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NONLOCAL DEGENERATE REACTION-DIFFUSION EQUATIONS WITH GENERAL NONLINEAR DIFFUSION TERM

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Abstract. We study a class of second-order nonlocal degenerate semilinear reaction-diffusion equations with general nonlinear diffusion term. Under a set of conditions on the general nonlinear diffusivity and nonlinear nonlocal source term, we prove global existence and uniqueness results in a subset of a Sobolev space. Furthermore, we prove nonexistence of smooth solution or blow-up of solution under some other set of conditions. Lastly, we give illustrative examples for which our results apply.

1. Introduction

We consider the degenerate semilinear parabolic second-order initial boundary value problem

$$u_t - (\phi(t,x,u)u_x)_x = f(u), \quad \text{in } (0,T] \times (0,a) \tag{1.1}$$

$$u(t,0) = 0, \quad u(t,a) = 0, \quad \text{in } (0,T] \tag{1.2}$$

$$u(0,x) = g(x), \quad x \in (0,a). \tag{1.3}$$

This equation is degenerate at the boundary, and its nonnegative nonlinear diffusivity $\phi(t,x,u) : [0,T] \times [0,a] \times \mathbb{R} \to \mathbb{R}$ and its nonlinear nonlocal source term $f(u) : \mathbb{R} \to \mathbb{R}$ (with $f(0) = 0$) satisfy some combinations of the following conditions:

$$\gamma u^p \le f(u), \quad \text{for some constant } p > 1 \tag{1.4}$$

$$0 \le \phi^u := \phi(t,x,u) \le B, \tag{1.5}$$

$$\lambda \le \phi^{u_0} := \phi(0,x,u(0,x)) \text{and } |\phi_x^{u_0}| \le B_0, \tag{1.6}$$

$$\|\partial_t \phi^u\|_{L^2[0,T;L^\infty(0,a)]} \le \sigma, \tag{1.7}$$

$$|\phi_u^u| \le L_1 \implies |\phi^u - \phi^v| \le L_1|u-v|, \tag{1.8}$$

$$|\phi_t^u| \le B_1 \text{ and } |\phi_t^u - \phi_t^v| \le L_2|u-v|, \tag{1.9}$$

$$|\phi_u^u - \phi_v^v| \le L_3|u-v|, \tag{1.10}$$

$$|f'(u)| \le L \implies |f(u) - f(v)| \le L|u-v|, \tag{1.11}$$

$$|f''(u)| \le L' \implies |f'(u) - f'(v)| \le L'|u-v|, \tag{1.12}$$

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for some strictly positive constants γ, B, B_0, λ, σ, L_1, L_2, L_3, L and L'. Note that $\partial_t \phi^u$ is the total derivative of ϕ^u, while ϕ^u_t is the partial derivative of ϕ^u with respect to the argument t only. The condition $f(0) = 0$ is necessary for compatibility; so that both sides of (1.1) vanish on the boundary.

Second-order parabolic equations describe the time-evolution of the density of some physical quantity u, say chemical concentration, temperature or electric potential, etc.

Nondegenerate reaction-diffusion equations, the case $\phi(t, x, u) > 0$, with nonlocal source have been considered by several authors; see for example [2, 5, 14, 15, 16, 17, 20, 22, 23]. Examples of authors who have investigated nondegenerate reaction-diffusion equations with local source terms are Cazenave and Lions [6], Friedman and McLeod [12], Giga and Kohn [13], and Ni et al. [15]. Among several authors who have investigated degenerate reaction-diffusion equations are Budd et al. [3], Budd et al. [4], Chen et al. [8], Chun and Li [7], Floater [11], and Souplet [20, 21]. The latter mentioned authors are concerned with the blow-up properties of the solutions to the various problems considered.

Equations (1.1)–(1.3) are considered, by Chen and Lihua [9], with the particular degenerate diffusion term $(x^\alpha u_x)_x$ $(0 < \alpha < 1)$ and the local source term $bf(u(x_o(t), t))$, where $b > 0$. The authors show that, under certain conditions, global solutions exist and are uniformly bounded for small b or small initial data; while the solutions blow up for large b or large initial data. Motivated by the work of Chen and Lihua [6], Sanni [18] considered (1.1)–(1.3) with the general diffusion term $(\phi(x)u_x)_x$ and the nonlocal source term $f(u)$. Under some conditions on the diffusion and the source terms, the author proved the existence and uniqueness of global weak solutions to the class of semilinear degenerate problems in some weighted Sobolev spaces. A further improvement on this research area was carried out by Sanni [19]; who considered a class of nonlocal degenerate reaction-diffusion equations with localized nonlinear diffusion term $\phi_u(t, x) := \int_0^x \phi(t, s, u(t, s))\, ds$. Under a set of conditions on the localized nonlinear diffusivity and nonlinear nonlocal source term, the author proved global existence and uniqueness result in the whole of some weighted Sobolev spaces. Furthermore, the author proved nonexistence of smooth solution or blow-up of solution under some other set of conditions.

The current work is an improvement on the paper [19]. The use of the nonlinear degenerate diffusion term introduces more difficulty in the analysis than in [19]. Under a set of conditions on the nonlinear diffusivity and nonlinear nonlocal source term, we prove global existence and uniqueness result in a subset of a Sobolev space. Furthermore, we prove nonexistence of smooth solution or blow-up of solution under another set of conditions.

The remaining part of this paper is organized as follows. In Section 2, we define the spaces used in this paper, give the definition of our weak solution and state some existing theorems. In Section 3, we construct Galerkin approximations for an auxiliary linear problem, obtain energy estimates and prove the existence of a unique weak solution to the linear problem. The existence of unique weak solutions to the nonlinear problems (1.1)–(1.3) is proved in subsection 4.1. The nonexistence of smooth solution or blow up of solution is proved in subsection 4.2. In Section 5, we give illustrative examples for which our results are applicable.

2. Preliminaries

We adopt the idea of not considering u as a function of x and t, but rather as a mapping $\mathbf{u} : [0,T] \to H_0^1(0,a)$ defined by

$$[\mathbf{u}(t)](x) := u(t,x) \quad (x \in (0,a),\ t \in [0,T]).$$

Let $L^2(0,a) := \{\mathbf{u} : (0,a) \to \mathbb{R} \quad \text{such that } \|\mathbf{u}\|_{L^2(0,a)} < \infty\}$ with the norm

$$\|\mathbf{u}\|_{L^2(0,a)} := \Big(\int_0^a \mathbf{u}^2 dx \Big)^{1/2} < \infty.$$

Let $L^\infty[0,T;L^2(0,a)]$ be the space of all measurable functions $u : [0,T] \to L^2(0,a)$ with the norm

$$\|\mathbf{u}\|_{L^\infty[0,T;L^2(0,a)]} := \operatorname{ess\,sup}_{0 \le t \le T} \|\mathbf{u}\|_{L^2(0,a)} < \infty.$$

Let $\Omega \subset \mathbb{R}^n$ and $H_0^1(\Omega) \cap H^k(\Omega) := \{\mathbf{u} : \Omega \to \mathbb{R} \text{ such that } \|\mathbf{u}\|_{H_0^1(\Omega)\cap H^k(\Omega)} < \infty\}$ with the norm

$$\|\mathbf{u}\|_{H_0^1(\Omega)\cap H^k(\Omega)} := \Big(\int_0^a \sum_{r=1}^k |\nabla^r \mathbf{u}|^2 dx \Big)^{1/2} < \infty.$$

Let $L^2[0,T;H_0^1(\Omega) \cap H^k(\Omega)]$ be the space of all measurable functions $\mathbf{u} : [0,T] \to H_0^1(\Omega) \cap H^k(\Omega)$ with the norm

$$\|\mathbf{u}\|_{L^2[0,T;H_0^1(\Omega)\cap H^k(\Omega)]} := \Big(\int_0^T \|\mathbf{u}\|^2_{H_0^1(\Omega)\cap H^k(\Omega)} dt \Big)^{1/2} < \infty.$$

Let $H^{*k}(\Omega)$ be the dual space of $H_0^1(\Omega) \cap H^k(\Omega)$ with the norm

$$\|u\|_{H^{*k}(\Omega)} := \sup\{\langle u, v\rangle : v \in H_0^1(\Omega) \cap H^k(\Omega),\ \|v\|_{H_0^1(\Omega)\cap H^k(\Omega)} \le 1\} < \infty,$$

where $\langle \cdot, \cdot \rangle$ is the pairing of $H_0^1(\Omega) \cap H^k(\Omega)$ with its dual.

Let $L^2[0,T;H^{*k}(\Omega)]$ be the space of all measurable functions $\mathbf{u} : [0,T] \to H^{*k}(\Omega)$ with the norm

$$\|\mathbf{u}\|_{L^2[0,T;H^{*k}(\Omega)]} := \Big(\int_0^T \|\mathbf{u}\|^2_{H^{*k}(\Omega)} dt \Big)^{1/2} < \infty.$$

Let $H_0^1(0,a)$ be the closure of the $C_c^\infty(0,a)$ in $H^1(0,a)$, with the norm

$$\|\mathbf{u}\|_{H_0^1(0,a)} := \Big(\int_0^a \mathbf{u}_x^2 dx \Big)^{1/2} < \infty.$$

Let $L^2[0,T;H_0^1(0,a)]$ be the space of all measurable functions $u : [0,T] \to H_0^1(0,a)$ with the norm

$$\|\mathbf{u}\|_{L^2[0,T;H_0^1(0,a)]} := \Big(\int_0^T \|\mathbf{u}\|^2_{H_0^1(0,a)} dt \Big)^{1/2} < \infty.$$

Let $L^\infty[0,T;H_0^1(0,a)]$ be the space of all measurable functions $u : [0,T] \to H_0^1(0,a)$ with the norm

$$\|\mathbf{u}\|_{L^\infty[0,T;H_0^1(0,a)]} := \operatorname{ess\,sup}_{0 \le t \le T} \|\mathbf{u}\|_{H_0^1(0,a)} < \infty.$$

Let $H^{-1}(0,a)$ be the dual space of $H_0^1(0,a)$ with the norm

$$\|u\|_{H^{-1}(0,a)} := \sup\{\langle u, v\rangle : v \in H_0^1(0,a),\ \|v\|_{H_0^1(0,a)} \le 1\} < \infty.$$

where $\langle\cdot,\cdot\rangle$ is the pairing of $H_0^1(0,a)$ with its dual. Let $L^2[0,T;H^{-1}(0,a)]$ be the space of all measurable functions $\mathbf{u}:[0,T]\to H^{-1}(0,a)$ with the norm

$$\|\mathbf{u}\|_{L^2[0,T;H^{-1}(0,a)]} := \Big(\int_0^T \|\mathbf{u}\|^2_{H^{-1}(0,a)}dt\Big)^{1/2} < \infty.$$

Let $H^*(0,a)$ be the dual space of $H_0^1(0,a)\cap H^2(0,a)$ with the norm

$$\|u\|_{H^*(0,a)} := \sup\{\langle u,v\rangle : v\in H_0^1(0,a)\cap H^2(0,a),\ \|v\|_{H_0^1(0,a)\cap H^2(0,a)}\le 1\} < \infty,$$

where $\langle\cdot,\cdot\rangle$ is the pairing of $H_0^1(0,a)\cap H^2(0,a)$ with its dual; and

$$\|\mathbf{u}\|_{H_0^1(0,a)\cap H^2(0,a)} := \Big(\int_0^a \mathbf{u}_x^2dx + \int_0^a \mathbf{u}_{xx}^2dx\Big)^{1/2} < \infty.$$

Let $L^2[0,T;H^*(0,a)$ be the space of all measurable functions $\mathbf{u}:[0,T]\to H^*(0,a)$ with the norm

$$\|\mathbf{u}\|_{L^2[0,T;H^*(0,a)} := \Big(\int_0^T \|\mathbf{u}\|^2_{H^*(0,a)}dt\Big)^{1/2} < \infty.$$

Let $C^{0,1/2}(0,a)$ be the Hölder space of bounded and continuous functions u with exponent 1/2, with the norm

$$\|u\|_{C^{0,1/2}(0,a)} := \|u\|_{C(0,a)} + [u]_{C^{0,1/2}(0,a)},$$

where

$$\|u\|_{C(0,a)} := \sup_{x\in(0,a)}|u(x)|$$

and the (1/2)th-Hölder seminorm is

$$[u]_{C^{0,1/2}(0,a)} := \sup_{x,y\in(0,a),\,x\neq y}\Big\{\frac{|u(x)-u(y)|}{|x-y|^{1/2}}\Big\}.$$

Definition 2.1. By a weak solution of the degenerate parabolic initial boundary value problem (1.1)–(1.3), we mean a function $\mathbf{u}$ such that

$$\begin{gathered}\mathbf{u}\in L^\infty[0,T;H_0^1(0,a)\cap H^2(0,a)],\quad \mathbf{u}'\in L^\infty[0,T;H_0^1(0,a)],\\ \mathbf{u}_x'\in L^2[0,T;H^*(0,a),\quad \mathbf{u}',\mathbf{u}''\in L^2[0,T;H^{-1}(0,a)],\end{gathered}\tag{2.1}$$

and that satisfies

$$\int_0^a \mathbf{u}'vdx + \int_0^a \phi(t,x,u)\mathbf{u}_xv_xdx = \int_0^a f(\mathbf{u})vdx,\tag{2.2}$$

for each $v\in H_0^1(0,a)$, a.e. $0\le t\le T$, and

$$\mathbf{u}(0)=g,\tag{2.3}$$

where $g\in H_0^1(0,a)\cap H^4(0,a)$.

The following Sobolev embedding is a special case of the theorem proved in [10].

Theorem 2.2. *If $u\in H^1(0,a)$, then $u\in C^{0,1/2}(0,a)$, and we have the estimate*

$$\|u\|_{C^{0,1/2}(0,a)}\le C\|u\|_{H^1(0,a)},\tag{2.4}$$

where $C=C(a)$ is a constant.

The next Poincaré-Friedrichs inequality is proved in [24].

Theorem 2.3. *Let $u \in H_0^1(\Omega)$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then there exists a constant $C = C(\Omega)$ such that*

$$\|u\|_{L^2(\Omega)} \leq C\|u\|_{H_0^1(\Omega)}. \tag{2.5}$$

The following Corollary to Theorem 2.2 follows easily from Theorem 2.3.

Corollary 2.4. *If $u \in H_0^1(0,a)$, then $u \in C^{0,1/2}(0,a)$, and we have the estimate*

$$\|u\|_{C^{0,1/2}(0,a)} \leq C\|u\|_{H_0^1(0,a)}, \tag{2.6}$$

for some constant $C = C(a)$.

The following theorem is a generalization of the theorem proved in [10].

Theorem 2.5. *Let $\mathbf{u} \in L^2[0,T; H_0^1(\Omega) \cap H^k(\Omega)]$ with $\mathbf{u}' \in L^2[0,T; H^{*k}(\Omega)]$.*

(i) Then $\mathbf{u} \in C[0,T; H^{k-1}(\Omega)]$ (after possibly being redefined on a set of measure zero).

