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GENERALIZED VAN DER POL EQUATION AND HILBERT'S 16TH PROBLEM

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Abstract. In this article, we study the bifurcation of limit cycles from the harmonic oscillator $\dot{x} = y$, $\dot{y} = -x$ in the system

$$\dot{x} = y, \quad \dot{y} = -x + \varepsilon f(y)\big(1 - x^2\big),$$

where ε is a small positive parameter tending to 0 and f is an odd polynomial of degree $2n + 1$, with n an arbitrary but fixed natural number. We prove that, the above differential system, in the global plane, for particularly chosen odd polynomials f of degree $2n + 1$ has exactly $n + 1$ limit cycles and that this number is an upper bound for the number of limit cycles for every case of an arbitrary odd polynomial f of degree $2n + 1$. More specifically, the existence of the limit cycles, which is the first of the main results in this work, is obtained by using the Poincaré's method, and the upper bound for the number of limit cycles can be derived from the work of Iliev [4]. We also investigate the possible relative positions of the limit cycles for this differential system, which is the second main problem studying in this work. In particular, we construct differential systems with n given limit cycles and one limit cycle whose position depends on the position of the previous n limit cycles. Finally, we give some examples in order to illustrate the general theory presented in this work.

1. Introduction

1.1. Generalized Van der Pol equation and statement of the main results.

In this article, we study the second part of Hilbert's 16th problem for a generalized Van der Pol equation. More specifically, we consider the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + \varepsilon f(y)\big(1 - x^2\big), \end{aligned} \tag{1.1}$$

where f is an odd polynomial of degree $2n + 1$, with n an arbitrary but fixed natural number and $0 < \varepsilon \ll 1$. System (1.1) reduces to the Van der Pol equation for $f(y) = y$. Our purpose here is to find an upper bound for the number of limit cycles for system (1.1), depending only on the degree of its polynomials and investigate their relative positions.

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System (1.1) is the generalized Van der Pol equation of the form

$$\ddot{x} - \varepsilon f(\dot{x})\big(1 - x^2\big) + x = 0, \tag{1.2}$$

where f is an odd polynomial of degree $2n+1$, with n an arbitrary but fixed natural number and $0 < \varepsilon \ll 1$. The problem is to find an upper bound for the number of limit cycles for equation (1.2), depending only on the degree $2n+1$ of the odd polynomial f and investigate their relative positions. We prove that the generalized Van der Pol equation (1.2) has exactly $n+1$ limit cycles for particularly chosen odd polynomials f of degree $2n+1$ and that this number is an upper bound for the number of limit cycles for every case of an arbitrary odd polynomial f of degree $2n+1$. Furthermore, we show how to construct these polynomials of equation (1.2) which attain that upper bound. On the possible relative positions of the $n+1$ limit cycles we show that there exists a limit cycle whose position depends on the position of the rest n limit cycles (actually, this limit cycle is close to the circle with the dependent radius (see Definition 1.13)).

Now, we state the main results of this article, which are the following theorems. The proofs of these theorems will be given in Section 3. For the definitions appear in these theorems, like the sets V^n, V^n_{n+1}, $n \in \mathbb{N}$, $n \geq 2$, V^1 and the dependent radius, see the next subsection. The first and second of our results, consider the system (1.1), with f an odd polynomial of degree $2n+1$, where $n \in \mathbb{N}$, $n \geq 2$.

Theorem 1.1. *Let $(\lambda_1, \lambda_2, \dots, \lambda_n) \in V^n$ be such that $(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}) \in V^n_{n+1}$, where λ_{n+1} is the dependent radius given by (1.6), if $n \in \mathbb{N}$, $n \geq 2$. Then the system (1.1), with $0 < \varepsilon \ll 1$ and*

$$\begin{aligned}
f(y) = {}& \tau y^{2n+1} + \dots + \tau(2n-2k+3)\dots(2n+1) \\
& \times \Big[1 - \frac{1}{2(n+2)} \sum_{i_1=1}^{n+1} \lambda_{i_1} + \frac{1}{4(n+1)(n+2)} \sum_{\substack{i_1,i_2=1 \\ i_1<i_2}}^{n+1} \lambda_{i_1}\lambda_{i_2} + \dots \\
& + \frac{1}{2^k(n-k+3)\dots(n+2)}(-1)^k \sum_{\substack{i_1,\dots,i_k=1 \\ i_1<\dots<i_k}}^{n+1} \lambda_{i_1}\dots\lambda_{i_k}\Big] y^{2(n-k)+1} \\
& + \dots + \tau\Big[\frac{1\cdot 3\dots(2n+1)}{2^{n+1}(n+2)!}(-1)^n \prod_{i_1=1}^{n+1} \lambda_{i_1}\Big] y,
\end{aligned} \tag{1.3}$$

where $\tau \in \mathbb{R} \setminus \{0\}$ and $1 \leq k \leq n-1$, has exactly the following $n+1$ limit cycles:

$$x^2 + y^2 = \lambda_1 + O(\varepsilon), \quad x^2 + y^2 = \lambda_2 + O(\varepsilon), \dots, \quad x^2 + y^2 = \lambda_{n+1} + O(\varepsilon).$$

Furthermore, (assuming from now on an ordering such that $\lambda_1 < \lambda_2 < \dots < \lambda_n < \lambda_{n+1}$, where now λ_{n+1} is not necessary the dependent radius) we have for the stability of the limit cycles that, if $\tau > 0$ (respectively $\tau < 0$),

$$x^2 + y^2 = \lambda_1 + O(\varepsilon), \quad x^2 + y^2 = \lambda_3 + O(\varepsilon), \dots, \quad x^2 + y^2 = \lambda_{n+1} + O(\varepsilon)$$

are stable (respectively unstable) and

$$x^2 + y^2 = \lambda_2 + O(\varepsilon), \quad x^2 + y^2 = \lambda_4 + O(\varepsilon), \dots, \quad x^2 + y^2 = \lambda_n + O(\varepsilon)$$

are unstable (respectively stable) for n even; and

$$x^2 + y^2 = \lambda_1 + O(\varepsilon), \quad x^2 + y^2 = \lambda_3 + O(\varepsilon), \dots, \quad x^2 + y^2 = \lambda_n + O(\varepsilon)$$

are unstable (respectively stable) and

$$x^2+y^2=\lambda_2+O(\varepsilon),\quad x^2+y^2=\lambda_4+O(\varepsilon),\dots,\quad x^2+y^2=\lambda_{n+1}+O(\varepsilon)$$

are stable (respectively unstable) for n odd.

Theorem 1.2. *For system* (1.1)*, where ε is small and f is an arbitrary odd polynomial of degree $2n+1$ we have that the number of $n+1$ limit cycles is an upper bound for the number of limit cycles. Moreover, from the set of all the odd polynomials, the polynomials f given by* (1.3)*, are the only that attain that upper bound.*

Our third and fourth results, concern the system (1.1), with f an odd polynomial of degree 3.

Theorem 1.3. *Let $\lambda_1\in V^1$. Then the system* (1.1)*, with $0<\varepsilon\ll 1$ and*

$$f(y)=\tau y^3-\tau\frac{1}{8}\lambda_1\lambda_2 y, \tag{1.4}$$

where $\tau\in\mathbb{R}\setminus\{0\}$ and λ_2 is the dependent radius given by (1.7)*, has exactly the following 2 limit cycles:*

$$x^2+y^2=\lambda_1+O(\varepsilon),\quad x^2+y^2=\lambda_2+O(\varepsilon).$$

Furthermore, (assuming from now on an ordering such that $\lambda_1<\lambda_2$, where now λ_2 is not necessary the dependent radius) we have for the stability of the limit cycles that, if $\tau>0$ (respectively $\tau<0$) $x^2+y^2=\lambda_1+O(\varepsilon)$ is unstable (respectively stable) and $x^2+y^2=\lambda_2+O(\varepsilon)$ is stable (respectively unstable).

Theorem 1.4. *For system* (1.1)*, where ε is small and f is an arbitrary odd polynomial of degree 3 we have that the number of 2 limit cycles is an upper bound for the number of limit cycles. Moreover, from the set of all the odd polynomials, the polynomials f given by* (1.4)*, are the only that attain that upper bound.*

Remark 1.5. It is important to note that the above theorems don't inform us which limit cycles we have for a differential equation of the form (1.1). That we succeed through these theorems is to construct differential equations of the form (1.1) with n given limit cycles and one limit cycle which is close to the circle with the dependent radius, for particularly chosen odd polynomials f of degree $2n+1$. So, we show how to construct differential equations of the form (1.1) that attain the upper bound of $n+1$ limit cycles, when the odd polynomial f is of degree $2n+1$. Evenly important it is still and one negative result which can be obtained by these theorems, that we know a priori which limit cycles we can't have for system (1.1) with odd polynomials f of degree $2n+1$. Substantially, we construct the set of all the possible limiting radii of limit cycles for the system (1.1) with odd polynomials f of degree $2n+1$. This is the set V^n_{n+1} which contains the Λ-points (see Definition 1.22).

Remark 1.6. It is surprising the connection between the dependent radius for a circle (see Definition 1.13) and the existence of one branch which can not separate from the rest branches for an algebraic curve. More specifically, relatively to the existence of such branch we refer the following of Hilbert's speech about the first part of Hilbert's 16th problem "As of the curves of degree 6, I have -admittedly in a rather elaborate way- convinced myself that the 11 branches, that they can have according to Harnack, never all can be separate, rather there must exist one branch, which have another branch running in its interior and nine branches running in its exterior, or opposite". Here, we have for the relative positions of limit cycles that the limit cycle which is close to the circle with the dependent radius can not lie wherever, contrary the position of this limit cycle depends on the position of the rest limit cycles. In this sense, we can say that the first and second part of Hilbert's 16th problem come closer.

Remark 1.7. I would mention for system (1.1) that by forcing the coefficients of an arbitrary odd polynomial to be those given in the Theorem 1.1 when $n \in \mathbb{N}$, $n \geq 2$ (respectively in the Theorem 1.3 when $n = 1$), do not allow us to put $n+1$ (respectively 2) limit cycles in arbitrary placements. The reason for this is the Theorem 1.2 (respectively the Theorem 1.4); in the statement of these theorems we see that the proposing polynomials f (given in Theorems 1.1 and 1.3) are the only that attain the upper bound of the $n+1$ limit cycles. Now it is easy to see that in the coefficients of these polynomials (unless in the first monomial in each case) appears the dependent radius, and this observation in turn implies that one limit cycle do not lie in arbitrary placements.

