Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 121, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE AND MULTIPLICITY OF PERIODIC SOLUTIONS GENERATED BY IMPULSES FOR SECOND-ORDER HAMILTONIAN SYSTEM

DAN ZHANG, QINGHUA WU, BINXIANG DAI

ABSTRACT. In this article, we study the existence of non-zero periodic solutions for Hamiltonian systems with impulsive conditions. By using a variational method and a variant fountain theorem, we obtain new criteria to guarantee that the system has at least one non-zero periodic solution or infinitely many non-zero periodic solutions. However, without impulses, there is no non-zero periodic solution for the system under our conditions.

1. INTRODUCTION

In this article, we consider the problem

$$\begin{aligned} \ddot{q}(t) &= f(t, q(t)), \quad \text{a.e. } t \in (s_{j-1}, s_j), \\ \Delta \dot{q}(s_j) &= G_j(q(s_j)), \quad j = 1, 2, \dots, m, \end{aligned} \tag{1.1}$$

where $j \in \mathbb{Z}$, $q \in \mathbb{R}^n$, $\Delta \dot{q}(s_j) = \dot{q}(s_j^+) - \dot{q}(s_j^-)$ with $\dot{q}(s_j^{\pm}) = \lim_{t \to s_j^{\pm}} \dot{q}(t)$, $f(t,q) = \operatorname{grad}_q F(t,q), F(t,q) \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), g_j(q) = \operatorname{grad}_q G_j(q), G_j(q) \in C^1(\mathbb{R}^n, \mathbb{R})$ for each $k \in \mathbb{Z}$ and there exist $m \in \mathbb{N}$ and $T \in \mathbb{R}^+$ such that $0 = s_0 < s_1 < s_2 < \dots < s_m = T$, $s_{j+m} = s_j + T$ and $g_{j+m} = g_j$ for all $j \in \mathbb{Z}$.

The second-order Hamiltonian system without impulse

$$\ddot{q}(t) = f(t, q(t)), \quad \text{a.e. } t \in \mathbb{R}. \tag{1.2}$$

had been studied widely, see [1, 5, 9, 15]. The authors studied the existence of periodic and homoclinic solutions for the system (1.2). Moreover, the existence of homoclinic solutions for other second-order Hamiltonian systems had been also studied in [3, 8, 12, 13, 14, 16, 17]. The main methods used for Hamiltonian systems are upper and lower solutions techniques, fixed point theorems and the coincidence degree theory of Mawhin in a special Banach space [2, 6, 7, 10, 11, 19] .

Under normal circumstances, the authors would give some additional conditions to impulsive functions when they studied the existence of solutions for the impulsive differential equations. But as the impulse force disappeared, many results remained to be established when the differential equations satisfied the same conditions. Therefore, impulse effect cannot be seen clearly. Based on this, in recent

---

2000 *Mathematics Subject Classification.* 34K50, 37K05.
Key words and phrases. Impulsive differential equations; critical point theory; periodic solution; variant fountain theorems.

Submitted February 18, 2014. Published May 2, 2014.

years, some scholars have launched the research on the existence of solutions generated by impulse [4, 18].

Solutions generated by impulses means that the solutions appear when the impulse is not zero , and disappear when the impulse is zero. Obviously, if the equations without impulse had only zero solution, the equations had non-zero solutions when the impulse is not zero, that is to say, the non-zero solutions are controlled by impulses. At present, the related research work on solutions generated by impulses is seldom, refer to literature [4, 18].

Han and Zhang [4] studied (1.1) and obtained the existence of non-zero periodic solutions generated by impulses. To obtain the existence of non-zero periodic solutions, they used the following conditions:

- (F1) $F(t,q) \geq \frac{1}{2} f(t,q)q > 0$ for all $t \in [0,T]$ and $q \in \mathbb{R}^n \setminus \{0\}$.
- (F2) $f(t,q) = \alpha q + w(t,q)$ for all $t \in [0,T]$ and $q \in \mathbb{R}^n$, where $\alpha > \frac{2}{\mu}$ for some $\mu > 2$, $w(t,q) = \operatorname{grad}_q W(t,q)$ and $W(t,q) \geq \frac{1}{2} w(t,q)q > 0$ for all $t \in [0,T]$ and $q \in \mathbb{R}^n$.
- (F3) $F(t,q) \geq \frac{1}{2} f(t,q)q \geq 0$ for all $t \in [0,T]$ and $q \in \mathbb{R}^n$.
- (G1) There exists $\mu > 2$ such that $g_j(q)q \leq \mu G_j(q) < 0$ for all $j = 1,2,\dots,m$ and $q \in \mathbb{R}^n \setminus \{0\}$.
- (G2) $g_j(q) = 2q + w_j(q)$, where $w_j(q) = \operatorname{grad}_q W_j(q)$ and satisfy that there exists $\mu > 2$ such that $w_j(q)q \leq \mu W_j(q) < 0$ for all $j = 1,2,\dots,m$ and $q \in \mathbb{R}^n$.

By using critical point theory, they obtained the following theorems.

