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GEODESICS OF QUADRATIC DIFFERENTIALS ON KLEIN SURFACES

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Abstract. The objective of this article is to establish the existence of a local Euclidean metric associated with a quadratic differential on a Klein surface, and to describe the shortest curve in the neighborhood of a holomorphic point.

1. Introduction

In this paper we develop a technique, based on similar results for Riemann surfaces, to determine the geodesics near holomorphic points of a quadratic differential on a Klein surface. Recent results about this topic are due to Andreian Cazacu [3], Boloşteanu [5], Bârză [4] and Roşiu [7]. The complex double of a Klein surface is an important tool in connection with the study of the geometric structure near critical points of a quadratic differential on a Klein surface. Effectively, we define a metric associated with a quadratic differential on a Klein surface, which corresponds to a symmetric metric on its double cover. We introduce special parameters, in terms of which the quadratic differential has particularly simple representations and we give explicit descriptions of the geodesics near the holomorphic points of a quadratic differential.

Throughout this article, by a Klein surface we mean a Klein surface with an empty boundary, which is not a Riemann surface.

2. Preliminaries

Suppose that X is a compact Klein surface, obtained from a compact surface by removing a finite number of points. We assume that X has hyperbolic type. Our study is based on the following theorem (see [6]) due to Klein.

Theorem 2.1. *Given a Klein surface X, its canonical (Riemann) double cover X_C admits a fixed point free symmetry σ, such that X is dianalytically equivalent with $X_C/\langle\sigma\rangle$, where $\langle\sigma\rangle$ is the group generated by σ. Conversely, given a pair (X_C, σ) consisting of a Riemann surface X and a symmetry σ , the orbit space $X_C/\langle\sigma\rangle$ admits a unique structure of Klein surface, such that $f : X_C \to X_C/\langle\sigma\rangle$ is a morphism of Klein surfaces, provided that one regards X_C as a Klein surface.*

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Following the next theorem we associate a surface NEC group with a compact Klein surface X. Details can be found in [7].

Theorem 2.2. *Let X be a compact Klein surface of algebraic genus $g \geq 2$. Then there exists a surface NEC group Γ, such that X and H/Γ are isomorphic as Klein surfaces. Moreover, the complex double (X_C, f, σ) is isomorphic with H/Γ^+, where Γ^+ is the canonical Fuchsian subgroup of Γ.*

Then Γ is the group of covering transformations of X and Γ^+ is the group of conformal covering transformations of X. If $\pi' : H \to H/\Gamma$ and $\pi : H \to H/\Gamma^+$ are the canonical projections, then we note with $\widehat{z}$ the local parameter near a point $\widehat{P} \in H$, $\widetilde{z}$ the local parameter near $\widetilde{P} \in X_C$ and z the local parameter near $P = f(\widetilde{P}) \in X$.

Given $A = \{(U_i, \phi_i) \mid i \in I\}$ the dianalytic atlas on X, we define $U_i' = U_i \times \{i\} \times \{1\}$ and $U_i'' = U_i \times \{i\} \times \{-1\}$, $i \in I$. Let $U \subset U_i \cap U_j \neq \emptyset$ be a connected component of $U_i \cap U_j$. Then, we identify $U \times \{i\} \times \{\delta\}$ with $U \times \{j\} \times \{\delta\}$, for $\delta = \pm 1$, if $\phi_j \circ \phi_i^{-1} : \phi_i(U) \to \mathbb{C}$ is analytic and $U \times \{i\} \times \{\delta\}$ with $U \times \{j\} \times \{-\delta\}$, for $\delta = \pm 1$, if $\phi_j \circ \phi_i^{-1} : \phi_i(U) \to \mathbb{C}$ is antianalytic. As in Alling, X_C is the quotient space of $X_0 = \cup_{i\in I} U_i' \cup \cup_{i \in I} U_i''$, $(i, j) \in I \times I$, with all the identifications above.

We consider $p : X_0 \to X_C$ the canonical projection and $\widetilde{U_i} = p(U_i' \cup U_i'')$, $i \in I$.