In order to see this more clearly consider for the system (1.1) the case where $n = 1$. Once we chose λ_1 from V^1, the dependent radius λ_2 follows from (1.7) will be positive (see Proposition 1.14) and different from the associated λ_1 (see Remark 1.17), and then for the system (1.1) with $n = 1$, the polynomial f given by (1.4) is the only that realizing the maximal number of 2 limit cycles, and are asymptotic to the circles $x^2 + y^2 = \lambda_1$ and $x^2 + y^2 = \lambda_2$ (note that for this circle the placement is not arbitrary, it depends on λ_1) as $\varepsilon \to 0$ (see Theorems 1.3 and 1.4). (See and Example 4.7.)

1.2. **Definitions.** In this subsection, we introduce some new definitions. These definitions are obtained by using technical integral expressions (see Remark 1.8) and properties of symmetric functions of the roots of polynomials (Vieta's formulas). The first of these definitions has an important role in the construction of the sinusoidal-type sets and also the advantage played by this definition along with the Definition 1.11 is going to be understandable in the proof of Proposition 1.14.

Remark 1.8. I adopt the sinusoidal terminology for the next two definitions, due to the formula

$$\int_0^{2\pi} \sin^{2n} t \, dt = \frac{1 \cdot 3 \dots (2n-3)(2n-1)}{2^{n-1} n!} \pi, \quad \text{for } \ n \in \mathbb{N}, \tag{1.5}$$

(see [9]) which gives the coefficients of the sums and products.

Definition 1.9 (sinusoidal-type numbers)**.** Let $\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n$ be distinct positive real numbers, where $n \in \mathbb{N}$, $n \geq 2$. We define for $n \in \mathbb{N}$, $n \geq 3$, the sinusoidal-type numbers of order n, associated to the $\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n$

$$\begin{aligned}
\bar{s}^n &:= 2(n+2) + \dots + \frac{(-1)^k}{2^{k-1}(n-k+3)\dots(n+1)} \sum_{\substack{i_1,\dots,i_k=1 \\ i_1<\dots<i_k}}^{n} \lambda_{i_1} \dots \lambda_{i_k} + \dots \\
&\quad + \frac{(-1)^n}{2^{n-2}(n+1)!} \prod_{i_1=1}^{n} \lambda_{i_1}, \\
\hat{s}^n &:= 2(n+1) + \dots + \frac{(-1)^k}{2^{k-1}(n-k+2)(n-k+3)\dots n} \sum_{\substack{i_1,\dots,i_k=1 \\ i_1<\dots<i_k}}^{n} \lambda_{i_1} \dots \lambda_{i_k} + \dots
\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{(-1)^n}{2^{n-1}n!}\prod_{i_1=1}^{n}\lambda_{i_1},\\
\tilde{s}^n &:= \frac{1}{4n(n+1)}\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n}\lambda_{i_1}\lambda_{i_2}+\dots\\
&\quad + \frac{(-1)^k(k-1)}{2^k(n-k+2)(n-k+3)\dots(n+1)}\sum_{\substack{i_1,\dots,i_k=1\\ i_1<\dots<i_k}}^{n}\lambda_{i_1}\dots\lambda_{i_k}+\dots\\
&\quad + \frac{(-1)^n(n-1)}{2^n(n+1)!}\prod_{i_1=1}^{n}\lambda_{i_1},
\end{aligned}
$$

where $2 \le k \le n-1$ for the $\bar{s}^n, \hat{s}^n$ and $3 \le k \le n-1$ for $\tilde{s}^n$. In the special case where $n = 2$, we define the sinusoidal-type numbers of second order, associated to the λ_1, λ_2

$$
\begin{gathered}
\bar{s}^2 := 8 + \frac{1}{6}\lambda_1\lambda_2,\\
\hat{s}^2 := 6 + \frac{1}{4}\lambda_1\lambda_2,\\
\tilde{s}^2 := \frac{1}{24}\lambda_1\lambda_2.
\end{gathered}
$$

For $n = 1$, we define the sinusoidal-type numbers of first order

$$
\bar{s}^1 := 6, \quad \hat{s}^1 := 4.
$$

For the sinusoidal-type numbers we have the following result. The proof is given in the Appendix.

Lemma 1.10. *For $n \in \mathbb{N}$, $n \ge 2$, we have that*

$$
\begin{gathered}
\bar{s}^n = \hat{s}^n \Longleftrightarrow \tilde{s}^n = 1,\\
\bar{s}^n > \hat{s}^n \Longleftrightarrow \tilde{s}^n < 1,\\
\bar{s}^n < \hat{s}^n \Longleftrightarrow \tilde{s}^n > 1,
\end{gathered}
$$

where $\bar{s}^n$, $\hat{s}^n$ and $\tilde{s}^n$ are the sinusoidal-type numbers of order n, with $n \in \mathbb{N}$, $n \ge 2$, of the Definition 1.9.

We continue with another definition. The role played by this definition is that the square roots of the coordinates $\lambda_1, \lambda_2, \dots, \lambda_n$ of the points $(\lambda_1, \lambda_2, \dots, \lambda_n)$, where $n \in \mathbb{N}$, $n \ge 2$, are going to be the limiting radii of the limit cycles which are asymptotic to circles of radii $\sqrt{\lambda_i}$ for $i = 1, 2, \dots, n$ centered at the origin when the small positive parameter of our system tending to 0. This will be done by forcing $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n}$ to be simple roots of the polynomial F defined in (2.5) (see Theorem 2.2). For this reason, we assume that the given $\lambda_1, \lambda_2, \dots, \lambda_n$ are all positive and with $\lambda_i \ne \lambda_j$ for all $i \ne j$ where $i, j = 1, 2, \dots, n$. In this way, we are going to construct n limit cycles for system (1.1). But Iliev in [4] proved that the maximal number of limit cycles due to polynomial perturbations of degree n of the harmonic oscillator is equal to $[\frac{n-1}{2}]$ (the largest integer less than or equal to $\frac{n-1}{2}$). Since in our case the polynomial perturbations are of degree $2n+3$ we can achieve $n+1$ limit cycles. Now, we see that we can have an additional limit cycle. About the position of this limit cycle we later give the definition of the dependent radius.

The same observation of all the above is valid and in the case where $n = 1$.

Definition 1.11 (sinusoidal-type sets)**.** For $n \in \mathbb{N}$, $n \geq 2$, we define the sinusoidal-type sets of order n as

$$\begin{aligned}
S_1^n &:= \Big\{(\lambda_1, \lambda_2, \dots, \lambda_n) : \lambda_i \neq \lambda_j \ \forall i \neq j \text{ where } i, j = 1, 2, \dots, n \text{ with } \lambda_i > 0 \\
&\qquad \forall i = 1, 2, \dots, n \text{ and } \sum_{i=1}^{n} \lambda_i < \bar{s}^n \text{ when } \tilde{s}^n > 1\Big\}, \\
S_2^n &:= \Big\{(\lambda_1, \lambda_2, \dots, \lambda_n) : \lambda_i \neq \lambda_j \ \forall i \neq j \text{ where } i, j = 1, 2, \dots, n \text{ with } \lambda_i > 0 \\
&\qquad \forall i = 1, 2, \dots, n \text{ and } \sum_{i=1}^{n} \lambda_i < \hat{s}^n \text{ when } \tilde{s}^n < 1\Big\}, \\
S_3^n &:= \Big\{(\lambda_1, \lambda_2, \dots, \lambda_n) : \lambda_i \neq \lambda_j \ \forall i \neq j \text{ where } i, j = 1, 2, \dots, n \text{ with } \lambda_i > 0 \\
&\qquad \forall i = 1, 2, \dots, n \text{ and } \sum_{i=1}^{n} \lambda_i < \bar{s}^n = \hat{s}^n \text{ when } \tilde{s}^n = 1\Big\}, \\
S_4^n &:= \Big\{(\lambda_1, \lambda_2, \dots, \lambda_n) : \lambda_i \neq \lambda_j \ \forall i \neq j \text{ where } i, j = 1, 2, \dots, n \text{ with } \lambda_i > 0 \\
&\qquad \forall i = 1, 2, \dots, n \text{ and } \sum_{i=1}^{n} \lambda_i > \bar{s}^n \text{ when } \tilde{s}^n < 1\Big\}, \\
S_5^n &:= \Big\{(\lambda_1, \lambda_2, \dots, \lambda_n) : \lambda_i \neq \lambda_j \ \forall i \neq j \text{ where } i, j = 1, 2, \dots, n \text{ with } \lambda_i > 0 \\
&\qquad \forall i = 1, 2, \dots, n \text{ and } \sum_{i=1}^{n} \lambda_i > \hat{s}^n \text{ when } \tilde{s}^n > 1\Big\}, \\
S_6^n &:= \Big\{(\lambda_1, \lambda_2, \dots, \lambda_n) : \lambda_i \neq \lambda_j \ \forall i \neq j \text{ where } i, j = 1, 2, \dots, n \text{ with } \lambda_i > 0 \\
&\qquad \forall i = 1, 2, \dots, n \text{ and } \sum_{i=1}^{n} \lambda_i > \bar{s}^n = \hat{s}^n \text{ when } \tilde{s}^n = 1\Big\},
\end{aligned}$$

where $\bar{s}^n$, $\hat{s}^n$ and $\tilde{s}^n$ are the sinusoidal-type numbers of order n, with $n \in \mathbb{N}$, $n \geq 2$, of the Definition 1.9. For $n = 1$, we define the sinusoidal-type sets of first order

$$\begin{gathered}
S_1^1 := \big\{\lambda_1 : \lambda_1 \in (0, 4)\big\}, \\
S_2^1 := \big\{\lambda_1 : \lambda_1 \in (6, +\infty)\big\}.
\end{gathered}$$

Definition 1.12. We define for $n \in \mathbb{N}$, $n \geq 2$, the set V^n as the set

$$V^n := \cup_{i=1}^{6} S_i^n.$$

For $n = 1$, we define the set V^1 as the set

$$V^1 := S_1^1 \cup S_2^1.$$

Now, we continue with the last statement of the observation that we made before the Definition 1.11. The positions of the n limit cycles have to satisfy an algebraic relation in order that there is an odd polynomial f, realizing the maximal number of limit cycles, and the position of the $(n+1)$-th limit cycle is estimated in terms of the positions of these n limit cycles. On this we have the following definition.

