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GROWTH OF MEROMORPHIC SOLUTIONS OF HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this article, we investigate the growth of meromorphic solutions of the differential equations

$$f^{(k)} + A_{k-1}f^{(k-1)} + \dots + A_0 f = 0,$$
$$f^{(k)} + A_{k-1}f^{(k-1)} + \dots + A_0 f = F,$$

where A_j, F $(j = 0, \dots, k-1)$ are meromorphic functions. When there exists one dominant coefficient with lower order less than $1/2$, we obtain some estimations of the hyper order and the hyper convergence exponent of zeros of meromorphic solutions of the above equations.

1. Introduction and statement of results

It is assumed that the reader is familiar with the standard notations and the fundamental results of the Nevanlinna theory [10, 13, 16]. Let $f(z)$ be a nonconstant meromorphic function in the complex plane. We use the symbols $\sigma(f)$ and $\mu(f)$ to denote the order and the lower order of f respectively, and use $\lambda(f)$ and $\lambda(1/f)$ to denote the convergence exponent of zeros and poles of f, respectively. In order to estimate the rate of growth of meromorphic function of infinite order more precisely, we recall the following definitions.

Definition 1.1 ([13]). *Let $f(z)$ be a nonconstant meromorphic function in the complex plane. Its hyper order $\sigma_2(f)$ is defined by*

$$\sigma_2(f) = \limsup_{r\to+\infty} \frac{\log^+ \log^+ T(r,f)}{\log r}.$$

Definition 1.2 ([13]). Let $f(z)$ be a nonconstant meromorphic function in the complex plane. Its hyper convergence exponent of zeros and distinct zeros of f(z) are respectively defined by

$$\lambda_2(f) = \limsup_{r\to+\infty} \frac{\log^+ \log^+ N(r,f)}{\log r}, \quad \overline{\lambda_2}(f) = \limsup_{r\to+\infty} \frac{\log^+ \log^+ \overline{N}(r,f)}{\log r}.$$

Consider the second-order linear differential equation

$$f'' + A(z)f' + B(z)f = 0, \tag{1.1}$$

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where $A(z), B(z)$ are entire functions. It is well known that every nonconstant solution f of (1.1) has infinite order if $\sigma(A) < \sigma(B)$. When the order of the coefficients of (1.1) are less than 1/2, Gundersen [8] proved the following result.

Theorem 1.3. *Suppose that $A(z), B(z)$ are entire functions. If $\sigma(B) < \sigma(A) < 1/2$, or $A(z)$ is transcendental and $\sigma(A) = 0$, $B(z)$ is a polynomial, then every nonconstant solution f of (1.1) satisfies $\sigma(f) = \infty$.*

Hellerstein, Miles and Rossi [11] investigated the case $\sigma(B) < \sigma(A) \le 1/2$, and also obtained that every nonconstant solution f of (1.1) satisfies $\sigma(f) = \infty$, which improved Theorem 1.3. Meanwhile, in [12] they also considered the linear differential equation

$$f^{(k)} + A_{k-1}f^{(k-1)} + \dots + A_0 f = F, \tag{1.2}$$

and obtained the following result.

Theorem 1.4. *Suppose that $A_0, A_1, \dots, A_{k-1}, F$ are entire functions. If there exists some $s \in \{0, 1, \dots, k-1\}$ such that $\max\{\sigma(F), \sigma(A_j) : j \ne s\} < \sigma(A_s) \le 1/2$, then every solution f of (1.2) is either a polynomial or a transcendental entire function of infinite order.*

When the coefficients $A_0, A_1, \dots, A_{k-1}$ and F are meromorphic functions, many authors investigated the value distribution of solutions of (1.2) and its corresponding homogeneous differential equation

$$f^{(k)} + A_{k-1}f^{(k-1)} + \dots + A_0 f = 0 \tag{1.3}$$

(see [1, 2, 5, 6, 7, 15]). Especially, Chen [5] obtained the following result.

Theorem 1.5. *Suppose that $A_0, A_1, \dots, A_{k-1}$ are meromorphic functions. If there exists some $A_s(0 \le s \le k-1)$ satisfying*

$$\max\Big\{\sigma(A_j)(j \ne s), \lambda(\frac{1}{A_s})\Big\} < \mu(A_s) \le \sigma(A_s) < 1/2,$$

then every transcendental meromorphic solution f whose poles are of uniformly bounded multiplicities, of (1.3) satisfies $\sigma_2(f) = \sigma(A_s)$, and every nontranscendental meromorphic solutions f is a polynomial with degree $\deg f \le s-1$.

