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SOLUTION TO NONLOCAL PROBLEMS OF PSEUDOHYPERBOLIC EQUATIONS

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ABSTRACT. In this article considers a nonlocal problem with integral condition for a fourth-order pseudohyperbolic equation. Existence and uniqueness of a generalized solution are proved.

1. Introduction

Currently, there is considerable interest in nonlocal problems for evolution equations. One reason for this lies in the fact that various phenomena of modern natural science can be described most conveniently in terms of nonlocal problems. Problems with nonlocal integral conditions form an important class of nonlocal problems. Recently, nonlocal boundary value problems with integral conditions have been actively studied. However, the majority of the works deals with second-order equations. The initial works devoted to nonlocal problems for second-order partial differential equations with integral conditions go back to Cannon [4] and Kamynin [9]. Note here some recent works: [1, 2, 3, 8, 12, 13, 16, 17, 18, 21]. See also references therein.

Pseudohyperbolic equations form important and interesting subclass of Sobolev type equations. Such equations may describe nonstationary waves in stratified and rotating liquid [14]. The starting point in studying of Sobolev type equations is [23]. Now there are a lot of works devoted to initial and boundary value problems for Sobolev type equations (see [24] and references therein). One of recent works dealing with some problems for pseudohyperbolic equations is [14]. In these work the author studies qualitative characteristics of solutions to initial-boundary value problems. On the other hand, various physical problems demand nonlocal conditions [5, 6, 7, 10, 11, 22].

Motivated by this, we consider a nonlocal problem with integral condition for a pseudohyperbolic equation.

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2. Results

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$, $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$. Consider an equation

$$Lu \equiv \frac{\partial^2}{\partial t^2}(u - \Delta u) - (a_{ij}(x,t)u_{x_i})_{x_j} + c(x,t)u = f(x,t) \tag{2.1}$$

and set a problem: Find a function $u(x,t)$ that is a solution of (2.1) in Q_T, satisfies initial data

$$u(x,0) = 0, \quad u_t(x,0) = 0 \tag{2.2}$$

and nonlocal condition

$$\Big(\frac{\partial^2}{\partial t^2}\frac{\partial u}{\partial \nu} + \frac{\partial u}{\partial N} + \int_\Omega K(x,y,t)u(y,t)dy\Big)\big|_{S_T} = 0. \tag{2.3}$$

We suppose throughout that repeated indices imply summation from 1 to n, $\nu(x) = (\nu_1, \dots, \nu_n)$ is outward unit normal vector to $\partial\Omega$ at the current point, $\frac{\partial u}{\partial N} = a_{ij}u_{x_i}\cos(\nu, x_j)$, $a_{ij} = a_{ji}$, $K(x,y,t)$ is a given function. We also use the following notation:

$$u_x = (u_{x_1}, \dots u_{x_n}), \quad u_x^2 = \sum_{i=1}^n u_{x_i}^2, \quad u_x v_x = \sum_{i=1}^n u_{x_i} v_{x_i}.$$

As the condition (2.3) does not look evident we give some explanations in an appendix at the end of the paper.

Let $W_2^1(Q_T)$ be the usual Sobolev space. We shall define

$$W(Q_T) = \{u : u \in W_2^1(Q_T),\, u_{xt} \in L_2(Q_T)\},$$
$$\hat{W}(Q_T) = \{v : v \in W(Q_T),\, v(x,T) = 0\}.$$

First we give a definition of a generalized solution to the problem using the standard method [15, p. 92]. To this end we multiply (2.1) by $v \in \hat{W}(Q_T)$ and integrate over Q_T. It follows from (2.2), (2.3) and an integration by parts that

$$\begin{aligned} &\int_0^T \int_\Omega (-u_t v_t - u_{xt}v_{xt} + a_{ij}u_{x_i}v_{x_j} + cuv)\,dx\,dt \\ &+ \int_0^T \int_{\partial\Omega} v(0,t) \int_\Omega K(x,y,t)u(x,t)\,dy\,ds\,dt \\ &= \int_0^T \int_\Omega fv\,dx\,dt. \end{aligned} \tag{2.4}$$

Definition 2.1. A function $u \in W(Q_T)$ is said to be a generalized solution to the problem (2.1)–(2.3) if $u(x,0) = 0$ and for every $v \in \hat{W}(Q_T)$ the identity (2.4) holds.