Using Schwarz reflection principle, we can associate an analytic structure on X_C by $\widetilde{A} = \{(\widetilde{U_i}, \widetilde{\phi_i}) : i \in I\}$, where

$$\widetilde{\phi_i} : \widetilde{U_i} \to \mathbb{C}, \quad \widetilde{\phi_i}(\widetilde{P}) = \begin{cases} \phi_i(P), & \text{when } \widetilde{P} \in U_i' \\ \overline{\phi_i(P)}, & \text{when } \widetilde{P} \in U_i''. \end{cases}$$

Remark 2.3. $\widetilde{A}' = \{(\widetilde{U}_i, \overline{\widetilde{\phi}_i}) : i \in I\}$is also an analytic atlas on X_C.

If $X_C = (X_C, \widetilde{A})$ and $\overline{X}_C = (X_C, \widetilde{A}')$, then $\overline{X}_C$ is X_C endowed with the second orientation.

Let $f : X_C \to X$, $f(\widetilde{P}) = \{\widetilde{P}, \sigma(\widetilde{P})\}$ be the covering projection, where $\sigma : X_C \to \overline{X}_C$ is an antianalytic involution, without fixed points. If $\widetilde{z}$ is a parameter near $\widetilde{P} \in X_C$, then $\overline{\widetilde{z}}$ is a parameter near $\sigma(\widetilde{P}) \in \overline{X}_C$.

Because two disjoint neighborhoods U_i' and U_i'' of X_C, lie over each neighborhood U_i of X, we can make the restriction at $U_i' \cup U_i''$ in the local study of the quadratic differentials on X_C.

As in Alling and Greenleaf [2], we associate with the dianalytic atlas A on X, the nonzero (holomorphic) quadratic differential $\varphi = (\varphi_i)_{i\in I}$ on X, in the local parameters $(z_i)_{i \in I}$, such that on each connected component U of $U_i \cap U_j$, the following transformation law

$$\varphi_i(z_i)(dz_i)^2 = \begin{cases} \varphi_j(z_j)(dz_j)^2, & \text{when } \phi_j \circ \phi_i^{-1} \text{ is analytic on } \phi_i(U) \\ \overline{\varphi_j(z_j)}(d\overline{z_j})^2, & \text{when } \phi_j \circ \phi_i^{-1} \text{ is antianalytic on } \phi_i(U) \end{cases}$$

holds whenever z_i and z_j are parameter values near the same point of X.

The family $\widetilde{\varphi} = (\widetilde{\varphi}_i)_{i\in I}$ of holomorphic function elements, in the local parameters $(\widetilde{z}_i)_{i \in I}$,

$$\widetilde{\varphi}_i(\widetilde{z}_i) = \begin{cases} \varphi_i(z_i), & \text{when } \widetilde{P} \in U_i' \\ \overline{\varphi_i(z_i)}, & \text{when } \widetilde{P} \in U_i'', \end{cases} \tag{2.1}$$

where $\widetilde{z_i}$ is a local parameter near $\widetilde{P}$, is a (holomorphic) quadratic differential on X_C, with respect to the analytic atlas $\widetilde{A}$ on X_C. By analogy, the family $\overline{\widetilde{\varphi}} = (\overline{\widetilde{\varphi_i}})_{i\in I}$ of holomorphic function elements, in the local parameters $(\overline{\widetilde{z_i}})_{i\in I}$ is a (holomorphic) quadratic differential on $\overline{X}_C$.

3. The natural parameter near holomorphic points

As a consequence of the above results, see [7], the order of a nonzero holomorphic quadratic differential φ on X at a point P is a dianalytic invariant, thus it does make sense to define the zeroes of the quadratic differential φ on a Klein surface. A holomorphic point is either a regular point or a zero.

In this section, we extend the notion of natural parameter near a holomorphic point of a quadratic differential on a Riemann surface to a Klein surface, which is studied in [9].

