delay differential equation

$$x'(t) = -Ax(t - \frac{\pi}{2}) - f(x(t - \frac{\pi}{2})), \quad x(t) \in \mathbb{R}^N. \tag{1.1}$$

We use the following assumptions:

(A1) A is a symmetric, nonnegative definite matrix, $\{(-1)^{j+1}(2j-1) : j \in \mathbb{N}\} \cap \sigma(A) = \emptyset$, where $\sigma(A)$ denotes the spectral of matrix A;

(F1) f is odd, i.e., $f(-x) = -f(x)$, for any $x \in \mathbb{R}^N$;

(F2) there exists a function $F \in C^1(\mathbb{R}^N, \mathbb{R})$ such that $F(0) = 0$ and the gradient of F is f; i.e., for any $x \in \mathbb{R}^N$, $\nabla F(x) = f(x)$;

(F3) there exist constants $s > 1$ and $a > 0$ such that

$$F(x) \le a(1 + |x|^s), \quad \forall x \in \mathbb{R}^N;$$

(F4) there exists $\mu > 2$ such that

$$0 < \mu F(x) \le (x, f(x)), \quad \forall\ x \in \mathbb{R}^N \setminus \{0\};$$

(F5) $(f(x), y)(x, y) \ge 0$ for all $x, y \in \mathbb{R}^N$;

(F6) $F(x) = F(y)$ and $(f(x), y) \le (f(x), x)$ if $|x| = |y|$;

(F7) $(f(x), y) \ne (f(y), x)$ if $|x| \ne |y|$ and $(x, y) \ne 0$.

Remark 1.1. Denote $M = \max_{|x|=1} F(x)$ and $m = \min_{|x|=1} F(x)$. Then (F4) implies that $F(x) \le M|x|^\mu$, when $|x| \le 1$; $F(x) \ge m|x|^\mu$, when $|x| \ge 1$.

Our main result reads as follows.

Theorem 1.2. *Assume that* (A1), (F1)–(F7) *hold. Then* (1.1) *has infinitely many pairs of periodic solutions. Also,* (1.1) *possesses a solution having* 2π *as its minimal period.*

The rest of this paper is organized as follows: in section 2, variation functional will be established and some useful lemmas will be given; in section 3, generalized Nehari manifold will be defined and our main results will be proved.

2. Preliminaries

The space $H = H^{1/2}(S^1, \mathbb{R}^N)$ consists of 2π-periodic vector-valued functions, which possess square integrable derivative of order $1/2$. For any $x \in H$, it has Fourier expansion

$$x(t) = \frac{a_0}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}} \sum_{j=1}^{\infty} (a_j \cos(jt) + b_j \sin(jt))$$

where $a_0 \in \mathbb{R}^N$, a_j, $b_j \in \mathbb{R}^N$, $j \in \mathbb{N}$. The space H is a Hilbert space with norm and inner product as follows

$$\|x\|_1^2 = |a_0|^2 + \sum_{j=1}^{\infty} j(|a_j|^2 + |b_j|^2),$$

$$\langle x, y\rangle_1 = (a_0, c_0) + \sum_{j=1}^{\infty} j[(a_j, c_j) + (b_j, d_j)],$$

where $y = c_0/\sqrt{2\pi} + 1/\sqrt{\pi}\sum_{j=1}^{\infty}(c_j \cos(jt) + d_j \sin(jt))$, $c_0 \in \mathbb{R}^N$, c_j, $d_j \in \mathbb{R}^N$, $j \in \mathbb{N}$.

Let $x, y \in L^2(S^1, \mathbb{R}^N)$. If for every $z \in C^\infty(S^1, \mathbb{R}^N)$,

$$\int_0^{2\pi} (x(t), z'(t))dt = -\int_0^{2\pi} (y(t), z(t))dt,$$

then y is called a weak derivative of x, denoted by $\dot{x}$.

We define the variational functional defined J on H as follows,

$$J(x) = \int_0^{2\pi} \big[\frac{1}{2}(\dot{x}(t - \frac{\pi}{2}), x(t)) - \frac{1}{2}(Ax(t), x(t)) - F(x(t))\big]dt. \tag{2.1}$$

Using the argument in [7], we can prove the following lemma.

Lemma 2.1. *Assume that* (A1), (F2), (F3) *hold. Then J is continuous differentiable on H and*

$$\langle J'(x), h\rangle = \int_0^{2\pi} [\frac{1}{2}(\dot{x}(t - \frac{\pi}{2}) - \dot{x}(t + \frac{\pi}{2}), h(t)) - (Ax(t), x(t)) - (f(x(t)), h(t))]dt,$$

for all $h \in H$. Moreover, $\varphi' : H \to H^*$ is a compact mapping defined as follows:

$$\langle \varphi'(x), h\rangle = \int_0^{2\pi} (f(x(t), h(t))dt, \quad \forall h \in H.$$

Set

$$E = \{x \in H : x(t - \pi) = -x(t)\}.$$

Then E is a closed subspace of H. If $x \in E$, it has Fourier expansion

$$x(t) = \frac{1}{\sqrt{\pi}} \sum_{j=1}^{\infty} [a_j \cos(2j-1)t + b_j \sin(2j-1)t].$$

It is easily to verify the following lemma.