Definition 1.13 (dependent radius). Let $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$. We call dependent radius representing with λ_{n+1} the quantity (when is defined) given by the formula

$$\lambda_{n+1} = \lambda_{n+1}(\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n) := \frac{\Xi}{\Psi}, \tag{1.6}$$

where

$$\begin{aligned}\Xi &= 2^{n+1}(n+2)! + \dots + (-1)^k 2^{n-k+1}(n-k+2)! \sum_{\substack{i_1,\dots,i_k=1\\ i_1<\dots<i_k}}^{n} \lambda_{i_1}\dots\lambda_{i_k} \\ &\quad + \dots + 4(-1)^n \prod_{i_1=1}^{n} \lambda_{i_1}\end{aligned}$$

and

$$\begin{aligned}\Psi &= 2^{n}(n+1)! + \dots + (-1)^k 2^{n-k}(n-k+1)! \sum_{\substack{i_1,\dots,i_k=1\\ i_1<\dots<i_k}}^{n} \lambda_{i_1}\dots\lambda_{i_k} \\ &\quad + \dots + (-1)^n \prod_{i_1=1}^{n} \lambda_{i_1},\end{aligned}$$

where $1 \leq k \leq n-1$. So, the dependent radius is the $(n+1)$-th radius associated to the radii $\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n$.

For $n = 1$, let $\lambda_1 \in \mathbb{R}$, then we call dependent radius representing with λ_2 the quantity (when is defined) given by the formula

$$\lambda_2 = \lambda_2(\lambda_1) := \frac{24 - 4\lambda_1}{4 - \lambda_1}. \tag{1.7}$$

So, in this case the dependent radius is the second radius associated to the radius λ_1.

For the dependent radius we have the following result. The proof is given in the Appendix.

Proposition 1.14. *If $(\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n) \in V^n$, $n \in \mathbb{N}$, $n \geq 2$, then the dependent radius $\lambda_{n+1} = \lambda_{n+1}(\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n)$, $n \in \mathbb{N}$, $n \geq 2$, is positive. If $n = 1$ and suppose that $\lambda_1 \in V^1$, then the dependent radius $\lambda_2 = \lambda_2(\lambda_1)$ is positive.*

On the other hand if $\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n$, $n \in \mathbb{N}$, $n \geq 2$, are distinct positive real numbers so that the dependent radius λ_{n+1}, $n \in \mathbb{N}$, $n \geq 2$, associated with the radii $\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n$ is positive, then $(\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n) \in V^n$, $n \in \mathbb{N}$, $n \geq 2$. If λ_1 is a positive real number so that the dependent radius λ_2 associated to the radius λ_1 is positive, then $\lambda_1 \in V^1$.

Remark 1.15. According to Proposition 1.14, the set V^n is the biggest set from which we can choose the points $(\lambda_1, \lambda_2, \dots, \lambda_n)$ so that the corresponding dependent radius λ_{n+1} given by (1.6), is positive if $n \in \mathbb{N}$, $n \geq 2$ and the set V^1 is the biggest set from which we can choose the numbers λ_1 so that the corresponding dependent radius λ_2 given by (1.7), is positive.

Now, is following the definition which has the central role. The advantage played by this definition is that the square roots of the coordinates $\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}$ (where λ_{n+1} is the dependent radius associated to the radii $\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n$)

of the points $(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1})$, where $n \in \mathbb{N}$, $n \geq 2$, are going to be the limiting radii of the limit cycles which are asymptotic to circles of radii $\sqrt{\lambda_i}$ for $i = 1, 2, \dots, n, n+1$ centered at the origin when the small positive parameter of our system tending to 0. This will be done by forcing $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n}, \sqrt{\lambda_{n+1}}$ to be all the simple roots of the polynomial F defined in (2.5) (see Theorem 2.2). For this reason, we assume that the given points $(\lambda_1, \lambda_2, \dots, \lambda_n)$ belong to V^n and we want for the corresponding dependent radius $\lambda_{n+1} = \lambda_{n+1}(\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n)$ (which from Proposition 1.14 is positive) to satisfy that $\lambda_{n+1} \neq \lambda_j$ for all $j = 1, 2, \dots, n$. In this way, we construct $n+1$ limit cycles for system (1.1), and so we achieve the maximal number of limit cycles due to polynomial perturbations of degree $2n+3$ of the harmonic oscillator (see [4]).

The same observation of all the above is valid and in the case where $n = 1$.

Definition 1.16. We define now for $n \in \mathbb{N}$, $n \geq 2$, the sets

$$\begin{aligned}
S^n_{1,n+1} := \big\{&(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}) : (\lambda_1, \lambda_2, \dots, \lambda_n) \in S^n_1,\ \lambda_{n+1} = \lambda_{n+1}(\lambda_1, \dots, \lambda_n),\\
&\lambda_{n+1} \neq \lambda_j\ \forall j = 1, 2, \dots, n\big\},\\
S^n_{2,n+1} := \big\{&(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}) : (\lambda_1, \lambda_2, \dots, \lambda_n) \in S^n_2,\ \lambda_{n+1} = \lambda_{n+1}(\lambda_1, \dots, \lambda_n),\\
&\lambda_{n+1} \neq \lambda_j\ \forall j = 1, 2, \dots, n\big\},\\
S^n_{3,n+1} := \big\{&(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}) : (\lambda_1, \lambda_2, \dots, \lambda_n) \in S^n_3,\ \lambda_{n+1} = \lambda_{n+1}(\lambda_1, \dots, \lambda_n),\\
&\lambda_{n+1} \neq \lambda_j\ \forall j = 1, 2, \dots, n\big\},\\
S^n_{4,n+1} := \big\{&(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}) : (\lambda_1, \lambda_2, \dots, \lambda_n) \in S^n_4,\ \lambda_{n+1} = \lambda_{n+1}(\lambda_1, \dots, \lambda_n),\\
&\lambda_{n+1} \neq \lambda_j\ \forall j = 1, 2, \dots, n\big\},\\
S^n_{5,n+1} := \big\{&(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}) : (\lambda_1, \lambda_2, \dots, \lambda_n) \in S^n_5,\ \lambda_{n+1} = \lambda_{n+1}(\lambda_1, \dots, \lambda_n),\\
&\lambda_{n+1} \neq \lambda_j\ \forall j = 1, 2, \dots, n\big\},\\
S^n_{6,n+1} := \big\{&(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}) : (\lambda_1, \lambda_2, \dots, \lambda_n) \in S^n_6,\ \lambda_{n+1} = \lambda_{n+1}(\lambda_1, \dots, \lambda_n),\\
&\lambda_{n+1} \neq \lambda_j\ \forall j = 1, 2, \dots, n\big\},
\end{aligned}$$

where $S^n_1, S^n_2, S^n_3, S^n_4, S^n_5$ and S^n_6 are the sinusoidal-type sets of order n, with $n \in \mathbb{N}$, $n \geq 2$, of the Definition 1.11 and λ_{n+1} is the dependent radius given by (1.6). For $n = 1$, we define the sets

$$\begin{aligned}
S^1_{1,2} &:= \big\{(\lambda_1, \lambda_2) : \lambda_1 \in S^1_1,\ \lambda_2 = \lambda_2(\lambda_1),\ \lambda_2 \neq \lambda_1\big\},\\
S^1_{2,2} &:= \big\{(\lambda_1, \lambda_2) : \lambda_1 \in S^1_2,\ \lambda_2 = \lambda_2(\lambda_1),\ \lambda_2 \neq \lambda_1\big\},
\end{aligned}$$

where S^1_1 and S^1_2 are the sinusoidal-type sets of first order of the Definition 1.11 and λ_2 is the dependent radius given by (1.7).

Remark 1.17. Notice that, if $\lambda_1 \in (0, 4)$, then the dependent radius λ_2 given by (1.7), belongs to $(6, +\infty)$ and so we have that $(\lambda_1, \lambda_2) \in S^1_{1,2}$. If $\lambda_1 \in (6, +\infty)$, then the dependent radius λ_2 given by (1.7), belongs to $(0, 4)$ and so we have that $(\lambda_1, \lambda_2) \in S^1_{2,2}$.

Remark 1.18. According to Remark 1.17, the sets $S^1_{1,2}$ and $S^1_{2,2}$, which defined as above, take the more simple form

$$\begin{aligned}
S^1_{1,2} &= \big\{(\lambda_1, \lambda_2) : \lambda_1 \in S^1_1,\ \lambda_2 = \lambda_2(\lambda_1)\big\},\\
S^1_{2,2} &= \big\{(\lambda_1, \lambda_2) : \lambda_1 \in S^1_2,\ \lambda_2 = \lambda_2(\lambda_1)\big\},
\end{aligned}$$

where S_1^1 and S_2^1 are the sinusoidal-type sets of first order of the Definition 1.11 and λ_2 is the dependent radius given by (1.7).

Definition 1.19. We define for $n \in \mathbb{N}$, $n \geq 2$, the set V_{n+1}^n as the set

$$V_{n+1}^n := \cup_{i=1}^6 S_{i,n+1}^n.$$

For $n = 1$, we define the set V_2^1 as the set

$$V_2^1 := S_{1,2}^1 \cup S_{2,2}^1.$$

Remark 1.20. In the set V_{n+1}^n, $n \in \mathbb{N}$, the positive dependent radius λ_{n+1}, $n \in \mathbb{N}$, is obviously different from $\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n$, $n \in \mathbb{N}$.

Remark 1.21. It is possible, for $n \in \mathbb{N}$, $n \geq 2$, the point $(\lambda_1, \lambda_2, \dots, \lambda_n) \in V^n$ but the point $(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}) \notin V_{n+1}^n$, where λ_{n+1} is the dependent radius given by (1.6), associated to the radii $\lambda_1, \lambda_2, \dots, \lambda_n$. (For example, it is easy to see that the point $(4, 6) \in V^2$; in particular belongs to S_3^2. We calculate the dependent radius λ_3, associated to the $4, 6$, which from Proposition 1.14 is positive and we have that $\lambda_3 = 6$. Now, the point $(4, 6, 6) \notin V_3^2$.)

For $n = 1$, according to Remark 1.17, if $\lambda_1 \in V^1$, then $(\lambda_1, \lambda_2) \in V_2^1$, where λ_2 is the dependent radius given by (1.7), associated to the radius λ_1.

Definition 1.22 (Λ-points (lambda points))**.** We will call the points

$$(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}) \in V_{n+1}^n, \quad n \in \mathbb{N},$$

the Λ-points.