Let P be a holomorphic point of the quadratic differential φ and (U_i, z_i), $i \in I$ be a dianalytic chart at P. Then, we can take U_i to be sufficiently small so that a single valued branch of $\Phi(z_i) = \int \sqrt{\varphi_i(z_i)}dz_i$ can be chosen. We introduce a local parameter

$$w_i = \Phi(z_i) = \int \sqrt{\varphi_i(z_i)}dz_i$$

in U_i, uniquely up to a transformation $w_i \to \pm w_i + \text{const.}$, such that on each connected component U of $U_i \cap U_j$, the following transformation law

$$w_i = \begin{cases} w_j, & \text{when } \phi_j \circ \phi_i^{-1} \text{ is analytic on } \phi_i(U) \\ \overline{w_j}, & \text{when } \phi_j \circ \phi_i^{-1} \text{ is antianalytic on } \phi_i(U) \end{cases}$$

holds whenever z_i and z_j are parameter values near the same point P of X.

By the definition of a quadratic differential on a Klein surface, we can see that the local parameter w_i is locally well defined, up to conjugation, see [2].

Remark 3.1. The quadratic differential φ on the Klein surface (X, A) has, in terms of the natural parameter w_i, the representation identically equal to one.

As for a Riemann surface, w_i will be called the natural parameter near P. Let w_i be the natural parameter in a neighborhood U_i of a holomorphic point P of the quadratic differential φ on X. Then, using the relation (2.1) it follows that

$$\widetilde{w_i} = \begin{cases} w_i, & \text{when } \widetilde{P} \in U_i' \\ \overline{w_i}, & \text{when } \widetilde{P} \in U_i'', \end{cases}$$

is the natural parameter in the corresponding neighborhood of the point $\widetilde{P} = p(P)$. The natural parameter on X_C is well defined, because the family $\widetilde{\varphi}$ is a quadratic differential on X_C, with respect to the analytic atlas $\widetilde{A}$ on X_C.

By analogy, $\overline{\widetilde{w_i}}$ is the natural parameter in the corresponding neighborhood of the point $\sigma(\widetilde{P})$.

Proposition 3.2. *The quadratic differential $\widetilde{\varphi}$ on the Riemann surface $(X_C, \widetilde{A})$ has, in terms of $\widetilde{w_i}$, the representation identically equal to one.*

Proof. If $\widetilde{P} \in U_i'$, the differential $d\widetilde{w_i}$ becomes $dw_i = \sqrt{\varphi(z_i)}dz_i$, therefore by squaring $(dw_i)^2 = \varphi(z_i)(dz_i)^2$ so, in terms of $\widetilde{w_i} = w_i$, the quadratic differential $\widetilde{\varphi}$ has the representation identically equal to one. A similar argument applies if

$\widetilde{P} \in U_i''$. The differential $d\widetilde{w_i}$ becomes $(d\overline{w_i})^2 = \overline{\varphi(z_i)}(d\overline{z_i})^2$ and in terms of $\widetilde{w_i} = \overline{w_i}$, the quadratic differential $\widetilde{\varphi}$ has the representation identically equal to one. $\square$

Remark 3.3. The quadratic differential $\overline{\widetilde{\varphi}}$ on the Riemann surface $\overline{X}_C$ has, in terms of $\overline{\widetilde{w_i}}$, the representation identically equal to one.

In the special case, when P is a zero of the quadratic differential φ, then as in Strebel [9, Theorem 6.2], we can introduce a local parameter ζ_i in the neighborhood U_i of P, $P \leftrightarrow \zeta_i = 0$, in terms of which the quadratic differential has the representation

$$(dw_i)^2 = \varphi(z_i)(dz_i)^2 = \big(\frac{n+2}{2}\big)^2 \zeta_i^n d\zeta_i^2. \tag{3.1}$$

As a function of ζ_i, the natural parameter becomes $w_i = \Phi(z_i) = (\zeta_i)^{\frac{n+2}{2}}$ and the corresponding natural parameter on X_C is

$$\widetilde{w_i} = \begin{cases} (\zeta_i)^{\frac{n+2}{2}}, & \text{when } \widetilde{P} \in U_i' \\ (\overline{\zeta_i})^{\frac{n+2}{2}}, & \text{when } \widetilde{P} \in U_i''. \end{cases} \tag{3.2}$$

3.1. **The φ-length of a curve.** In this section, we introduce the φ-metric associated with a quadratic differential on a Klein surface and we study its relation with the corresponding $\widetilde{\varphi}$-metric on its canonical (Riemann) double cover X_C, see [9].