Lemma 2.2. *Assume that* (A1), (F1)–(F4) *hold. Then the critical points of J restricted to E are critical points of J on the whole space H.*

For the rest of this article, J is considered as a functional restricted to E. For simplicity, we denote J the restriction of J to E.

Define an operator on E by extending the bilinear forms

$$\langle Lx, y\rangle_1 = \int_0^{2\pi} [(\dot{x}(t-\frac{\pi}{2}), y(t)) - (Ax(t), y(t))]dt.$$

It is easy to verify that L is a linear, bounded and self-adjoint operator. Suppose that there exists a function $x \in E \setminus \{0\}$ such that $Lx = \nu x$, where $\nu \in \mathbb{R}$. Then for any $y \in E$, we have $\langle Lx, y\rangle_1 = \nu\langle x, y\rangle_1$. By a direct computation,

$$\langle Lx, y\rangle_1 = \sum_{j=1}^{\infty}(-1)^{j+1}(2j-1)[(a_j, c_j) + (b_j, d_j)] - \sum_{j=1}^{\infty}[(Aa_j, c_j) + (Ab_j, d_j)],$$

$$\nu\langle x, y\rangle_1 = \nu\sum_{j=1}^{\infty}(2j-1)[(a_j, c_j) + (b_j, d_j)].$$

For any $j \in \mathbb{N}$, take $y(t) = 1/\sqrt{\pi}\cos[(2j-1)t]e_i$ and $y(t) = 1/\sqrt{\pi}\sin[(2j-1)t]e_i$, where $\{e_i : i = 1, 2, \dots, N\}$ denotes the canonical basis of $\mathbb{R}^N$. Then the theory of Fourier series implies that

$$(-1)^{j+1}(2j-1)a_j - Aa_j = \nu(2j-1)a_j$$

and

$$(-1)^{j+1}(2j-1)b_j - Ab_j = \nu(2j-1)b_j.$$

Thus, for some $j \in \mathbb{N}$, $\nu(2j-1)$ is an eigenvalue of $(-1)^{j+1}(2j-1)I - A$. By the definition of the space E, one can check that ν is an eigenvalue of L if and only if $\nu(2j-1)$ is an eigenvalue of $(-1)^{j+1}(2j-1)I - A$ for some $j \in \mathbb{N}$.

Since A is a symmetric matrix, all eigenvalues are real numbers. It follows that eigenvalues of matrix $(-1)^{j+1}(2j-1)I - A$ and then operator L are real numbers. Since $\{(-1)^{j+1}(2j-1) : j \in \mathbb{N}\} \cap \sigma(A) = \emptyset$, then $0 \notin \sigma(L)$. Denote the eigenvalues of L on E by

$$\dots \le \lambda_{-2} \le \lambda_{-1} < 0 < \lambda_1 \le \lambda_2 \le \dots.$$

Let $\{\overline{e}_{\pm j}\}_{j\in\mathbb{N}}$ be the eigenvectors of L corresponding to $\{\lambda_{\pm j}\}_{j\in\mathbb{N}}$, respectively. Define

$$E^+ = \overline{\operatorname{span}\{\overline{e}_j : j \in \mathbb{N}\}}, \quad E^- = \overline{\operatorname{span}\{\overline{e}_{-j} : j \in \mathbb{N}\}}.$$

Hence there exists an orthogonal decomposition $E = E^+ \oplus E^-$. Clearly, $x \in E$ can be written as $x = x^+ + x^-$, where $x^+ \in E^+$ and $x^- \in E^-$. Define an equivalent inner product in E, denoted by $\langle\cdot,\cdot\rangle$ and defined by

$$\langle x, y\rangle = \langle Lx^+, y^+\rangle_1 - \langle Lx^-, y^-\rangle_1.$$

Hence, we have

$$\int_0^{2\pi} [(\dot{x}(t-\frac{\pi}{2}), x(t)) - (Ax(t), x(t))]dt = \langle Lx, x\rangle_1 = \|x^+\|^2 - \|x^-\|^2,$$

where $\|\cdot\|$ denotes the norm induced by $\langle\cdot,\cdot\rangle$.

Now, J can be rewritten as

$$J(x) = \frac{1}{2}\|x^+\|^2 - \frac{1}{2}\|x^-\|^2 - \varphi(x), \quad \forall x \in E.$$

At the end of this section, we state a useful lemma.

Lemma 2.3 ([24])**.** *If X is infinite-dimensional and S is the unit sphere of X, $\psi \in C^1(S,\mathbb{R})$ is even, bounded below and satisfies (PS) condition, then ψ has infinitely many pairs of critical points.*

3. Main results and their proofs

Define the generalized Nehari Manifold

$$\mathcal{M} = \{x \in E \setminus E^- : \langle J'(x), x\rangle = 0 \text{ and } \langle J'(x), y\rangle = 0 \text{ for all } y \in E^-\}.$$

Proposition 3.1. *Nontrivial critical points of J belong to $E \setminus E^-$.*

Proof. Assume that $x_0 \in E^-$ is a nontrivial critical point of J. Then $J'(x_0) = 0$. It follows that

$$\langle J'(x_0), x_0\rangle = -\|x_0\|^2 - \int_0^{2\pi} (f(x_0(t)), x_0(t))dt = 0,$$

which implies that

$$\int_0^{2\pi} (f(x_0(t)), x_0(t))dt = -\|x_0\|^2 \le 0. \tag{3.1}$$

On the other hand, $\int_0^{2\pi}(f(x_0(t)), x_0(t))dt > 0$ since $(f(x), x) > 0$, which contradicts with (3.1). □