1.3. Known results on Liénard equations.

We now continue with known results on Liénard equations. Such equations are very challenging and many questions about these are still open. Note that Smale [11] proposed to study Hilbert's 16th problem restricted to these special classes. This is Smale's 13th problem. We mention several interesting works here. The Liénard equation

$$\ddot{x} + g(x)\dot{x} + x = 0, \tag{1.8}$$

where g is a polynomial, is another generalization of the Van der Pol equation. Equation (1.8) can be studied in a phase plane as a system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - g(x)y, \end{aligned} \tag{1.9}$$

or in the so-called Liénard plane as

$$\begin{aligned} \dot{x} &= y - G(x), \\ \dot{y} &= -x, \end{aligned} \tag{1.10}$$

where $G(x) = \int_0^x g(s)\,ds$. The systems (1.9) and (1.10) are analytically conjugate. We observe that system (1.1) is not of the form of Liénard's equation (1.9), except when $f(y) = y$. Obviously, for $f(y) \neq y$, (1.9) can not reduce to (1.1). So, in general, (1.1) is not a special case of (1.9) and (1.9) is not a special case of (1.1).

Liénard [6] proved that, if G is a continuous odd function, which has a unique positive root at $x = a$ and is monotone increasing for $x \geq a$, then (1.10) has a unique limit cycle. Rychkov [10] proved that, if G is an odd polynomial of degree 5, then (1.10) has at most two limit cycles.

Lins, de Melo and Pugh [7] have studied the Liénard equation (1.10), where G is a polynomial of degree d. They proved that, if $G(x) = a_3x^3 + a_2x^2 + a_1x$, then

(1.10) has at most one limit cycle. In fact, they gave a complete classification of the phase space of the cubic Liénard's equation, in terms of some explicit algebraic conditions on the coefficients of G. Also, using a method due to Poincaré they proved that, if $d = 2n+1$ or $2n+2$, then for any $k \in \mathbb{N}_0$ with $0 \le k \le n$ there exists a polynomial $G(x) = a_d x^d + \dots + a_1 x$ such that the system (1.10) has exactly k closed orbits. Motivated by this, they conjectured that the maximum number of limit cycles for system (1.10), where G is a polynomial of degree n would be equal to $[\frac{n-1}{2}]$.

However, in [3] it has been proven by Dumortier, Panazzolo and Roussarie the existence of classical Liénard equations (1.10) of degree 7 with at least 4 limit cycles. This easily implied the existence of classical Liénard equations of degree n, $n \ge 7$, with $[\frac{n-1}{2}] + 1$ limit cycles. The counterexamples were proven to occur in systems

$$\begin{gathered}\dot{x} = y - \Big(x^7 + \sum_{i=2}^{6} c_i x^i\Big),\\ \dot{y} = \varepsilon(b - x),\end{gathered}$$

for small $\varepsilon > 0$. Recently, in [8] it has been proven by De Maesschalck and Dumortier the existence of classical Liénard equations (1.10) of degree 6 having 4 limit cycles. It implies the existence of classical Liénard equations of degree n, $n \ge 6$, having at least $[\frac{n-1}{2}] + 2$ limit cycles.

Ilyashenko and Panov [5] proved that, if

$$G(x) = x^n + \sum_{i=1}^{n-1} a_i x^i, \quad |a_i| \le C, \quad C \ge 4, \quad n \ge 5,$$

and suppose that n is odd, then the number $L(n, C)$ of limit cycles of (1.10) admits the upper estimate

$$L(n, C) \le \exp(\exp C^{14n}).$$

Caubergh and Dumortier [2] proved that the maximal number of limit cycles for (1.10) of even degree is finite when restricting the coefficients to a compact, thus proving the existential part of Hilbert's 16th problem for Liénard equations when restricting the coefficients to a compact set.

2. Elementary remarks about small perturbation of a Hamiltonian system

We consider the system

$$\begin{gathered}\dot{x} = y + \varepsilon f_1(x, y),\\ \dot{y} = -x + \varepsilon f_2(x, y),\end{gathered} \tag{2.1}$$

where $0 < \varepsilon \ll 1$ and f_1, f_2 are C^1 functions of x and y, which is a perturbation of the linear harmonic oscillator

$$\begin{gathered}\dot{x} = y,\\ \dot{y} = -x,\end{gathered}$$

which has all the solutions periodic with:

$$x^0(t) = A\cos(t - t_0) \quad \text{and} \quad y^0(t) = -A\sin(t - t_0).$$

In general, the phase curves of (2.1) are not closed and it is possible to have the form of a spiral with a small distance of order ε between neighboring turns.

In order to decide if the phase curve approaches the origin or recedes from it, we consider the function (mechanic energy)

$$E(x,y)=\frac{1}{2}\big(x^2+y^2\big).$$

It is easy to compute the derivative of the energy and it is proportional to ε:

$$\frac{d}{dt}E(x,y)=x\dot{x}+y\dot{y}=\varepsilon\big(xf_1(x,y)+yf_2(x,y)\big)=:\varepsilon\dot{E}(x,y). \tag{2.2}$$

We want information for the sign of the quantity

$$\int_0^{T(\varepsilon)}\varepsilon\dot{E}\big(x^\varepsilon(t),y^\varepsilon(t)\big)dt=:\Delta E, \tag{2.3}$$

which corresponds to the change of energy of $(x^\varepsilon(t),y^\varepsilon(t))$ in one complete turn: $y^\varepsilon(0)=y^\varepsilon(T(\varepsilon))=0$. Using the theorem of continuous dependence on parameters in ODEs, one can prove the following lemma (see [1]):

Lemma 2.1. *For* (2.3) *we have*

$$\Delta E=\varepsilon\int_0^{2\pi}\dot{E}\big(A\cos(t-t_0),-A\sin(t-t_0)\big)dt+o(\varepsilon). \tag{2.4}$$

Let

$$F(A):=\int_0^{2\pi}\dot{E}\big(x^0(t),y^0(t)\big)dt, \tag{2.5}$$

and we write (2.4) as

$$\Delta E=\varepsilon\Big[F(A)+\frac{o(\varepsilon)}{\varepsilon}\Big].$$

Using the implicit function theorem, one can prove the following theorem, which is the Poincaré's method (see [1]):

Theorem 2.2. *If the function F given by* (2.5)*, has a positive simple root A_0, namely*

$$F(A_0)=0 \quad \textit{and} \quad F'(A_0)\neq 0,$$

then (2.1) *has a periodic solution with amplitude $A_0+O(\varepsilon)$ for $0<\varepsilon\ll 1$.*

3. Proofs of Theorems 1.1, 1.2, 1.3 and 1.4

Proof of Theorem 1.1. From (2.2) we have

$$\dot{E}(x,y)=yf(y)\big(1-x^2\big), \tag{3.1}$$

where f is the polynomial introduced in (1.3). Substituting (3.1) into (2.5), we obtain that

$$F(A)=\int_0^{2\pi}y^0(t)f(y^0(t))\big(1-(x^0(t))^2\big)dt, \tag{3.2}$$

where f is the polynomial introduced in (1.3). We insert the definition of f given by (1.3) in (3.2) to obtain

$$\begin{aligned} F(A)=\tau\int_0^{2\pi}&\Big[(y^0(t))^{2(n+1)}+(2n+1)\Big(1-\frac{1}{2(n+2)}\sum_{i_1=1}^{n+1}\lambda_{i_1}\Big)(y^0(t))^{2n}\\ &+\dots+\Big(\frac{1\cdot 3\dots(2n+1)}{2^{n+1}(n+2)!}(-1)^n\prod_{i_1=1}^{n+1}\lambda_{i_1}\Big)(y^0(t))^2\Big]\big(1-(x^0(t))^2\big)dt. \end{aligned} \tag{3.3}$$

Substituting $x^0(t) = A\cos(t-t_0)$ and $y^0(t) = -A\sin(t-t_0)$ into (3.3) we get

$$\begin{aligned}
F(A) = \tau A^2 \int_0^{2\pi} \Big[& A^{2n}\sin^{2(n+1)}(t-t_0) \\
& + (2n+1)\Big(1 - \frac{1}{2(n+2)}\sum_{i_1=1}^{n+1}\lambda_{i_1}\Big)A^{2(n-1)}\sin^{2n}(t-t_0) \\
& + \dots + \Big(\frac{1\cdot 3\dots(2n+1)}{2^{n+1}(n+2)!}(-1)^n\prod_{i_1=1}^{n+1}\lambda_{i_1}\Big)\sin^2(t-t_0)\Big] \\
& \times \big[1 - A^2 + A^2\sin^2(t-t_0)\big]\,dt,
\end{aligned}$$

whence, after multiplying the terms in the two brackets we get

$$\begin{aligned}
F(A) = \tau A^2 \int_0^{2\pi} \Big[& A^{2n}\sin^{2(n+1)}(t-t_0) - A^{2(n+1)}\sin^{2(n+1)}(t-t_0) \\
& + A^{2(n+1)}\sin^{2(n+2)}(t-t_0) \\
& + (2n+1)\Big(1 - \frac{1}{2(n+2)}\sum_{i_1=1}^{n+1}\lambda_{i_1}\Big)A^{2(n-1)}\sin^{2n}(t-t_0) \\
& - (2n+1)\Big(1 - \frac{1}{2(n+2)}\sum_{i_1=1}^{n+1}\lambda_{i_1}\Big)A^{2n}\sin^{2n}(t-t_0) \\
& + (2n+1)\Big(1 - \frac{1}{2(n+2)}\sum_{i_1=1}^{n+1}\lambda_{i_1}\Big)A^{2n}\sin^{2(n+1)}(t-t_0) \\
& + \dots + \Big(\frac{1\cdot 3\dots(2n+1)}{2^{n+1}(n+2)!}(-1)^n\prod_{i_1=1}^{n+1}\lambda_{i_1}\Big)\sin^2(t-t_0) \\
& - \Big(\frac{1\cdot 3\dots(2n+1)}{2^{n+1}(n+2)!}(-1)^n\prod_{i_1=1}^{n+1}\lambda_{i_1}\Big)A^2\sin^2(t-t_0) \\
& + \Big(\frac{1\cdot 3\dots(2n+1)}{2^{n+1}(n+2)!}(-1)^n\prod_{i_1=1}^{n+1}\lambda_{i_1}\Big)A^2\sin^4(t-t_0)\Big]\,dt.
\end{aligned}$$