Theorem 1.1 ([4, Theorem 1]). *If F is T-periodic in t and satisfies* (F1), *$g_j(q)$ satisfies* (G1) *for all $j = 1,2,\dots,m$, then* (1.1) *possesses at least one non-zero periodic solution generated by impulses.*

Corollary 1.2 ([4, Corollary 1]). *If F is T-periodic in t and satisfies* (F2), *$g_j(q)$ satisfies* (G1) *for all $j = 1,2,\dots,m$, then* (1.1) *possesses at least one non-zero periodic solution generated by impulses.*

Theorem 1.3 ([4, Theorem 2]). *If F is T-periodic in t and satisfies (F3), $g_j(q)$ satisfies* (G2) *for all $j = 1,2,\dots,m$, then* (1.1) *possesses at least one non-zero periodic solution generated by impulses.*

They only obtained that system (1.1) has at least non-zero periodic solution generated by impulses when $f(t,x)$ is asymptotically linear or sublinear. However, there is no work on studying the existence of infinitely many solutions generated by impulses for system (1.1) when f is asymptotically linear or sublinear. As a result, the goal of this article is to fill the gap in this area. By using a variant fountain theorem, the results that the system (1.1) has infinitely many periodic solutions generated by impulses will be obtained. Meanwhile, we will also consider some cases which are not included in [4].

This article is organized as follows. In Section 2, we present some preliminaries. In Section 3, we give the main results and their proofs. Some examples are presented to illustrate our main results in the last section.

2. Preliminaries

First, we introduce some notation and some definitions. Let E be a Banach space with the norm $\|\cdot\|$ and $E = \overline{\oplus_{j\in N} X_j}$ with $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \oplus_{j=0}^{k} X_j$, $Z_k = \overline{\oplus_{j=k}^{\infty} X_j}$, $B_k = \{u \in Y_k : \|u\| \leq \rho_k\}$, $N_k = \{u \in Z_k : \|u\| =$

$r_k\}$, here $\rho_k > r_k > 0$. Consider the following C^1-functional $\varphi_\lambda : E \to \mathbb{R}$ defined by

$$\varphi_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in [1,2].$$

Assumed that:

((C1) φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Furthermore, $\varphi_\lambda(-u) = \varphi_\lambda(u)$ for all $(\lambda,u) \in [1,2] \times E$.

(C2) $B(u) \geq 0$ for all $u \in E$; $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$; or

(C2') $B(u) \leq 0$ for all $u \in E$; $B(u) \to -\infty$ as $\|u\| \to \infty$.

For $k \geq 2$, define $\Gamma_k = \{\gamma \in C(B_k, E) : \gamma \text{ is odd}; \gamma|\partial B_k = id\}$,

$$\begin{aligned} c_k(\lambda) &:= \inf_{u\in\Gamma_k} \max_{u\in B_k} \varphi_\lambda(u) \\ b_k(\lambda) &:= \inf_{u\in Z_k, \|u\|=r_k} \varphi_\lambda(u) \\ a_k(\lambda) &:= \max_{u\in Y_k, \|u\|=\rho_k} \varphi_\lambda(u) \end{aligned}$$

Theorem 2.1 ([19, Theorem 2.1]). *Assume that* (C1) *and* (C2) *(or* (C2')*) hold. If $b_k(\lambda) > a_k(\lambda)$ for all $\lambda \in [1,2]$, then $c_k(\lambda) \geq b_k(\lambda)$ for all $\lambda \in [1,2]$. Moreover, for a.e. $\lambda \in [1,2]$, there exists a sequence $\{u_n^k(\lambda)\}_{n=1}^\infty$ such that $\sup_n \|u_n^k(\lambda)\| < \infty, \varphi'_\lambda(u_n^k(\lambda)) \to 0$ and $\varphi_\lambda(u_n^k(\lambda)) \to c_k(\lambda)$ as $n \to \infty$. In particular, if $\{u_n^k\}$ has a convergent subsequence for every k, then φ_1 has infinitely many nontrivial critical points $\{u_k\} \subset E \setminus \{0\}$ satisfying $\varphi_1(u_k) \to 0^-$ as $k \to \infty$.*

Let $E = \{q : \mathbb{R} \to \mathbb{R}^n$ is absolutely continuous, $\dot{q} \in L^2((0,T), \mathbb{R}^n)$ and $q(t) = q(t+T)$ for $t \in \mathbb{R}\}$, equipped with the norm

$$\|q\| = \Big(\int_0^T |\dot{q}(t)|^2 dt\Big)^{1/2}.$$

It is easy to verify that E is a reflexive Banach space. We define the norm in $C([0,T], \mathbb{R}^n)$ as $\|q\|_\infty = \max_{t\in[0,T]} |q(t)|$. Since E is continuously embedded into $C([0,T], \mathbb{R}^n)$, then there exists a constant $C > 0$ such that

$$\|q\|_\infty \leq C\|q\|, \quad \forall q \in E. \tag{2.1}$$

For each $q \in E$, consider the functional φ defined on E by

$$\begin{aligned} \varphi(q) &= \frac{1}{2}\int_0^T |\dot{q}(t)|^2 dt + \int_0^T F(t, q(t))dt + \sum_{j=1}^m G_j(q(s_j)) \\ &= \frac{1}{2}\|q\|^2 + \int_0^T F(t, q(t))dt + \sum_{j=1}^m G_j(q(s_j)). \end{aligned} \tag{2.2}$$

Suppose F is T-periodic in t and $g_j(s)$ are continuous for $j = 1, 2, \dots, m$, then φ is differentiable at any $q \in E$ and

$$\varphi'(q)p = \int_0^T \dot{q}(t)\dot{p}(t)dt + \int_0^T f(t, q(t))p(t)dt + \sum_{j=1}^m g_j(q(s_j))p(s_j), \tag{2.3}$$

for any $p \in E$. Obviously, φ' is continuous.