Theorem 2.2. *If the function $K(x,y,t)$ is continuous in $\bar{\Omega} \times \bar{Q}_T$,*

$$f \in L_2(Q_T), \quad c \in C(\bar{Q}_T), \quad a_{ij}, \frac{\partial a_{ij}}{\partial t} \in C(\bar{Q}_T),$$
$$\forall (x,t) \in \bar{Q}_T,\ \gamma\xi^2 \le a_{ij}(x,t)\xi_i\xi_j \le \mu\xi^2,\ \gamma > 0,$$

then there exists a unique generalized solution to the problem (2.1)–(2.3).

Proof. First we prove the uniqueness. To this end we obtain a number of inequalities and then use Gronwall's lemma. We prove the existence part in several steps. First, we construct approximations of the generalized solution by the Faedo-Galerkin method. Then we obtain a priori estimates to guarantee convergence of approximations. Finally, we show that the limit of approximations is the required solution.

Uniqueness. Suppose that u_1 and u_2 are two different solutions to (2.1)–(2.3). Then $u = u_1 - u_2$ satisfies $u(x,0) = 0$ and the identity

$$\begin{aligned} &\int_0^T \int_\Omega (-u_t v_t - u_{xt} v_{xt} + a_{ij} u_{x_i} v_{x_j} + cuv)\,dx\,dt \\ &+ \int_0^T \int_{\partial\Omega} v(x,t) \int_\Omega K(x,y,t) u(y,t)\,dy\,ds\,dt = 0 \end{aligned} \tag{2.5}$$

holds for every $v \in \hat{W}(Q_T)$. For an arbitrary $\tau \in [0,T]$, take v as

$$v(x,t) = \begin{cases} \int_\tau^t u(x,\eta) d\eta, & 0 \le t \le \tau, \\ 0, & \tau \le t \le T. \end{cases} \tag{2.6}$$

It is easy to see that $v \in \hat{W}(Q_T)$, as well $v_t = u$, $v_{xt} = u_x$ in $Q_\tau = \Omega \times (0,\tau)$.

Substitute $v(x,t)$ from (2.6) in (2.5) and express u in terms of v and its derivatives. As a result we obtain the equality

$$\begin{aligned} &\int_0^\tau \int_\Omega (-v_{tt} v_t + a_{ij} v_{x_i t} v_{x_j} - v_{ttx} v_{xt} + c v_t v)\,dx\,dt \\ &+ \int_0^\tau \int_{\partial\Omega} v(x,t) \int_\Omega K(x,y,t) v_t(y,t)\,dy\,ds\,dt = 0. \end{aligned}$$

After integrating by parts first three terms, we obtain

$$\begin{aligned} &\frac{1}{2} \int_\Omega [v_t^2(x,\tau) + a_{ij}(x,0) v_{x_i}(x,0) v_{x_j}(x,0) + v_{xt}^2(x,\tau)] dx \\ &= \int_0^\tau \int_\Omega c(x,t) v(x,t) v_t(x,t)\,dx\,dt - \frac{1}{2} \int_0^\tau \int_\Omega \frac{\partial a_{ij}}{\partial t} v_{x_i} v_{x_j}\,dx\,dt \\ &\quad + \int_0^\tau \int_{\partial\Omega} v(x,t) \int_\Omega K(x,y,t) v_t(y,t)\,dy\,ds\,dt. \end{aligned} \tag{2.7}$$

Our next aim is to derive an estimate of a right-hand side of (2.7). Taking into account hypotheses of the theorem we can see that there exists positive number c_0 such that

$$\max_{\bar{Q}_T} \{ \big|\frac{\partial a_{ij}}{\partial t}\big|,\ |c| \} \le c_0.$$

Let

$$k = \max_{\bar{Q}_T} \int_\Omega K^2(x,y,t) dy, \quad \omega = \int_{\partial\Omega} ds.$$

Applying the Cauchy inequality we obtain

$$\begin{gathered} \big| \int_0^\tau \int_\Omega c v_t v\,dx\,dt \big| \le \frac{c_0}{2} \int_0^\tau \int_\Omega (v_t^2 + v^2)\,dx\,dt; \\ \frac{1}{2} \big| \int_0^\tau \int_\Omega \frac{\partial a_{ij}}{\partial t} v_{x_i} v_{x_j}\,dx\,dt \big| \le c_0 \int_0^\tau \int_\Omega v_x^2\,dx\,dt; \end{gathered}$$

$$\Big|\int_0^\tau\int_{\partial\Omega} v\int_\Omega K v_t\,dy\,ds\,dt\Big| \le \frac{1}{2}\int_0^\tau\int_{\partial\Omega} v^2\,ds\,dt + \frac{\omega k}{2}\int_0^\tau\int_\Omega v_t^2\,dy\,dt.$$