Using now (1.5), we finally obtain

$$\begin{aligned}
F(A) = \pi\tau A^2\frac{1\cdot 3\dots(2n+1)}{2^{n+1}(n+2)!}\Big[& -A^{2(n+1)} + \sum_{i_1=1}^{n+1}\lambda_{i_1}A^{2n} - \sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n+1}\lambda_{i_1}\lambda_{i_2}A^{2(n-1)} \\
& + \dots - (-1)^k \sum_{\substack{i_1,\dots,i_k=1\\ i_1<\dots<i_k}}^{n+1}\lambda_{i_1}\dots\lambda_{i_k}A^{2(n-k+1)} - \dots \\
& - (-1)^n \sum_{\substack{i_1,\dots,i_n=1\\ i_1<\dots<i_n}}^{n+1}\lambda_{i_1}\dots\lambda_{i_n}A^2 + (-1)^n\prod_{i_1=1}^{n+1}\lambda_{i_1}\Big].
\end{aligned}$$

We show now that $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n}, \sqrt{\lambda_{n+1}}$ are roots of the polynomial F. Let

$$\begin{aligned} W(A) := A^{2(n+1)} - \sum_{i_1=1}^{n+1} \lambda_{i_1} A^{2n} + \sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n+1} \lambda_{i_1}\lambda_{i_2} A^{2(n-1)} - \dots \\ + (-1)^n \sum_{\substack{i_1,\dots,i_n=1\\ i_1<\dots<i_n}}^{n+1} \lambda_{i_1}\dots\lambda_{i_n} A^2 - (-1)^n \prod_{i_1=1}^{n+1} \lambda_{i_1}, \end{aligned}$$

namely, we write

$$F(A) = -\pi\tau A^2 \frac{1\cdot 3\dots(2n+1)}{2^{n+1}(n+2)!} \cdot W(A).$$

Now, it suffices to show that $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n}, \sqrt{\lambda_{n+1}}$ are roots of the polynomial W. Without loss of generality we consider the quantity $\sqrt{\lambda_1}$. For $W\big(\sqrt{\lambda_1}\,\big)$ we have

$$\begin{aligned} W\big(\sqrt{\lambda_1}\,\big) &= \lambda_1^{n+1} - \lambda_1^n \sum_{i_1=1}^{n+1} \lambda_{i_1} + \lambda_1^{n-1} \sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n+1} \lambda_{i_1}\lambda_{i_2} - \dots - (-1)^n \prod_{i_1=1}^{n+1} \lambda_{i_1} \\ &= \lambda_1^{n+1} - \lambda_1^{n+1} - \lambda_1^n \sum_{i_1=2}^{n+1} \lambda_{i_1} + \lambda_1^n \sum_{i_1=2}^{n+1} \lambda_{i_1} + \lambda_1^{n-1} \sum_{\substack{i_1,i_2=2\\ i_1<i_2}}^{n+1} \lambda_{i_1}\lambda_{i_2} \\ &\quad - \lambda_1^{n-1} \sum_{\substack{i_1,i_2=2\\ i_1<i_2}}^{n+1} \lambda_{i_1}\lambda_{i_2} - \lambda_1^{n-2} \sum_{\substack{i_1,i_2,i_3=2\\ i_1<i_2<i_3}}^{n+1} \lambda_{i_1}\lambda_{i_2}\lambda_{i_3} \\ &\quad + \lambda_1^{n-2} \sum_{\substack{i_1,i_2,i_3=2\\ i_1<i_2<i_3}}^{n+1} \lambda_{i_1}\lambda_{i_2}\lambda_{i_3} + \dots + (-1)^n \prod_{i_1=1}^{n+1} \lambda_{i_1} - (-1)^n \prod_{i_1=1}^{n+1} \lambda_{i_1} \\ &= 0. \end{aligned}$$

So, $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n}, \sqrt{\lambda_{n+1}}$ are roots of the polynomial W and therefore and for the polynomial F.

Now, using Theorem 2.2, it suffices to show that $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n}, \sqrt{\lambda_{n+1}}$ are not roots of the polynomial W'; therefore they are not roots and for polynomial F'. For the derivative of W we have that

$$\begin{aligned} W'(A) = 2A\Big[(n+1)A^{2n} - n \sum_{i_1=1}^{n+1} \lambda_{i_1} A^{2(n-1)} + (n-1) \sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n+1} \lambda_{i_1}\lambda_{i_2} A^{2(n-2)} \\ - \dots + (-1)^n \sum_{\substack{i_1,\dots,i_n=1\\ i_1<\dots<i_n}}^{n+1} \lambda_{i_1}\dots\lambda_{i_n}\Big]. \end{aligned}$$

Now, we have that one root of W' is $A = 0$ and we also have another $2n$ roots. From those $2n$ roots, n are positive and the other n are negative (these roots are opposite numbers). Let

$$G(A) := (n+1)A^{2n} - n\sum_{i_1=1}^{n+1}\lambda_{i_1}A^{2(n-1)} + (n-1)\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n+1}\lambda_{i_1}\lambda_{i_2}A^{2(n-2)} - \dots + (-1)^n\sum_{\substack{i_1,\dots,i_n=1\\ i_1<\dots<i_n}}^{n+1}\lambda_{i_1}\dots\lambda_{i_n},$$

namely, we write

$$W'(A) = 2A\cdot G(A).$$

Now, we check if the roots $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n}, \sqrt{\lambda_{n+1}}$ of the polynomial W are possible to be roots and for the polynomial W', therefore and for the polynomial G. Without loss of generality we consider the root $\sqrt{\lambda_1}$ of W. For $G\big(\sqrt{\lambda_1}\,\big)$ we have

$$\begin{aligned}
G\big(\sqrt{\lambda_1}\,\big) &= (n+1)\lambda_1^n - n\lambda_1^{n-1}\sum_{i_1=1}^{n+1}\lambda_{i_1} + (n-1)\lambda_1^{n-2}\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n+1}\lambda_{i_1}\lambda_{i_2} \\
&\quad - \dots + (-1)^n\sum_{\substack{i_1,\dots,i_n=1\\ i_1<\dots<i_n}}^{n+1}\lambda_{i_1}\dots\lambda_{i_n} \\
&= \lambda_1^n - \lambda_1^{n-1}\sum_{i_1=2}^{n+1}\lambda_{i_1} + \lambda_1^{n-2}\sum_{\substack{i_1,i_2=2\\ i_1<i_2}}^{n+1}\lambda_{i_1}\lambda_{i_2} - \lambda_1^{n-3}\sum_{\substack{i_1,i_2,i_3=2\\ i_1<i_2<i_3}}^{n+1}\lambda_{i_1}\lambda_{i_2}\lambda_{i_3} \\
&\quad + \dots - (-1)^n\lambda_1\sum_{\substack{i_1,\dots,i_n=2\\ i_1<\dots<i_n}}^{n+1}\lambda_{i_1}\dots\lambda_{i_n} + (-1)^n\lambda_2\lambda_3\dots\lambda_n\lambda_{n+1} \\
&= (\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\dots(\lambda_1-\lambda_n)(\lambda_1-\lambda_{n+1}).
\end{aligned}$$

Obviously, $W'\big(\sqrt{\lambda_1}\big)$ is not zero since in the set V_{n+1}^n we have that $\lambda_1 \neq \lambda_j$ for $j = 2, 3, \dots, n, n+1$. Similarly, none of the $\sqrt{\lambda_2}, \sqrt{\lambda_3}, \dots, \sqrt{\lambda_n}, \sqrt{\lambda_{n+1}}$ is a root of W'.

Therefore, we have that the roots $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n}, \sqrt{\lambda_{n+1}}$ of W are not roots of W'. Finally, none of the roots $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n}, \sqrt{\lambda_{n+1}}$ of F is a root of F'. That is essential so that the $n+1$ simple roots of F create $n+1$ limit cycles. Hence, from Poincaré's method (see Theorem 2.2) it follows that (1.1), with f be the polynomial introduced in (1.3), has at least $n+1$ limit cycles, and are asymptotic to circles of radius $\sqrt{\lambda_i}$ for $i = 1, 2, \dots, n+1$ centered at the origin as $\varepsilon \to 0$.

Let now prove that the number of limit cycles for system (1.1), with ε small and f be the polynomial introduced in (1.3), is exactly $n+1$. The proof of this can be derived from the work of Iliev [4] since it constitutes a special case of the Theorem 1 proved there. Actually, applying this theorem from [4] for the special case $k = 1$, since the degree of (1.1) is $2n+3$ we can obtain at most $n+1$ limit cycles. Finally, combining this result with the result that (1.1), with f be the polynomial introduced in (1.3), has at least $n+1$ limit cycles we get the desired result, namely that the number of limit cycles for system (1.1), with ε small and f be the polynomial introduced in (1.3) is exactly $n+1$.

Now, concerning the stability of the limit cycles we have the following.

From now on we will suppose for the coordinates $\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}$ of the points $(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}) \in V^n_{n+1}$, $n \in \mathbb{N}$, $n \geq 2$, an ordering such that $\lambda_1 < \lambda_2 < \dots < \lambda_n < \lambda_{n+1}$. We can always achieve this since the positive real numbers $\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}$ are distinct. Note that in this order with λ_{n+1} we do not necessary mean the dependent radius. Since

$$G\big(\sqrt{\lambda_1}\,\big) = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\dots(\lambda_1 - \lambda_n)(\lambda_1 - \lambda_{n+1}),$$

we have that $W'\big(\sqrt{\lambda_1}\big) < 0$ if n is odd and that $W'\big(\sqrt{\lambda_1}\big) > 0$ if n is even.