(ii) The mapping $t \mapsto \|\mathbf{u}(t)\|^2_{H^{k-1}(\Omega)}$ is absolutely continuous, with

$$\frac{d}{dt}\|\mathbf{u}(t)\|^2_{H^{k-1}(\Omega)} = 2\langle \mathbf{u}'(t), \mathbf{u}(t)\rangle \tag{2.7}$$

for a.e. $0 \leq t \leq T$.

(iii) Furthermore, we have the estimate

$$\max_{0\leq t\leq T} \|\mathbf{u}(t)\|_{H^{k-1}(\Omega)} \leq C\left(\|\mathbf{u}(t)\|_{L^2[0,T;H_0^1(\Omega)\cap H^k(\Omega)]} + \|\mathbf{u}'(t)\|_{L^2[0,T;H^{*k}(\Omega)]}\right), \tag{2.8}$$

where $C = C(T)$.

Proof. We establish the proof in 3 steps.

Step 1. As in [10], we extend $\mathbf{u}$ to a larger interval $[-\beta, T+\beta]$ for $\beta > 0$, and define the regularization $\mathbf{u}^{m^{-1}} = \eta_{m^{-1}} * \mathbf{u}$ (where $\eta_{m^{-1}}$ is the usual mollifier on $\mathbb{R}$ and $m \geq 1$). Thus for m^{-1}, n^{-1},

$$\begin{aligned} &\frac{d}{dt}\|\mathbf{u}^{m^{-1}}(t) - \mathbf{u}^{n^{-1}}(t)\|^2_{H^{k-1}(\Omega)} \\ &= 2\left(\mathbf{u}^{m^{-1}\prime}(t) - \mathbf{u}^{n^{-1}\prime}(t), \mathbf{u}^{m^{-1}}(t) - \mathbf{u}^{n^{-1}}(t)\right)_{H^{k-1}(\Omega)}, \end{aligned}$$

where $(\cdot,\cdot)_{H^{k-1}(\Omega)}$ denotes the inner product in $H^{k-1}(\Omega)$. Integrating this equation over $[s,t]$ yields

$$\begin{aligned} &\|\mathbf{u}^{m^{-1}}(t) - \mathbf{u}^{n^{-1}}(t)\|^2_{H^{k-1}(\Omega)} \\ &= \|\mathbf{u}^{m^{-1}}(s) - \mathbf{u}^{n^{-1}}(s)\|^2_{H^{k-1}(\Omega)} \\ &\quad + 2\int_s^t \langle \mathbf{u}^{m^{-1}\prime}(\tau) - \mathbf{u}^{n^{-1}\prime}(\tau), \mathbf{u}^{m^{-1}}(\tau) - \mathbf{u}^{n^{-1}}(\tau)\rangle d\tau, \end{aligned} \tag{2.9}$$

for all $0 \leq s, t \leq T$; where $\langle\cdot,\cdot\rangle$ denotes the pairing of the space $H_0^1(\Omega) \cap H^k(\Omega)$ with its dual. Next, fix $s \in (0,T)$ for which

$$\mathbf{u}^{m^{-1}}(s) \to \mathbf{u}(s) \quad \text{in } H^{k-1}(\Omega), \text{ as } m \to \infty.$$

Consequently, (2.9) implies

$$\limsup_{m,n\to\infty} \sup_{0\leq t\leq T} \|\mathbf{u}^{m^{-1}}(t) - \mathbf{u}^{n^{-1}}(t)\|^2_{L^2(0,a)}$$

$$
\begin{aligned}
&\leq \lim_{m,n\to\infty} \int_0^T \|\mathbf{u}^{m^{-1}}{}'(\tau) - \mathbf{u}^{n^{-1}}{}'(\tau)\|^2_{H^{*k}(\Omega)} d\tau \\
&\quad + \lim_{m,n\to\infty} \int_0^T \|\mathbf{u}^{m^{-1}}(\tau) - \mathbf{u}^{n^{-1}}(\tau)\|^2_{H_0^1(\Omega)\cap H^k(\Omega)} d\tau = 0.
\end{aligned}
$$

It follows that the smoothed functions $\{\mathbf{u}^{m^{-1}}\}_{m=1}^\infty$ is a Cauchy sequence which converges in $C[0,T;H^{k-1}(\Omega)]$ to $\mathbf{v} \in C[0,T;H^{k-1}(\Omega)]$. Since $\mathbf{u}^{m^{-1}}(t) \to \mathbf{u}(t)$ for a.e. t, we conclude that $\mathbf{u} = \mathbf{v}$ a.e.

Step 2. Hence, we have analogous equation to (2.9), namely:

$$
\|\mathbf{u}^{m^{-1}}(t)\|^2_{H^{k-1}(\Omega)} = \|\mathbf{u}^{m^{-1}}(s)\|^2_{H^{k-1}(\Omega)} + 2\int_s^t \langle \mathbf{u}^{m^{-1}}{}'(\tau), \mathbf{u}^{m^{-1}}(\tau)\rangle d\tau. \tag{2.10}
$$

We send m to ∞ in the last equation, and identify $\mathbf{u}$ with $\mathbf{v}$ above, to obtain

$$
\|\mathbf{u}(t)\|^2_{H^{k-1}(\Omega)} = \|\mathbf{u}(s)\|^2_{H^{k-1}(\Omega)} + 2\int_s^t \langle \mathbf{u}'(\tau), \mathbf{u}(\tau)\rangle d\tau, \tag{2.11}
$$

for all $0 \leq s, t \leq T$. By the fundamental theorem of Lebesgue integral calculus, due to Lebesgue (see page 129 of [1]), (2.11) implies that the mapping $t \mapsto \|\mathbf{u}(t)\|^2_{H^{k-1}(\Omega)}$ is absolutely continuous. Differentiating (2.11) yields (2.7) for a.e. $0 \leq t \leq T$.

Step 3. Integrate (2.11) with respect to s over $[0,t]$, to obtain

$$
\begin{aligned}
&t\|\mathbf{u}(t)\|^2_{H^{k-1}(\Omega)} \\
&= \int_0^t \|\mathbf{u}(s)\|^2_{H^{k-1}(\Omega)} ds + 2\int_0^t \int_s^t \langle \mathbf{u}'(\tau), \mathbf{u}(\tau)\rangle d\tau ds \\
&\leq \int_0^T \|\mathbf{u}(t)\|^2_{H^{k-1}(\Omega)} dt + 2T\int_0^T |\langle \mathbf{u}', \mathbf{u}\rangle| dt \\
&\leq \int_0^T \|\mathbf{u}\|^2_{H^{k-1}(\Omega)} dt + 2T\int_0^T \|\mathbf{u}\|_{L^2[0,T;H_0^1(\Omega)\cap H^k(\Omega)]} \|\mathbf{u}'\|_{L^2[0,T;H^{*k}(\Omega)]} dt \\
&\leq (1+T)\|\mathbf{u}\|^2_{L^2[0,T;H_0^1(\Omega)\cap H^k(\Omega)]} + T\|\mathbf{u}'\|^2_{L^2[0,T;H^{*k}(\Omega)]},
\end{aligned}
$$

where we used Young inequality and a simplification. From whence (2.8) follows. □

Remark 2.6. It is trivial to show that, if $s \in L^\infty[0,T;L^2(0,a)]$ and f satisfies the Lipschitz condition (1.11), with $f(0) = 0$, then we have the estimate

$$
\|f(s)\|_{L^\infty[0,T;L^2(0,a)]} \leq L\|s\|_{L^\infty[0,T;L^2(0,a)]} < \infty, \tag{2.12}
$$

so that $f(s) \in L^\infty[0,T;L^2(0,a)]$.

Remark 2.7. *We will use the following equivalent form to* (1.1) *in most of our analysis:*

$$
u_t - (\phi^{u_0} u_x)_x = \Big(\int_0^t \partial_r \phi^u dr u_x\Big)_x + f(u), \quad \text{in } (0,T] \times (0,a). \tag{2.13}
$$

Remark 2.8. Notice that, by applying Hölder inequality, (1.7) implies

$$
\int_0^t |\partial_r \phi^u| dr \leq \sqrt{T}\|\partial_t \phi^u\|_{L^2[0,T;L^\infty(0,a)]} \leq \sigma\sqrt{T}, \quad \text{for } 0 \leq t \leq T. \tag{2.14}
$$

3. Auxiliary linear problem

Consider the degenerate linear parabolic initial boundary value problem

$$u_t - (\phi(t,x,s)u_x)_x = f(s(t,x)), \text{ in } (0,T] \times (0,a) \tag{3.1}$$

$$u(t,0) = 0, \quad u(t,a) = 0, \quad \text{in } (0,T] \tag{3.2}$$

$$u(0,x) = g(x), \quad x \in (0,a), \tag{3.3}$$

where $s \in L^\infty[0,T;L^2(0,a)]$ is a known function, with $\partial_t s \in L^2[0,T;L^2(0,a)]$, $\partial_t\phi^s \in L^2[0,T,L^\infty(0,a)]$ and $s_0 := s(0,x) \in H_0^1(0,a) \cap H^2(0,a)$.

The definition of our weak solution for (3.1)–(3.3) is the same as in the Definition 2.1 with $\phi(t,x,\mathbf{u})$ and $f(\mathbf{u})$ replaced by $\phi(t,x,s)$ and $f(s)$ respectively. We shall build a weak solution of our degenerate parabolic problem (3.1)–(3.3) by constructing some finite-dimensional approximations (Galerkin approximations), before passing to limits.

3.1. **Construction of approximate solution.** Assume that the functions $w_k = w_k(x)$ $(k = 1,\dots)$ are smooth,

$$\{w_k\}_{k=1}^\infty \text{ is an orthogonal basis of } H_0^1(0,a), \tag{3.4}$$

$$\{w_k\}_{k=1}^\infty \text{ is an orthonormal basis of } L^2(0,a). \tag{3.5}$$

(We can for example take $\{w_k\}_{k=1}^\infty$ to be the complete set of appropriately normalized eigenfunctions for $-\frac{\partial^2}{\partial x^2}$ in $H_0^1(0,a)$).

We fix a positive integer m; and look for function $\mathbf{u}^m : [0,T] \to H_0^1(0,a)$ of the form

$$\mathbf{u}^m := \sum_{k=1}^m d_m^k(t) w_k, \tag{3.6}$$

where we intend to select the coefficients $d_m^k(t)$ $(0 \le t \le T)$, $k = 1,\dots,m$ so that

$$d_m^k(0) = \int_0^a g w_k dx, \quad (k = 1,\dots,m), \tag{3.7}$$

$$\int_0^a \mathbf{u}^{m\prime} w_k dx + \int_0^a \phi(t,x,s)\mathbf{u}_x^m (w_k)_x dx = \int_0^a f(s) w_k dx. \tag{3.8}$$

We now construct approximate solutions.

Theorem 3.1. *There exists a unique function $\mathbf{u}^m$ of the form (3.6) satisfying (3.7)–(3.8) for each integer $m = 1,2,\dots$.*

Proof. From (3.5), if $\mathbf{u}^m$ has the structure (3.6), then

$$\int_0^a \mathbf{u}^{m\prime} w_k dx = \int_0^a \sum_{k=1}^m d_m^k{}'(t) w_k^2 dx = d_m^k{}'(t). \tag{3.9}$$

Note that, by (1.5),

$$0 \le \int_0^a \phi(t,x,s)|\mathbf{u}_x^m (w_k)_x| dx = d_m^k(t)\int_0^a \phi(t,x,s)|(w_k)_x|^2 dx := d_m^k(t) h^k(t),$$

where $h^k(t) := \int_0^a \phi(t,x,s)|(w_k)_x|^2 dx \le B\int_0^a |(w_k)_x|^2 dx < \infty$ $(k = 1,\dots,m)$. Further, define $f^k(t) := \int_0^a f(s) w_k dx$. Thus (3.8) becomes

$$d_m^k{}'(t) + h^k(t) d_m^k(t) = f^k(t), \tag{3.10}$$

subject to the condition (3.7). By the standard existence theory for ordinary differential equations, there exists a unique absolutely continuous function $d_m(t) = (d_m^1(t), \dots, d_m^m(t))$, which satisfies (3.7) and (3.10) for a.e. $0 \le t \le T$. Thus $\mathbf{u}^m$ defined by (3.6) solves (3.8) uniquely for a.e. $0 \le t \le T$. □

3.2. **Energy estimates.**

Theorem 3.2. *Let $g \in H_0^1(0,a) \cap H^4(0,a)$ and the conditions (1.5)–(1.9) and (1.11)–(1.12) be satisfied. Then there exists a constant $C > 0$ such that*

$$\begin{aligned} &\sup_{0\le t\le T}\Big(\|\mathbf{u}^m(t)\|^2_{H_0^1(0,a)\cap H^2(0,a)} + \|\mathbf{u}^{m\prime}(t)\|^2_{H_0^1(0,a)}\Big) \\ &+ \|\mathbf{u}_x^{m\prime}\|^2_{L^2[0,T;H^*(0,a)} + \|\mathbf{u}^{m\prime}\|^2_{L^2[0,T;H^{-1}(0,a)]} + \|\mathbf{u}^{m\prime\prime}\|^2_{L^2[0,T;H^{-1}(0,a)]} \\ &\le C\Big(\|s\|^2_{L^2[0,T;L^2(0,a)]} + \|\partial_t s\|^2_{L^2[0,T;L^2(0,a)]} + \|g\|^2_{H^4(0,a)} + \|s_0\|^2_{H^2(0,a)}\Big), \end{aligned} \tag{3.11}$$

for $m = 1, 2, \dots$; where $C = C(L, L', a, \lambda, \sigma, B, B_0, c_1)$, where c_1 is the principal eigenvalue of $\frac{\partial^2}{\partial x^2}$. In particular, if $s = g$, then we have the estimate

$$\begin{aligned} &\sup_{0\le t\le T}\Big(\|\mathbf{u}^m(t)\|^2_{H_0^1(0,a)\cap H^2(0,a)} + \|\mathbf{u}^{m\prime}(t)\|^2_{H_0^1(0,a)}\Big) \\ &+ \|\mathbf{u}_x^{m\prime}\|^2_{L^2[0,T;H^*(0,a)} + \|\mathbf{u}^{m\prime}\|^2_{L^2[0,T;H^{-1}(0,a)]} + \|\mathbf{u}^{m\prime\prime}\|^2_{L^2[0,T;H^{-1}(0,a)]} \\ &\le C\|g\|^2_{H^4(0,a)} =: \Lambda. \end{aligned} \tag{3.12}$$

Proof. We split the proof in ten steps.