As by hypotheses $\partial\Omega$ is smooth then (see [15, p. 77])

$$\int_{\partial\Omega} v^2 ds \le c_1\int_\Omega (v_x^2+v^2)dx$$

and we obtain the inequality

$$\begin{aligned}
&\int_\Omega [v_t^2(x,\tau)+a_{ij}(x,0)v_{x_i}(x,0)v_{x_j}(x,0)+v_{xt}^2(x,\tau)]dx\\
&\le c_2\int_0^\tau\int_\Omega (v_t^2+v_x^2+v^2)\,dx\,dt,
\end{aligned}\tag{2.8}$$

where c_2 depends only on c_0, c_1, k, and ω.

Introduce now the functions $w_i(x,t)=\int_0^t u_{x_i}(x,\eta)d\eta$. By (2.6),

$$v_{x_i}(x,t)=w_i(x,t)-w_i(x,\tau),\quad v_{x_i}(x,0)=-w_i(x,\tau).$$

Furthermore, for a.e. $x\in\Omega$,

$$\int_0^\tau v^2dt=\int_0^\tau\Big(\int_\tau^t u(x,\eta)d\eta\Big)^2dt \le \tau^2\int_0^\tau u^2dt.$$

Thus, from (2.8), it follows that

$$\begin{aligned}
&\int_\Omega [u^2(x,\tau)+a_{ij}(x,0)w_i(x,\tau)w_j(x,\tau)+u_x^2(x,\tau)]dx\\
&\le 2c_2\int_0^\tau\int_\Omega [(1+\tau^2/2)u^2+\sum_{i=1}^n w_i^2]\,dx\,dt + 2c_2\tau\int_\Omega\sum_{i=1}^n w_i^2(x,\tau)dx.
\end{aligned}$$

Note that $a_{ij}(x,0)w_i(x,\tau)w_j(x,\tau)\ge\gamma w^2$. As τ is arbitrary we choose it in such a way that an inequality $\gamma-2c_2\tau>0$ holds. Let $\gamma-2c_2\tau\ge\gamma/2$. Then for every $\tau\in[0,\frac{\gamma}{4c_2}]$,

$$\int_\Omega [u^2(x,\tau)+\sum_{i=1}^n w_i^2(x,\tau)+u_x^2]dx \le c_3\int_0^\tau\int_\Omega (u^2(x,t)+\sum_{i=1}^n w_i^2(x,t))\,dx\,dt,$$

with $c_3=c_2\max\{1+\tau^2,2\}/\min\{1,\gamma/2\}$, and in particular,

$$\int_\Omega [u^2(x,\tau)+\sum_{i=1}^n w_i^2(x,\tau)]dx \le c_3\int_0^\tau\int_\Omega (u^2(x,t)+\sum_{i=1}^n w_i^2(x,t))\,dx\,dt.$$

Now by Gronwall's lemma we conclude that, for $\tau\in[0,\frac{\gamma}{4c_2}]$,

$$\int_\Omega (u^2(x,\tau)+\sum_{i=1}^n w_i^2(x,\tau))dx\le 0.$$

It follows immediately that $u(x,\tau)=0$ for $\tau\in[0,\frac{\gamma}{4c_2}]$.

Following [15] we repeat these arguments for $\tau\in[\frac{\gamma}{4c_2},\frac{\gamma}{2c_2}]$ and then continue this procedure. It follows that $u(x,\tau)=0$ for all $\tau\in[0,T]$. It implies that there exists at most one solution to (2.1)–(2.3).

Existence. Let $w_k(x)\in C^2(\bar\Omega)$ be a basis in $W_2^1(\Omega)$. We define the approximations

$$u^m(x,t)=\sum_{k=1}^m c_k(t)w_k(x),\tag{2.9}$$

where $c_k(t)$ are solutions to the Cauchy problem

$$\begin{aligned}
&\int_\Omega (u^m_{tt} w_p + a_{ij} u^m_{x_i} w_{px_j} + u^m_{x_i tt} w_{px_j} + c u^m w_p) dx \\
&+ \int_{\partial\Omega} w_p(x) \int_\Omega K(x,y,t) u^m \, dy \, ds \\
&= \int_\Omega f w_p \, dx,
\end{aligned} \tag{2.10}$$

$$c_k(0) = 0, \quad c'_k(0) = 0. \tag{2.11}$$

We write the Cauchy problem (2.10)–(2.11) such that:

$$\begin{gathered}
\sum_{k=1}^m c''_k(t) A_{kp} + \sum_{k=1}^m c_k(t) B_{kp}(t) = f_p(t), \\
A_{kp} = (w_k, w_p)_{W_2^1(\Omega)},
\end{gathered} \tag{2.12}$$

where

$$\begin{aligned}
B_{kp}(t) = &\int_\Omega [a_{ij}(x,t) w_{kx_i} w_{px_j} + c(x,t) w_k w_p] dx \\
&+ \int_{\partial\Omega} w_p(x) \int_\Omega K(x,y,t) w_k(y) \, dy \, ds, \\
f_p(t) = &\int_\Omega f(x,t) w_p(x) dx.
\end{aligned}$$

Note that the matrix $\|(w_k, w_j)_{W_2^1(\Omega)}\|$ is Gramian matrix as the functions w_k are linearly independent, hence the system (2.12) is normal. Under the hypothesis of the theorem coefficients A_{kp}, B_{kp} are bounded and $f_j \in L_1(0,T)$. Thus the Cauchy problem has a unique solution $c_k \in W_2^2(0,T)$ for every m and all approximations (2.9) are defined.

Next, we need a priori estimates to pass to the limit as $m \to \infty$.

Multiplying (2.10) by $c'_p(t)$, summing from $p = 1$ to $p = m$ and integrating with respect to t from 0 to τ, we obtain

$$\begin{aligned}
&\int_0^\tau \int_\Omega (u^m_{tt} u^m_t + a_{ij} u^m_{x_i} u^m_{x_j t} + u^m_{x_i tt} u^m_{x_j t} + c u^m u^m_t) \, dx \, dt \\
&+ \int_0^\tau \int_\Omega \frac{\partial a_{ij}}{\partial x_i} u^m_{x_i} u^m_t \, dx \, dt + \int_0^\tau \int_{\partial\Omega} u^m_t \int_\Omega K(x,y,t) u^m \, dy \, ds \, dt \\
&= \int_0^\tau \int_\Omega f(x,t) u^m_t(x,t) \, dx \, dt.
\end{aligned} \tag{2.13}$$

Integrating by parts on the first term of the left-hand side of (2.13), we obtain

$$\begin{aligned}
&\int_\Omega [(u^m_t)^2 + a_{ij} u^m_{x_i} u^m_{x_j} + (u^m_{xt})^2]\big|_{t=\tau} dx \\
&= 2\int_0^\tau \int_\Omega f u^m_t \, dx \, dt - 2\int_0^\tau \int_\Omega c u^m u^m_t \, dx \, dt + \int_0^\tau \int_\Omega \frac{\partial a_{ij}}{\partial t} u^m_{x_i} u^m_{x_j} \, dx \, dt \\
&\quad - 2\int_0^\tau \int_{\partial\Omega} u^m_t(x,t) \int_\Omega K(x,y,t) u^m(x,t) \, dy \, ds \, dt.
\end{aligned} \tag{2.14}$$

Consider the right-hand side of (2.14) and focus our attention on the term generated by nonlocal conditions. By applying Cauchy and Cauchy-Bunyakovskii inequalities, we obtain

$$
\begin{aligned}
&\big|2\int_0^\tau\int_{\partial\Omega} u_t^m(x,t)\int_\Omega K(x,y,t)u^m(y,t)\,dy\,ds\,dt\big|\\
&\le \int_0^\tau\int_{\partial\Omega}(u_t^m(x,t))^2\,ds\,dt + k\omega\int_0^\tau\int_\Omega (u^m(x,t))^2\,dx\,dt,
\end{aligned}
$$

where $k=\max_{[0,T]}\int_\Omega K^2(x,y,t)dy$, $\omega=\int_{\partial\Omega}ds$.