We denote

$$S = \{x \in E : \|x\| = 1\}, \quad S^+ = S \cap E^+, \tag{3.2}$$

and for any $x \in E$, denote

$$E(x) = \mathbb{R}x \oplus E^- \equiv \mathbb{R}x^+ \oplus E^-, \quad \widehat{E}(x) = \mathbb{R}^* x \oplus E^- \equiv \mathbb{R}^* x^+ \oplus E^-.$$

Proposition 3.2. *For any $x \in E$, there exists R_1 large enough such that $J(y) \le 0$ for all $y \in \widehat{E}(x) \setminus B_{R_1}(0)$, where $B_r(0)$ denotes the circle in E with center 0 and radius r.*

Proof. Suppose, to the opposite, that there exists a sequence $\{x_n\} \subset \widehat{E}(x)$ such that $J(x_n) > 0$ and $\|x_n\| \to \infty$ as $n \to \infty$. Let

$$y_n = \frac{x_n}{\|x_n\|}, \quad z = \frac{x^+}{\|x^+\|}.$$

Then $\|y_n\| = 1$. Obviously, there exists a sequence $\{s_n\}$ such that $y_n^+ = s_n z$. Passing to a subsequence if necessary, $\{y_n\}$ converges weakly to some point, denoted by y_0. Suppose that $y_0 \ne 0$. Since $\{y_n\}$ converges weakly to y_0, then $\{y_n\}$ converges strongly to y_0 in $L^\mu(S^1, \mathbb{R}^N)$. It follows that $\{y_n\}$ is convergence in measure to y_0 (cf. [25, Theorem 4.2.2]). Thus there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\{y_{n_k}\}$ converges almost everywhere to $\{y_0\}$ (cf. [25, Theorem 3.2.3]). For any $\delta > 0$, there exists a measurable subset M_δ of $[0,1]$ such that $\text{meas}\{[0,1] - M_\delta\} < \delta$, where $\text{meas}\{M\}$ denotes the length of set M, and $\{y_{n_k}\}$ converges uniformly to y_0 (cf. [25, Theorem 3.2.8]). Choosing $0 < \delta_0 < 1/2$, then $\text{meas}\{M_{\delta_0}\} \ge 1/2$. It follows from (F4) and Remark 1.1 that

$$\begin{aligned}\int_0^{2\pi} \frac{F(\|x_{n_k}\| \cdot y_{n_k}(t))}{\|x_{n_k}\|^2} dt &\ge \int_{M_{\delta_0}} \frac{F(\|x_{n_k}\| \cdot y_{n_k}(t))}{\|x_{n_k}\|^2} dt \\ &\ge \int_{M_{\delta_0}} \frac{m\|x_{n_k}\|^\mu \cdot |y_{n_k}(t)|^\mu + M}{\|x_{n_k}\|^2} dt \to +\infty,\end{aligned}$$

as $k \to \infty$. Consequently,

$$0 \le \frac{J(x_{n_k})}{\|x_{n_k}\|^2} = \frac{1}{2}\|y^+_{n_k}\|^2 - \frac{1}{2}\|y^-_{n_k}\|^2 - \int_0^{2\pi} \frac{F(\|x_{n_k}\| \cdot y_{n_k}(t))}{\|x_{n_k}\|^2} dt \to -\infty, \tag{3.3}$$

which is a contradiction. Thus $y_0 = 0$ and $y_n \rightharpoonup 0$. Since $F(x) \ge 0$, we obtain from (3.3) that $\|y_n^+\| > \|y_n^-\|$. If $y_n^+ \to 0$, then also $y_n^- \to 0$. Hence, $y_n = y_n^+ + y_n^- \to 0$, which contradicts with the fact that $\|y_n\| = 1$. Thus $y_n^+ \to 0$ and $\|y_n^+\| \ge \lambda$ for some $\lambda > 0$, possibly after passing to a subsequence. Thus $\|y_n^+\| = \|s_n z\| = s_n$ is bounded and bounded away from 0. Passing to a subsequence if necessary, $y_n^+ \to sz$ for some $s > 0$, which contradicts with the fact that $y_n \rightharpoonup 0$. Thus, there exists R_1 large enough such that $J(x) \le 0$ for all $\widehat{E}(x) \setminus B_{R_1}(0)$. □

Lemma 3.3. *For every $c > 0$, J satisfies $(PS)_c$ condition on $\widehat{E}(x)$.*

Proof. Let $\{x_n\} \subset \widehat{E}(x)$ be a $(PS)_c$ sequence. Because of Proposition 3.2, $\{x_n\}$ is bounded. Since $\varphi' : H \to H^*$ is compact and

$$J'(x_n) = x_n^+ - x_n^- - \varphi'(x_n) \to 0,$$

we conclude that $\{x_n^+ - x_n^-\}$ has a convergent subsequence. Thus $\{x_n\}$ has a convergent subsequence. □

Lemma 3.4. *$\widehat{E}(x) \cap \mathcal{M} \neq \emptyset$ for any $x \in E \setminus E^-$.*