Step 1. In this step and the next, we estimate some initial values. Estimate for $\|\mathbf{u}^m(0)\|^2_{H^2(0,a)}$. Let c_k be the eigenvalue corresponding to the eigenvector w_k. Then we have

$$-(w_k)_{xx} = c_k w_k, \quad k = 1, 2, \dots, \quad (0 < c_1 \le c_2 \le c_3 \dots). \tag{3.13}$$

Notice that the definition of $\mathbf{u}^m$ defined by (3.6) implies, in particular, that $\mathbf{u}^m_{xx} = 0$, $\mathbf{u}^m_{xxxx} = 0$, and $\mathbf{u}^m_{xxxxxx} = 0$ on $\partial(0,a)$. We thus deduce, by repeated integration by parts, the following estimates:

$$\|\mathbf{u}^m(0)\|^2_{L^2(0,a)} \le c_1^{-4}\int_0^a \mathbf{u}^m(0)\mathbf{u}^m_{xxxxxxxx}(0) = c_1^{-4}\|\mathbf{u}^m_{xxxx}(0)\|^2_{L^2(0,a)}. \tag{3.14}$$

$$\|\mathbf{u}^m_x(0)\|^2_{L^2(0,a)} \le -c_1^{-3}\int_0^a \mathbf{u}^m_x(0)\mathbf{u}^m_{xxxxxxx}(0) = c_1^{-3}\|\mathbf{u}^m_{xxxx}(0)\|^2_{L^2(0,a)}. \tag{3.15}$$

$$\|\mathbf{u}^m_{xx}(0)\|^2_{L^2(0,a)} \le c_1^{-2}\int_0^a \mathbf{u}^m_{xx}(0)\mathbf{u}^m_{xxxxxx}(0) = c_1^{-2}\|\mathbf{u}^m_{xxxx}(0)\|^2_{L^2(0,a)}. \tag{3.16}$$

$$\|\mathbf{u}^m_{xxx}(0)\|^2_{L^2(0,a)} \le -c_1^{-1}\int_0^a \mathbf{u}^m_{xxx}(0)\mathbf{u}^m_{xxxxx}(0) = c_1^{-1}\|\mathbf{u}^m_{xxxx}(0)\|^2_{L^2(0,a)}. \tag{3.17}$$

where c_1 is the principal eigenvalue of (3.13). Using (3.14)–(3.17), we thus have

$$\|\mathbf{u}^m(0)\|^2_{H^4(0,a)} \le C(c_1)\|\mathbf{u}^m_{xxxx}(0)\|^2_{L^2(0,a)} = C(c_1)\int_0^a \mathbf{u}^m(0)\mathbf{u}^m_{xxxxxxxx}(0)dx, \tag{3.18}$$

by integrating by parts repeatedly. Now

$$\mathbf{u}^m_{xxxxxxxx}(0) \in \operatorname{span}\{w_k\}_{k=1}^m, \quad \int_0^a \mathbf{u}^m(0)w_k dx = d_m^k(0) = \int_0^a g w_k dx.$$

Consequently, integrating by parts repeatedly, we have

$$\begin{aligned}\|\mathbf{u}^m(0)\|^2_{H^4(0,a)} &\le C(c_1)\int_0^a g\mathbf{u}^m_{xxxxxxxx}(0)dx\\ &= C(c_1)\int_0^a g_{xxxx}\mathbf{u}^m_{xxxx}(0)dx\\ &\le \frac{1}{2}\|\mathbf{u}^m(0)\|^2_{H^4(0,a)} + C(c_1)\|g\|^2_{H^4(0,a)},\end{aligned}$$

where we used the Cauchy inequality. Simplifying the last inequality,

$$\|\mathbf{u}^m(0)\|_{H^4(0,a)} \le C(c_1)\|g\|_{H^4(0,a)}. \tag{3.19}$$

Step 2. Estimate for $\|\mathbf{u}^{m\prime}(0)\|^2_{H^2(0,a)}$. Take (3.8) on t=0 and use (3.13) to deduce

$$\begin{aligned}&\int_0^a \mathbf{u}^{m\prime}(0)c_k^{-2}(w_k)_{xxxx}dx\\ &= -\int_0^a \phi^{s_0}\mathbf{u}^m_x(0)c_k^{-2}(w_k)_{xxxxx}dx + \int_0^a f(s_0)c_k^{-2}(w_k)_{xxxx}dx.\end{aligned} \tag{3.20}$$

Multiply by $c_k^2 {d_m^k}'(0)$ and sum over from $k=1$ to m to deduce

$$\int_0^a \mathbf{u}^{m\prime}(0)\mathbf{u}^{m\prime}{}_{xxxx}(0)dx = -\int_0^a \phi^{s_0}\mathbf{u}^m_x(0)\mathbf{u}^{m\prime}{}_{xxxxx}(0)dx + \int_0^a f(s_0)\mathbf{u}^{m\prime}{}_{xxxx}dx. \tag{3.21}$$

Notice that $\mathbf{u}^{m\prime}$ derived from (3.6) implies that ${\mathbf{u}^m_{xx}}' = 0$ and ${\mathbf{u}^m_{xxxx}}'(0) = 0$ on $\partial(0,a)$. We thus have the following analogous estimates to (3.19):

$$\|\mathbf{u}^{m\prime}(0)\|^2_{H^2(0,a)} \le C(c_1)\|\mathbf{u}^{m\prime}{}_{xx}(0)\|^2_{L^2(0,a)} = C(c_1)\int_0^a \mathbf{u}^{m\prime}(0)\mathbf{u}^{m\prime}{}_{xxxx}(0)dx.$$

Using (3.21) in the last inequality, we obtain

$$\begin{aligned}&\|\mathbf{u}^{m\prime}(0)\|^2_{H^2(0,a)}\\ &\le C(c_1)\Big(-\int_0^a \phi^{s_0}\mathbf{u}^m_x(0)\mathbf{u}^{m\prime}{}_{xxxxx}(0)dx + \int_0^a f(s_0)\mathbf{u}^{m\prime}{}_{xxxx}(0)dx\Big)\\ &\le C(c_1)\Big(B\int_0^a [\mathbf{u}^m_x(0)\mathbf{u}^{m\prime}{}_{xxxxx}(0)]^+\,dx - B\int_0^a [\mathbf{u}^m_x(0)\mathbf{u}^{m\prime}{}_{xxxxx}(0)]^-\,dx\\ &\quad + \int_0^a f(s_0)\mathbf{u}^{m\prime}{}_{xxxx}dx\Big)\\ &\le C(c_1,B)\Big(\int_0^a [\mathbf{u}^m_{xxxx}(0)\mathbf{u}^{m\prime}{}_{xx}(0)]^+\,dx - \int_0^a [\mathbf{u}^m_{xxxx}(0)\mathbf{u}^{m\prime}{}_{xx}(0)]^-\,dx\\ &\quad + \int_0^a [(s_0)_{xx}f'(s_0) + (s_0)_x f''(s_0)]\mathbf{u}^{m\prime}{}_{xx}(0)dx\Big), \quad \text{(integrating by parts)}\\ &\le C(c_1,B)\Big(\int_0^a |\mathbf{u}^m_{xxxx}(0)\mathbf{u}^{m\prime}{}_{xx}(0)|dx + \int_0^a [(s_0)_{xx}f'(s_0)\\ &\quad + (s_0)_x f''(s_0)]\mathbf{u}^{m\prime}{}_{xx}(0)dx\Big)\\ &\le C(c_1,B,L,L')\left(\|\mathbf{u}^m(0)\|_{H^4(0,a)}\|\mathbf{u}^{m\prime}(0)\|_{H^2(0,a)} + \|s_0\|_{H^2(0,a)}\|\mathbf{u}^{m\prime}(0)\|_{H^2(0,a)}\right)\\ &\quad \text{(using Hölder inequality, (1.11), (1.12) and simplifying)}\\ &\le \epsilon\|\mathbf{u}^{m\prime}(0)\|^2_{H^2(0,a)} + \epsilon^{-1}C(c_1,B,L,L')\Big(\|\mathbf{u}^m(0)\|^2_{H^4(0,a)} + \|s_0\|^2_{H^2(0,a)}\Big),\end{aligned}$$

where we used the Cauchy inequality with ϵ Choosing ϵ sufficiently small and simplifying, yield:

$$\begin{aligned}\|\mathbf{u}^{m\prime}(0)\|^2_{H^2(0,a)} &\le C(c_1,B,L,L')\Big(\|\mathbf{u}^m(0)\|^2_{H^4(0,a)}+\|s_0\|^2_{H^2(0,a)}\Big)\\ &\le C(c_1,B,L,L')\Big(\|g\|^2_{H^4(0,a)}+\|s_0\|^2_{H^2(0,a)}\Big),\end{aligned}\tag{3.22}$$

where we have employed (3.19)

Step 3. Multiplying (3.8) by ${d^k_m}'(t)$, summing over $k=1,\dots,m$, recalling (3.6) and integrating by parts, we deduce

$$\begin{aligned}&2\|\mathbf{u}^{m\prime}(t)\|^2_{L^2(0,a)}+\frac{d}{dt}\Big(\int_0^a \phi^{s_0}|\mathbf{u}^m_x(t)|^2dx\Big)\\ &=-2\int_0^a\Big(\int_0^t\partial_r\phi^s dr\Big)\mathbf{u}^m_x\mathbf{u}^{m\prime}_x dx+2\int_0^a f(s)\mathbf{u}^{m\prime}dx\\ &\le\sqrt{T}\sigma\Big(\|\mathbf{u}^m_x(t)\|^2_{L^2(0,a)}+\|\mathbf{u}^{m\prime}_x(t)\|^2_{L^2(0,a)}\Big)+\|\mathbf{u}^{m\prime}(t)\|^2_{L^2(0,a)}+\int_0^a|f(s)|^2dx,\end{aligned}\tag{3.23}$$

using (2.14) and Cauchy inequality. Simplifying (3.23), integrating over $[0,t]$ and using (1.5) and (1.6), we deduce

$$\begin{aligned}&\int_0^t\|\mathbf{u}^{m\prime}(r)\|^2_{L^2(0,a)}dr+\lambda\|\mathbf{u}^m_x(t)\|^2_{L^2(0,a)}\\ &\le\sqrt{T}\sigma\int_0^t\|\mathbf{u}^m_x(r)\|^2_{L^2(0,a)}dr+\sqrt{T}\sigma\int_0^t\|\mathbf{u}^{m\prime}_x(r)\|^2_{L^2(0,a)}dr\\ &\quad+\|f(s)\|^2_{L^2[0,T;L^2(0,a)]}+B\|\mathbf{u}^m_x(0)\|^2_{L^2(0,a)}.\end{aligned}\tag{3.24}$$

Step 4. For a fixed $m\ge 1$, define $\bar{\mathbf{u}}^m:=\mathbf{u}^{m\prime}$ and differentiate (3.8) with respect to t to obtain

$$\begin{aligned}&\int_0^a\bar{\mathbf{u}}^{m\prime}w_k dx+\int_0^a\phi^{s_0}\bar{\mathbf{u}}^m_x(w_k)_x\,dx\\ &=-\int_0^a\Big(\int_0^t\partial_r\phi^s dr\Big)\bar{\mathbf{u}}^m_x(w_k)_x\,dx-\int_0^a\partial_t\phi^s\mathbf{u}^m_x(w_k)_x dx+\int_0^a\partial_t f(s)w_k\,dx\\ &=-\int_0^a\Big(\int_0^t\partial_r\phi^s dr\Big)\bar{\mathbf{u}}^m_x(w_k)_x\,dx-\int_0^a\partial_t\phi^s\mathbf{u}^m_x(w_k)_x\,dx+\int_0^a\partial_t s f'(s)w_k\,dx\end{aligned}\tag{3.25}$$

for $k=1,\dots,m$. Multiplying by $d^{k\prime}_m(t)$, summing over $k=1,\dots,m$, and using (1.6), (1.11) and (2.14), we deduce

$$\begin{aligned}&\frac{d}{dt}\Big(\|\bar{\mathbf{u}}^m\|^2_{L^2(0,a)}\Big)+2\lambda\|\bar{\mathbf{u}}^m_x(t)\|^2_{L^2(0,a)}\\ &\le-2\int_0^a\Big(\int_0^t\partial_r\phi^s dr\Big)|\bar{\mathbf{u}}^m_x|^2dx-2\int_0^a\partial_t\phi^s\mathbf{u}^m_x\bar{\mathbf{u}}^m_x dx+2\int_0^a f'(s)\partial_t s\bar{\mathbf{u}}^m dx\\ &\quad+\|\bar{\mathbf{u}}^m(0)\|^2_{L^2(0,a)}\\ &\le 2\sqrt{T}\sigma\|\bar{\mathbf{u}}^m_x(t)\|^2_{L^2(0,a)}+\epsilon\|\bar{\mathbf{u}}^m_x(t)\|^2_{L^2(0,a)}+(4\epsilon)^{-1}\|\partial_t\phi^s\|^2_{L^\infty(0,a)}\|\mathbf{u}^m_x(t)\|^2_{L^2(0,a)}\\ &\quad+L\|\partial_t s\|^2_{L^2(0,a)}+L\|\bar{\mathbf{u}}^m(t)\|^2_{L^2(0,a)}.\end{aligned}$$

Integrating over $[0,t]$, we deduce

$$
\begin{aligned}
&\|\bar{\mathbf{u}}^m(t)\|^2_{L^2(0,a)} + 2\lambda\int_0^t \|\bar{\mathbf{u}}^m_x(r)\|^2_{L^2(0,a)}dr\\
&\le (2\sqrt{T}\sigma+\epsilon)\int_0^t \|\bar{\mathbf{u}}^m_x(r)\|^2_{L^2(0,a)}dr + (4\epsilon)^{-1}\int_0^t \|\partial_r\phi^s\|^2_{L^\infty(0,a)}(r)\|\mathbf{u}^m_x(r)\|^2_{L^2(0,a)}dr\\
&\quad + L\int_0^t \|\bar{\mathbf{u}}^m(r)\|^2_{L^2(0,a)}dr + L\|\partial_t s\|^2_{L^2[0,T;L^2(0,a)]} + \|\bar{\mathbf{u}}^m(0)\|^2_{L^2(0,a)}.
\end{aligned}
\tag{3.26}
$$

Step 5. Combining (3.24) and (3.26), choosing $T,\epsilon>0$ sufficiently small and simplifying, we deduce

$$
\begin{aligned}
&\|\bar{\mathbf{u}}^m(t)\|^2_{L^2(0,a)} + \|\mathbf{u}^m_x(t)\|^2_{L^2(0,a)} + \int_0^t \|\bar{\mathbf{u}}^m_x(r)\|^2_{L^2(0,a)}dr\\
&\le C(\lambda,B,L)\Big[\int_0^t \big(1+\|\partial_r\phi^s\|^2_{L^\infty(0,a)}(r)\big)\Big(\|\bar{\mathbf{u}}^m(r)\|^2_{L^2(0,a)} + \|\mathbf{u}^m_x(r)\|^2_{L^2(0,a)}\Big)dr\\
&\quad + \|f(s)\|^2_{L^2[0,T;L^2(0,a)]} + \|\partial_t s\|^2_{L^2[0,T;L^2(0,a)]} + \|\mathbf{u}^m_x(0)\|^2_{L^2(0,a)} + \|\bar{\mathbf{u}}^m(0)\|^2_{L^2(0,a)}\Big]\\
&\le C(\lambda,B,L)\Big[\int_0^t \big(1+\|\partial_r\phi^s\|^2_{L^\infty(0,a)}(r)\big)\Big(\|\bar{\mathbf{u}}^m(r)\|^2_{L^2(0,a)} + \|\mathbf{u}^m_x(r)\|^2_{L^2(0,a)}\Big)dr\\
&\quad + \|s\|^2_{L^2[0,T;L^2(0,a)]} + \|\partial_t s\|^2_{L^2[0,T;L^2(0,a)]} + \|\mathbf{u}^m(0)\|^2_{H^2(0,a)} + \|\mathbf{u}^{m\prime}(0)\|^2_{H^2(0,a)}\Big],
\end{aligned}
\tag{3.27}
$$

where we have employed (2.12) in the last inequality. Extracting appropriate inequality from (3.27) and applying Gronwall inequality we deduce

$$
\begin{aligned}
&\|\bar{\mathbf{u}}^m(t)\|^2_{L^2(0,a)} + \|\mathbf{u}^m_x(t)\|^2_{L^2(0,a)}\\
&\le e^{C(T+\sigma^2)}C^2(T+\sigma^2)\Big(\|s\|^2_{L^2[0,T;L^2(0,a)]} + \|\partial_t s\|^2_{L^2[0,T;L^2(0,a)]}\\
&\quad + \|\mathbf{u}^m(0)\|^2_{H^2(0,a)} + \|\mathbf{u}^{m\prime}(0)\|^2_{H^2(0,a)}\Big)\\
&\le 2C^2\sigma^2e^{2C\sigma^2}\Big(\|s\|^2_{L^2[0,T;L^2(0,a)]} + \|\partial_t s\|^2_{L^2[0,T;L^2(0,a)]}\\
&\quad + \|\mathbf{u}^m(0)\|^2_{H^2(0,a)} + \|\mathbf{u}^{m\prime}(0)\|^2_{H^2(0,a)}\Big),
\end{aligned}
\tag{3.28}
$$

for sufficiently small $T>0$. Using (3.28) in (3.27), and employing (3.19) and (3.22), we deduce

$$
\begin{aligned}
&\sup_{0\le t\le T}\Big(\|\mathbf{u}^{m\prime}(t)\|^2_{L^2(0,a)} + \|\mathbf{u}^m_x(t)\|^2_{L^2(0,a)}\Big) + \|\mathbf{u}^{m\prime}\|^2_{L^2[0,T;H^1_0(0,a)]}\\
&\le C\Big(\|s\|^2_{L^2[0,T;L^2(0,a)]} + \|\partial_t s\|^2_{L^2[0,T;L^2(0,a)]} + \|g\|^2_{H^4(0,a)} + \|s_0\|^2_{H^2(0,a)}\Big),
\end{aligned}
\tag{3.29}
$$

where $C=C(c_1,\lambda,B,L,L',\sigma)$.