Lemma 2.2 ([4, Lemma 1]). *If $q \in E$ is a critical point of the functional φ, then q is a T-periodic solution of system* (1.1).

Definition 2.3 ([7, P 81])**.** Let E be a real reflexive Banach space. For any sequence $\{q_n\} \subset E$, if $\{\varphi(q_n)\}$ is bounded and $\varphi'(q_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence, then we say φ satisfies the Palais-Smale condition (denoted by the PS condition for short).

Theorem 2.4 ([11, Theorem 2.7])**.** *Let E be a real Banach space and $\varphi \in C^1(E, \mathbb{R})$ satisfying the (PS) condition. If φ is bounded from below, then*

$$c = \inf_E \varphi$$

is a critical value of φ.

3. Main results

In this article, we will use the following conditions:

(V1) There exists a constant $\delta_1 \in [0, \frac{1}{2TC^2})$ such that $f(t, q)q \geq -\delta_1 |q|^2$ for all $t \in [0, T]$ and $q \in \mathbb{R}^n$.

(V2) $F(t, q) \geq \frac{1}{\beta} f(t, q)q$ for all $t \in [0, T]$ and $q \in \mathbb{R}^n$, where $\beta > 1$.

(V3) There exist constants $\delta_2 > 0$ and $\gamma \in [0, 2)$ such that $F(t, q) \geq -\delta_2 |q|^\gamma$ for all $t \in [0, T]$ and $q \in \mathbb{R}^n$.

(S1) $G_j(q) \geq 0$ for all $j = 1, 2, \dots, m$ and $q \in \mathbb{R}^n$.

(S2) There cannot exist a constant q such that $g_j(q) = 0$ for all $j = 1, 2, \dots, m$.

(S3) There exist constants $a_j, b_j > 0$ and $\gamma_j \in [0, 1)$ such that $|g_j(q)| \leq a_j + b_j |q|^{\gamma_j}$ for all $j = 1, 2, \dots, m$ and $q \in \mathbb{R}^n$.

(S4) There exist constants $a_j, b_j > 0$ and $\alpha_j > 1$ such that $|g_j(q)| \leq a_j + b_j |q|^{\alpha_j}$ for all $j = 1, 2, \dots, m$ and $q \in \mathbb{R}^n$.

Theorem 3.1. *If F is T-periodic in t and satisfies* (V1)–(V2), *$g_j(q)$ satisfies* (S1)–(S2) *for all $j = 1, 2, \dots, m$, then system* (1.1) *possesses at least one non-zero periodic solution generated by impulses.*

Proof. It follows form the conditions (V1)–(V2), (S1)–(S2) and (2.3) that

$$\begin{aligned}
\varphi(q) &= \frac{1}{2}\int_0^T |\dot{q}(t)|^2 dt + \int_0^T F(t, q(t)) dt + \sum_{j=1}^m G_j(q(s_j)) \\
&\geq \frac{1}{2}\|q\|^2 + \frac{1}{\beta}\int_0^T f(t, q(t))q(t) dt \\
&\geq \frac{1}{2}\|q\|^2 - \frac{\delta_1}{\beta}\int_0^T |q(t)|^2 dt \\
&\geq \frac{1}{2}\|q\|^2 - \frac{\delta_1 T}{\beta}\|q\|_\infty^2 \\
&\geq \frac{1}{2}\|q\|^2 - \frac{\delta_1 T C^2}{\beta}\|q\|^2 \\
&= \left(\frac{1}{2} - \frac{\delta_1 T C^2}{\beta}\right)\|q\|^2.
\end{aligned} \tag{3.1}$$

Since $\delta_1 \in [0, 1/(2TC^2))$ and $\beta > 1$, we have $(\frac{1}{2} - \frac{\delta_1 T C^2}{\beta}) > 0$. The inequality (3.1) implies that $\lim_{\|q\| \to \infty} \varphi(q) = +\infty$. So φ is a functional bounded from below.

Next we prove that φ satisfies the Palais-Smale condition. Let $\{q_n\}$ be a sequence in E such that $\{\varphi(q_n)\}$ is bounded and $\varphi'(q_n) \to 0$ as $n \to \infty$. Then there exists a constant C_0 such that $|\varphi(q_n)| \le C_0$. We first prove that $\{q_n\}$ is bounded. By (3.1), one has

$$C_0 \ge \varphi(q_n) \ge (\frac{1}{2} - \frac{\delta_1 T C^2}{\beta})\|q_n\|^2.$$

Since $(\frac{1}{2} - \frac{\delta_1 T C^2}{\beta}) > 0$, it follows that $\{q_n\}$ is bounded in E. Going if necessary to a subsequence, we can assume that there exists $q \in E$ such that $q_n \rightharpoonup q$ in E, $q_n \to q$ on $C([0,T), \mathbb{R}^n)$ as $n \to +\infty$. Hence

$$\begin{gathered}
(\varphi'(q_n) - \varphi'(q))(q_n - q) \to 0, \\
\int_0^T [f(t,q_n) - f(t,q)](q_n(t) - q(t))dt \to 0, \\
\sum_{j=1}^m [g_j(q_n(s_j)) - g_j(q(s_j))](q_n(s_j) - q(s_j)) \to 0,
\end{gathered}$$

as $n \to +\infty$. Moreover, an easy computation shows that

$$\begin{aligned}
&(\varphi'(q_n) - \varphi'(q))(q_n - q) \\
&= \|q_n - q\|^2 + \int_0^T [f(t,q_n) - f(t,q)](q_n(t) - q(t))dt \\
&\quad + \sum_{j=1}^m [g_j(q_n(s_j)) - g_j(q(s_j))](q_n(s_j) - q(s_j)).
\end{aligned}$$

So $\|q_n - q\| \to 0$ as $n \to +\infty$, which implies that $\{q_n\}$ converges strongly to q in E. Therefore, φ satisfies the Palais-Smale condition. According to Theorem 2.4, there is a critical point q of φ, i.e. q is a periodic solution of system (1.1). The condition (S2) means q is non-zero. So system (1.1) possesses a non-zero periodic solution.