Proof. Let $x \in E \setminus E^-$. Since $\widehat{E}(x) = \widehat{E}(x^+) = \widehat{E}(x^+/\|x^+\|)$, we assume that $x \in S^+$. Let $y \in \widehat{E}(x) \cap E^+$. Since

$$J(y) = \frac{1}{2}\|y\|^2 - \int_0^{2\pi} F(y(t))dt,$$

it follows from Remark 1.1 that there exists $r_0 > 0$ small enough such that $J(y) \ge \|y\|^2/4$ for $\|y\| < r_0$. On the other hand, Proposition 3.2 implies that

$$0 < \frac{r_0^2}{16} \le \sup_{y \in \widehat{E}(x)} J(y) = \sup_{y \in \widehat{E}(x), \|y\| \le R_1} J(y) < \infty.$$

Since J satisfies the $(PS)_c$ condition for every $c > 0$, the supremum $\sup_{y \in \widehat{E}(x)} J(y)$ is attained at some point $\overline{x}$. Since $J(0) = 0$, then $\overline{x} \neq 0$ and $\overline{x} \in \mathcal{M}$. □

Now, we give two lemmas. The idea comes from [3, 24].

Lemma 3.5. *If $x \in \mathcal{M}$, then $J(x + y) < J(x)$ whenever $x + y \in \widehat{E}(x)$, $y \neq 0$. Hence, x is the unique global maximum of $J|_{\widehat{E}(x)}$.*

Proof. Let $x + y = (1 + s)x + z$, where $s \ge -1$, $z \in E^-$. Then

$$\begin{aligned}
&J(x) - J(x + y) \\
&= \frac{1}{2}\langle x, x\rangle - \int_0^{2\pi} F(x(t))dt - \frac{1}{2}\langle (x + y), (x + y)\rangle + \int_0^{2\pi} F(x(t) + y(t))dt \\
&= -\frac{s^2 + 2s}{2}\langle x, x\rangle - (1 + s)\langle x, z\rangle - \frac{1}{2}\langle z, z\rangle - \int_0^{2\pi} F(x(t))dt \\
&\quad + \int_0^{2\pi} F(x(t) + y(t))dt
\end{aligned}$$

$$= -\langle x, s(\frac{s}{2}+1)x + (1+s)z\rangle + \frac{1}{2}\|z\|^2 - \int_0^{2\pi} F(x(t))dt + \int_0^{2\pi} F(x(t)+y(t))dt$$

Since $x \in \mathcal{M}$, $\langle J'(x), x\rangle = \langle J'(x), z\rangle = 0$. It follows that $\langle J'(x), s(s/2+1)x + (1+s)z)\rangle = 0$; i.e.,

$$\langle x, s(\frac{s}{2}+1)x + (1+s)z\rangle - \int_0^{2\pi} (f(x(t)), s(\frac{s}{2}+1)x(t) + (1+s)z(t))dt = 0.$$

Thus

$$\begin{aligned} J(x) - J(x+y) = \frac{1}{2}\|z\|^2 - \int_0^{2\pi} (f(x(t)), s(\frac{s}{2}+1)x(t) + (1+s)z(t))dt \\ - \int_0^{2\pi} F(x(t))dt + \int_0^{2\pi} F(x(t)+y(t))dt \end{aligned}$$

The result follows if

$$F(x(t)+y(t)) - F(x(t)) - (f(x(t)), s(s/2+1)x(t) + (1+s)z(t)) \geq 0.$$

This will be proved in the following lemma. □

Lemma 3.6. *Assume that* (F4)–(F7) *hold. Let* $x, z \in \mathbb{R}^N$ *and* $s \in \mathbb{R}$ *such that* $s \geq -1$ *and let* $y = z + sx$. *Then*

$$F(x+y) - F(x) - (f(x), s(\frac{s}{2}+1)x + (1+s)z) \geq 0.$$

Proof. Set $u = (1+s)x + z$. Then $u = x + y$. Denote

$$g(s) := F((1+s)x + z)) - F(x) - (f(x), s(\frac{s}{2}+1)x + (1+s)z).$$

We only need check that $g(s) \geq 0$.

Case I: $(x, u) \leq 0$. It follows from (F4) and (F5) that

$$\begin{aligned} g(s) &= F(u) - F(x) - (f(x), s(\frac{s}{2}+1)x + (1+s)(u - (1+s)x)) \\ &> F(u) - \frac{1}{2}(f(x), x) + (\frac{s^2}{2} + s + 1)(f(x), x) - (1+s)(f(x), u) \\ &= F(u) + \frac{1}{2}(s+1)^2(f(x), x) - (1+s)(f(x), u) \geq 0. \end{aligned}$$