Step 6. As in [10], fix any $v\in H^1_0(0,a)$, such that $\|v\|_{H^1_0(0,a)}\le 1$, and set $v=v^1+v^2$, where $v^1\in\operatorname{span}\{w_k\}_{k=1}^m$ and $\int_0^a v^2w_k dx=0$, $(k=1,\dots,m)$. Thus $\|v^1\|_{H^1_0(0,a)}\le\|v\|_{H^1_0(0,a)}\le 1$, since $\{w_k\}_{k=0}^\infty$ are orthogonal in $H^1_0(0,a)$. Hence, using (3.8), we deduce for a.e. $0\le t\le T$ that

$$
\int_0^a \mathbf{u}^{m\prime}v^1dx + \int_0^a \phi(t,x,s)\mathbf{u}^m_x v^1_x dx = \int_0^a f(s)v^1dx. \tag{3.30}
$$

Using Hölder inequality and the last equality, (3.6) implies

$$\begin{aligned}
\langle \mathbf{u}^{m\prime}, v^1\rangle &= \int_0^a \mathbf{u}^{m\prime} v dx = \int_0^a \mathbf{u}^{m\prime} v^1 dx\\
&= \int_0^a f(s) v^1 dx - \int_0^a \phi(t,x,s)\mathbf{u}_x^m v_x^1 dx\\
&\le \|f(s)\|_{L^2(0,a)}\|v^1\|_{L^2(0,a)} + B\|\mathbf{u}^m\|_{H_0^1(0,a)}\|v^1\|_{H_0^1(0,a)}, \quad \text{(using (1.5))}\\
&\le C(a)\|f(s)\|_{L^2(0,a)}\|v^1\|_{H_0^1(0,a)} + B\|\mathbf{u}^m\|_{H_0^1(0,a)}\|v^1\|_{H_0^1(0,a)},
\end{aligned}$$

by Poincaré-Friedrichs inequality (Theorem 2.3). Therefore,

$$|\langle \mathbf{u}^{m\prime}, v^1\rangle| \le C(a,B)\Big(\|f(s)\|_{L^2(0,a)} + \|\mathbf{u}^m\|_{H_0^1(0,a)}\Big), \tag{3.31}$$

since $\|v^1\|_{H_0^1(0,a)} \le 1$. We thus have

$$\|\mathbf{u}^{m\prime}\|_{H^{-1}(0,a)} \le C(a,B)\Big(\|f(s)\|_{L^2(0,a)} + \|\mathbf{u}^m\|_{H_0^1(0,a)}\Big), \tag{3.32}$$

using (1.5). We can easily deduce

$$\begin{aligned}
&\|\mathbf{u}^{m\prime}\|^2_{L^2[0,T;H^{-1}(0,a)]}\\
&\le C(a,B)\Big(T\|\mathbf{u}\|^2_{L^\infty[0,T;H_0^1(0,a)]} + \|f(s)\|^2_{L^2[0,T;L^2(0,a)]}\Big)\\
&\le C(a,B)\Big(\|\mathbf{u}\|^2_{L^\infty[0,T;H_0^1(0,a)]} + \|f(s)\|^2_{L^2[0,T;L^2(0,a)]}\Big)\\
&\quad \text{for sufficiently small } T>0\\
&\le C\Big(\|s\|^2_{L^2[0,T;L^2(0,a)]} + \|\partial_t s\|^2_{L^2[0,T;L^2(0,a)]} + \|g\|^2_{H^4(0,a)} + \|s_0\|^2_{H^2(0,a)}\Big),
\end{aligned} \tag{3.33}$$

where $C = C(c_1, L, a, B, \sigma, \lambda)$ and where we have employed (2.12) and (3.29).

Step 7. Next, we show that $\mathbf{u}'' \in L^2[0,T;H^{-1}(0,a)]$. We employ once more the function v of Step 4. Using (3.8), we deduce for a.e. $0 \le t \le T$ that

$$\int_0^a \mathbf{u}^{m\prime\prime} v^1 dx + \int_0^a \phi(t,x,s)\mathbf{u}_x^{m\prime} v_x^1 dx = -\int_0^a \partial_t\phi(t,x,s)\mathbf{u}_x^m v_x^1 + \int_0^a \partial_t s f'(s) v^1 dx. \tag{3.34}$$

Thus, (3.6) implies

$$\begin{aligned}
&\langle \mathbf{u}^{m\prime\prime}, v^1\rangle\\
&= \int_0^a \mathbf{u}^{m\prime\prime} v dx = \int_0^a \mathbf{u}^{m\prime\prime} v^1 dx\\
&= -\int_0^a \phi(t,x,s)\mathbf{u}_x^{m\prime} v_x^1 dx - \int_0^a \partial_t\phi(t,x,s)\mathbf{u}_x^m v_x^1 dx + \int_0^a \partial_t s f'(s) v^1 dx\\
&\le B\|\mathbf{u}^{m\prime}(t)\|_{H_0^1(0,a)}\|v^1\|_{H_0^1(0,a)} + \|\partial_t\phi^s\|_{L^\infty(0,a)}\|\mathbf{u}^m(t)\|_{H_0^1(0,a)}\|v^1\|_{H_0^1(0,a)}\\
&\quad + LC(a)\|\partial_t s\|_{L^2(0,a)}\|v^1\|_{H_0^1(0,a)},\\
&\text{(using (1.5), (1.11), Hölder and Poincaré-Friedrichs inequalities)}\\
&\le C(a,L,B)\Big(\|\mathbf{u}^{m\prime}(t)\|_{H_0^1(0,a)} + \|\partial_t\phi^s\|_{L^\infty(0,a)}\|\mathbf{u}^m(t)\|_{H_0^1(0,a)} + \|\partial_t s\|_{L^2(0,a)}\Big),
\end{aligned}$$

since $\|v^1\|_{H_0^1(0,a)} \le 1$. Therefore,

$$\|\mathbf{u}^{m\prime\prime}\|_{H^{-1}(0,a)} \le C(a,L,B)\Big(\|\mathbf{u}^{m\prime}(t)\|_{H_0^1(0,a)} + \|\partial_t\phi^s\|_{L^\infty(0,a)}\sup_{0\le t\le T}\|\mathbf{u}^m(t)\|_{H_0^1(0,a)} + \|\partial_t s\|_{L^2(0,a)}\Big). \tag{3.35}$$

Integrating (3.35) over $[0,T]$, we deduce

$$\begin{aligned}
&\|\mathbf{u}^{m\prime\prime}\|_{L^2[0,T;H^{-1}(0,a)]}\\
&\le C(a,L,B)\Big(\|\mathbf{u}^{m\prime}(t)\|_{L^2[0,T;H_0^1(0,a)]} + \|\partial_t s\|_{L^2[0,T;L^2(0,a)]}\\
&\quad + \|\partial_t\phi^s\|_{L^2[0,T,L^\infty(0,a)]}\sup_{0\le t\le T}\|\mathbf{u}^m(t)\|_{H_0^1(0,a)}\Big)\\
&\le C\Big(\|s\|^2_{L^2[0,T;L^2(0,a)]} + \|\partial_t s\|^2_{L^2[0,T;L^2(0,a)]} + \|g\|^2_{H^4(0,a)} + \|s_0\|^2_{H^2(0,a)}\Big),
\end{aligned} \tag{3.36}$$

where $C = C(a,B,\lambda,L,\sigma)$; and we have employed (3.29).

Step 8. We estimate $\|\mathbf{u}^m_{xx}\|_{L^\infty[0,T;L^2(0,a)]}$. Now, (3.13) implies that $(w_k)_{xx}(0) = (w_k)_{xx}(a) = 0$. Using the Remark 2.8, (3.8), (3.13) and integration by parts, we deduce

$$\begin{aligned}
&-\int_0^a \phi^{s_0}\mathbf{u}^m_x(c_k^{-1}(w_k)_{xx})_x dx\\
&= \int_0^a \phi^{s_0}_x\mathbf{u}^m_x c_k^{-1}(w_k)_{xx}dx + \int_0^a \phi^{s_0}\mathbf{u}^m_{xx}c_k^{-1}(w_k)_{xx}dx\\
&\le \int_0^a \mathbf{u}^{m\prime}c_k^{-1}(w_k)_{xx}dx + \sqrt{T}\sigma\int_0^a |\mathbf{u}^m_{xx}c_k^{-1}(w_k)_{xx}|dx - \int_0^a f(s)c_k^{-1}(w_k)_{xx}dx.
\end{aligned}$$

Multiplying by $c_k d_m^k(t)$ and summing from $k=1$ to m, we deduce

$$\begin{aligned}
\int_0^a \mathbf{u}^m_{xx}|^2dx &\le \int_0^a \phi^{s_0}|\mathbf{u}^m_{xx}|^2dx\\
&\le -\int_0^a \phi^{s_0}_x\mathbf{u}^m_x\mathbf{u}^m_{xx}dx + \sqrt{T}\sigma\int_0^a|\mathbf{u}^m_{xx}|^2dx\\
&\quad + \int_0^a \mathbf{u}^{m\prime}\mathbf{u}^m_{xx}dx - \int_0^a f(s)\mathbf{u}^m_{xx}dx\\
&\le \sqrt{T}\sigma|\mathbf{u}^m_{xx}(t)\|^2_{L^2(0,a)} + 3\epsilon\|\mathbf{u}^m_{xx}(t)\|^2_{L^2(0,a)}\\
&\quad + \frac{1}{4\epsilon}\Big(\|\mathbf{u}^{m\prime}\|^2_{L^2(0,a)} + B_0^2\|\mathbf{u}^m_x\|^2_{L^2(0,a)} + \|f(s)\|^2_{L^2(0,a)}\Big),
\end{aligned}$$

where we used (1.6), (2.14) and Cauchy inequality (with and without ϵ). Choosing $T,\epsilon>0$ sufficiently small in the last inequality and simplifying, we deduce

$$\begin{aligned}
&\sup_{0\le t\le T}\|\mathbf{u}^m_{xx}(t)\|^2_{L^2(0,a)}\\
&\le C\Big(\|\mathbf{u}^m\|^2_{L^\infty[0,T;H_0^1(0,a)]} + \|\mathbf{u}^{m\prime}\|^2_{L^\infty[0,T;L^2(0,a)]} + \|f(s)\|^2_{L^\infty[0,T;L^2(0,a)]}\Big)\\
&\le C\Big(\|s\|^2_{L^2[0,T;L^2(0,a)]} + \|\partial_t s\|^2_{L^2[0,T;L^2(0,a)]} + \|g\|^2_{H^4(0,a)} + \|s_0\|^2_{H^2(0,a)}\Big),
\end{aligned} \tag{3.37}$$

where $C = C(a,B,B_0,\lambda,\sigma,L,L',c_1)$; and we have employed (1.6), (1.7), (2.12), and (3.29).

Step 9. Estimate for $\|\mathbf{u}_x^{m\prime}\|_{L^2[0,T;H^*(0,a)}$. For $k=1,\dots,m$, (3.8) and (3.13) imply that

$$-\int_0^a \mathbf{u}^{m\prime}(w_k)_{xx}dx - \int_0^a \phi(t,x,s)\mathbf{u}_x^m(w_k)_{xxx}dx = -\int_0^a f(s)(w_k)_{xx}dx, \quad (3.38)$$

which, on integrating the left hand side by parts, one deduces

$$\int_0^a \mathbf{u}_x^{m\prime}(w_k)_x dx \le B\int_0^a |\mathbf{u}_{xx}^m(w_k)_{xx}|dx - \int_0^a f(s)(w_k)_{xx}dx, \quad (3.39)$$

where we used (1.5). Fix any $v\in H_0^1(0,a)\cap H^2(0,a)$ such that $\|v\|_{H_0^1(0,a)\cap H^2(0,a)}\le 1$ and set $v=v^1+v^2$, where $v_{xx}^1\in \operatorname{span}\{w_k\}_{k=1}^m$ and $\int_0^a v_{xx}^2 w_k dx=0$, $(k=1,\dots,m)$. Consequently, $\|v^1\|_{H_0^1(0,a)\cap H^2(0,a)}\le \|v\|_{H_0^1(0,a)\cap H^2(0,a)}\le 1$, since the functions $\{w_k\}_{k=0}^\infty$ are orthogonal in $H_0^1(0,a)$. Hence, using (3.39), we have

$$\int_0^a \mathbf{u}_x^{m\prime}v_x^1 dx \le B\int_0^a |\mathbf{u}_{xx}^m v_{xx}^1|dx - \int_0^a f(s)v_{xx}^1 dx. \quad (3.40)$$

Using Hölder inequality and the last equality, (3.6) implies

$$\begin{aligned}
\langle \mathbf{u}_x^{m\prime}, v_x^1\rangle &= \int_0^a \mathbf{u}_x^{m\prime}v_x = \int_0^a \mathbf{u}_x^{m\prime}v_x^1 dx\\
&\le B\int_0^a |\mathbf{u}_{xx}^m v_{xx}^1|dx - \int_0^a f(s)v_{xx}^1 dx\\
&\le \|v_{xx}^1\|_{L^2(0,a)}\big(B\|\mathbf{u}_{xx}^m\|_{L^2(0,a)} + \|f(s)\|_{L^2(0,a)}\big)\\
&\le \|v_{xx}^1\|_{H_0^1(0,a)\cap H^2(0,a)}\big(B\|\mathbf{u}_{xx}^m\|_{L^2(0,a)} + \|f(s)\|_{L^2(0,a)}\big).
\end{aligned}$$

Consequently, we have

$$|\langle \mathbf{u}_x^{m\prime}, v_x^1\rangle| \le \left(B\|\mathbf{u}_{xx}^m\|_{L^2(0,a)} + \|f(s)\|_{L^2(0,a)}\right), \quad (3.41)$$

since $\|v^1\|_{H_0^1(0,a)\cap H^2(0,a)}\le 1$. Therefore,

$$\|\mathbf{u}_x^{m\prime}\|_{H^*(0,a)} \le \left(B\|\mathbf{u}_{xx}^m\|_{L^2(0,a)} + \|f(s)\|_{L^2(0,a)}\right). \quad (3.42)$$

Squaring both sides of (3.42), integrating with respect to t from $t=0$ to $t=T$ and simplifying, we deduce

$$\begin{aligned}
&\|\mathbf{u}_x^{m\prime}\|_{L^2[0,T;H^*(0,a)}^2\\
&\le 2B^2\|\mathbf{u}^m\|_{L^\infty[0,T;H_0^1(0,a)\cap H^2(0,a)]}^2 + 2T\|f(s)\|_{L^\infty[0,T;L^2(0,a)]}^2\\
&\le C\Big(\|s\|_{L^2[0,T;L^2(0,a)]}^2 + \|\partial_t s\|_{L^2[0,T;L^2(0,a)]}^2 + \|g\|_{H^4(0,a)}^2 + \|s_0\|_{H^2(0,a)}^2\Big),
\end{aligned} \quad (3.43)$$

where $C=C(a,B,B_0,\lambda,\sigma,L,L',c_1)$; and $T>0$ is chosen sufficiently small; and we have employed (2.12), (3.29) and (3.37).