Let w_i be the natural parameter in a neighborhood U_i of a holomorphic point P of the quadratic differential φ on X. We define the length element of the φ-metric, $|dw_i|$, such that on each connected component U of $U_i \cap U_j$, the transformation law

$$|dw_i| = \begin{cases} |dw_j|, & \text{when } \phi_j \circ \phi_i^{-1} \text{ is analytic on } \phi_i(U) \\ |d\overline{w_j}|, & \text{when } \phi_j \circ \phi_i^{-1} \text{ is antianalytic on } \phi_i(U) \end{cases}$$

holds whenever z_i and z_j are parameter values near the same point P of X.

Remark 3.4. The length element of the φ-metric is a dianalytic invariant, namely $|dw_i| = |d\overline{w_i}| = \sqrt{|\varphi(z_i)|}|dz_i|$.

The length element of the $\widetilde{\varphi}$ -metric is defined by

$$|d\widetilde{w_i}| = \begin{cases} |dw_i|, & \text{when } \widetilde{P} \in U_i' \\ |d\overline{w_i}|, & \text{when } \widetilde{P} \in U_i''. \end{cases} \tag{3.3}$$

Let U_i be a neighborhood of a regular point P of the quadratic differential φ on X and $f^{-1}(U_i) = U_i' \cup U_i''$. Let γ be a piecewise smooth curve in U_i. The curve γ has exactly two liftings at $f^{-1}(U_i)$. If γ has the initial point P and $\widetilde{\gamma}$ is the lifting of γ at $\widetilde{P} \in U_i'$, then $\sigma(\widetilde{\gamma})$ is the lifting of γ at $\widetilde{P} \in U_i''$.

Remark 3.5. Because $\sigma : X_C \to \overline{X}_C$ is an antianalytic involution, without fixed points, we get that if $\widetilde{\gamma}$ is the lifting of γ at $\widetilde{P} \in U_i''$, then $\sigma(\widetilde{\gamma})$ is the lifting of γ at $\widetilde{P} \in U_i'$.

The curves $\widetilde{\gamma}$ and $\sigma(\widetilde{\gamma})$ are called symmetric on X_C. Next, we will identify $\widetilde{\gamma}$ and $\sigma(\widetilde{\gamma})$, with their images in the complex plane, from the corresponding charts.

The curve γ is mapped by a branch of $\Phi(z_i) = \int \sqrt{\varphi_i(z_i)}dz_i$ onto a curve γ_0 in the w-plane. The Euclidean length of γ_0 does not depend on the branch of Φ

which we have chosen. The φ-length of γ can be computed, in terms of the natural parameter w_i, by

$$l_\varphi(\gamma) = \int_{\gamma_0} |dw_i| = \int_\gamma \sqrt{|\varphi_i(z_i)|}|dz_i|,$$

then the φ-length of γ is the Euclidean length of γ_0.

Thus, the natural parameter is the local isometry between the φ- metric and the Euclidean metric.

Remark 3.6. By the definition of the quadratic differential φ on X, we obtain that the φ-length of γ is a dianalytic invariant.

From (3.3), we deduce that the $\widetilde{\varphi}$-length of $\widetilde{\gamma}$ is

$$l_{\widetilde{\varphi}}(\widetilde{\gamma}) = \begin{cases} l_\varphi(\gamma), & \text{when } \widetilde{\gamma} \in U_i' \\ l_{\overline{\varphi}}(\gamma), & \text{when } \widetilde{\gamma} \in U_i'', \end{cases}$$

where $l_{\overline{\varphi}}(\gamma) = \int_\gamma \sqrt{|\overline{\varphi_i(z_i)}|}|d\overline{z_i}|$.