Case II: $(x, u) > 0$. Obviously, $g(-1) > F(z) > 0$. It follows from Remark 1.1 that $F(sx) \geq s^{\mu}F(x)$ when $s \geq 1$. Thus $g(s) \to +\infty$ as $s \to +\infty$. By a directly computation,

$$g'(s) = (f(u), x) - (f(x), u).$$

If there exists s_1 such that $g(s_1) < 0$, then there exists s_2 such that $g'(s_2) = (f(u), x) - (f(x), u) = 0$. It follows from (F7) that $|x| = |u|$. Hence (F6) implies that

$$\begin{aligned} g(s) &:= F((1+s)x + z)) - F(x) - (f(x), s(\frac{s}{2}+1)x + (1+s)z) \\ &= F(u) - F(x) - (f(x), (1+s)u - (\frac{s^2}{2} + s + 1)x) \\ &\geq -(1+s)(f(x), u) + (\frac{s^2}{2} + s + 1)(f(x), x) \end{aligned}$$

$$= \frac{s^2}{2}(f(x), x) \geq 0.$$

This completes the proof. □

From Lemmas 3.4 and 3.5, we know that for each $x \in E \setminus E^-$ there exists a unique nontrivial critical point $\widehat{m}(x)$ of $J|_{\widehat{E}(x)}$. Moreover, $\widehat{m}(x)$ is the unique global maximum of $J|_{\widehat{E}(x)}$. Let

$$\widehat{m} : E \setminus E^- \to \mathcal{M}, \quad m := \widehat{m}|_{S^+} : S^+ \to \mathcal{M}.$$
$$\widehat{\Psi} : E^+ \setminus \{0\} \to \mathbb{R}, \quad \widehat{\Psi}(x) := J(\widehat{m}(x)), \quad \Psi := \widehat{\Psi}|_{S^+}.$$

Lemma 3.7. *Assume that* (A1), (F1)–(F7) *hold. Then Conditions* (B1)–(B3) *in* [24] *hold.*

Proof. It follows from (F4) and Lemmas 2.1, 3.4, 3.5 that (B1) and (B2) hold.

Now, we verify that (B3) holds. For any $x \in E \setminus E^-$, Proposition 3.2 implies that

$$\sup_{y \in \widehat{E}(x)} J(y) \geq r_0^2/16 > 0.$$

Since $J(\widehat{m}(x)^+) \geq J(\widehat{m}(x))$, then there exists $\delta > 0$ such that $\widehat{m}(x)^+ \geq \delta$.

Let $W \in E \setminus \{0\}$ be a weakly compact set and let $\{x_n\} \subset W$. Then, after passing to a subsequence if necessary, $x_n \rightharpoonup x \neq 0$. Thus, $x_n(t) \to x(t)$ a.e. for $t \in [0, 2\pi]$. If $s_n \to \infty$ as $n \to \infty$, then $|s_n x_n(t)| \to \infty$ as $n \to \infty$. It follows that

$$\frac{\varphi(s_n x_n)}{s_n^2} = \int_0^{2\pi} \frac{F(s_n x_n(t))}{s_n^2 |x_n(t)|^2} |x_n(t)|^2 dt \to \infty, \quad \text{as } n \to \infty.$$

Hence $\varphi(sx)/s^2 \to \infty$ uniformly for x on weakly compact subsets of $E \setminus \{0\}$ as $s \to \infty$. Let $W' \subset S^+$ be a compact set and let $\{y_n\} \subset W'$. Since $J(sy)/s^2 = 1/2 - \varphi(sy_n)/s^2$, then $\{J(sy_n)\}$ must be bounded from above. Thus $\{J(\widehat{m}(y_n))\}$ is bounded from above. Hence $\{\widehat{m}(y_n)\}$ is bounded from above and (B3) holds. □

Lemma 3.8 ([24])**.** *The mapping $\widehat{m}$ is continuous and m is a homeomorphism between S^+ and $\mathcal{M}$.*

Lemma 3.9 ([24])**.** *(1) $\widehat{\Psi}, \Psi \in C^1(S^+, \mathbb{R})$ and*

$$\langle \widehat{\Psi}'(x), y \rangle = \frac{\|\widehat{m}(x)^+\|}{\|x\|} \langle \Psi'(\widehat{m}(x)), y \rangle \quad \text{for all } x, y \in E^+, x \neq 0,$$
$$\langle \Psi'(x), y \rangle = \|m(x)^+\| \langle \Psi'(m(x)), y \rangle \quad \text{for all } y \in T_x(S^+),$$

where $T_x(S^+)$ is the tangent space of S^+ at x.
(2) If $\{x_n\}$ is a (PS) sequence for Ψ, then $\{m(x_n)\}$ is a (PS) sequence for J. If $\{x_n\} \subset \mathcal{M}$ is a bounded (PS) sequence for J, then $\{m^{-1}(x_n)\}$ is a (PS) sequence for Ψ.
(3) x is a critical point of Ψ if and only if $m(x)$ is a nontrivial critical point of J. Moreover, the corresponding values of Ψ and J coincide and $\inf_{S^+} \Psi = \inf_{\mathcal{M}} J$.
(4) If J is even, then so is Ψ.

Lemma 3.10. *J satisfies the (PS) condition on $\mathcal{M}$.*

Proof. If $x \in \mathcal{M}$, by Lemmas 3.4 and 3.5,

$$J(x) = \sup_{y \in \widehat{E}(x)} J(y) \geq \frac{r_0^2}{16} > 0.$$

Suppose that $\{x_n\} \subset \mathcal{M}$ is a (PS) sequence. Suppose that $\{x_n\}$ is unbounded. Then, passing to a subsequence, $\|x_n\| \to \infty$ as $n \to \infty$. Set

$$y_n = \frac{x_n}{\|x_n\|}, \quad z_n = \frac{x_n^+}{\|x_n^+\|}.$$

Similarly as in the proof of Proposition 3.4, we can prove that $y_n \rightharpoonup 0$ and $\|y_n^+\| \geq \lambda$ for some $\lambda > 0$, possibly after passing to a subsequence. By the assumption of $\{x_n\}$, there exists $d > 0$ such that

$$d \geq J(x_n) \geq J(sy_n^+) \geq \frac{1}{2}s^2\lambda^2 - \varphi(sy_n^+) \to \frac{1}{2}s^2\lambda^2, \tag{3.4}$$

for all $s > 0$, which is a contradiction. So $\{x_n\}$ is bounded. Similarly as in the proof of Lemma 3.3, $\{x_n\}$ has a convergent subsequence. □