Step 10. We finally estimate $\|\mathbf{u}^{m\prime}\|_{L^\infty[0,T;H_0^1(0,a)]}$. Multiply (3.25) by ${d_m^k}''(t)$, sum from $k=1$ to m and integrate by parts to deduce

$$\begin{aligned}
&2\int_0^a |\bar{\mathbf{u}}^{m\prime}|^2dx + \frac{d}{dt}\Big(\int_0^a \phi^{s_0}|\bar{\mathbf{u}}_x^m|^2dx\Big)\\
&= -\int_0^a \left(\int_0^t \partial_r\phi^s dr\right)|\bar{\mathbf{u}}_x^m|^2dx + \int_0^a \partial_t\phi^s|\bar{\mathbf{u}}_x^m|^2dx + 2\int_0^a \partial_t\phi_x^s\mathbf{u}_x^m\bar{\mathbf{u}}^{m\prime}dx\\
&\quad + 2\int_0^a \partial_t\phi^s\mathbf{u}_{xx}^m\bar{\mathbf{u}}^{m\prime}dx + 2\int_0^a \partial_t sf'(s)\bar{\mathbf{u}}^{m\prime}dx\\
&\le \big(\sqrt{T}\sigma + \|\partial_t\phi^s\|_{L^\infty(0,a)}\big)\int_0^a |\bar{\mathbf{u}}_x^m|^2dx + 3\epsilon\int_0^a |\bar{\mathbf{u}}^{m\prime}|^2dx\\
&\quad + \epsilon^{-1}\Big(\|\partial_t\phi_x^s\|^2_{L^\infty(0,a)}\|\mathbf{u}^m\|^2_{L^\infty[0,T;H_0^1(0,a)]}\\
&\quad + \|\partial_t\phi^s\|^2_{L^\infty(0,a)}\|\mathbf{u}_{xx}^m\|^2_{L^\infty[0,T;L^2(0,a)]} + L^2\|\partial_t s\|^2_{L^2(0,a)}\Big),
\end{aligned} \tag{3.44}$$

where used the Cauchy inequality with ϵ. Choosing $\epsilon > 0$ sufficiently small, simplifying and integrating over $[0,t]$, we deduce

$$\begin{aligned}
\|\mathbf{u}^{m\prime}(t)\|_{H_0^1(0,a)} \le C(\lambda,\sigma,L)\Big(&\int_0^t \Big(\sqrt{T}\sigma + \|\partial_r\phi^s(r)\|_{L^\infty(0,a)}\Big)\|\mathbf{u}^{m\prime}(r)\|^2_{H_0^1(0,a)}dr\\
&+ \|\mathbf{u}^m\|^2_{L^\infty[0,T;H_0^1(0,a)]} + \|\mathbf{u}_{xx}^m\|^2_{L^\infty[0,T;L^2(0,a)]}\\
&+ \|\partial_t s\|^2_{L^2[0,T;L^2(0,a)]} + \|\mathbf{u}^{m\prime}(0)\|^2_{H_0^1(0,a)}\Big).
\end{aligned}$$

Applying the Gronwall inequality to the last inequality, we deduce

$$\begin{aligned}
&\sup_{0\le t\le T}\|\mathbf{u}^{m\prime}(t)\|^2_{H_0^1(0,a)}\\
&\le C(\lambda,\sigma,L)\Big(\|\mathbf{u}^m\|^2_{L^\infty[0,T;H_0^1(0,a)]} + \|\mathbf{u}_{xx}^m\|^2_{L^\infty[0,T;L^2(0,a)]}\\
&\quad + \|\partial_t s\|^2_{L^2[0,T;L^2(0,a)]} + \|\mathbf{u}^{m\prime}(0)\|^2_{H^2(0,a)}\Big), \quad \text{(for some fixed } T>0)\\
&\le C\Big(\|s\|^2_{L^2[0,T;L^2(0,a)]} + \|\partial_t s\|^2_{L^2[0,T;L^2(0,a)]} + \|g\|^2_{H^4(0,a)} + \|s_0\|^2_{H^2(0,a)}\Big),
\end{aligned} \tag{3.45}$$

where $C = C(a,B,B_0,\lambda,\sigma,L,L',c_1)$ and we have used (3.22), (3.29) and (3.37).

Using (3.29), (3.33), (3.36), (3.37) and (3.45), we deduce (3.11), as desired. The particular case follows readily by substituting $s=g$ in (3.11) to deduce (3.12). □

3.3. Existence of unique solution to the auxiliary linear problem.

Theorem 3.3. *The auxiliary linear problem* (3.1)–(3.3) *has a unique solution*

$$\begin{gathered}
\mathbf{u} \in L^\infty[0,T;H_0^1(0,a)\cap H^2(0,a)], \quad \mathbf{u}' \in L^\infty[0,T;H_0^1(0,a)],\\
\mathbf{u}_x \in L^2[0,T;H^*(0,a), \quad \mathbf{u}',\mathbf{u}'' \in L^2[0,T;H^{-1}(0,a)].
\end{gathered} \tag{3.46}$$

Proof. The proof is split in four steps.

Step 1. Notice that the energy estimates (3.11) imply that the sequences:

$$\begin{gathered}
\{\mathbf{u}^m\}_{m=1}^\infty \text{ is bounded in } L^\infty[0,T;H_0^1(0,a)\cap H^2(0,a)],\\
\{\mathbf{u}^{m\prime}\}_{m=1}^\infty \text{ is bounded in } L^\infty[0,T;H_0^1(0,a)],\\
\{\mathbf{u}_x^{m\prime}\}_{m=1}^\infty \text{ is bounded in } L^2[0,T;H^*(0,a),
\end{gathered}$$

$$\{\mathbf{u}^{m\prime}\}_{m=1}^{\infty} \text{ is bounded in } L^2[0,T;H^{-1}(0,a)],$$
$$\{\mathbf{u}^{m\prime\prime}\}_{m=1}^{\infty} \text{ is bounded in } L^2[0,T;H^{-1}(0,a)].$$

Consequently, there exists a subsequence $\{\mathbf{u}^{m_l}\}_{l=1}^{\infty} \subset \{\mathbf{u}^m\}_{m=1}^{\infty}$ and a function $\mathbf{u} \in L^{\infty}[0,T;H_0^1(0,a)\cap H^2(0,a)]$, with $\mathbf{u}' \in L^{\infty}[0,T;H_0^1(0,a)]$, $\mathbf{u}_x \in L^2[0,T;H^*(0,a)$ $\mathbf{u}' \in L^2[0,T;H^{-1}(0,a)]$ and $\mathbf{u}'' \in L^2[0,T;H^{-1}(0,a)]$ such that

$$\mathbf{u}^{m_l} \rightharpoonup \mathbf{u} \quad \text{in } L^{\infty}[0,T;H_0^1(0,a)\cap H^2(0,a)], \tag{3.47}$$
$$\mathbf{u}^{m_l\prime} \rightharpoonup \mathbf{u}' \quad \text{in } L^{\infty}[0,T;H_0^1(0,a)], \tag{3.48}$$
$$\mathbf{u}_x^{m_l} \rightharpoonup \mathbf{u}_x \quad \text{in } L^2[0,T;H^*(0,a), \tag{3.49}$$
$$\mathbf{u}^{m_l\prime} \rightharpoonup \mathbf{u}' \quad \text{in } L^2[0,T;H^{-1}(0,a)], \tag{3.50}$$
$$\mathbf{u}^{m_l\prime\prime} \rightharpoonup \mathbf{u}'' \quad \text{in } L^2[0,T;H^{-1}(0,a)]. \tag{3.51}$$

Step 2. Next we fix an integer M and choose a function $v \in C^1[0,T;H_0^1(0,a)]$ of the form

$$\mathbf{v} = \sum_{k=1}^{M} d^k(t)w_k, \quad [10] \tag{3.52}$$

where $\{d^k\}_{k=1}^M$ are given smooth functions. We choose $m \geq M$, sum over $k = 1,\dots,M$, and then integrate with respect to t, to get

$$\int_0^T\int_0^a \mathbf{u}^{m\prime}\mathbf{v}\,dx\,dt + \int_0^T\int_0^a \phi(t,x,s)\mathbf{u}_x^m\mathbf{v}_x\,dx\,dt = \int_0^T\int_0^a f(s)\mathbf{v}\,dx\,dt. \tag{3.53}$$

Using (1.5), it is trivial to show that

$$\begin{aligned} &\Big|\int_0^T\int_0^a \phi(t,x,s)\,(\mathbf{u}_x^m-\mathbf{u}_x)\,\mathbf{v}_x\,dx\,dt\Big| \\ &\leq BT\|\mathbf{u}_x^m-\mathbf{u}_x\|_{L^{\infty}[0,T;H_0^1(0,a)]}\|v\|_{C^1[0,T;H_0^1(0,a)]}. \end{aligned} \tag{3.54}$$

We set $m = m_l$ in (3.53), employ (3.54), recall (3.47)–(3.48) and pass to weak limits, to get

$$\int_0^T\int_0^a \mathbf{u}'\mathbf{v}\,dx\,dt + \int_0^T\int_0^a \phi(t,x,s)\mathbf{u}_x\mathbf{v}_x\,dx\,dt = \int_0^T\int_0^a f(s)\mathbf{v}\,dx\,dt. \tag{3.55}$$

Since the functions of the form (3.52) are dense in $L^2[0,T;H_0^1(0,a)]$, (3.55) holds for all functions in this space. In particular

$$\int_0^a \mathbf{u}'v dx + \int_0^a \phi(t,x,s)\mathbf{u}_x v_x dx = \int_0^a f(s)v dx, \tag{3.56}$$

for each $v \in H_0^1(0,a)$ and a.e. $0 \leq t \leq T$.

Step 3. We now prove that $\mathbf{u}(0) = g$. Note that (3.55) is equivalent to

$$\begin{aligned} &-\int_0^T\int_0^a \mathbf{v}'\mathbf{u}\,dx\,dt + \int_0^T\int_0^a \phi(t,x,s)\mathbf{u}_x\mathbf{v}_x\,dx\,dt \\ &= \int_0^T\int_0^a f(s)\mathbf{v}\,dx\,dt + \int_0^a \mathbf{u}(0)\mathbf{v}(0)dx, \end{aligned} \tag{3.57}$$

for each $\mathbf{v} \in C^1[0,T;H_0^1(0,a)]$ with $\mathbf{v}(T) = 0$. Similarly, (3.53) is equivalent to

$$\begin{aligned} &-\int_0^T\int_0^a \mathbf{v}'\mathbf{u}^m\,dx\,dt + \int_0^T\int_0^a \phi(t,x,s)\mathbf{u}_x^m\mathbf{v}_x\,dx\,dt \\ &= \int_0^T\int_0^a f(s)\mathbf{v}\,dx\,dt + \int_0^a \mathbf{u}^m(0)\mathbf{v}(0)dx. \end{aligned} \tag{3.58}$$

Setting $m = m_l$ in (3.58) and using once again (3.47)–(3.48), we deduce

$$\begin{aligned} &-\int_0^T\int_0^a \mathbf{v}'\mathbf{u}\,dx\,dt + \int_0^T\int_0^a \phi(t,x,s)\mathbf{u}_x\mathbf{v}_x\,dx\,dt \\ &= \int_0^T\int_0^a f(s)\mathbf{v}\,dx\,dt + \int_0^a g\mathbf{v}(0)dx, \end{aligned} \tag{3.59}$$

since $\mathbf{u}_{m_l}(0) \to g \in L^2(0,a)$. Comparing (3.57) and (3.59), we conclude that $\mathbf{u}(0) = g$, as $\mathbf{v}(0)$ is arbitrary.

Step 4. We now prove uniqueness. Let $\mathbf{u}$ and $\bar{\mathbf{u}}$ be two solutions of (3.1)–(3.3). Then $\mathbf{u} - \bar{\mathbf{u}}$ satisfies

$$\int_0^a (\mathbf{u} - \bar{\mathbf{u}})'v dx - \int_0^a \phi(t,x,s)(\mathbf{u} - \bar{\mathbf{u}})_x v_x\,dx = 0, \tag{3.60}$$

$$(\mathbf{u} - \bar{\mathbf{u}})(0) = 0, \tag{3.61}$$

for each $v \in H_0^1(0,a)$. Setting $v = \mathbf{u} - \bar{\mathbf{u}}$ in (3.60) we deduce

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u} - \bar{\mathbf{u}}\|^2\right) = -\int_0^a \phi(t,x,s)|\left(\mathbf{u} - \bar{\mathbf{u}}\right)_x|^2 dx \le 0. \tag{3.62}$$

Integrating (3.62) over $[0,t]$ and applying (3.61), we conclude that $\mathbf{u} \equiv \bar{\mathbf{u}}$. □

4. Main results

4.1. **Existence of unique solutions to the nonlinear problems.** We deduce from Theorem 2.5 and Corollary 2.4, after u is possibly redefined on a set of measure zero, that (2.1) implies that $\mathbf{u} \in C[0,T;C^{0,1/2}(0,a)]$ with $\mathbf{u}' \in C[0,T;L^2(0,a)]$.