In [7], the authors pointed out that the condition that the multiplicity of poles of the solution f is uniformly bounded in Theorem 1.5 is necessary (see Remark 3.1). The above obtained results are related to the question: what conditions on coefficients $A_j(j = 0, \dots, k-1)$ will guarantee that every transcendental solution of (1.2) or (1.3) is of infinite order? From Theorems 1.3–1.5, we know that the answer is affirmative, if there exists one dominant coefficient A_s such that $\mu(A_s) \le \sigma(A_s) < 1/2$. In this paper, we continue to investigate the above question. In the following results, we estimate the hyper order, the hyper convergence exponent of zeros of transcendental meromorphic solutions of (1.2) or (1.3) under the condition that the dominant coefficient A_s satisfying $\mu(A_s) < 1/2$.

Theorem 1.6. *Suppose that $A_0, A_1, \dots, A_{k-1}, F$ are meromorphic functions of finite order. If there exists some $A_s(0 \le s \le k-1)$ such that*

$$b = \max\Big\{\sigma(F), \sigma(A_j)(j \ne s), \lambda\Big(\frac{1}{A_s}\Big)\Big\} < \mu(A_s) < 1/2, \tag{1.4}$$

Then

(i) *Every transcendental meromorphic solution f whose poles are of uniformly bounded multiplicities, of* (1.2) *satisfies $\mu(A_s) \le \sigma_2(f) \le \sigma(A_s)$. Furthermore, if $F \not\equiv 0$, then we have $\mu(A_s) \le \overline{\lambda_2}(f) = \lambda_2(f) = \sigma_2(f) \le \sigma(A_s)$.*

(ii) *If $s \ge 2$, then every nontranscendental meromorphic solution f of* (1.2) *is a polynomial with degree* $\deg f \le s-1$. *If $s = 0$ or 1, then every nonconstant solution of* (1.2) *is transcendental.*

Corollary 1.7. *Suppose that $A_0, A_1, \dots, A_{k-1}, F(\not\equiv 0)$ are meromorphic functions. If there exists some $A_s (0 \le s \le k-1)$ such that*

$$\max\Big\{\sigma(F), \sigma(A_j)(j \ne s), \lambda\Big(\frac{1}{A_s}\Big)\Big\} < \mu(A_s) = \sigma(A_s) < 1/2,$$

then every transcendental meromorphic solution f whose poles are of uniformly bounded multiplicities, of (1.2) *satisfies $\overline{\lambda_2}(f) = \lambda_2(f) = \sigma_2(f) = \sigma(A_s)$.*

Corollary 1.8. *Suppose that $A_0, A_1, \dots, A_{k-1}, F(\not\equiv 0)$ are entire functions. If there exists some $A_s (0 \le s \le k-1)$ such that*

$$\max\{\sigma(F), \sigma(A_j)(j \ne s)\} < \mu(A_s) = \sigma(A_s) < 1/2,$$

then every transcendental solution f of (1.2) *satisfies $\overline{\lambda_2}(f) = \lambda_2(f) = \sigma_2(f) = \sigma(A_s)$, and every nontranscendental solution f is a polynomial with degree* $\deg f \le s-1$.

Remark 1.9. From Corollary 1.8, we obtain the precise estimation of the growth of transcendental solutions in Theorem 1.4 when $\mu(A_s) = \sigma(A_s) < 1/2$.

When F is of infinite order, we obtain the following results.

Theorem 1.10. *Suppose that $A_0, A_1, \dots, A_{k-1}$, $Q(\not\equiv 0)$ are meromorphic functions of finite order, P is a transcendental entire function, such that*

$$\max\Big\{\sigma(P), \sigma(Q), \sigma(A_j)(1 \le j \le k-1), \lambda\Big(\frac{1}{A_0}\Big)\Big\} < \mu(A_0) < \frac{1}{2}. \tag{1.5}$$

Then every solution f of the equation

$$f^{(k)} + A_{k-1} f^{(k-1)} + \dots + A_0 f = Q e^P \tag{1.6}$$

is transcendental, and every transcendental meromorphic solution f whose poles are of uniformly bounded multiplicities, of (1.6) *satisfies $\mu(A_0) \le \overline{\lambda_2}(f) = \lambda_2(f) = \sigma_2(f) \le \sigma(A_0)$.*