Proof of Theorem 1.2. Set

$$\bar{c} = \inf_{x \in \mathcal{M}} J(x) = \inf_{x \in E \setminus F} \max_{y \in \widehat{E}(x)} J(y) = \inf_{x \in S^+} \max_{y \in \widehat{E}(x)} J(y). \tag{3.5}$$

Obviously, $\bar{c} \geq r_0^2/16 > 0$.

Let $\{y_n\} \subset S^+$ be a (PS) sequence for Ψ. Set $x_n = m(y_n)$ for $n \in \mathbb{N}$. Then $\{x_n\}$ is a (PS) sequence for J. According to Lemma 3.10, passing to a subsequence if necessary, $x_n \to x_0$ and $y_n \to m^{-1}(x_0)$. Thus Ψ satisfies (PS) condition.

Let $\{y_n\}$ be a minimizing sequence for Ψ. By Ekeland's variational principle, we may assume that $\Psi'(y_n) \to 0$. By the (PS) condition, $y_n \to y_0$. Hence $x_0 = m(y_0)$ is a critical point for J. Since $J(0) = 0$, x_0 is a nonconstant solution of (1.1).

Since $F(-x) = F(x)$, then both J and Ψ are even. Since $\inf_S \Psi = \inf_{\mathcal{M}} J = \bar{c} > 0$, Ψ is bounded from below. Since Ψ satisfies (PS) condition, it follows from Lemma 2.3 that Ψ and then J has infinitely many pairs of solutions. □

Lemma 3.11. *The minimal period of x_0 is 2π.*

Proof. Suppose, to the opposite, that x_0 has minimal period $2\pi/m$, where $m > 1$ is an integer.

Claim: m is odd. If there exists $k \in \mathbb{N}$ such that $m = 2k$, then

$$-x_0(t) = x_0(t - \pi) = x_0(t - \frac{2\pi}{m} \cdot k) = x_0(t),$$

which implies $x_0(t) \equiv 0$. This contradicts with the fact that x_0 is a nonconstant solution. Thus m must be odd.

Denote $y_0(t) = x_0(t/m)$. Obviously, $y_0(t)$ has minimal period 2π.

Case I: $m = 4k + 1$ for some $k \in \mathbb{N}$. Then

$$x_0(t - \frac{\pi}{2m}) = x_0(t - \frac{\pi}{2m} - \frac{2\pi}{m}k) = x_0(t - \frac{4k+1}{2m}\pi) = x_0(t - \frac{\pi}{2}).$$

It follows that

$$x_0(t - \frac{\pi}{m}) = x_0(t - \frac{\pi}{2m} - \frac{\pi}{2m}) = x_0(t - \pi) = -x_0(t), \tag{3.6}$$

Thus

$$y_0(t-\pi)=x_0(\frac{t-\pi}{m})=x_0(\frac{t}{m}-\frac{\pi}{m})=-x_0(\frac{t}{m})=-y_0(t), \tag{3.7}$$

which implies that $y_0\in E$. Since $x_0\in\mathcal{M}$, $y_0\in E\setminus E^-$. Denote $\overline{y}_0:=\widehat{m}(y_0)=sy_0^++z$, where $z\in E^-$ and $s>0$. Then $J(\overline{y}_0)\geq J(x_0)=\inf_{x\in\mathcal{M}}J(x)$. Setting $\overline{z}(t/m)=z(t)$, we have $\overline{y}_0(t)=sx_0^+(t/m)+\overline{z}(t/m)$. Also

$$\overline{z}(\frac{t}{m}-\frac{\pi}{2})=z(t-\frac{m\pi}{2})=z(t-2k\pi-\frac{\pi}{2})=z(t-\frac{\pi}{2})=\overline{z}(\frac{t}{m}-\frac{\pi}{2m}).$$

Thus $\overline{z}(t/m-\pi/(2m))=\overline{z}(t/m-\pi/2)$.

Let $y(t)=sx_0^+(t)+\overline{z}(t)$. Then $y\in\widehat{E}(x_0)$. Computing directly,