Finally, let us verify that system (1.2) possesses only a zero periodic solution. Suppose $q(t)$ is a periodic solution of system (1.2), then $q(0) = q(T)$. According to condition (V1), we have

$$\begin{aligned}
0 &= -\int_0^T \ddot{q} q dt + \int_0^T f(t,q) q dt \\
&= \int_0^T |\dot{q}|^2 dt + \int_0^T f(t,q) q dt - \dot{q} q|_0^T \\
&\ge \|q\|^2 - \delta_1 \int_0^T |q|^2 dt \\
&\ge \|q\|^2 - \delta_1 T \|q\|_\infty^2 \\
&\ge \|q\|^2 - \delta_1 T C^2 \|q\|^2 \\
&\ge (1 - \delta_1 T C^2)\|q\|^2.
\end{aligned}$$

Since $\delta_1 \in [0, 1/(2TC^2))$, which implies system (1.2) does not possess any non-trivial periodic solution. □

Remark 3.2. Condition (F1) guarantee that the conditions (V1) and (V2) hold, but the reverse is not true. For example, take

$$F(t,q) = -\frac{1}{8TC^2} h(t) q^2,$$

where $h : \mathbb{R} \to (0,1]$ is continuous with period T. Then

$$qf(t,q) = -\frac{1}{4TC^2}h(t)q^2 \geq -\frac{1}{4TC^2}|q|^2.$$

Take $\delta_1 = \frac{1}{4TC^2}, \beta = 2$. It is easy to check that conditions (V1) and (V2) are satisfied, but (F1) cannot be satisfied. Meanwhile, conditions (S1) and (S2) consider the case that $G_j(s)$ are positive functions. Let

$$G_j(q) = \kappa(j)(q^2 + \mathrm{e}^q),$$

where $\kappa : \mathbb{Z} \to \mathbb{R}^+$ is positive and m-periodic in $\mathbb{Z}$. Obviously, $g_j(s)$ can satisfies conditions (S1) and (S2) for all $j = 1, 2, \dots, m$. In [4], they only consider the case that $G_j(q) < 0$, so we extend and improve Theorem 1.1.

Next, we assume that the impulsive functions $g_j(q)$ are sublinear.

Theorem 3.3. *If F is T-periodic in t and satisfies* (V1)–(V2)*, $g_j(s)$ satisfies conditions (S2) and (S3) for all $j = 1, 2, \dots, m$, then system* (1.1) *possesses at least one non-zero periodic solution generated by impulses.*

Proof. It follows form the conditions (V1)–(V2), (S2)–(S3) and (2.3) that

$$\begin{aligned}
\varphi(q) &= \frac{1}{2}\int_0^T |\dot{q}(t)|^2 dt + \int_0^T F(t, q(t))dt + \sum_{j=1}^m G_j(q(s_j)) \\
&\geq \frac{1}{2}\|q\|^2 - \frac{\delta_1}{\beta}\int_0^T |q|^2 dt + \sum_{j=1}^m G_j(q(s_j)) \\
&\geq \frac{1}{2}\|q\|^2 - \frac{\delta_1 T}{\beta}\|q\|_\infty^2 - \sum_{j=1}^m [a_j\|q\|_\infty + b_j\|q\|_\infty^{\gamma_j+1}] \\
&\geq (\frac{1}{2} - \frac{\delta_1 TC^2}{\beta})\|q\|^2 - \sum_{j=1}^m [a_j C\|q\| + b_j C^{\gamma_j+1}\|q\|^{\gamma_j+1}].
\end{aligned} \tag{3.2}$$

Since $(\frac{1}{2} - \frac{\delta_1 TC^2}{\beta}) > 0$ and $\gamma_j \in [0,1)$, the above inequality implies $\lim_{\|q\|\to\infty}\varphi(q) = +\infty$. So the functional φ is bounded from below.

Next we prove that φ satisfies the Palais-Smale condition. Let $\{q_n\}$ be a sequence in E such that $\{\varphi(q_n)\}$ is bounded and $\varphi'(q_n) \to 0$ as $n \to \infty$. Then there exists a constant C_0 such that $|\varphi(q_n)| \leq C_0$. We first prove that $\{q_n\}$ is bounded. By (3.2), one has

$$C_0 \geq \varphi(q_n) \geq (\frac{1}{2} - \frac{\delta_1 TC^2}{\beta})\|q_n\|^2 - \sum_{j=1}^m [a_j C\|q_n\| + b_j C^{\gamma_j+1}\|q_n\|^{\gamma_j+1}].$$