Proposition 3.7. *The symmetric curves $\widetilde{\gamma}$ and $\sigma(\widetilde{\gamma})$ have the same $\widetilde{\varphi}$-length, namely $l_{\widetilde{\varphi}}(\widetilde{\gamma}) = l_{\widetilde{\varphi}}(\sigma(\widetilde{\gamma})) = l_\varphi(\gamma)$. Therefore, the $\widetilde{\varphi}$-metric is a symmetric metric on X_C.*

Proof. We assume without loss of generality, that $\widetilde{\gamma} \in U_i'$. Then, $\sigma(\widetilde{\gamma}) \in U_i''$. By definition, the $\widetilde{\varphi}$-length of $\widetilde{\gamma}$ is

$$l_{\widetilde{\varphi}}(\widetilde{\gamma}) = l_\varphi(\gamma) = \int_{\gamma_0} |dw_i| = \int_\gamma \sqrt{|\varphi_i(z_i)|}|dz_i|$$

and the $\widetilde{\varphi}$-length of $\sigma(\widetilde{\gamma})$ is

$$l_{\widetilde{\varphi}}(\sigma(\widetilde{\gamma})) = l_{\overline{\varphi}}(\gamma) = \int_{\gamma_0} |d\overline{w_i}| = \int_\gamma \sqrt{|\overline{\varphi_i(z_i)}|}|d\overline{z_i}|$$

where γ_0 is the image of γ by a branch of Φ. Thus, $l_{\widetilde{\varphi}}(\widetilde{\gamma}) = l_{\widetilde{\varphi}}(\sigma(\widetilde{\gamma}))$. □

Next, we consider a point from the surface as being its image through the corresponding local chart. The φ-distance between two points z_1 and z_2 in U_i is, by definition,

$$d_\varphi(z_1, z_2) = \inf_\gamma l_\varphi(\gamma) = \inf_\gamma \int_\gamma \sqrt{|\varphi_i(z_i)|}|dz_i|$$

where γ ranges over the piecewise smooth curves in U_i joining z_1 and z_2. The φ-distance depends of the domain which is selected.

Because X_C is compact, any two points $\widetilde{z_1}$ and $\widetilde{z_2}$ can be connected with a shortest curve whose length is the $\widetilde{\varphi}$-distance between $\widetilde{z_1}$ and $\widetilde{z_2}$. We note this distance with $d_{\widetilde{\varphi}}(\widetilde{z_1}, \widetilde{z_2})$.

Using the definition of the $\widetilde{\varphi}$-length of a curve, we can observe that

$$d_{\widetilde{\varphi}}(\widetilde{z_1}, \overline{\widetilde{z_2}}) = d_{\widetilde{\varphi}}(\overline{\widetilde{z_1}}, \widetilde{z_2}) \quad \text{and} \quad d_{\widetilde{\varphi}}(\widetilde{z_1}, \widetilde{z_2}) = d_{\widetilde{\varphi}}(\overline{\widetilde{z_2}}, \overline{\widetilde{z_1}}).$$

A piecewise smooth curve is called geodesic if it is locally shortest. The similar notion defined on Riemann surfaces is studied in [9].

A straight arc with respect to the quadratic differential φ is a smooth curve γ along which

$$\arg(dw)^2 = \arg \varphi(z)(dz)^2 = \theta = const., \quad 0 \le \theta < 2\pi.$$

Remark 3.8. A straight arc only contains regular points of φ.

3.2. **The φ-metric near holomorphic points.** Given a holomorphic quadratic differential φ on a Klein surface X, a φ-disk is a region which is mapped homeomorphically onto a disk in the complex plane by a branch of Φ. The φ-radius of a φ-disk is the Euclidean radius of the corresponding disk in the complex plane.

We want to determine the shortest curve (in the φ-metric) near holomorphic points. In this case the φ-length of a curve γ has sense and is finite, even if γ goes through a zero of φ. Further, we consider a point from the surface X or X_C as being its image through the corresponding local charts.

The following theorem is an extension of Strebel's Theorem 5.4., see [9], from Riemann surfaces to Klein surfaces.