$$\begin{aligned}
&J(\overline{y}_0)\\
&=\frac{1}{2}\int_0^{2\pi}(\dot{\overline{y}}_0(t-\frac{\pi}{2}),\overline{y}_0(t))dt-\frac{1}{2}\int_0^{2\pi}(A\overline{y}_0(t),\overline{y}_0(t))dt-\int_0^{2\pi}F(\overline{y}_0(t))dt\\
&=\frac{1}{2m}\int_0^{2\pi}(s\dot{x}_0^+(\frac{t}{m}-\frac{\pi}{2m})+\dot{\overline{z}}(\frac{t}{m}-\frac{\pi}{2m}),sx_0^+(\frac{t}{m})+\overline{z}(\frac{t}{m}))dt\\
&\quad-\frac{1}{2}\int_0^{2\pi}(sAx_0^+(\frac{t}{m})+A\overline{z}(\frac{t}{m}),sx_0^+(\frac{t}{m})+\overline{z}(\frac{t}{m}))dt-\int_0^{2\pi}F(sx_0^+(\frac{t}{m})+\overline{z}(\frac{t}{m}))dt\\
&=\frac{1}{2m}\int_0^{2\pi}(s\dot{x}_0^+(\frac{t}{m}-\frac{\pi}{2})+\dot{\overline{z}}(\frac{t}{m}-\frac{\pi}{2}),sx_0^+(\frac{t}{m})+\overline{z}(\frac{t}{m}))dt\\
&\quad-\frac{1}{2}\int_0^{2\pi}(sAx_0^+(\frac{t}{m})+A\overline{z}(\frac{t}{m}),sx_0^+(\frac{t}{m})+\overline{z}(\frac{t}{m}))dt-\int_0^{2\pi}F(sx_0^+(\frac{t}{m})+\overline{z}(\frac{t}{m}))dt\\
&=\frac{1}{2m}\int_0^{2\pi}(s\dot{x}_0^+(\tau-\frac{\pi}{2})+\dot{\overline{z}}(\tau-\frac{\pi}{2}),sx_0^+(\tau)+\overline{z}(\tau))dt\\
&\quad-\frac{1}{2}\int_0^{2\pi}(sAx_0^+(\tau)+A\overline{z}(\tau),sx_0^+(\tau)+\overline{z}(\tau))dt-\int_0^{2\pi}F(sx_0^+(\tau)+\overline{z}(\tau))dt\\
&=\frac{1}{2m}\int_0^{2\pi}(\dot{y}(\tau-\frac{\pi}{2}),y(\tau))d\tau-\frac{1}{2}\int_0^{2\pi}(Ay(\tau),y(\tau))dt-\int_0^{2\pi}F(y(\tau))d\tau.
\end{aligned}$$

Since $J(\overline{y}_0)>0$ and A is a nonnegative definite matrix, then

$$\frac{1}{2m}\int_0^{2\pi}(\dot{y}(\tau-\frac{\pi}{2}),y(\tau))d\tau>\frac{1}{2}\int_0^{2\pi}(Ay(\tau),y(\tau))dt+\int_0^{2\pi}F(y(\tau))d\tau>0.$$

It follows that

$$\begin{aligned}
J(\overline{y}_0)&=\frac{1}{2m}\int_0^{2\pi}(\dot{y}(\tau-\frac{\pi}{2}),y(\tau))d\tau-\frac{1}{2}\int_0^{2\pi}(Ay(\tau),y(\tau))dt-\int_0^{2\pi}F(y(\tau))d\tau\\
&<\frac{1}{2}\int_0^{2\pi}(\dot{y}(\tau-\frac{\pi}{2}),y(\tau)d\tau-\frac{1}{2}\int_0^{2\pi}(Ay(\tau),y(\tau))dt-\int_0^{2\pi}F(y(\tau))d\tau\\
&=J(y)\leq J(x_0)=\inf_{x\in\mathcal{M}}J(x),
\end{aligned}$$

which contradicts with the fact that $J(\overline{y}_0)\geq\inf_{x\in\mathcal{M}}J(x)$.

Case II: $m=4k-1$ for some $k\in\mathbb{N}$. Then

$$x_0(t-\frac{\pi}{2m})=x_0(t-\frac{\pi}{2m}+\frac{2\pi}{m}k)=x_0(t+\frac{4k-1}{2m}\pi)=x_0(t+\frac{\pi}{2})=-x_0(t-\frac{\pi}{2}).$$

It follows that

$$x_0(t-\frac{\pi}{m}) = x_0(t-\frac{\pi}{2m}-\frac{\pi}{2m}) = x_0(t-\pi) = -x_0(t),$$

Thus

$$y_0(t-\pi) = x_0(\frac{t-\pi}{m}) = x_0(\frac{t}{m}-\frac{\pi}{m}) = -x_0(\frac{t}{m}) = -y_0(t),$$

which implies that $y_0 \in E$. Since

$$\begin{aligned}
0 &\leq \|y_0^+\|^2 \\
&= \int_0^{2\pi} (\dot{y}_0^+(t-\frac{\pi}{2}), y_0^+(t))dt - \int_0^{2\pi} (Ay_0^+(t), y_0^+(t))dt \\
&= \frac{1}{m}\int_0^{2\pi} (\dot{x}_0^+(\frac{t}{m}-\frac{\pi}{2m}), x_0^+(\frac{t}{m}))dt - \int_0^{2\pi} (Ax_0^+(\frac{t}{m}), x_0^+(\frac{t}{m}))dt \\
&= -\frac{1}{m}\int_0^{2\pi} (\dot{x}_0^+(\frac{t}{m}-\frac{\pi}{2}), x_0^+(\frac{t}{m}))dt - \int_0^{2\pi} (Ax_0^+(\frac{t}{m}), x_0^+(\frac{t}{m}))dt \\
&= -\frac{1}{m}\int_0^{2\pi} (\dot{x}_0^+(\tau-\frac{\pi}{2}), x_0^+(\tau))dt - \int_0^{2\pi} (Ax_0^+(\tau), x_0^+(\tau))dt \\
&= -\frac{1}{m}\|x_0^+\|^2 - (1-\frac{1}{m})\int_0^{2\pi} (Ax_0^+(\tau), x_0^+(\tau))dt \leq 0,
\end{aligned}$$

it follows that $x_0^+ = 0$. Then $x_0 = x_0^+ + x_0^- = x_0^- \in E^-$, which contradicts the fact that all nontrivial critical points belong to $E \setminus E^-$. Consequently, x_0 has minimal period 2π. □

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References

[1] A. Ambrosetti, G. Mancini; *Solutions of minimal period for a class of convex Hamiltonian systems*, Math. Ann., 255(1981), 405-421.