As the boundary $\partial\Omega$ is smooth [15], we have

$$
\int_{\partial\Omega}(u_t^m)^2 ds \le c_1\int_\Omega[(u_{xt}^m)^2+(u_t^m)^2]dx.
$$

Hence

$$
\begin{aligned}
&2\big|\int_0^\tau\int_\Omega (u_t^m(x,t))\int_\Omega K(x,y,t)u^m(y,t)\,dy\,ds\,dt\big|\\
&\le c_1\int_0^\tau\int_\Omega[(u_t^m)^2+(u_{xt}^m)^2]\,dx\,dt + k\omega\int_0^\tau\int_\Omega (u^m)^2\,dx\,dt.
\end{aligned}
\tag{2.15}
$$

Continue our estimates of right-hand side of (2.14). As mentioned above there exists $c_0>0$ such that $|a_{ijt}|,|c|\le c_0$ and $a_{ij}\xi_i\xi_j\ge\gamma\xi^2$ with $\gamma>0$. Now we apply Cauchy inequality to estimate the second and the third terms in the right-hand side of (2.14) and obtain

$$
\begin{gathered}
2\big|\int_0^\tau\int_\Omega cu^m u_t^m\,dx\,dt\big| \le c_0\int_0^\tau\int_\Omega((u^m)^2+(u_t^m)^2)\,dx\,dt,\\
2\big|\int_0^\tau\int_\Omega f u_t^m\,dx\,dt\big| \le \int_0^\tau\int_\Omega f^2\,dx\,dt + \int_0^\tau\int_\Omega (u_t^m)^2\,dx\,dt.
\end{gathered}
$$

With this result, from (2.14) and (2.15), we can now obtain

$$
\begin{aligned}
&\int_\Omega[(u_t^m)^2+\gamma(u_x^m)^2+(u_{xt}^m)^2]\big|_{t=\tau}dx\\
&\le c_2\int_0^\tau\int_\Omega[(u^m)^2+(u_t^m)^2+(u_x^m)^2+(u_{xt}^m)^2]\,dx\,dt + \int_0^\tau\int_\Omega f^2(x,t)\,dx\,dt
\end{aligned}
\tag{2.16}
$$

It easy to see that the relation

$$
u^m(x,\tau)=\int_0^\tau u_t^m(x,t)dt + u^m(x,0)
$$

implies (as $u^m(x,0)=0$) the inequality

$$
\int_0^l (u^m(x,\tau))^2\,dx\,dt \le \tau\int_0^\tau\int_\Omega (u_t^m(x,t))^2\,dx\,dt.
$$

Adding this inequality to (2.16), we obtain

$$
\begin{aligned}
&m_0\int_\Omega[(u^m)^2+(u_t^m)^2+(u_x^m)^2+(u_{xt}^m)^2]\big|_{t=\tau}dx\\
&\le M\int_0^\tau\int_\Omega[(u^m)^2+(u_t^m)^2+(u_x^m)^2+(u_{xt}^m)^2]\,dx\,dt\\
&\quad + N\int_0^l[(u^m(x,0))^2+(u_t^m(x,0))^2+(u_x^m(x,0))^2+(u_{xt}^m(x,0))^2]dx\\
&\quad + \int_0^\tau\int_\Omega f^2(x,t)\,dx\,dt,
\end{aligned}
\tag{2.17}
$$

where $M > 0$, $N > 0$ depend only on c_0, c_1, ω, γ, T. By Gronwall's lemma, we conclude that for all $m \geq 1$,

$$\|u^m\|_{W(Q_T)} \leq P, \tag{2.18}$$

where $P > 0$ and does not depend on m.

Note that $W(Q_T)$ is Hilbert space. Therefore, because of (2.18), we can extract from $\{u^m\}$ a subsequence that convergence weakly in $W(Q_T)$ and uniformly with respect to $t \in [0,T]$ in the norm of $L_2(\Omega)$ to $u \in W(Q_T)$. We need only to show that this limit function is a required generalized solution.

Initial condition $u(x,0) = 0$ is fulfilled as $u^m(x,t) \to u(x,t)$ in $L_2(\Omega)$ uniformly for every $t \in [0,T]$ and $u^m(x,0) \to 0$ in $L_2(\Omega)$. To show that (2.4) is valid we multiply (2.10) by $d_p \in C^1[0,T]$, $d_p(T) = 0$, take sum from $p = 1$ to $p = m$ and integrate with respect to t from 0 to T. This leads us to the equality

$$\begin{aligned}&\int_0^T \int_\Omega \left(u^m_{tt}\eta + a_{ij}u^m_{x_i}\eta_{x_j} + u^m_{x_itt}\eta_{x_j} + cu^m\eta\right) dx\,dt + \int_0^T \int_{\partial\Omega} \eta \int_\Omega K u^m \,dy\,ds\,dt\\ &= \int_0^T \int_\Omega f\eta\,dx\,dt.\end{aligned}$$