Theorem 3.9. *Let P be a regular point P of φ. Then there exists a neighborhood U of P such that any two points P_1 and P_2 in U can be joined by a uniquely determined shortest curve γ (in the φ-metric) in U. The geodesic γ is the pre-image of a straight line segment under a branch of Φ.*

Proof. Let U_0 be the largest φ-disk around a regular point P. Choose a branch Φ_0 of Φ in U_0, such that $\Phi_0(P) = 0$. If the radius of U_0 is r, let V be the disk $|w| < \frac{r}{2}$ and $U = \Phi_0^{-1}(V)$. Then $w = \Phi_0(z) = \int \sqrt{\varphi_0(z)}dz$ is the natural parameter in U, where $\varphi_0(z)(dz)^2$ is the local representation of the quadratic differential on X.

Let P_1 and P_2 be two points from U and $w_i = \Phi_0(P_i)$, $i = 1, 2$. We lift P_1 and P_2 to X_C. Let $\widetilde{P}_i$ and $\sigma(\widetilde{P}_i)$ be the two points of X_C which lie over the same point P_i of X, $i = 1, 2$.

We notice that γ does not pass through any zeroes of φ. If γ is an arbitrary curve joining P_1 and P_2 in U, then either $\widetilde{\gamma}$ preserves the orientation or $\widetilde{\gamma}$ changes the orientation in a point of it.

In the first case, either $\widetilde{\gamma}$ is contained in U' or $\widetilde{\gamma}$ is contained in U''.

If $\widetilde{\gamma}$ is contained in U', by Strebel's Theorem 5.4., see [9], $\widetilde{P_1}$ and $\widetilde{P_2}$ can be joined by a uniquely determined shortest arc $\widetilde{\gamma}$ on X_C (in the $\widetilde{\varphi}$-metric). The arc $\widetilde{\gamma}$ is the pre-image of the straight line segment joining w_1 and w_2 in the w-plane, under Φ_0. Then $l_\varphi(\gamma) = l_{\widetilde{\varphi}}(\widetilde{\gamma}) = \int_\gamma \sqrt{|\varphi_0(z)|}|dz| = |w_2 - w_1|$.

Similarly, if $\widetilde{\gamma}$ is contained in U'', both $\sigma(\widetilde{P_1})$ and $\sigma(\widetilde{P_2})$ are in U'' and the uniquely determined shortest arc $\widetilde{\gamma}$ is the pre-image of the straight line segment joining $\overline{w_1}$ and $\overline{w_2}$ in the w-plane, under $\overline{\Phi_0}$. Then $l_\varphi(\gamma) = l_{\widetilde{\varphi}}(\sigma(\widetilde{\gamma})) = \int_\gamma \sqrt{|\overline{\varphi_0(z)}|}|d\overline{z}| = |\overline{w_2} - \overline{w_1}|$.

In the second case, because γ does not pass through a zero of φ, then we consider the direct analytic continuation of Φ_0^{-1} along the straight line segment joining w_1 and $\overline{w_2}$ in the w-plane. Thus, the points $\widetilde{P_1}$ and $\sigma(\widetilde{P_2})$ are joined by the shortest curve $\widetilde{\gamma}$ with respect to the $\widetilde{\varphi}$-metric on X_C, which is composed of straight arcs of the above type and the length of $\widetilde{\gamma}$ is the sum of the lengths of the component straight arcs. Applying the above results, we obtain $l_\varphi(\gamma) = l_{\widetilde{\varphi}}(\widetilde{\gamma}) = |\overline{w_2} - w_1|$.

In conclusion, if $|w_2 - w_1| \le |\overline{w_2} - w_1|$, the geodesic γ is the pre-image of the straight line segment $[w_1, w_2]$ under Φ_0 and if $|\overline{w_2} - w_1| \le |w_2 - w_1|$, the geodesic γ is the pre-image of the straight line segment $[w_1, \overline{w_2}]$ under a branch of Φ. $\square$

Corollary 3.10. *The φ-distance between P_1 and P_2 ,*

$$d_\varphi(z_1, z_2) = \min(|w_2 - w_1|, |\overline{w_2} - w_1|)$$

thus $d_\varphi(z_1, z_2) = \min\Big(d_{\widetilde{\varphi}}(\widetilde{z_1}, \widetilde{z_2}), d_{\widetilde{\varphi}}(\widetilde{z_1}, \overline{\widetilde{z_2}})\Big)$.