[2] K. Brown, Y. P. Zhang; *The Nehari manifold for a semilinear elliptic equation with a sign-change weight function*, J. Differential Equations, 193(2003), 481-499.

[3] G. W. Chen, S. W. Ma; *Periodic solutions for Hamiltonian systems without Ambrosetti-Rabinowitz condition and spectrum* 0, J. Math. Anal. Appl., 379(2011), 842-851.

[4] S. N. Chow, J. Mallet-Paret; *The Fuller index and global Hopf bifurcation*, J. Differential Equations, 29(1978), 66-85.

[5] G. H. Fei; *Multiple periodic solutions of differential delay equations via hamiltonian systems (I)*, Nonlinear Anal., 65(2006), 25-39.

[6] G. H. Fei; *Multiple periodic solutions of differential delay equations via hamiltonian systems (II)*, Nonlinear Anal., 65(2006), 40-58.

[7] Z. M. Guo, J. S. Yu; *Multiplicity results for periodic solutions to delay differential equations via critical point theory*, J. Differential Equations, 218(2005), pp. 15-35.

[8] Z. M. Guo, J. S. Yu; *Multiplicity results on period solutions to higher dimensional differential equations with multiple delays*, J. Dyn. Diff. Equat., 23(2011), 1029-1052.

[9] G. Jones; *The existence of periodic solutions of $f'(x) = -af(x(t-1))[1+f(x)]$*, J. Math. Anal. Appl., 5(1962), pp. 435-450.

[10] J. Kaplan, J. Yorke; *Ordinary differential equations which yield periodic solution of delay equations*, J. Math. Anal. Appl., 48(1974), 317-324.

[11] W. Krawcewicz, J. S. Yu, H. F. Xiao; *Multiplicity of periodic solutions to symmetric delay differential equations*, Journal of Fixed Point theory and its Applications, 13(2013), 103-141.

[12] J. B. Li, X. Z. He; *Multiple periodic solutions of differential delay equations created by asymptotically linear hamiltonian systems*, Nonlinear Anal., 31(1998), 45-54.
[13] J. B. Li, X. Z. He; *Proof and generalization of kaplan-yorke's conjecture on periodic solution of differential delay equations*, Sci. China (Ser.A), 42(1999), 957-964.
[14] J. B. Li, X. Z. He, Z. Liu; *Hamiltonian symmetric groups and multiple periodic solutions of differential delay equations*, Nonlinear Anal., 35(1999), 457-474.
[15] J. B. Li, Z. R. Liu, X. Z. He; *Periodic solutions of some differential delay equations created by hamiltonian systems*, Bull. Austral. Math. Soc., 60(1999), 377-390.
[16] P. Magrone; *Critical Point Methods for Indefinite Nonlinear Elliptic Equations and Hamiltonian Systems*, Universitá Degli Studi di Roma, PhD Thesis, (2001).
[17] J. Mallet-Paret, G. Sell; *The Poincare-Bendixson theorem for monotone cyclic feedback systems with delay*, J. Differential Equations, 125(1996), 441-489.
[18] J. Mawhin; *Equivalence theorems for nonlinear operator equations and coincidence degree theory for some mappings in locally convex topological vector space*, J. Differential Equations, 12(1972), 610-636.
[19] J. Mawhin, M. Willem; *Critical Point Theory and Hamiltonian systems*, New York: Spinger, (1989).
[20] Z. Nehari; *On a class of nonlinear second-order differential equations*, Trans. Amer. Math. Soc., 95(1960), 101-123.
[21] Z. Nehari; *Characteristic values associated with a class of non-linear second-order differential equations*, Acta Math., 105(1961), 141-175.
[22] R. Nussbaum; *Periodic solutions of special differential delay equations: an example in nonlinear functional analysi*s, Proc. Royal Soc. Edinburgh, 81(1978), 131-151.
[23] A. Pankov; *On decay of solutions to nonlinear schrodinger equations*, Proc. Amer. Math. Soc., 136(2008), 2565-2570.
[24] A. Szukin, T. Weth; *The Method of Nehari Manifold, In Handbook of Nonconver Analysis and Applications*, International Prees of Boston, (2010).
[25] D. X. Xia, Z. R. Wu, S. Z. Yan, W. C. Shu; *Theory of real variable functions and functional analysis*, Beijing: Higher Education Press, (1986).
[26] J. S. Yu; *A note on periodic solutions of the delay differential equation* $x'(t) = -f(x(t-1))$, Proceeding of the American Mathematical Society, 141 (2012), 1281-1288.
[27] J. S. Yu, H. F. Xiao; *Multiple periodic solutions with minimal period* 4 *of the delay differential equation* $\dot{x} = -f(t, x(t-1))$, J. Differential Equations 254 (2013), 2158-2172.

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