Since $(\frac{1}{2} - \frac{\delta_1 TC^2}{\beta}) > 0$, we know that $\{q_n\}$ is bounded in E. Then, as the proof of Theorem 3.1, we can prove φ satisfies the Palais-Smale condition. According to Theorem 2.4, there is a critical point q of φ, i.e. q is a periodic solution of system (1.1). The condition (S2) means q is non-zero. Similar to the proof of Theorem 3.1, we know system (1.2) does not possess any non-trivial periodic solution. So system (1.1) possesses a non-zero periodic solution generated by impulses. $\square$

Example 3.4. Let

$$F(t,q) = h(t)q^6,$$

where $h:\mathbb{R}\to\mathbb{R}^+$ is continuous with period T. Then $f(t,q)=6h(t)q^5$. It is easy to check that $f(t,q)$ satisfies the conditions (V1) and (V2). Let

$$G_j(q)=(q^{\frac{5}{3}}+q+\frac{1}{2}\sin 2q),$$

then $g_j(q)=(\frac{5}{3}q^{\frac{2}{3}}+1+\cos 2q)=\frac{5}{3}q^{\frac{2}{3}}+2\cos^2 q$. Hence, $0<g_j(q)\le(2+\frac{5}{3}|q|^{\frac{2}{3}})$ for all $q\in\mathbb{R}^n$, the conditions (S2) and (S3) are satisfied. Therefore, according to Theorem 3.3, system (1.1) possesses at least one non-zero periodic solution generated by impulses.

Remark 3.5. Obviously, $G_j(q)=(q^{\frac{5}{3}}+q+\frac{1}{2}\sin 2q)$ cannot satisfy the condition (G1). So example 3.4 cannot obtain the existence of non-zero periodic solutions in [4].

Theorem 3.6. *If F is T-periodic in t and satisfies (V1) and (V3), $g_j(q)$ satisfies conditions (S2) and (S3) for all $j=1,2,\dots,m$, then system* (1.1) *possesses at least one non-zero periodic solution generated by impulses.*

Proof. It follows form the conditions (V1) and (V3), (S2)–(S3) and (2.2) that

$$\begin{aligned}
\varphi(q)&\ge\frac{1}{2}\int_0^T|\dot{q}|^2dt-\delta_2\int_0^T|q|^\gamma dt+\sum_{j=1}^m G_j(q(s_j))\\
&\ge\frac{1}{2}\|q\|^2-\delta_2T\|q\|_\infty^\gamma-\sum_{j=1}^m[a_j\|q\|_\infty+b_j\|q\|_\infty^{\gamma_j+1}]\\
&\ge\frac{1}{2}\|q\|^2-\delta_2TC^\gamma\|q\|^\gamma-\sum_{j=1}^m[a_jC\|q\|+b_jC^{\gamma_j+1}\|q\|^{\gamma_j+1}].
\end{aligned}\tag{3.3}$$

Since $\gamma\in[0,2)$ and $\gamma_j\in[0,1)$, the above inequality implies that $\lim_{\|q\|\to\infty}\varphi(q)=+\infty$. So the functional φ is bounded from below.

Next we prove that φ satisfies the Palais-Smale condition. Let $\{q_n\}$ be a sequence in E such that $\{\varphi(q_n)\}$ is bounded and $\varphi'(q_n)\to 0$ as $n\to\infty$. Then there exists a constant C_0 such that $|\varphi(q_n)|\le C_0$. We first prove that $\{q_n\}$ is bounded. By (3.3), we get

$$C_0\ge\varphi(q_n)\ge\frac{1}{2}\|q_n\|^2-\delta_2TC^\gamma\|q_n\|^\gamma-\sum_{j=1}^m[a_jC\|q_n\|+b_jC^{\gamma_j+1}\|q_n\|^{\gamma_j+1}].$$

Since $\gamma\in[0,2)$ and $\gamma_j\in[0,1)$ it follows that $\{q_n\}$ is bounded in E. Then, as the proof of Theorem 3.1, we can prove φ satisfies the Palais-Smale condition. According to Theorem 2.4, there is a critical point q of φ; i.e., q is a periodic solution of system (1.1). The condition (S2) means q is non-zero. Similar to the proof of Theorem 3.1, we know system (1.2) does not possess any non-trivial periodic solution. So system (1.1) possesses a non-zero periodic solution. □

Example 3.7. Let

$$F(t,q)=h(t)[q^{4/3}+e^{(q^2)}],$$

where $h:\mathbb{R}\to\mathbb{R}^+$ is continuous with period T. Then $f(t,q)=h(t)[\frac{4}{3}q^{1/3}+2q\mathrm{e}^{(q^2)}]$. An easy computation shows that $F(t,q)>0$, $qf(t,q)\ge 0$, so the conditions (V1) and (V3) are satisfied. We also take $G_j(q)$ the same as in Example 3.7, then according to Theorem 3.6, system (1.1) possesses at least one non-zero periodic solution generated by impulses.

Remark 3.8. Obviously, $F(t,q)$ cannot satisfy the conditions (F1) or (F2). So Example 3.7 cannot obtain the existence of non-zero periodic solutions in [4].