The following theorem is an extension of Strebel's Theorem 8.1., see [9], from Riemann surfaces to Klein surfaces.

Theorem 3.11. *Let P be a zero of φ of order n. Then there exists a neighborhood U of P such that any two points P_1 and P_2 in U can be joined by a uniquely determined shortest curve in U (in the φ-metric). The geodesic γ is either the pre-image of a straight line segment under a branch of Φ or is composed of the pre-images of two radii under branches of Φ.*

Proof. Let P be zero of φ of order n. Using (3.1), in the neighborhood U of P, there is a parameter ζ, $P \leftrightarrow \zeta = 0$ such that the local representation of φ is $\varphi(z)(dz)^2 = (\frac{n+2}{2})^2\zeta^n(d\zeta)^2$. Thus the corresponding natural parameter is $w = \Phi(\zeta) = \zeta^{\frac{n+2}{2}}$. The function Φ maps each one of the sector

$$\{\zeta \in C | \frac{2\pi}{n+2}k \le arg\zeta \le \frac{2\pi}{n+2}(k+1), k = 0, 1, \dots, n+1\}$$

onto an upper or lower half-plane.

The φ-length of the radius of the circle $|\zeta| = \rho$ is equal to $\rho^{\frac{n+2}{2}}$. Let V be the disk $|w| < \frac{1}{2}\rho^{\frac{n+2}{2}}$ and $U = \Phi_0^{-1}(V)$.

Let P_1 and P_2 be two points in U and $w_i = \Phi_0(P_i)$, $i = 1, 2$. We lift P_1 and P_2 to X_C. Let $\widetilde{P_i}$ and $\sigma(\widetilde{P_i})$ be the two points of X_C which lie over the same point P_i of X, $i = 1, 2$.

If γ is an arbitrary curve joining P_1 and P_2 in U, then either $\widetilde{\gamma}$ preserves the orientation or $\widetilde{\gamma}$ changes the orientation in a point of it. We may assume, without loss of generality, that $z_1 \neq 0$ and $\arg z_1 = 0$.

In the first case, either $\widetilde{\gamma}$ is contained in U' or $\widetilde{\gamma}$ is contained in U''.

If $\widetilde{\gamma}$ is contained in U', by Strebel's Theorem 8.1., see [9] and (3.2), $\widetilde{P_1}$ and $\widetilde{P_2}$ can be joined by a uniquely determined shortest curve $\widetilde{\gamma}$ on X_C (in the $\widetilde{\varphi}$-metric). Furthermore,

(a) if $|\arg\widetilde{\zeta} - \arg\widetilde{\zeta_1}| \le \frac{2\pi}{n+2}$ for any $\widetilde{\zeta} \in \widetilde{\gamma}$, $\widetilde{\zeta} \neq 0$, then $\widetilde{\gamma}$ is the pre-image of the straight line segment joining w_1 and w_2 in the w-plane, under Φ_0. Hence $l_\varphi(\gamma) = l_{\widetilde{\varphi}}(\widetilde{\gamma}) = \int_\gamma \sqrt{|\varphi_0(z)|}|dz| = |w_2 - w_1|$.

(b) If there is a $\widetilde{\zeta} \in \widetilde{\gamma}$ such that $|\arg\widetilde{\zeta} - \arg\widetilde{\zeta_1}| > \frac{2\pi}{n+2}$, then the curve $\widetilde{\gamma}$ is composed, in terms of $\widetilde{\zeta}$, of two radii enclosing angles $\ge \frac{2\pi}{n+2}$. Hence $l_\varphi(\gamma) = l_{\widetilde{\varphi}}(\widetilde{\gamma}) = |w_1| + |w_2|$.