So using the fact that, if $F'\big(\sqrt{\lambda_i}\big) < 0$ the limit cycle $x^2 + y^2 = \lambda_i + O(\varepsilon)$ is stable and if $F'\big(\sqrt{\lambda_i}\big) > 0$ the limit cycle is unstable we have for the stability of the $n+1$ limit cycles that, if $\tau > 0$ (respectively $\tau < 0$)

$$x^2 + y^2 = \lambda_1 + O(\varepsilon), \quad x^2 + y^2 = \lambda_3 + O(\varepsilon), \dots, \quad x^2 + y^2 = \lambda_{n+1} + O(\varepsilon)$$

are stable (respectively unstable) and

$$x^2 + y^2 = \lambda_2 + O(\varepsilon), \quad x^2 + y^2 = \lambda_4 + O(\varepsilon), \dots, \quad x^2 + y^2 = \lambda_n + O(\varepsilon)$$

are unstable (respectively stable) for n even; and

$$x^2 + y^2 = \lambda_1 + O(\varepsilon), \quad x^2 + y^2 = \lambda_3 + O(\varepsilon), \dots, \quad x^2 + y^2 = \lambda_n + O(\varepsilon)$$

are unstable (respectively stable) and

$$x^2 + y^2 = \lambda_2 + O(\varepsilon), \quad x^2 + y^2 = \lambda_4 + O(\varepsilon), \dots, \quad x^2 + y^2 = \lambda_{n+1} + O(\varepsilon)$$

are stable (respectively unstable) for n odd. The proof is complete. $\square$

Proof of Theorem 1.2. As we already saw, according to Theorem 1 from [4] the number of $n+1$ limit cycles is an upper bound for the number of limit cycles for system (1.1), where ε is small and f is an arbitrary odd polynomial of degree $2n+1$.

Now, it is easy to see that for system (1.1), where f is an arbitrary odd polynomial of degree $2n+1$, the associated F given by (2.5) is an even polynomial of degree $2n+4$, with 0 as a double root. Therefore, in general the polynomial F has at most $n+1$ simple positive roots. Furthermore, since V^n, $n \in \mathbb{N}$, $n \geq 2$ is the biggest set from which we can choose the points $(\lambda_1, \lambda_2, \dots, \lambda_n)$ so that the dependent radius λ_{n+1} given by (1.6), is positive if $n \in \mathbb{N}$, $n \geq 2$ (see Remark 1.15) and F as we showed has at most $n+1$ simple positive roots, we must choose the points $(\lambda_1, \lambda_2, \dots, \lambda_n, \lambda_{n+1}) \in V^n_{n+1}$, in order the polynomial F has exactly $n+1$ simple positive roots, and thus, from the set of all the odd polynomials, the polynomials f given by (1.3) are the only such that the system (1.1) attains the upper bound of the $n+1$ limit cycles. The proof is complete. $\square$

Proof of Theorem 1.3. The proof is identical as in Theorem 1.1; the only modification is that the polynomial f given by (1.3) will be replaced by the polynomial f introduced in (1.4). $\square$

Proof of Theorem 1.4. The proof is identical as in Theorem 1.2; the only modification is that the case where $n \in \mathbb{N}$, $n \geq 2$ will be replaced by $n = 1$. $\square$

4. Examples

In this section we illustrate the general theory of this work by some examples.

Example 4.1. We consider $\lambda_1 = 4$, $\lambda_2 = 5$. These λ_1, λ_2 are distinct and positive. We have according to Definition 1.9 that the sinusoidal-type numbers of second order, associated to the values $4, 5$ are

$$\begin{gathered}\bar{s}^2 := 8 + \frac{1}{6}\lambda_1\lambda_2 = 8 + \frac{1}{6}\cdot 4\cdot 5 = \frac{34}{3},\\ \hat{s}^2 := 6 + \frac{1}{4}\lambda_1\lambda_2 = 6 + \frac{1}{4}\cdot 4\cdot 5 = 11,\\ \tilde{s}^2 := \frac{1}{24}\lambda_1\lambda_2 = \frac{1}{24}\cdot 4\cdot 5 = \frac{5}{6}.\end{gathered}$$

Since $\lambda_1 + \lambda_2 = 4 + 5 = 9$, we have that $(4,5) \in S_2^2$ and therefore $(4,5) \in V^2$. We calculate the dependent radius λ_3, associated to the values $4, 5$, which from Proposition 1.14 is positive and we have that

$$\lambda_3 := \frac{192 - 24(\lambda_1 + \lambda_2) + 4\lambda_1\lambda_2}{24 - 4(\lambda_1 + \lambda_2) + \lambda_1\lambda_2} = \frac{192 - 216 + 80}{24 - 36 + 20} = 7.$$

So, we have the Λ-point $(4,5,7)$ which belongs to the set V_3^2.

Now, using Theorem 1.1, for $\tau = 16$, we have that the system

$$\begin{gathered}\dot{x} = y,\\ \dot{y} = -x + \varepsilon\big(16y^5 - 80y^3 + 175y\big)\big(1 - x^2\big),\end{gathered} \tag{4.1}$$

with $0 < \varepsilon \ll 1$ has exactly, the limit cycles $x^2 + y^2 = 4 + O(\varepsilon)$, $x^2 + y^2 = 5 + O(\varepsilon)$ and $x^2 + y^2 = 7 + O(\varepsilon)$.

Since $4 < 5 < 7$, the limit cycles $x^2 + y^2 = 4 + O(\varepsilon)$, $x^2 + y^2 = 7 + O(\varepsilon)$ are stable and the limit cycle $x^2 + y^2 = 5 + O(\varepsilon)$ is unstable.

From Theorem 1.1 we have for the system (4.1) that, if we change τ from 16 to -16 the unstable limit cycle $x^2 + y^2 = 5 + O(\varepsilon)$ becomes stable and the stable limit cycles $x^2 + y^2 = 4 + O(\varepsilon)$, $x^2 + y^2 = 7 + O(\varepsilon)$ become unstable.

Example 4.2. We consider $\lambda_1 = 4, \lambda_2 = 16$. These λ_1, λ_2 are distinct and positive. We have according to Definition 1.9 that the sinusoidal-type numbers of second order, associated to the $4, 16$ are

$$\begin{gathered}\bar{s}^2 := 8 + \frac{1}{6}\lambda_1\lambda_2 = 8 + \frac{1}{6}\cdot 4\cdot 16 = \frac{56}{3},\\ \hat{s}^2 := 6 + \frac{1}{4}\lambda_1\lambda_2 = 6 + \frac{1}{4}\cdot 4\cdot 16 = 22,\\ \tilde{s}^2 := \frac{1}{24}\lambda_1\lambda_2 = \frac{1}{24}\cdot 4\cdot 16 = \frac{8}{3}.\end{gathered}$$

Since $\lambda_1 + \lambda_2 = 4 + 16 = 20$, we have that $(4,16) \notin S_1^2$, $(4,16) \notin S_5^2$ and therefore $(4,16) \notin V^2$. Therefore from Proposition 1.14 the dependent radius λ_3, associated to the $4, 16$ is not positive.

So, according to the Theorem 1.1 it does not exist a system of the form

$$\begin{gathered}\dot{x} = y,\\ \dot{y} = -x + \varepsilon\big(a_0y^5 + a_1y^3 + a_2y\big)\big(1 - x^2\big),\end{gathered}$$

where $0 < \varepsilon \ll 1$ and $a_0, a_1, a_2 \in \mathbb{R}$, which has exactly three limit cycles whereof the two of them have the equations $x^2 + y^2 = 4 + O(\varepsilon)$, $x^2 + y^2 = 16 + O(\varepsilon)$.

Example 4.3. We consider $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$. These $\lambda_1, \lambda_2, \lambda_3$ satisfy our assertions, since $\lambda_i \neq \lambda_j$ for all $i \neq j$ where $i, j = 1, 2, 3$ and are positive. We have according to Definition 1.9 that the sinusoidal-type numbers of third order, associated to the $1, 2, 3$ are

$$\begin{aligned}
\bar{s}^3 &:= 10 + \frac{1}{8}(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) - \frac{1}{48}\lambda_1\lambda_2\lambda_3 = 10 + \frac{11}{8} - \frac{1}{8} = \frac{45}{4}, \\
\hat{s}^3 &:= 8 + \frac{1}{6}(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) - \frac{1}{24}\lambda_1\lambda_2\lambda_3 = 8 + \frac{11}{6} - \frac{1}{4} = \frac{115}{12}, \\
\tilde{s}^3 &:= \frac{1}{48}(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) - \frac{1}{96}\lambda_1\lambda_2\lambda_3 = \frac{11}{48} - \frac{1}{16} = \frac{1}{6}.
\end{aligned}$$

Since $\lambda_1 + \lambda_2 + \lambda_3 = 1 + 2 + 3 = 6$, we have that $(1, 2, 3) \in S_2^3$ and therefore $(1, 2, 3) \in V^3$. We calculate the dependent radius λ_4, associated to the $1, 2, 3$, which from Proposition 1.14 is positive and we have that

$$\lambda_4 := \frac{1920 - 192(\lambda_1 + \lambda_2 + \lambda_3) + 24(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) - 4\lambda_1\lambda_2\lambda_3}{192 - 24(\lambda_1 + \lambda_2 + \lambda_3) + 4(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) - \lambda_1\lambda_2\lambda_3} = \frac{504}{43}.$$

So, we have the Λ-point $(1, 2, 3, 504/43)$ which belongs to the set V_4^3.

Now, using Theorem 1.1, for $\tau = 43/8$, we have that the system

$$\begin{gathered}
\dot{x} = y, \\
\dot{y} = -x + \varepsilon\Big(\frac{43}{8}y^7 - \frac{581}{20}y^5 + \frac{5887}{128}y^3 - \frac{1323}{64}y\Big)\big(1 - x^2\big),
\end{gathered} \tag{4.2}$$

with $0 < \varepsilon \ll 1$ has exactly, the limit cycles $x^2 + y^2 = 1 + O(\varepsilon)$, $x^2 + y^2 = 2 + O(\varepsilon)$, $x^2 + y^2 = 3 + O(\varepsilon)$ and $x^2 + y^2 = (504/43) + O(\varepsilon)$.

Since $1 < 2 < 3 < 504/43$, the limit cycles $x^2 + y^2 = 1 + O(\varepsilon)$, $x^2 + y^2 = 3 + O(\varepsilon)$ are unstable and the limit cycles $x^2 + y^2 = 2 + O(\varepsilon)$, $x^2 + y^2 = (504/43) + O(\varepsilon)$ are stable.

From Theorem 1.1 we have for the system (4.2) that, if we change τ from $43/8$ to $-43/8$ the unstable limit cycles $x^2 + y^2 = 1 + O(\varepsilon)$, $x^2 + y^2 = 3 + O(\varepsilon)$ become stable and the stable limit cycles $x^2 + y^2 = 2 + O(\varepsilon)$, $x^2 + y^2 = (504/43) + O(\varepsilon)$ become unstable.