Corollary 3.9. *If F is T-periodic in t and satisfies* (V1) *and* (V3)*, $g_j(q)$ satisfies conditions* (S1) *and* (S2) *for all $j = 1, 2, \dots, m$, then system* (1.1) *possesses at least one non-zero periodic solution generated by impulses.*

Theorem 3.10. *If F is T-periodic in t and satisfies* (F1)*, $g_j(q)$ satisfies conditions* (G1) *and* (S4) *for all $j = 1, 2, \dots, m$. Moreover if $f(t,q)$, $g_j(q)$ are odd about q, then system* (1.1) *possesses infinitely many periodic solutions generated by impulses.*

To apply Theorem 2.1 and to prove Theorem 3.10, we define the functionals A, B and φ_λ on our working space E by

$$A(q) = \frac{1}{2}\|q\|^2 + \int_0^T F(t, q(t))dt,$$

$$B(q) = -\sum_{j=1}^{m} G_j(q(s_j)),$$

and

$$\varphi_\lambda(q) = A(q) - \lambda B(q) = \frac{1}{2}\|q\|^2 + \int_0^T F(t, q(t))dt + \lambda \sum_{j=1}^{m} G_j(q(s_j)), \tag{3.4}$$

for all $q \in E$ and $\lambda \in [1, 2]$. Clearly, we know that $\varphi_\lambda(q) \in C^1(E, \mathbb{R})$ for all $\lambda \in [1, 2]$. We choose a completely orthonormal basis $\{e_j\}$ of E and define $X_j := \mathbb{R}e_j$. Then Z_k, Y_k can be defined as that in the beginning of Section 2. Note that $\varphi_1 = \varphi$, where φ is the functional defined in (2.3).

Remark 3.11. If (F1) holds, $f(t,q)$ is T-periodic in t, then there exist constants $d_1, d_2 > 0$ such that

$$F(t,q) \le d_1|q|^2 + d_2, \quad \forall q \in \mathbb{R}^n.$$

Assume that (G1) holds, then there exist constants $a, b > 0$ such that

$$G_j(q) \ge -a|q|^\mu, \quad \text{for } 0 < |q| \le 1, \tag{3.5}$$

$$G_j(q) \le -b|q|^\mu, \quad \text{for } |q| \ge 1. \tag{3.6}$$

The assumption $g_{j+m} = g_j$ and (3.5)-(3.6) imply that there exists $M > 0$ such that

$$G_j(q) \le -b|q|^\mu + M, \quad \forall q \in \mathbb{R}^n. \tag{3.7}$$

Lemma 3.12. *Under the assumptions of Theorem 3.10, $B(q) \ge 0$ and $A(q) \to \infty$ as $\|q\| \to \infty$ for all $q \in E$.*

Proof. By (F1) and (G1), for any $q \in E$, we have

$$B(q) = -\sum_{j=1}^{m} G_j(q(s_j)) \ge 0,$$

$$A(q) = \frac{1}{2}\|q\|^2 + \int_0^T F(t, q(t))dt \ge \frac{1}{2}\|q\|^2.$$

Which implies that $A(q) \to \infty$ as $\|q\| \to \infty$. $\square$

Lemma 3.13. *Under the assumptions of Theorem 3.10, there exists a sequence $\rho_k > 0$ large enough such that*

$$a_k(\lambda) := \max_{q\in Y_k, \|q\|=\rho_k} \varphi_\lambda(q) \le 0,$$

for all $\lambda \in [1,2]$.

Proof. By (F1) and (G1), for any $q \in Y_k$, we have

$$\begin{aligned} \varphi_\lambda(q) &\le \frac{1}{2}\|q\|^2 + \int_0^T (d_1|q|^2 + d_2)dt + \lambda \sum_{j=1}^m (-b|q|^\mu + C_2) \\ &= \frac{1}{2}\|q\|^2 + d_1 \int_0^T |q|^2 dt + d_2 T - \lambda \sum_{j=1}^m b|q|^\mu + \lambda M m, \end{aligned} \tag{3.8}$$

Let $q = rw$, $r > 0, w \in Y_k$ with $\|w\| = 1$, we have

$$\varphi_\lambda(rw) \le \frac{1}{2}r^2 + r^2 d_1 \int_0^T |w|^2 dt + d_2 T - \lambda r^\mu \sum_{j=1}^m b|w|^\mu + \lambda M m. \tag{3.9}$$

Since $\mu > 2$, for $\|q\| = \rho_k = r$ large enough, we have $\varphi_\lambda(q) \le 0$; i.e., $a_k(\lambda) := \max_{q\in Y_k,\|q\|=\rho_k} \varphi_\lambda(q) \le 0$ for all $\lambda \in [1,2]$. □

Lemma 3.14. *Under the assumptions of Theorem 3.10, there exists $r_k > 0, \widetilde{b}_k \to \infty$ such that*

$$b_k(\lambda) := \inf_{q\in Z_k, \|q\|=r_k} \varphi_\lambda(q) \ge \widetilde{b}_k,$$

for all $\lambda \in [1,2]$.

Proof. Set $\beta_k := \sup_{q\in Z_k,\|q\|=1} \|q\|_\infty$. Then $\beta_k \to 0$ as $k \to \infty$. Indeed, it is clear that $0 < \beta_{k+1} \le \beta_k$, so that $\beta_k \to \beta \ge 0$, as $k \to \infty$. For every $k \ge 0$, there exists $q_k \in Z_k$ such that $\|q_k\| = 1$ and $\|q_k\|_\infty > \beta_k/2$. By definition of Z_k, $q_k \rightharpoonup 0$ in E. Then it implies that $q_k \to 0$ in $C([0,T),\mathbb{R}^n)$. Thus we have proved that $\beta = 0$.