Analogously, for $\widetilde{\gamma}$ contained in U'', we have:

(c) If $|arg\widetilde{\zeta} - arg\widetilde{\zeta_1}| \le \frac{2\pi}{n+2}$ for any $\widetilde{\zeta} \in \widetilde{\gamma}$, $\widetilde{\zeta} \neq 0$, then $\widetilde{\gamma}$ is the pre-image of the straight line segment joining $\overline{w_1}$ and $\overline{w_2}$ in the w-plane, under $\overline{\Phi_0}$. Hence $l_\varphi(\gamma) = l_{\widetilde{\varphi}}(\widetilde{\gamma}) = \int_\gamma \sqrt{|\overline{\varphi_0(z)}|}|d\overline{z}| = |\overline{w_2} - \overline{w_1}|$.

(d) If there is a $\widetilde{\zeta} \in \widetilde{\gamma}$ such that $|\arg\widetilde{\zeta} - \arg\widetilde{\zeta_1}| > \frac{2\pi}{n+2}$, then the curve $\widetilde{\gamma}$ is composed, in terms of $\widetilde{\zeta}$, of two radii enclosing angles greater than or equal $\frac{2\pi}{n+2}$. Hence $l_\varphi(\gamma) = l_{\widetilde{\varphi}}(\widetilde{\gamma}) = |\overline{w_1}| + |\overline{w_2}|$.

In the second case, we have:

(e) If $|\arg\widetilde{\zeta} - \arg\widetilde{\zeta_1}| \le \frac{2\pi}{n+2}$ for any $\widetilde{\zeta} \in \widetilde{\gamma}$, $\widetilde{\zeta} \neq 0$ we consider the direct analytic continuation of Φ_0^{-1} along the straight line segment joining w_1 and $\overline{w_2}$ in the w-plane. The points $\widetilde{P_1}$ and $\sigma(\widetilde{P_2})$ can be joined by a uniquely determined shortest curve $\widetilde{\gamma}$ with respect to the $\widetilde{\varphi}$-metric on X_C which is composed of straight arcs of the above type. Then, $l_\varphi(\gamma) = l_{\widetilde{\varphi}}(\widetilde{\gamma}) = |\overline{w_2} - w_1|$.

(f) If there is a $\widetilde{\zeta} \in \widetilde{\gamma}$ such that $|\arg \widetilde{\zeta} - \arg \widetilde{\zeta_1}| > \frac{2\pi}{n+2}$, then the curve $\widetilde{\gamma}$ is composed, in terms of $\widetilde{\zeta}$, of two radii enclosing angles greater than or equal to $\frac{2\pi}{n+2}$. Hence $l_\varphi(\gamma) = l_{\widetilde{\varphi}}(\widetilde{\gamma}) = |w_1| + |\overline{w_2}|$.

In conclusion, if $|\arg \widetilde{\zeta} - \arg \widetilde{\zeta_1}| \leq \frac{2\pi}{n+2}$, then the geodesic γ is the pre-image of one of the straight line segments, $[w_1, w_2]$ or $[w_1, \overline{w_2}]$, namely the one that has the smallest Euclidean length, under a branch of Φ and if there is a $\widetilde{\zeta} \in \widetilde{\gamma}$ such that $|\arg \widetilde{\zeta} - \arg \widetilde{\zeta_1}| > \frac{2\pi}{n+2}$, then the geodesic γ is composed either of the pre-images of the radii $[0, w_1]$ and $[0, w_2]$ or of the pre-images of the radii $[0, w_1]$ and $[0, \overline{w_2}]$, namely those that have the smallest sum of the Euclidean lengths, under branches of Φ. □

Corollary 3.12. *The φ-distance between the points P_1 and P_2,*

$$d_\varphi(z_1, z_2) = \min\big(|w_2 - w_1|, |\overline{w_2} - w_1|, |w_1| + |w_2|\big),$$

thus $d_\varphi(z_1, z_2) = \min(d_{\widetilde{\varphi}}(\widetilde{z_1}, \widetilde{z_2}), d_{\widetilde{\varphi}}(\widetilde{z_1}, \overline{\widetilde{z_2}}))$

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