Theorem 4.1. *Suppose conditions* (1.5)–(1.12) *hold. Then, there exist unique weak solutions to the problems* (1.1)–(1.3).

Proof. The proof consists of nine steps.

Step 1. Banach fixed point theorem will be applied in the space

$$X := \{\mathbf{U} \in C[0,T;C^{0,1/2}(0,a)] : \mathbf{U}' \in C[0,T;L^2(0,a)]\},$$

equipped with the norm

$$\|\mathbf{U}\|_X := \max_{0\le t\le T}\left(\|\mathbf{U}(t)\|_{C^{0,1/2}(0,a)} + \|\mathbf{U}'(t)\|_{L^2(0,a)}\right).$$

A fixed point argument to (1.1)–(1.3) is

$$w_t - (\phi(t,x,u)w_x)_x = f(u), \quad \text{in } [0,T]\times(0,a) \tag{4.1}$$

$$w(t,0) = 0, \quad w(t,a) = 0, \quad \text{in } (0,T] \tag{4.2}$$

$$w(0,x) = g(x), \quad x \in (0,a). \tag{4.3}$$

For a given function $\mathbf{u} \in X$, Theorems 3.2 and 3.3 ensure the existence and uniqueness of the solution to (4.1)–(4.3), namely:

$$\mathbf{w} \in L^\infty[0,T; H_0^1(0,a) \cap H^2(0,a)], \quad \mathbf{w}' \in L^\infty[0,T; H_0^1(0,a)],$$
$$\mathbf{w}'_x \in L^2[0,T; H^*(0,a), \quad \mathbf{w}', \mathbf{w}'' \in L^2[0,T; H^{-1}(0,a)],$$

with the estimate

$$\begin{aligned}
&\sup_{0\le t\le T} \Big(\|\mathbf{w}(t)\|^2_{H_0^1(0,a)\cap H^2(0,a)} + \|\mathbf{w}'(t)\|^2_{H_0^1(0,a)}\Big) + \|\mathbf{w}'_x\|^2_{L^2[0,T;H^*(0,a)} \\
&\quad + \|\mathbf{w}'\|^2_{L^2[0,T;H^{-1}(0,a)]} + \|\mathbf{w}''\|^2_{L^2[0,T;H^{-1}(0,a)]} \\
&\le C\Big(\|\mathbf{u}\|_X^2 + \|g\|^2_{H^4(0,a)}\Big),
\end{aligned} \tag{4.4}$$

where $C = C(L, a, \lambda, B, \sigma, c_1)$. Notice in particular that, if $u = g$, then we have the estimate

$$\begin{aligned}
&\sup_{0\le t\le T} \Big(\|\mathbf{w}(t)\|^2_{H_0^1(0,a)\cap H^2(0,a)} + \|\mathbf{w}'(t)\|^2_{H_0^1(0,a)}\Big) + \|\mathbf{w}'_x\|^2_{L^2[0,T;H^*(0,a)} \\
&\quad + \|\mathbf{w}'\|^2_{L^2[0,T;H^{-1}(0,a)]} + \|\mathbf{w}''\|^2_{L^2[0,T;H^{-1}(0,a)]} \\
&\le C\|g\|^2_{H^4(0,a)} =: \Lambda
\end{aligned} \tag{4.5}$$

Step 2. Now, define a mapping $M : X \to X$ by setting $M[\mathbf{u}] = \mathbf{w}$ whenever $\mathbf{w}$ is derived from $\mathbf{u}$ via (4.1)–(4.3). We will show that M is a contraction mapping for sufficiently small time $T > 0$. We choose $\mathbf{u}, \tilde{\mathbf{u}} \in X$ and define $M[\mathbf{u}] = \mathbf{w}$, $M[\tilde{\mathbf{u}}] = \tilde{\mathbf{w}}$. For two weak solutions $\mathbf{w}, \tilde{\mathbf{w}} \in X$ of (4.1)-(4.3), we deduce

$$\begin{aligned}
&\int_0^a (\mathbf{w} - \tilde{\mathbf{w}})' v + \int_0^a \phi(0, x, \mathbf{u}(0))(\mathbf{w} - \tilde{\mathbf{w}})_x v_x dx \\
&= -\int_0^a \Big(\int_0^t \partial_r \phi(r, x, \mathbf{u}(r)) dr\Big)(\mathbf{w} - \tilde{\mathbf{w}})_x v_x \, dx \\
&\quad - \int_0^a (\phi(t, x, \mathbf{u}) - \phi(t, x, \tilde{\mathbf{u}}))\, \tilde{\mathbf{w}}_x v_x \, dx + \int_0^a (f(\mathbf{u}) - f(\tilde{\mathbf{u}})) v \, dx,
\end{aligned} \tag{4.6}$$

$$(\mathbf{w} - \tilde{\mathbf{w}})(0) = 0, \tag{4.7}$$

for each $v \in H_0^1(0,a)$. Setting $v = (\mathbf{w} - \tilde{\mathbf{w}})'$ in (4.6) and using (1.6)–(1.8), we deduce

$$\begin{aligned}
&2\int_0^t \|\mathbf{w}'(r) - \tilde{\mathbf{w}}'(r)\|^2_{L^2(0,a)} dr + \lambda \|(\mathbf{w}(t) - \tilde{\mathbf{w}}(t))\|^2_{H_0^1(0,a)} \\
&\le -2\int_0^t \int_0^a \Big(\int_0^r \partial_p \phi(p, x, \mathbf{u}(p)) dp\Big)(\mathbf{w}_x(r) - \tilde{\mathbf{w}}_x(r))(\mathbf{w}'_x(r) - \tilde{\mathbf{w}}'_x(r)) \, dx \, dr \\
&\quad - 2\int_0^t \int_0^a \Big(\phi^{\mathbf{u}(r)} - \phi^{\tilde{u}(r)}\Big) \tilde{\mathbf{w}}_x(r)(\mathbf{w}'_x(r) - \tilde{\mathbf{w}}'_x(r)) \, dx \, dr \\
&\quad + 2\int_0^a (f(\mathbf{u}) - f(\tilde{\mathbf{u}}))(\mathbf{w} - \tilde{\mathbf{w}})' dx \\
&\le \sqrt{T}\sigma \int_0^t \Big(\|\mathbf{w}(r) - \tilde{\mathbf{w}}(r)\|^2_{H_0^1(0,a)} + \|\mathbf{w}'(r) - \tilde{\mathbf{w}}'(r)\|^2_{H_0^1(0,a)}\Big) dr \\
&\quad + \epsilon^{-1} C(L_1) \|\tilde{\mathbf{w}}\|^2_{L^\infty[0,T;H_0^1(0,a)\cap H^2(0,a)]} \int_0^t \|\mathbf{u}(r) - \tilde{\mathbf{u}}(r)\|^2_{C^{0,1/2}(0,a)} dr
\end{aligned}$$

$$+ \epsilon \int_0^t \|\mathbf{w}'(r) - \tilde{\mathbf{w}}'(r)\|_{H_0^1(0,a)}^2 dr + \epsilon \int_0^t \|\mathbf{w}'(r) - \tilde{\mathbf{w}}'(r)\|_{L^2(0,a)}^2 dr$$
$$+ \epsilon^{-1} C(a, L_1) \int_0^t \|\mathbf{u}(r) - \tilde{\mathbf{u}}(r)\|_{C^{0,1/2}(0,a)}^2 dr,$$

where we used the Cauchy inequality with and without ϵ.

Step 3. We next differentiate (4.1)–(4.2) to deduce:

$$\begin{aligned} &\int_0^a (\mathbf{w} - \tilde{\mathbf{w}})'' v dx + \int_0^a \phi(0, x, \mathbf{u}(0))(\mathbf{w} - \tilde{\mathbf{w}})_x' v_x, dx \\ &= -\int_0^a \partial_t \phi^{\mathbf{u}} (\mathbf{w} - \tilde{\mathbf{w}})_x v_x dx - \int_0^a \left(\partial_t \phi^{\mathbf{u}} - \partial_t \phi^{\tilde{\mathbf{u}}}\right) \tilde{\mathbf{w}}_x \mathbf{v}_x dx \\ &\quad - \int_0^a \left(\phi^{\mathbf{u}} - \phi^{\tilde{\mathbf{u}}}\right) \tilde{\mathbf{w}}_x' \mathbf{v}_x dx + \int_0^a \left[\mathbf{u}' f'(\mathbf{u}) - \tilde{\mathbf{u}}' f'(\tilde{\mathbf{u}})\right] v dx, \end{aligned} \tag{4.8}$$

$$\partial_t (\mathbf{w} - \tilde{\mathbf{w}})(0) = 0, \tag{4.9}$$

for each $v \in H_0^1(0,a)$. Setting $v = (\mathbf{w} - \tilde{\mathbf{w}})'$ in (4.9) and using (1.6), we deduce

$$\begin{aligned} &\|\mathbf{w}'(t) - \tilde{\mathbf{w}}'(t)\|_{L^2(0,a)}^2 + 2\lambda \int_0^t \|\mathbf{w}'(r) - \tilde{\mathbf{w}}'(r)\|_{H_0^1(0,a)}^2 dr \\ &\le -2 \int_0^t \int_0^a \partial_r \phi^{\mathbf{u}(r)} (\mathbf{w}_x(r) - \tilde{\mathbf{w}}_x(r))(\mathbf{w}_x'(r) - \tilde{\mathbf{w}}_x'(r))\, dx\, dr \\ &\quad - 2 \int_0^t \int_0^a \left(\partial_r \phi^{\mathbf{u}(r)} - \partial_r \phi^{\tilde{\mathbf{u}}(r)}\right) \tilde{\mathbf{w}}_x(r)(\mathbf{w}_x'(r) - \tilde{\mathbf{w}}_x'(r))\, dx\, dr \\ &\quad - 2 \int_0^t \int_0^a \left(\phi^{\mathbf{u}(r)} - \phi^{\tilde{\mathbf{u}}(r)}\right) \tilde{\mathbf{w}}_x'(r)(\mathbf{w}_x'(r) - \tilde{\mathbf{w}}_x'(r))\, dx\, dr \\ &\quad + 2 \int_0^t \int_0^a \left[\mathbf{u}'(r) f'(\mathbf{u}(r)) - \tilde{\mathbf{u}}' f'(\tilde{\mathbf{u}}(r))\right] (\mathbf{w}'(r) - \tilde{\mathbf{w}}'(r))\, dx\, dr. \end{aligned} \tag{4.10}$$

We estimate the terms on the right side of (4.10) using (1.5), (1.8)–(1.12); and by applying Hölder, Cauchy and Sobolev inequalities, as appropriate, thus:

$$\begin{aligned} &- 2 \int_0^t \int_0^a \partial_r \phi^{\mathbf{u}(r)} (\mathbf{w}_x(r) - \tilde{\mathbf{w}}_x(r))(\mathbf{w}_x'(r) - \tilde{\mathbf{w}}_x'(r))\, dx\, dr \\ &\le 2 \int_0^t \|\partial_r \phi^{\mathbf{u}(r)}\|_{L^\infty(0,a)} \|\mathbf{w}(r) - \tilde{\mathbf{w}}(r)\|_{H_0^1(0,a)} \|\mathbf{w}'(r) - \tilde{\mathbf{w}}'(r)\|_{H_0^1(0,a)} dr \\ &\le \epsilon \int_0^t \|\mathbf{w}'(r) - \tilde{\mathbf{w}}'(r)\|_{H_0^1(0,a)} dr \\ &\quad + \frac{1}{4\epsilon} \int_0^t \|\partial_r \phi^{\mathbf{u}(r)}\|_{L^\infty(0,a)}^2 \|\mathbf{w}(r) - \tilde{\mathbf{w}}(r)\|_{H_0^1(0,a)}^2 dr. \end{aligned} \tag{4.11}$$

$$\begin{aligned} &- 2 \int_0^t \int_0^a \left(\partial_r \phi^{\mathbf{u}(r)} - \partial_r \phi^{\tilde{\mathbf{u}}(r)}\right) \tilde{\mathbf{w}}_x(r)(\mathbf{w}_x'(r) - \tilde{\mathbf{w}}_x'(r))\, dx\, dr \\ &= -2 \int_0^t \int_0^a \left(\phi_r^{\mathbf{u}(r)} - \phi_r^{\tilde{\mathbf{u}}(r)} + \phi_{\mathbf{u}}^{\mathbf{u}(r)} \mathbf{u}'(r) - \phi_{\tilde{\mathbf{u}}}^{\tilde{\mathbf{u}}(r)} \tilde{\mathbf{u}}'(r)\right) \tilde{\mathbf{w}}_x(r)(\mathbf{w}_x'(r) \\ &\quad - \tilde{\mathbf{w}}_x'(r))\, dx\, dr \end{aligned}$$

$$
\begin{aligned}
&= -2\int_0^t\int_0^a \Big[\phi_r^{\mathbf{u}(r)} - \phi_r^{\tilde{\mathbf{u}}(r)} + \phi_{\mathbf{u}}^{\mathbf{u}(r)}(\mathbf{u}'(r) - \tilde{\mathbf{u}}'(r)) \\
&\quad + \tilde{\mathbf{u}}'(r)(\phi_{\mathbf{u}}^{\mathbf{u}(r)} - \phi_{\tilde{\mathbf{u}}}^{\tilde{\mathbf{u}}(r)})\Big]\tilde{\mathbf{w}}_x(r)(\mathbf{w}_x'(r) - \tilde{\mathbf{w}}_x'(r))\,dx\,dr \\
&\le 2\int_0^t \Big[C(a, L_2)\|\mathbf{u}(r) - \tilde{\mathbf{u}}(r)\|_{C^{0,1/2}(0,a)} + L_1\|\mathbf{u}'(r) - \tilde{\mathbf{u}}'(r)\|_{L^2(0,a)} \\
&\quad + \|\tilde{\mathbf{u}}\|_X L_3\|\mathbf{u}(r) - \tilde{\mathbf{u}}(r)\|_{C^{0,1/2}(0,a)}\Big]\|\tilde{\mathbf{w}}_x(r)\|_{C^{0,1/2}(0,a)}\|\mathbf{w}'(r) - \tilde{\mathbf{w}}'(r)\|_{H_0^1(0,a)}dr \\
&\le \epsilon\int_0^t \|\mathbf{w}'(r) - \tilde{\mathbf{w}}'(r)\|^2_{H_0^1(0,a)}dr \\
&\quad + \epsilon^{-1}TC(a, L_1, L_2, L_3)(1 + \|\tilde{\mathbf{u}}\|_X^2)\|\tilde{\mathbf{w}}_x\|^2_{L^\infty[0,T;H_0^1(0,a)\cap H^2(0,a)]} \\
&\quad \times\int_0^t \Big(\|\mathbf{u}(r) - \tilde{\mathbf{u}}(r)\|^2_{C^{0,1/2}(0,a)} + \|\mathbf{u}'(r) - \tilde{\mathbf{u}}'(r)\|^2_{L^2(0,a)}\Big)\,dr.
\end{aligned}
$$