Denote $\eta(x,t) = \sum_{p=1}^m d_p(t)w_p(x)$. After integrating by parts the terms containing u^m_{tt} and u^m_{xtt}, we obtain

$$\begin{aligned}&\int_0^T \int_\Omega \left(-u^m_t\eta_t - u^m_{xt}\eta_{xt} + a_{ij}u^m_{x_i}\eta_{x_j} + cu^m\eta\right) dx\,dt\\ &+ \int_0^T \int_{\partial\Omega} \eta(x,t) \int_\Omega K(x,y,t)u^m(y,t)\,dy\,ds\,dt\\ &= \int_0^T \int_\Omega f\eta\,dx\,dt.\end{aligned} \tag{2.19}$$

Taking into account the convergence proved above one can pass to the limit in (2.19) as $m \to \infty$ for any fixed η. Denote the set of functions $\eta = \sum_{p=1}^m d_p(t)w_p(x)$ by $\mathcal{N}_m$. As $\cup_{m=1}^\infty \mathcal{N}_m$ is dense in $\hat{W}$ [15], it follows that the limit relation is fulfilled for every function $v \in \hat{W}(Q_T)$, hence, u is the solution of (2.1)–(2.3). $\square$

Remark 2.3. We use homogeneous initial data (2.2) and a nonlocal condition (2.3) for technical reasons. This involves no loss of generality but simplifies computational work. Nonhomogeneous conditions with usual properties can also be considered . In fact, suppose

$$\begin{gathered}u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x),\\ \frac{\partial^2}{\partial t^2}\frac{\partial u}{\partial \nu} + \frac{\partial u}{\partial N} + \int_\Omega K(x,y,t)u(y,t)dy = g(x,t), \quad (x,t) \in S_T.\end{gathered}$$

The identity (2.4) becomes

$$\begin{aligned}&\int_0^T \int_\Omega (-u_tv_t - u_{xt}v_{xt} + a_{ij}u_{x_i}v_{x_j} + cuv)\,dx\,dt + \int_0^T \int_{\partial\Omega} v(0,t) \int_\Omega Ku\,dy\,ds\,dt\\ &= \int_0^T \int_\Omega fv\,dx\,dt + \int_\Omega (\psi v + \psi_x v_x)dx + \int_0^T \int_{\partial\Omega} gv\,ds\,dt.\end{aligned}$$

If $\varphi, \psi \in W_2^2(Q_T), g \in L_2(\partial Q_T)$, we are able to obtain necessary a priori estimates as above.

3. Appendix

Here we give some reasons of arising nonlocal condition (2.3). Consider very simple particular case of (2.1),

$$u_{tt} - u_{ttxx} - u_{xx} + c(x,t)u = 0 \tag{3.1}$$

and set a following problem: Find a solution to (3.1) in the domain $Q_T = (0,l) \times (0,T)$ such that

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x), \quad u_x(0,t) = 0, \tag{3.2}$$

$$\int_0^l u(x,t)dx = 0. \tag{3.3}$$

Note that (3.3) is a nonlocal condition of the first kind. On physical grounds this one or more general integral conditions of the form $\int_0^l K(x,t)u(x,t)\,dx$ are very natural (see [22]–[10]) but give rise some difficulties when we try to prove a solvability of a nonlocal problem (see [20, 21] and references therein). One method has been advanced for overcoming these difficulties in [20], [21] for hyperbolic equations. The main idea of the procedure is as follows. We reduce the nonlocal condition of the first kind to a certain nonlocal condition of the second kind. This method may be applied in a similar way to the problem (3.1)–(3.3). We show it in brief.

Let $u(x,t)$ be a solution to (3.1)–(3.3). Integrating (3.1) with respect to x from 0 to l we obtain

$$u_{ttx}(l,t) - u_x(l,t) - \int_0^l c(x,t)u(x,t)dx = 0. \tag{3.4}$$

It is easy to see that (3.4) is a nonlocal condition of the second kind as this relation involves terms outside the integral.

If we assume now that $u(x,t)$ satisfies (3.1), (3.2), (3.4) and the compatibility conditions

$$\int_0^l \varphi(x)dx = 0, \quad \int_0^l \psi(x)dx = 0,$$

then after integrating (3.1) with respect to x from 0 to l we obtain

$$\frac{d^2}{dt^2}\int_0^l u(x,t)dx = 0.$$

The compatibility conditions give us zero initial data for an unknown function $\int_0^l u(x,t)dx$, hence $\int_0^l u(x,t)dx = 0$.

Now it may be concluded that problems (3.1)–(3.3) and (3.1), (3.2), (3.4) are equivalent.

In addition, we can now consider the nonlocal condition (2.3) as a generalization of (3.4).

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