By (F1) and (S4), for any $q \in Z_k$, we have

$$\begin{aligned} \varphi_\lambda(q) &\ge \frac{1}{2}\|q\|^2 + \lambda \sum_{j=1}^m G_j(q(s_j)) \\ &\ge \frac{1}{2}\|q\|^2 - 2\sum_{j=1}^m (a_j|q| + b_j|q|^{\alpha_j+1}) \\ &\ge \frac{1}{2}\|q\|^2 - 2\sum_{j=1}^m (a_j\|q\|_\infty + b_j\|q\|_\infty^{\alpha_j+1}) \\ &\ge \frac{1}{2}\|q\|^2 - 2\sum_{j=1}^m (a_j\beta_k\|q\| + b_j\beta_k^{\alpha_j+1}\|q\|^{\alpha_j+1}). \end{aligned} \tag{3.10}$$

Let

$$r_k = (8\sum_{j=1}^m (a_j\beta_k + b_j\beta_k^{\alpha_j+1}))^{\frac{1}{1-\alpha_j}}.$$

Since $\alpha_j > 2$, then $r_k \to \infty$ as $k \to \infty$. Evidently, there exists an positive integer $k_0 > \bar{n} + 1$ such that

$$r_k > 1, \quad \forall k \ge k_0.$$

For any $k \geq k_0$, let $\|q\| = r_k > 1$, we have

$$
\begin{aligned}
2\sum_{j=1}^{m}(a_j\beta_k\|q\| + b_j\beta_k^{\alpha_j+1}\|q\|^{\alpha_j+1}) &\leq 2\sum_{j=1}^{m}(a_j\beta_k + b_j\beta_k^{\alpha_k+1})\|q\|^{\alpha_j+1}\\
&= \frac{1}{4}r_k^{1-\alpha_j}r_k^{\alpha_j+1} = \frac{1}{4}r_k^2.
\end{aligned}
$$

Combining this with (3.10), straightforward computation shows that

$$
b_k(\lambda) := \inf_{q\in Z_k,\|q\|=r_k}\varphi_\lambda(q) \geq \frac{1}{4}r_k^2 = \widetilde{b}_k \to \infty,
$$

as $k \to \infty$ for all $\lambda \in [1,2]$. $\square$

Proof of Theorem 3.10. Evidently, the condition (C1) in Theorem 2.1 holds. By Lemmas 3.12, 3.13, 3.14 and Theorem 2.1, there exist a sequence $\lambda_n \to 1$ as $n \to \infty$, $\{q_n(k)\}_{n=1}^{\infty} \subset E$ such that $\varphi_{\lambda_n}^{'}(q_n(k)) \to 0, \varphi_{\lambda_n}(q_n(k)) \to c_k \in [\widetilde{b}_k, \widetilde{c}_k]$, where $\widetilde{c}_k = \sup_{q\in B_k}\varphi_1(q)$

For the sake of simplicity, in what follows we set $q_n = q_n(k)$ for all $n \in \mathbb{N}$.

Now we show that $\{q_n\}_{n=1}^{\infty}$ is bounded in E. Indeed, by (F1), (g_1) and (3.4), we have

$$
\begin{aligned}
\mu\varphi_{\lambda_n}(q_n) - \varphi'_{\lambda_n}(q_n)q_n &= (\frac{\mu}{2}-1)\|q_n\|^2 + \mu\int_0^T F(t,q_n)dt - \int_0^T f(t,q_n)q_n dt\\
&\quad + \lambda_n[\mu\sum_{j=1}^{m}G_j(q_n(s_j)) - \sum_{j=1}^{m}g_j(q_n(s_j))q_n(s_j)]\\
&\geq (\frac{\mu}{2}-1)\|q_n\|^2.
\end{aligned}
$$

Since $\mu > 2$, the above inequality implied that $\{q_n\}$ is bounded in E.

Finally, we show that $\{q_n\}$ possesses a strong convergent subsequence in E. In fact, in view of the boundedness of $\{q_n\}$, without loss of generality, we may assume $q_n \rightharpoonup q_0$ as $n \to \infty$, for some $q_0 \in E$, then $q_n \to q_0$ on $C([0,T),\mathbb{R}^n)$. Moreover, an easy computation shows that

$$
\begin{aligned}
&(\varphi'_{\lambda_n}(q_n) - \varphi'_{\lambda_n}(q_0))(q_n - q_0)\\
&= \|q_n - q_0\|^2 + \int_0^T [f(t,q_n(t)) - f(t,q_0(t))](q_n(t) - q_0(t))dt\\
&\quad + \lambda_n\sum_{j=1}^{m}[g_j(q_n(s_j)) - g_j(q_0(s_j))](q_n(s_j) - q_0(s_j)).
\end{aligned}
$$

So $\|q_n - q_0\| \to 0$ as $n \to +\infty$, which implies that $\{q_n\}$ converges strongly to q_0 in E and $\varphi'_1(q_0) = 0$. Hence, $\varphi = \varphi_1$ has a critical point q_0 with $\varphi_1(q_0) \in [\widetilde{b}_k, \widetilde{c}_k]$. Consequently, we obtain infinitely many periodic solutions since $\widetilde{b}_k \to \infty$. Similar to the proof of Theorem 3.1, we know system (1.2) does not possess any non-trivial periodic solution. Therefore, all the non-zero periodic solutions we obtain are generated by impulses. $\square$