$$
\begin{aligned}
&-2\int_0^t\int_0^a \Big(\phi^{\mathbf{u}(r)} - \phi^{\tilde{\mathbf{u}}(r)}\Big)\,\tilde{\mathbf{w}}_x'(\mathbf{w}(r) - \tilde{\mathbf{w}}(r))_x'\,dx\,dr \\
&\le L_1\int_0^a |\mathbf{u}(r) - \tilde{\mathbf{u}}(r)||\tilde{\mathbf{w}}_x'(r)||\mathbf{w}_x'(r) - \tilde{\mathbf{w}}_x'(r)|\,dx\,dr \\
&\le 2L_1\int_0^t \|\mathbf{u}(r) - \tilde{\mathbf{u}}(r)\|_{C^{0,1/2}(0,a)}\|\tilde{\mathbf{w}}_x'(r)\|_{L^2(0,a)}\|\mathbf{w}_x'(r) - \tilde{\mathbf{w}}_x'(r)\|_{L^2(0,a)}dr \\
&\le \epsilon^{-1}\|\tilde{\mathbf{w}}'\|^2_{L^\infty[0,T;H_0^1(0,a)]}\int_0^t \|\mathbf{u}(r) - \tilde{\mathbf{u}}(r)\|^2_{C^{0,1/2}(0,a)}dr \\
&\quad + \epsilon\int_0^t \|\mathbf{w}'(r) - \tilde{\mathbf{w}}'(r)\|^2_{H_0^1(0,a)}dr.
\end{aligned} \tag{4.12}
$$

$$
\begin{aligned}
&2\int_0^t\int_0^a [\mathbf{u}'(r)f'(\mathbf{u}(r)) - \tilde{\mathbf{u}}'(r)f'(\tilde{\mathbf{u}})]\,(\mathbf{w} - \tilde{\mathbf{w}})'\,dx\,dr \\
&= 2\int_0^t\int_0^a f'(\mathbf{u}(r))(\mathbf{u}'(r) - \tilde{\mathbf{u}}'(r)) + \tilde{\mathbf{u}}'(r)(f'(\mathbf{u}(r)) \\
&\quad - f'(\tilde{\mathbf{u}}(r)))](\mathbf{w}'(r) - \tilde{\mathbf{w}}'(r))\,dx\,dr \\
&\le 2L\int_0^t\int_0^a |\mathbf{u}'(r) - \tilde{\mathbf{u}}'(r)||\mathbf{w}'(r) - \tilde{\mathbf{w}}'(r)|\,dx\,dr \\
&\quad + 2L'\int_0^t\int_0^a |\tilde{\mathbf{u}}'(r)||\mathbf{u}(r) - \tilde{\mathbf{u}}(r)||\mathbf{w}'(r) - \tilde{\mathbf{w}}'(r)|\,dx\,dr \\
&\le C(a, L, L')(1 + \|\tilde{\mathbf{u}}\|_X^2)\int_0^t \Big(\|\mathbf{u}(r) - \tilde{\mathbf{u}}(r)\|^2_{C^{0,1/2}(0,a)} \\
&\quad + \|\mathbf{u}'(r) - \tilde{\mathbf{u}}'(r)\|^2_{L^2(0,a)}\Big)dr + \epsilon\int_0^t \|\mathbf{w}'(r) - \tilde{\mathbf{w}}'(r)\|^2_{L^2(0,a)}dr.
\end{aligned} \tag{4.13}
$$

Using estimates (4.11)–(4.13) in (4.10), combining the ensuing inequality with (4.9), choosing $T, \epsilon > 0$ sufficiently small and simplifying, we deduce

$$
\|\mathbf{w}'(t) - \tilde{\mathbf{w}}'(t)\|^2_{L^2(0,a)} + \|\mathbf{w}(t) - \tilde{\mathbf{w}}(t)\|^2_{H_0^1(0,a)}
$$

$$\begin{aligned}
&\le C\Big[\int_0^t \big(1+\|\partial_r\phi^{\mathbf{u}(r)}\|^2_{L^\infty(0,a)}\big)\Big(\|\mathbf{w}'(r)-\tilde{\mathbf{w}}'(r)\|^2_{L^2(0,a)} \\
&\quad + \|\mathbf{w}(r)-\tilde{\mathbf{w}}(r)\|^2_{H_0^1(0,a)}\Big)dr + \big(1+\|\tilde{\mathbf{u}}\|_X^2\big)\Big(1+\|\tilde{\mathbf{w}}\|^2_{L^\infty[0,T;H_0^1(0,a)\cap H^2(0,a)]} \\
&\quad + \|\tilde{\mathbf{w}}'\|^2_{L^\infty[0,T;H_0^1(0,a)]}\Big)\int_0^t \big(\|\mathbf{u}(r)-\tilde{\mathbf{u}}(r)\|^2_{C^{0,1/2}(0,a)} + \|\mathbf{u}'(r)-\tilde{\mathbf{u}}'(r)\|^2_{L^2(0,a)}\big)dr\Big] \\
&\le C\Big[\int_0^t \big(1+\|\partial_r\phi^{\mathbf{u}(r)}\|^2_{L^\infty(0,a)}\big)\Big(\|\mathbf{w}'(r)-\tilde{\mathbf{w}}'(r)\|^2_{L^2(0,a)} \\
&\quad + \|\mathbf{w}(r)-\tilde{\mathbf{w}}(r)\|^2_{H_0^1(0,a)}\Big)dr + \Big(\|\tilde{\mathbf{u}}\|_X^2 + \|g\|^2_{H^4(0,a)}\Big)^2 \\
&\quad \times \int_0^t \Big(\|\mathbf{u}(r)-\tilde{\mathbf{u}}(r)\|^2_{C^{0,1/2}(0,a)} + \|\mathbf{u}'(r)-\tilde{\mathbf{u}}'(r)\|^2_{L^2(0,a)}\Big)\,dr\Big],
\end{aligned} \tag{4.14}$$

where $C = C(a, L, L', L_1, L_2, L_3, \lambda, \sigma, c_1)$ and we have estimated

$$\|\tilde{\mathbf{w}}\|^2_{L^\infty[0,T;H_0^1(0,a)\cap H^2(0,a)]} + \|\tilde{\mathbf{w}}'\|^2_{L^\infty[0,T;H_0^1(0,a)]}$$

by the bound in (4.4). Applying Gronwall inequality to (4.14), we deduce

$$\begin{aligned}
&\|\mathbf{w}'(t)-\tilde{\mathbf{w}}'(t)\|^2_{L^2(0,a)} + \|\mathbf{w}(t)-\tilde{\mathbf{w}}(t)\|^2_{H_0^1(0,a)} \\
&\le C\big[C(T+\sigma^2)e^{C(T+\sigma^2)}+1\big]\Big(\|\tilde{\mathbf{u}}\|_X^2 + \|g\|^2_{H^4(0,a)}\Big)^2 \\
&\quad \times \int_0^t \Big(\|\mathbf{u}(r)-\tilde{\mathbf{u}}(r)\|^2_{C^{0,1/2}(0,a)} + \|\mathbf{u}'(r)-\tilde{\mathbf{u}}'(r)\|^2_{L^2(0,a)}\Big)dr.
\end{aligned} \tag{4.15}$$

Step 4. Since the bound in (4.15) is not uniform, we will solve the problem in a bounded subset K of X, defined by

$$K := \{\mathbf{u} : \mathbf{u} - g \in X_0,\ \|\mathbf{u}\|_X \le 2\sqrt{\Lambda}\},$$

where X_0 is the set where the initial and boundary values of $\mathbf{u} - g$ are zero; and Λ is the bound in (4.5). Notice that for $\mathbf{u}, \tilde{\mathbf{u}} \in K$, (4.15) gives

$$\begin{aligned}
&\|\mathbf{w}'(t)-\tilde{\mathbf{w}}'(t)\|^2_{L^2(0,a)} + \|\mathbf{w}(t)-\tilde{\mathbf{w}}(t)\|^2_{H_0^1(0,a)} \\
&\le C\Big[C(T+\sigma^2)e^{C(T+\sigma^2)}+1\Big]\Big(4\Lambda + \|g\|^2_{H^4(0,a)}\Big)^2 \\
&\quad \times \int_0^t \Big(\|\mathbf{u}(r)-\tilde{\mathbf{u}}(r)\|^2_{C^{0,1/2}(0,a)} + \|\mathbf{u}'(r)-\tilde{\mathbf{u}}'(r)\|^2_{L^2(0,a)}\Big)\,dr.
\end{aligned} \tag{4.16}$$

We will show that if $T > 0$ is sufficiently small, then

$$M[K] \subseteq K \text{ and } \|M[\mathbf{u}] - M[\tilde{\mathbf{u}}]\|_X \le \frac{1}{2}\|\mathbf{u} - \tilde{\mathbf{u}}\|_X,$$

for all $\mathbf{u}, \tilde{\mathbf{u}} \in K$. Notice that (4.16) implies

$$\begin{aligned}
&\|\mathbf{w}'(t)-\tilde{\mathbf{w}}'(t)\|^2_{L^2(0,a)} + \|\mathbf{w}(t)-\tilde{\mathbf{w}}(t)\|^2_{H_0^1(0,a)} \\
&\le C[C(T+\sigma^2)e^{C(T+\sigma^2)}+1]T\Big(4\Lambda + \|g\|^2_{H^4(0,a)}\Big)^2\|\mathbf{u}-\tilde{\mathbf{u}}\|_X^2.
\end{aligned} \tag{4.17}$$

Maximizing the left side of (4.17) and applying the Corollary (2.4), we have

$$\|\mathbf{w}(t)-\tilde{\mathbf{w}}(t)\|_X^2 \le C\Big[C(T+\sigma^2)e^{C(T+\sigma^2)}+1\Big]T\Big(4\Lambda + \|g\|^2_{H^4(0,a)}\Big)^2\|\mathbf{u}-\tilde{\mathbf{u}}\|_X^2. \tag{4.18}$$

The definition of the mapping M and the estimate (4.5) imply that

$$M[g] \le \sqrt{\Lambda} < 2\sqrt{\Lambda}; \tag{4.19}$$

so that $M[g] \in K$. Thus the set K is not empty. For an arbitrary $\mathbf{u} \in K$, using (4.18)–(4.19), we deduce

$$\begin{aligned} \|M[\mathbf{u}]\|_X &= \|M[g]\|_X + \|M[\mathbf{u}] - M[g]\|_X \\ &\le \sqrt{\Lambda} + \Big(C \Big[C(T+\sigma^2)e^{C(T+\sigma^2)} + 1 \Big] T \big(4\Lambda + \|g\|^2_{H^4(0,a)} \big)^2 (4\Lambda) \Big)^{1/2} \\ &\le 2\sqrt{\Lambda}, \end{aligned}$$

for sufficiently small $T > 0$; so that $\|M[\mathbf{u}]\|_X \subseteq K$. Since $\mathbf{u}$ was arbitrarily chosen, it follows that $M[K] \subseteq K$. Furthermore, if $T > 0$ is chosen sufficiently small so that

$$\Big(C \Big[C(T+\sigma^2)e^{C(T+\sigma^2)} + 1 \Big] T \big(4\Lambda + \|g\|^2_{H^4(0,a)} \big)^2 \Big)^{1/2} \le \frac{1}{2}, \tag{4.20}$$

then we have that

$$\|M[\mathbf{u}] - M[\tilde{\mathbf{u}}]\|_X \le \frac{1}{2}\|\mathbf{u} - \tilde{\mathbf{u}}\|_X,$$

for all $\mathbf{u}, \tilde{\mathbf{u}} \in K$; so that M is a strict contraction mapping for $T > 0$ sufficiently small such that (4.20) holds.

Step 5. Now write $\mathbf{u}^0 = g$. For $k = 0, 1, 2, \dots$, inductively define $\mathbf{u}^{k+1} \in L^2[0,T; H^1_0(0,a)]$, with $\mathbf{u}^{k+1\prime} \in L^2[0,T; H^{-1}(0,a)]$ to be the unique weak solution of the linear boundary value problem

$$\frac{\partial}{\partial t}\big(u^{k+1}\big) - (\phi(t,x,u^k)u^{k+1}_x)_x = f(u^k), \quad \text{in } (0,T] \times (0,a) \tag{4.21}$$

$$u^{k+1}(t,0) = 0, \quad u^{k+1}(t,a) = 0, \quad \text{in } (0,T] \tag{4.22}$$

$$u^{k+1}(0,x) = g(x) \quad x \in (0,a). \tag{4.23}$$

Notice that Theorem 3.3 justifies our definition of $\mathbf{u}^{k+1}$ as the unique weak solution of (4.21)–(4.23). Consequently, by the definition of the mapping M, we deduce for $k = 0, 1, 2, \dots$,

$$\mathbf{u}^{k+1} = M[\mathbf{u}^k].$$

Since M is contractive (for sufficiently small $T > 0$), there exists $\mathbf{u} \in X$ such that

$$\lim_{k\to\infty} \mathbf{u}^{k+1} = \lim_{k\to\infty} M[\mathbf{u}^k] = M[\mathbf{u}] = \mathbf{u}. \tag{4.24}$$

Step 6. Using (4.4), we obtain

$$\begin{aligned} &\sup_{0\le t\le T} \Big(\|\mathbf{u}^{k+1}(t)\|^2_{H^1_0(0,a)\cap H^2(0,a)} + \|\mathbf{u}^{k+1\prime}(t)\|^2_{L^\infty[0,T;H^1_0(0,a)]} \Big) \\ &+ \|\mathbf{u}^{k+1\prime}_x\|^2_{L^2[0,T;H^*(0,a)} + \|\mathbf{u}^{k+1\prime}\|^2_{L^2[0,T;H^{-1}(0,a)]} + \|\mathbf{u}^{k+1\prime\prime}\|^2_{L^2[0,T;H^{-1}(0,a)]} \\ &\le C\Big(\|\mathbf{u}^k\|^2_X + \|g\|^2_{H^4(0,a)} \Big), \end{aligned} \tag{4.25}$$

from where using (4.24) and sending k to infinity on the right-hand side we deduce

$$\sup_k \|\mathbf{u}^{k+1}(t)\|_{L^\infty[0,T;H^1_0(0,a)\cap H^2(0,a)]} < \infty, \tag{4.26}$$

$$\sup_k \|\mathbf{u}^{k+1\prime}\|_{L^\infty[0,T;H^1_0(0,a)]} < \infty, \tag{4.27}$$

$$\sup_k \|\mathbf{u}_x^{k+1\prime}\|_{L^2[0,T;H^*(0,a)} < \infty, \tag{4.28}$$

$$\sup_k \|\mathbf{u}^{k+1\prime}\|_{L^2[0,T;H^{-1}(0,a)]} < \infty, \tag{4.29}$$

$$\sup_k \|\mathbf{u}^{k+1\prime}\|_{L^2[0,T;H^{-1}(0,a)]} < \infty. \tag{4.30}$$