Example 4.4. We consider $\lambda_i = i$ for $i = 1, 2, \dots, 6$. These $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$ satisfy our assertions, since $\lambda_i \neq \lambda_j$ for all $i \neq j$ where $i, j = 1, 2, 3, 4, 5, 6$ and are positive. It is easy to show, after some calculations, that $(1, 2, 3, 4, 5, 6) \in V^6$. We calculate the dependent radius λ_7, associated to the $1, 2, 3, 4, 5, 6$, which from Proposition 1.14 is positive and we have that $\lambda_7 = 13337/690$. So, we have the Λ-point $(1, 2, 3, 4, 5, 6, 13337/690)$ which belongs to the set V_7^6.

Therefore, from Theorem 1.1 exists a system of the form (1.1), where $0 < \varepsilon \ll 1$ and f is an odd polynomial of degree 13, which has exactly the limit cycles: $x^2 + y^2 = 1 + O(\varepsilon)$, $x^2 + y^2 = 2 + O(\varepsilon)$, $x^2 + y^2 = 3 + O(\varepsilon)$, $x^2 + y^2 = 4 + O(\varepsilon)$, $x^2 + y^2 = 5 + O(\varepsilon)$, $x^2 + y^2 = 6 + O(\varepsilon)$, $x^2 + y^2 = (13337/690) + O(\varepsilon)$.

Example 4.5. We consider $\lambda_1 = 7, \lambda_2 = 701/100$. It is easy to show, after some calculations, that $(7, 701/100) \in V^2$. We calculate the dependent radius λ_3, associated to the $7, 701/100$, which from Proposition 1.14 is positive and we have that $\lambda_3 = 5204/1703$. So, we have the Λ-point $(7, 701/100, 5204/1703)$ which belongs to the set V_3^2.

Now, using Theorem 1.1, for $\tau = -2179840$, we have that the system

$$\begin{gathered} \dot{x} = y, \\ \dot{y} = -x + \varepsilon\big(-2179840y^5 + 12351224y^3 - 25536028y\big)\big(1 - x^2\big), \end{gathered} \tag{4.3}$$

with $0 < \varepsilon \ll 1$ has exactly, the limit cycles $x^2 + y^2 = 7 + O(\varepsilon)$, $x^2 + y^2 = (701/100) + O(\varepsilon)$ and $x^2 + y^2 = (5204/1703) + O(\varepsilon)$.

Since $5204/1703 < 7 < 701/100$, the limit cycles $x^2 + y^2 = (5204/1703) + O(\varepsilon)$, $x^2 + y^2 = (701/100) + O(\varepsilon)$ are unstable and the limit cycle $x^2 + y^2 = 7 + O(\varepsilon)$ is stable.

Since $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$ have very small difference ($|\sqrt{\lambda_1} - \sqrt{\lambda_2}| = |\frac{10\sqrt{7}-\sqrt{701}}{10}| \simeq 0.0019$), the qualitative and quantitative image that one gets using a program, may give the misimpression that system (4.3) has a semistable limit cycle. This happens because the stable limit cycle $x^2 + y^2 = 7 + O(\varepsilon)$ lies close enough to the unstable limit cycle $x^2 + y^2 = (701/100) + O(\varepsilon)$. This of course is prospective since a priori we have chosen the λ_1 and λ_2 so as to be close enough the one to the other. So, the two limit cycles $x^2 + y^2 = 7 + O(\varepsilon)$ and $x^2 + y^2 = (701/100) + O(\varepsilon)$, create "one system with one pseudosemistable limit cycle" as we can say, since the two limit cycles together behave like a semistable limit cycle.

Remark 4.6. It is easy to see, according to Remark 1.17, that system (1.1) with $n = 1$ can't have "a system with a pseudosemistable limit cycle" as we mean above "the system with a pseudosemistable limit cycle".

Example 4.7. We consider $\lambda_1 = 7$. This λ_1 belongs to S_2^1. We know from Remark 1.17 that the point $(7, \lambda_2) \in S_{2,2}^1$, where λ_2 is the dependent radius associated to the 7.

We calculate the dependent radius λ_2, associated to the 7, (which from Proposition 1.14 is positive) and we have that

$$\lambda_2 := \frac{24 - 4\lambda_1}{4 - \lambda_1} = \frac{24 - 28}{4 - 7} = \frac{4}{3}.$$

So, we have the Λ-point $(7, 4/3)$ which belongs to the set V_2^1.

Now, using Theorem 1.3, for $\tau = 6$, we have that the system

$$\begin{gathered} \dot{x} = y, \\ \dot{y} = -x + \varepsilon\big(6y^3 - 7y\big)\big(1 - x^2\big), \end{gathered} \tag{4.4}$$

with $0 < \varepsilon \ll 1$ has exactly, the limit cycles $x^2 + y^2 = 7 + O(\varepsilon)$ and $x^2 + y^2 = (4/3) + O(\varepsilon)$.

Since $4/3 < 7$, the limit cycle $x^2 + y^2 = (4/3) + O(\varepsilon)$ is unstable and the limit cycle $x^2 + y^2 = 7 + O(\varepsilon)$ is stable.

From Theorem 1.3 we have for the system (4.4) that, if we change τ from 6 to -6 the unstable limit cycle $x^2 + y^2 = (4/3) + O(\varepsilon)$ becomes stable and the stable limit cycle $x^2 + y^2 = 7 + O(\varepsilon)$ becomes unstable.

Appendix

Proof of Lemma 1.10. Clearly, for $n \in \mathbb{N}$, $n \geq 3$,

$$2(n+2) + \frac{1}{2(n+1)} \sum_{\substack{i_1,i_2=1 \\ i_1<i_2}}^{n} \lambda_{i_1}\lambda_{i_2} - \frac{1}{4n(n+1)} \sum_{\substack{i_1,i_2,i_3=1 \\ i_1<i_2<i_3}}^{n} \lambda_{i_1}\lambda_{i_2}\lambda_{i_3} + \dots$$

$$
\begin{aligned}
&+\frac{(-1)^k}{2^{k-1}(n-k+3)\dots(n+1)}\sum_{\substack{i_1,\dots,i_k=1\\ i_1<\dots<i_k}}^{n}\lambda_{i_1}\dots\lambda_{i_k}+\dots+\frac{(-1)^n}{2^{n-2}(n+1)!}\prod_{i_1=1}^{n}\lambda_{i_1}\\
&=2(n+1)+\frac{1}{2n}\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n}\lambda_{i_1}\lambda_{i_2}-\frac{1}{4(n-1)n}\sum_{\substack{i_1,i_2,i_3=1\\ i_1<i_2<i_3}}^{n}\lambda_{i_1}\lambda_{i_2}\lambda_{i_3}+\dots\\
&\quad+\frac{(-1)^k}{2^{k-1}(n-k+2)(n-k+3)\dots n}\sum_{\substack{i_1,\dots,i_k=1\\ i_1<\dots<i_k}}^{n}\lambda_{i_1}\dots\lambda_{i_k}+\dots+\frac{(-1)^n}{2^{n-1}n!}\prod_{i_1=1}^{n}\lambda_{i_1}
\end{aligned}
$$

if and only if

$$
\begin{aligned}
&\frac{1}{4n(n+1)}\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n}\lambda_{i_1}\lambda_{i_2}-\frac{2}{8(n-1)n(n+1)}\sum_{\substack{i_1,i_2,i_3=1\\ i_1<i_2<i_3}}^{n}\lambda_{i_1}\lambda_{i_2}\lambda_{i_3}+\dots\\
&+\frac{(-1)^k(k-1)}{2^k(n-k+2)(n-k+3)\dots(n+1)}\sum_{\substack{i_1,\dots,i_k=1\\ i_1<\dots<i_k}}^{n}\lambda_{i_1}\dots\lambda_{i_k}+\dots\\
&+\frac{(-1)^n(n-1)}{2^n(n+1)!}\prod_{i_1=1}^{n}\lambda_{i_1}=1,
\end{aligned}
$$

and the first equivalence has been proved. Similarly, one can prove and the rest two equivalences. For $n=2$, (i.e. for the sinusoidal-type numbers of second order), it is easy to see that the above equivalences hold. The proof of the lemma is complete. □

Proof of Proposition 1.14. To prove that the dependent radius λ_{n+1}, $n\in\mathbb{N}$, $n\ge 2$, associated to the radii $\lambda_1,\lambda_2,\dots,\lambda_n$ is positive when $(\lambda_1,\lambda_2,\dots,\lambda_n)\in V^n$, $n\in\mathbb{N}$, $n\ge 2$, it suffices to show that both numerator and denominator of (1.6) are of the same sign.

We will check the case where $(\lambda_1,\lambda_2,\dots,\lambda_{n-1},\lambda_n)\in S_1^n$, $n\in\mathbb{N}$, $n\ge 2$. Similarly, one can prove and the other cases.

In the set S_1^n, $n\in\mathbb{N}$, $n\ge 2$, we have $\sum_{i_1=1}^{n}\lambda_{i_1}<\bar{s}^n$. Now, for $n\in\mathbb{N}$, $n\ge 3$, using the definition of $\bar{s}^n$ and multiplying the last inequality by $-2^n(n+1)!$ we have

$$
2^{n+1}(n+2)!-2^n(n+1)!\sum_{i_1=1}^{n}\lambda_{i_1}+2^{n-1}n!\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n}\lambda_{i_1}\lambda_{i_2}-\dots+4(-1)^n\prod_{i_1=1}^{n}\lambda_{i_1}>0,
$$

which shows that the numerator of (1.6) is positive.

Since $\sum_{i_1=1}^{n}\lambda_{i_1}<\bar{s}^n$ in the set S_1^n, we have that $-\sum_{i_1=1}^{n}\lambda_{i_1}>-\bar{s}^n$. Using this observation we obtain the first inequality for the denominator of (1.6)

$$
\begin{aligned}
&2^n(n+1)!-2^{n-1}n!\sum_{i_1=1}^{n}\lambda_{i_1}+2^{n-2}(n-1)!\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n}\lambda_{i_1}\lambda_{i_2}-\dots+(-1)^n\prod_{i_1=1}^{n}\lambda_{i_1}\\
&>2^n(n+1)!-2^{n-1}n!\bar{s}^n+2^{n-2}(n-1)!\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n}\lambda_{i_1}\lambda_{i_2}-\dots+(-1)^n\prod_{i_1=1}^{n}\lambda_{i_1}
\end{aligned}
$$

$$
\begin{aligned}
&= 2^n(n+1)! - 2^n n!(n+2) - 2^{n-2}\frac{n!}{n+1}\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n} \lambda_{i_1}\lambda_{i_2} + \dots - \frac{2(-1)^n}{n+1}\prod_{i_1=1}^{n}\lambda_{i_1}\\
&\quad + 2^{n-2}(n-1)!\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n} \lambda_{i_1}\lambda_{i_2} - \dots + (-1)^n\prod_{i_1=1}^{n}\lambda_{i_1}\\
&= -2^n n! + 2^n n!\tilde{s}^n > -2^n n! + 2^n n! = 0.
\end{aligned}
$$

Here, we have used in the first equality the definition of $\bar{s}^n$ for $n \in \mathbb{N}$, $n \geq 3$ and in the last inequality that $\tilde{s}^n > 1$ in the set S_1^n.