Example 3.15. Let $F(t,q) = h(t)q^{6/5}$, where $h : \mathbb{R} \to \mathbb{R}^+$ is continuous with period T. Then $f(t,q) = \frac{6}{5}h(t)q^{1/5}$. It is easy to know that $f(t,q)q = \frac{6}{5}h(t)q^{6/5} > 0$

and $\frac{1}{2}f(t,q)q = \frac{3}{5}h(t)q^{6/5} \le F(t,q)$, so $f(t,q)$ satisfies the condition (F1). Let

$$G_j(q) = -\kappa(j)(q^4 + q^6),$$

where $\kappa : \mathbb{Z} \to (0,10]$ is positive and m-periodic in $\mathbb{Z}$. Then $g_j(q) = -\kappa(j)(4q^3 + 6q^5) \le 100(1+|q|^5)$. For $g_j(q)q = -\kappa(j)(4q^4+6q^6) \le -4\kappa(j)(q^4+q^6)$, take $\mu = 4$, it is easy to show that conditions (G1) and (S4) are satisfied. Moreover, $f(t,q), g_j(q)$ are odd about q. Therefore, according to Theorem 3.10, system (1.1) has infinitely many periodic solutions generated by impulses.

In [4], by Theorem 1.1, Example 3.15 we can only have the existence of at least one non-zero periodic solution. In this article, we obtain the existence of infinitely many periodic solutions.

Acknowledgements. This work is supported by the construct program of the key discipline in Hunan University of Science and Engineering, the Natural Science Foundation of Hunan Province, China (No. 14JJ3134) and the National Natural Science Foundation of China (No. 11271371). The authors are grateful to the anonymous referees for their useful suggestions.

References

[1] Vittorio Coti Zelati, I. Ekeland, Eric Sr.; *A variational approach to homoclinic orbits in Hamiltonian system*, Math. Ann. 288 (1990) 133-160.
[2] C. De Coster, P. Habets; *Upper and lower solutions in the theory of ODE boundary value problems: classical and recent results*, in: F.Zanolin (Ed.), Nonlinear Analysis and Boundary Value Problems for Ordinary Differential Equations CISM-ICMS,vol. 371, Springer, New York, 1996, pp. 1-78.
[3] Y. Ding; *Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems*, Nonlinear Anal. 25 (1995) 1095-1113.
[4] X. Han, H. Zhang; *Periodic and homoclinic solutions generated by impulses for asymptotically linear and sublinear Hamiltonian system*, J. Comput. Appl. Math. 235 (2011) 1531-1541.
[5] Mare Izydorek; *Joanna Janczewska, Homoclinic solutions for nonautonomous second order Hamiltonian systems with a coercive potential*, J. Differential Equations 335 (2007) 1119-1127.
[6] V. Lakshmikantham, D. D. Bainov, P. S. Simeonov; *Theory of Impulsive Differential Equations*, World Science, Singapore, 1989.
[7] J. Mawhin, M. Willem; *Critical Point Theory and Hamiltonian Systems*, Springer-Verlag, Berlin, 1989. 2.
[8] Z. Ou, C. Tang; *Existence of homoclinic solutions for the second order Hamiltonian systems*, J. Math. Anal. Appl. 291 (2004) 203-213.
[9] P. H. Rabinowitz; *Periodic solutions of Hamiltonian systems*, Comm. Pure Appl. 31 (1978) 157-184.
[10] P. H. Rabinowitz; *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conference Series in Mathematics,vol. 65,American Mathematical Society,Providence, RI, 1986.
[11] P. H. Rabinowitz; *Variational methods for Hamiltonian systems*, in:Handbook of Dynamical Systems, vol. 1, north-Holland, 2002, Part 1, Chapter 14, pp.1091-1127.
[12] X. Tang, L. Xiao; *Homoclinic solutions for non-autonomous second-order Hamiltonian systems with a coercive potential*, J. Math. Anal. Appl. 351 (2009) 586-594.
[13] X. Tang, X. Lin; *Homoclinic solutions for a class of second order Hamiltonian systems*, J. Math. Anal. Appl. 354 (2009) 539-549.
[14] J. Yang, F. Zhang; *Infinitely many homoclinic orbits for the second-order Hamiltonian systems with super-quadratic potentials*, Nonlinear Anal. Real World Appl. 10 (2009) 1417-1423.
[15] Z. H. Zhang, R. Yuan; *Homoclinic solutions for a class of non-autonomous subquadratic second-order Hamiltonian systems*, Nonlinear Anal.RWA 71(2009) 4125-4130.
[16] Z. Zhang, R. Yuan; *Homoclinic solutions for some second-order non-autonomous systems*, Nonlinear Anal. 71 (2009) 5790-5798.

[17] Q. Zhang, C. Liu; *Infinitely many homoclinic solutions for second order Hamiltonian systems*, Nonlinear Anal. 72 (2010) 894-903.

[18] H. Zhang, Z. Li; *Periodic and homoclinic solutions generated by impulses*, Nonlinear. Analy. RWA 12 (2011) 39-51.

[19] W. Zou; *Variant fountain theorems and their applications*, Manuscripta Math. 104 (2001) 343-358.

Dan Zhang
Institute of Computational Mathematics, Department of Mathematics, Hunan University of Science and Engineering, Yongzhou, 425100 Hunan, China
E-mail address: zhang11dan@126.com

Qinghua Wu
Institute of Computational Mathematics, Department of Mathematics, Hunan University of Science and Engineering, Yongzhou, 425100 Hunan, China
E-mail address: jackwqh@163.com

Binxiang Dai (corresponding author)
School of Mathematics and Statistics, Central South University, Changsha, 410083 Hunan, China
E-mail address: binxiangdai@126.com