The inequalities (4.26)–(4.30) imply the existence of a subsequence $\{\mathbf{u}^{k_j}\}_{j=1}^{\infty} \subset \{\mathbf{u}^k\}_{k=1}^{\infty}$ and a function $\mathbf{u} \in L^2[0,T;H_0^1(0,a)]$, with $\mathbf{u}' \in L^\infty[0,T;H_0^1(0,a)]$, $\mathbf{u}_x' \in L^2[0,T;H^*(0,a)$ and $\mathbf{u}' \in L^2[0,T;H^{-1}(0,a)]$ and $\mathbf{u}'' \in L^2[0,T;H^{-1}(0,a)]$, such that

$$\mathbf{u}^{k_j} \rightharpoonup \mathbf{u} \quad \text{in } L^\infty[0,T;H_0^1(0,a)\cap H^2(0,a)], \tag{4.31}$$

$$\mathbf{u}^{k_j\prime} \rightharpoonup \mathbf{u}' \quad \text{in } L^\infty[0,T;H_0^1(0,a)], \tag{4.32}$$

$$\mathbf{u}_x^{m_l\prime} \rightharpoonup \mathbf{u}_x' \quad \text{in } L^2[0,T;H^*(0,a), \tag{4.33}$$

$$\mathbf{u}^{k_j\prime} \rightharpoonup \mathbf{u}' \quad \text{in } L^2[0,T;H^{-1}(0,a)], \tag{4.34}$$

$$\mathbf{u}^{k_j\prime\prime} \rightharpoonup \mathbf{u}'' \quad \text{in } L^2[0,T;H^{-1}(0,a)]. \tag{4.35}$$

Further, we deduce

$$\|f(\mathbf{u}^k)\|_{L^\infty[0,T;L^2(0,a)]} \le C\|\mathbf{u}^k\|_X \tag{4.36}$$

for some constant $C > 0$. Using (4.24), we take the limit on the right side of (4.36) to conclude

$$\sup_k \|f(\mathbf{u}^k)\|_{L^\infty[0,T;L^2(0,a)]} < \infty. \tag{4.37}$$

Then there existence of a subsequence $\{f(\mathbf{u}^{k_j})\}_{j=1}^{\infty} \in L^\infty[0,T;L^2(0,a)]$ and function $f(\mathbf{u}) \in L^\infty[0,T;L^2(0,a)]$ such that

$$f(\mathbf{u}^{k_j}) \rightharpoonup f(\mathbf{u}) \text{ in } L^\infty[0,T;L^2(0,a)].$$

Step 7. We next verify that $\mathbf{u}$ is a weak solution of (1.1)–(1.3). For brevity, we take the subsequence $\{\mathbf{u}^{k_j}\}_{j=1}^{\infty}$ of the last step as the sequence $\{\mathbf{u}^k\}_{k=0}^{\infty}$. Fix $v \in L^2[0,T;H_0^1(0,a)]$. Since $\mathbf{u}^{k+1}$ is the unique weak solution of (4.21)–(4.23), we have

$$\int_0^T\int_0^a \mathbf{u}^{k+1\prime} v\,dx\,dt + \int_0^T\int_0^a \phi(t,x,\mathbf{u}^k)\mathbf{u}_x^{k+1} v_x\,dx\,dt = \int_0^T\int_0^a f(\mathbf{u}^k) v\,dx\,dt, \tag{4.38}$$

$$\mathbf{u}^{k+1}(0) = g. \tag{4.39}$$

Passage to limit is not immediately apparent in the second term on the left side of (4.38). Notice that

$$\begin{aligned}
&\Big|\int_0^T\int_0^a \left[\phi(t,x,\mathbf{u}^k)\mathbf{u}_x^{k+1} - \phi(t,x,\mathbf{u})\mathbf{u}_x\right] v_x\,dx\,dt\Big| \\
&\Big|\int_0^T\int_0^a \left[\phi(t,x,\mathbf{u}^k)\left(\mathbf{u}_x^{k+1} - \mathbf{u}_x\right) + \mathbf{u}_x\left(\phi(t,x,\mathbf{u}^k) - \phi(t,x,\mathbf{u})\right)\right] v_x\,dx\,dt\Big| \\
&\le \int_0^T \Big(B\int_0^a \left|\mathbf{u}_x^{k+1} - \mathbf{u}_x\right| |v_x|\,dx + L_1\int_0^a \left|\mathbf{u}^k - \mathbf{u}\right| |\mathbf{u}_x|\,|v_x|\,dx\Big)dt \\
&\quad \text{(Using (1.5) and (1.8))}
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^T \Big(B\int_0^a \left|\mathbf{u}_x^{k+1}-\mathbf{u}_x\right| |v_x|\,dx + L_1\|\mathbf{u}^k-\mathbf{u}\|_{C^{0,1/2}(0,a)} \int_0^a |\mathbf{u}_x|\,|v_x|\,dx\Big) dt\\
&\le \sqrt{T}\|v\|_{L^2[0,T;H_0^1(0,a)]}\Big(B\left\|\mathbf{u}^{k+1}-\mathbf{u}\right\|_{L^\infty[0,T;H_0^1(0,a)]}\\
&\quad + L_1\|\mathbf{u}\|_{L^\infty[0,T;H_0^1(0,a)]}\left\|\mathbf{u}^k-\mathbf{u}\right\|_{L^\infty[0,T;H_0^1(0,a)]}\Big) \to 0 \quad \text{as } k\to\infty,
\end{aligned}
\tag{4.40}
$$

where we applied Corollary 2.4. Using (4.31), (4.32), the deduction on $\{f(\mathbf{u}^{k_j})\}_{j=1}^\infty$ in the last step and (4.40), we send $k\to\infty$ in (4.38)–(4.39) to obtain

$$
\int_0^T\int_0^a \mathbf{u}'v\,dx\,dt + \int_0^T\int_0^a \phi(t,x,\mathbf{u})(x)\mathbf{u}_x v_x\,dx\,dt = \int_0^T\int_0^a f(\mathbf{u})v\,dx\,dt, \tag{4.41}
$$

$$
\mathbf{u}(0)=g, \tag{4.42}
$$

from where we deduce (2.2)–(2.3) as desired.

Step 8. For any given $T>0$, we select $T_1>0$ sufficiently small that

$$
\Big(C\Big[C(T_1+\sigma^2)e^{C(T_1+\sigma^2)}+1\Big]T_1\Big(4\Lambda+\|g\|_{H^4(0,a)}^2\Big)^2\Big)^{1/2} \le \frac{1}{2}.
$$

Banach fixed point theorem can be applied to find a weak solution $\mathbf{u}\in K$ of the problem (1.1)–(1.3) on the time interval $[0,T_1]$. Since $\mathbf{u}(t)\in K$ for a.e. $0\le t\le T$, we can upon redefining T_1 if necessary assume that $\mathbf{u}(T_1)\in K$. The argument above can be used to extend our solution to the time interval $[T_1,2T_1]$. We continue after finitely many steps to construct a weak solution of (1.1)–(1.3) existing on the whole interval $[0,T]$.

Step 9. Lastly, we prove uniqueness. Suppose there exists two weak solutions $\mathbf{u}$ and $\tilde{\mathbf{u}}$ of (1.1)–(1.3). Then we have $\mathbf{w}=\mathbf{u}$, $\tilde{\mathbf{w}}=\tilde{\mathbf{u}}$ in (4.16); and using Corollary 2.4 we deduce:

$$
\begin{aligned}
&\|\mathbf{u}(t)-\tilde{\mathbf{u}}(t)\|_{C^{0,1/2}(0,a)}^2 + \|\mathbf{u}'(t)-\tilde{\mathbf{u}}'(t)\|_{L^2(0,a)}^2\\
&\le \|\mathbf{u}(t)-\tilde{\mathbf{u}}(t)\|_{H_0^1(0,a)}^2 + \|\mathbf{u}'(t)-\tilde{\mathbf{u}}'(t)\|_{L^2(0,a)}^2\\
&\le C[C(T+\sigma^2)e^{C(T+\sigma^2)}+1]\Big(4\Lambda+\|g\|_{H^4(0,a)}^2\Big)^2\\
&\quad\times\int_0^t\Big(\|\mathbf{u}(r)-\tilde{\mathbf{u}}(r)\|_{C^{0,1/2}(0,a)}^2 + \|\mathbf{u}'(r)-\tilde{\mathbf{u}}'(r)\|_{L^2(0,a)}^2\Big)dr,
\end{aligned}
\tag{4.43}
$$

for $t\in[0,T]$. Gronwall inequality applied to (4.43) implies that $\mathbf{u}\equiv\tilde{\mathbf{u}}$. □

4.2. Nonexistence or blow-up of solution.

Theorem 4.2. *Let the conditions* (1.4) *and* (1.5) *hold. Assume further that*

- (i) w_1 *is a smooth eigenfunction* [10] *corresponding to the principal eigenvalue* $\mu_1>0$ *of* $-\frac{d^2}{dx^2}$ *in* $H_0^1(0,a)$,
- (ii) $w_1>0$ *in* $(0,a)$ *and* $\int_0^a w_1dx=1$, *without loss of generality.*

Define

$$
\eta(t) := \int_0^a u(t,x)w_1(x)dx \quad (0\le t\le T),
$$

$$
t^* := \frac{1}{(p-1)B\mu_1}\log\Big(\frac{\gamma\eta(0)^{p-1}}{\gamma\eta(0)^{p-1}-B\mu_1}\Big).
$$

If $\eta(0) = \int_0^a g w_1 dx > \left(\frac{B\mu_1}{\gamma}\right)^{\frac{1}{p-1}}$*, then, for sufficiently large* $T > 0$*, there does not exist a smooth solution* u *of* (1.1)–(1.3)*. Either the solution is not smooth enough to justify some calculation; or*

$$\lim_{t_* \to t^*} \int_0^a u(x,t) w_1 dx = \infty$$

for some $0 < t_* \le t^* \le T$*; in which case the solution blows-up.*

Proof. If we replace β and $\psi(x)$ of [19] by B and 1 respectively, then the proof of the theorem is precisely the same as that of [19]. $\square$

5. Illustrative examples

Note that if (1.5) and (1.7); and the first inequalities in (1.6), (1.8), (1.11) and (1.12) are satisfied, we can deduce from (3.29) that $u_t \in L^2[0,T; H_0^1(0,a)]$. On the other hand, we have the following result.

Theorem 5.1. *Assume that the first inequalities in* (1.8) *and* (1.9) *are satisfied, namely;* $\phi_u^u \le L_1$ *and* $\phi_t^u \le B_1$*, for some strictly positive constants* L_1 *and* B_1*. If* $u_t \in L^2[0,T; H_0^1(0,a)]$*, then there exists a constant* $\sigma > 0$ *such that* $\|\partial_t \phi^u\|_{L^2[0,T,L^\infty(0,a)]} \le \sigma$*.*

Proof. By Corollary 2.4, we have

$$\begin{aligned} |\partial_t \phi^u| &= |\phi_t^u + \phi_u^u u_t| \le |\phi_t^u| + |\phi_u^u||u_t| \\ &\le B_1 + L_1 \|u_t\|_{C^{0,1/2}(0,a)} \le B_1 + L_1 C(a) \|u_t\|_{H_0^1(0,a)} \end{aligned} \tag{5.1}$$

Maximizing the left side of (5.1) on $[0,a]$, squaring both sides and integrating over $[0,T]$ we deduce

$$\|\partial_t \phi^u\|_{L^2[0,T,L^\infty(0,a)]} \le C(B_1, L_1, T)\left(1 + \|u_t\|_{L^2[0,T;H_0^1(0,a)]}\right) =: \sigma. \tag{5.2}$$

$\square$

Example 5.2. Consider

$$u_t - \Big(\frac{(1 + x^2 + t^\alpha \sin^2 u)\sin\frac{\pi x}{a}}{t + \sin\frac{\pi x}{a}} u_x\Big)_x = \sqrt{1+u^2} - 1, \text{ in } (0,T] \times (0,a) \tag{5.3}$$

$$u(t,0) = 0, \ u(t,a) = 0, \quad \text{in } (0,T] \tag{5.4}$$

$$u(0,x) = \sin\frac{\pi x}{a}, \quad x \in (0,a), \tag{5.5}$$

where $\alpha \ge 2$. We have that

$$\phi(t,x,u) := \frac{(1 + x^2 + t^\alpha \sin^2 u)\sin\frac{\pi x}{a}}{t + \sin\frac{\pi x}{a}} \ge 0,$$

Equation (5.3) is degenerate at the boundary $\partial(0,a)$, $f(0) = 0$, and $g := \sin\frac{\pi x}{a} \in H_0^1(0,a) \cap H^4(0,a)$. Now, the hypothesis of Theorem 5.1 is satisfied with $L_1 := 2T^{\alpha-1}$ and $B_1 := \alpha T^{\alpha-2}$. Hence, assuming a priori that $u_t \in L^2[0,T; H_0^1(0,a)]$, we deduce, using Theorem 5.1 that

$$\|\partial_t \phi^u\|_{L^2[0,T,L^\infty(0,a)]} \le C(T)\left(1 + \|u_t\|_{L^2[0,T;H_0^1(0,a)]}\right) =: \sigma.$$

The functions ϕ^u and $f(u)$ satisfy the remaining conditions of (1.5)–(1.12) with $B := 1 + a^2 + T^\alpha$, $\lambda := 1$, $B_0 := 2a$, $L_2 := 2(\alpha+1)T^{\alpha-2}$, $L_3 := 2T^{\alpha-1}$, $L := 1$ and

$L' := 2$. Consequently, the existence and uniqueness of weak solution to (5.3)–(5.5) is guaranteed by Theorem 4.1.

Example 5.3. We next give an example where there does not exist a smooth solution or where the solution blows up. Consider the problem

$$u_t - (\sin^2 u u_x)_x = u^2, \quad \text{in } [0,T] \times (0,1) \tag{5.6}$$

$$u(t,0) = 0, \quad u(t,1) = 0, \quad \text{in } (0,T] \tag{5.7}$$

$$u(0,x) = 6\pi \sin \pi x, \quad x \in (0,1). \tag{5.8}$$

Note that the conditions of Theorem 4.2 are satisfied, with $B := 1$, $\gamma := 1$, and $p := 2 > 1$. Now $w_1 := \frac{\pi}{2} \sin \pi x$ is an eigenfunction corresponding to the principal eigenvalue $\mu_1 := \pi^2$ of $-u''$ in $H_0^1(0,1)$. Note that w_1 is smooth and that

$$w_1 > 0 \text{ in } (0,1), \quad \int_0^1 w_1 dx = 1. \tag{5.9}$$

Suppose u is a smooth solution of (5.6)–(5.8). Since $g := 6\pi \sin \pi x \geq 0$, $g \not\equiv 0$, we have $u > 0$ within $(0,T] \times (0,1)$ by the strong maximum principle. Define

$$\eta(t) := \int_0^1 u(t,x) w_1 dx = \int_0^1 u(t,x) \Big(\frac{\pi}{2} \sin \pi x\Big) dx \quad (0 \leq t \leq T). \tag{5.10}$$

We have

$$\eta(0) = \int_0^1 g w_1 dx = \int_0^1 6\pi \sin \pi x \big(\frac{\pi}{2} \sin \pi x\big) dx = \frac{3}{2}\pi^2 > \frac{B\mu_1}{\gamma} = \pi^2.$$

Consequently, by Theorem 4.2, either there cannot exist a smooth solution of (5.6)–(5.8); or else

$$\lim_{t_* \to t^*} \int_0^1 u(x,t) \big(\frac{\pi}{2} \sin \pi x\big) dx = \infty$$

for some $0 < t_* \leq t^*$, where

$$t^* := \frac{1}{\pi^2} \log 3;$$

in which case the solution blows-up in a finite time.

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