So, we proved that both numerator and denominator of (1.6) when $n \in \mathbb{N}$, $n \geq 3$, are positive, which show that the dependent radius λ_{n+1} is positive if $(\lambda_1, \dots, \lambda_n) \in S_1^n$, $n \in \mathbb{N}$, $n \geq 3$.

In the case where $n = 2$, it is easy to show that the dependent radius λ_3 associated to the radii λ_1, λ_2 is positive if $(\lambda_1, \lambda_2) \in S_1^2$, since in that case both numerator and denominator of (1.6) with $n = 2$, are positive.

If $n = 1$, it is easy to show that the dependent radius λ_2 associated to the radius λ_1 is positive if $\lambda_1 \in V^1$, since in that case both numerator and denominator of (1.7) are of the same sign.

Let us now show the inverse. Let $\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n$, $n \in \mathbb{N}$, $n \geq 2$, be distinct positive real numbers so that the dependent radius λ_{n+1}, $n \in \mathbb{N}$, $n \geq 2$, associated to the radii $\lambda_1, \lambda_2, \dots, \lambda_n$ is positive.

First, we examine the case where both numerator and denominator of λ_{n+1} are positive.

Since we suppose that the numerator of λ_{n+1} is positive, we have that

$$
2^{n+1}(n+2)! - 2^n(n+1)!\sum_{i_1=1}^{n}\lambda_{i_1} + 2^{n-1}n!\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n} \lambda_{i_1}\lambda_{i_2} - \dots + 4(-1)^n\prod_{i_1=1}^{n}\lambda_{i_1} > 0,
$$

and dividing this inequality by $-2^n(n+1)!$ we have that $\sum_{i_1=1}^{n}\lambda_{i_1} < \bar{s}^n$.

Since we suppose that the denominator of λ_{n+1} is positive, we have that

$$
2^n(n+1)! - 2^{n-1}n!\sum_{i_1=1}^{n}\lambda_{i_1} + 2^{n-2}(n-1)!\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n} \lambda_{i_1}\lambda_{i_2} - \dots + (-1)^n\prod_{i_1=1}^{n}\lambda_{i_1} > 0,
$$

and dividing this inequality by $-2^{n-1}n!$ we have that $\sum_{i_1=1}^{n}\lambda_{i_1} < \hat{s}^n$.

Now, we have the following possibilities: $\bar{s}^n = \hat{s}^n$ or $\bar{s}^n > \hat{s}^n$ or $\bar{s}^n < \hat{s}^n$, where $n \in \mathbb{N}$, $n \geq 2$.

In the case where $\bar{s}^n = \hat{s}^n$ we know from Lemma 1.10 that $\tilde{s}^n = 1$ and hence $(\lambda_1, \lambda_2, \dots, \lambda_n) \in S_3^n$, $n \in \mathbb{N}$, $n \geq 2$.

In the case where $\bar{s}^n > \hat{s}^n$ we know from Lemma 1.10 that $\tilde{s}^n < 1$ and hence $(\lambda_1, \lambda_2, \dots, \lambda_n) \in S_2^n$, $n \in \mathbb{N}$, $n \geq 2$.

In the case where $\bar{s}^n < \hat{s}^n$ we know from Lemma 1.10 that $\tilde{s}^n > 1$ and hence $(\lambda_1, \lambda_2, \dots, \lambda_n) \in S_1^n$, $n \in \mathbb{N}$, $n \geq 2$.

Let us now examine the case where both numerator and denominator of λ_{n+1} are negative.

Since we suppose that the numerator of λ_{n+1} is negative, we have that

$$2^{n+1}(n+2)! - 2^n(n+1)!\sum_{i_1=1}^{n}\lambda_{i_1} + 2^{n-1}n!\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n}\lambda_{i_1}\lambda_{i_2} - \dots + 4(-1)^n\prod_{i_1=1}^{n}\lambda_{i_1} < 0,$$

and dividing this inequality by $-2^n(n+1)!$ we have that $\sum_{i_1=1}^{n}\lambda_{i_1} > \bar{s}^n$.

Since we suppose that the denominator of λ_{n+1} is negative, we have that

$$2^n(n+1)! - 2^{n-1}n!\sum_{i_1=1}^{n}\lambda_{i_1} + 2^{n-2}(n-1)!\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^{n}\lambda_{i_1}\lambda_{i_2} - \dots + (-1)^n\prod_{i_1=1}^{n}\lambda_{i_1} < 0,$$

and dividing this inequality by $-2^{n-1}n!$ we have that $\sum_{i_1=1}^{n}\lambda_{i_1} > \hat{s}^n$.

Now, we have the following possibilities: $\bar{s}^n = \hat{s}^n$ or $\bar{s}^n > \hat{s}^n$ or $\bar{s}^n < \hat{s}^n$, where $n \in \mathbb{N}$, $n \geq 2$.

In the case where $\bar{s}^n = \hat{s}^n$ we know from Lemma 1.10 that $\tilde{s}^n = 1$ and hence $(\lambda_1, \lambda_2, \dots, \lambda_n) \in S_6^n$, $n \in \mathbb{N}$, $n \geq 2$.

In the case where $\bar{s}^n > \hat{s}^n$ we know from Lemma 1.10 that $\tilde{s}^n < 1$ and hence $(\lambda_1, \lambda_2, \dots, \lambda_n) \in S_4^n$, $n \in \mathbb{N}$, $n \geq 2$.

In the case where $\bar{s}^n < \hat{s}^n$ we know from Lemma 1.10 that $\tilde{s}^n > 1$ and hence $(\lambda_1, \lambda_2, \dots, \lambda_n) \in S_5^n$, $n \in \mathbb{N}$, $n \geq 2$.

We have thus proved the inverse of Proposition 1.14 for $n \in \mathbb{N}$, $n \geq 2$. In fact we proved a stronger result. Let $\lambda_1, \lambda_2, \dots, \lambda_n$, $n \in \mathbb{N}$, $n \geq 2$, be distinct positive real numbers. Supposing that both numerator and denominator of the positive dependent radius λ_{n+1}, $n \in \mathbb{N}$, $n \geq 2$, associated to the radii $\lambda_1, \lambda_2, \dots, \lambda_n$ are positive, then $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \cup_{i=1}^{3} S_i^n$, $n \in \mathbb{N}$, $n \geq 2$. If we suppose that both numerator and denominator of the positive dependent radius λ_{n+1}, $n \in \mathbb{N}$, $n \geq 2$, associated to the radii $\lambda_1, \lambda_2, \dots, \lambda_n$ are negative, then $(\lambda_1, \lambda_2, \dots, \lambda_n) \in \cup_{i=4}^{6} S_i^n$, $n \in \mathbb{N}$, $n \geq 2$.

If $n = 1$, let λ_1 be a positive real number so that the dependent radius λ_2 associated to the radius λ_1 is positive.

First, we examine the case where both numerator and denominator of λ_2 are positive. In that case we have for the numerator that $24 - 4\lambda_1 > 0$ which implies that $\lambda_1 < 6$ and for the denominator that $4 - \lambda_1 > 0$ which implies that $\lambda_1 < 4$. Combining the last two results about λ_1, we have that $0 < \lambda_1 < 4$ and hence $\lambda_1 \in S_1^1$.

Let now examine the case where both numerator and denominator of λ_2 are negative. In that case we have for the numerator that $24 - 4\lambda_1 < 0$ which implies that $\lambda_1 > 6$ and for the denominator that $4 - \lambda_1 < 0$ which implies that $\lambda_1 > 4$. Combining the last two results about λ_1, we have that $6 < \lambda_1 < +\infty$ and hence $\lambda_1 \in S_2^1$.

So, we proved that, if λ_1 is a positive real number so that the dependent radius λ_2 associated to the radius λ_1 is positive, then $\lambda_1 \in V^1$. The proof of the proposition is complete. $\square$

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References

[1] V. I. Arnol'd; *Ordinary Differential Equations*, Springer-Verlag, 1992.
[2] M. Caubergh, F. Dumortier; *Hilbert's 16th problem for classical Liénard equations of even degree*, J. Differential Equations **244** (2008), 1359–1394.
[3] F. Dumortier, D. Panazzolo, R. Roussarie; *More limit cycles than expected in Liénard equations*, Proc. Amer. Math. Soc. **135** (2007), 1895–1904.
[4] I. D. Iliev; *The number of limit cycles due to polynomial perturbations of the harmonic oscillator*, Math. Proc. Cambridge Philos. Soc. **127** (1999), 317–322.
[5] Yu. Ilyashenko, A. Panov; *Some upper estimates of the number of limit cycles of planar vector fields with applications to Liénard equations*, Mosc. Math. J. **1** (2001), 583–599.
[6] A. Liénard; *Étude des oscillations entretenues*, Revue Génerale de l'Électricité **23** (1928), 901–912, 946–954.
[7] A. Lins, W. de Melo, C. C. Pugh; *On Liénard's equation*, Geometry and Topology (Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976), Lecture Notes in Math. **597**, Springer, Berlin, (1977), 335–357.
[8] P. De Maesschalck, F. Dumortier; *Classical Liénard equations of degree $n \geq 6$ can have $\left[\frac{n-1}{2}\right] + 2$ limit cycles*, J. Differential Equations **250** (2011), 2162–2176.
[9] S. Negrepontis, S. Giotopoulos, E. Giannakoulias; *Infinitesimal Calculus II a*, (in Greek), Symmetria, Athens, 1993.
[10] G. S. Rychkov; *The maximum number of limit cycles of the system $\dot{x} = y - \sum_{i=0}^{2} a_i x^{2i+1}$, $\dot{y} = -x$ is two*, Differ. Equ. **11** (1975), 390–391.
[11] S. Smale; *Mathematical problems for the next century*, Math. Intelligencer **20** (1998), 7–15.

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