Corollary 1.11. *Suppose that $A_0, A_1, \dots, A_{k-1}$, $Q(\not\equiv 0)$ are entire functions of finite order, P is a transcendental entire function, such that*

$$\max\{\sigma(P), \sigma(Q), \sigma(A_j)(1 \le j \le k-1)\} < \mu(A_0) < \frac{1}{2}.$$

Then every solution f of (1.6) *satisfies $\mu(A_0) \le \overline{\lambda_2}(f) = \lambda_2(f) = \sigma_2(f) \le \sigma(A_0)$.*

Corollary 1.12. *Suppose that $A_0, A_1, \dots, A_{k-1}$, $Q(\not\equiv 0)$ are entire functions of finite order, P is a transcendental entire function, such that*

$$\max\{\sigma(P), \sigma(Q), \sigma(A_j)(1 \le j \le k-1)\} < \mu(A_0) = \sigma(A_0) < \frac{1}{2}.$$

Then every solution f of (1.6) *satisfies $\overline{\lambda_2}(f) = \lambda_2(f) = \sigma_2(f) = \sigma(A_0)$.*

2. Preliminaries

Lemma 2.1 ([9]). *Let $f(z)$ be a transcendental meromorphic function, and let $\Gamma = \{(k_1, j_1), \dots, (k_m, j_m)\}$ be a finite set of distinct pairs of integers such that $k_i > j_i \geq 0$ $(i = 1, \dots, m)$. Let $\alpha > 1$ be a given constant. Then there exists a set $E \subset (1, +\infty)$ that has a finite logarithmic measure, and a constant $B > 0$ depending only on α and Γ, such that for all z with $|z| = r \notin [0, 1] \cup E$ and $(k, j) \in \Gamma$, we have*

$$\Big|\frac{f^{(k)}(z)}{f^{(j)}(z)}\Big| \leq B\Big(\frac{T(\alpha r, f)}{r}\log^{\alpha} r \cdot \log T(\alpha r, f)\Big)^{k-j}.$$

It is known that the Wiman-Valiron theory (see [13]) is an important tool while considering the value distribution theory of entire solutions of differential equations. In [5], using the Wiman-Valiron theory for entire functions and the Hadamard factorization theorem for meromorphic functions, the author obtained the following Wiman-Valiron theory for meromorphic functions, which is a generalization of [14, Lemma 5]. Their proofs are quite parallel.

Lemma 2.2 ([5]). *Let $f(z) = g(z)/d(z)$ be a meromorphic function, where $g(z)$ and $d(z)$ are entire functions, such that*

$$\mu(g) = \mu(f) = \mu \leq \sigma(g) = \sigma(f) \leq +\infty, \quad \lambda(d) = \sigma(d) = \lambda\Big(\frac{1}{f}\Big) < \mu.$$

Then there exists a set $E \subset (1, +\infty)$ that has a finite logarithmic measure, such that for point z with $|z| = r \notin [0, 1] \cup E$ and $|g(z)| = M(r, g)$, we have

$$\frac{f^{(n)}(z)}{f(z)} = \Big(\frac{v_g(r)}{z}\Big)^n (1 + o(1)), (n \geq 1),$$

where $v_g(r)$ denotes the central index of $g(z)$.

Lemma 2.3 ([5]). *Let $f(z) = g(z)/d(z)$ be a meromorphic function, where $g(z)$ and $d(z)$ are entire functions, such that*

$$\mu(g) = \mu(f) = \mu \leq \sigma(g) = \sigma(f) \leq +\infty, \quad \lambda(d) = \sigma(d) = \lambda\Big(\frac{1}{f}\Big) < \mu.$$

Then there exists a set $E \subset (1, +\infty)$ that has a finite logarithmic measure, such that for point z with $|z| = r \notin [0, 1] \cup E$ and $|g(z)| = M(r, g)$, we have

$$\Big|\frac{f(z)}{f^{(s)}(z)}\Big| \leq r^{2s},$$

where s is a positive integer.

Since [5, Lemma 2.3] is published in Chinese, for the convenience of the non-Chinese readers, we show the following proof of [5, Lemma 2.3].

Proof. By Lemma 2.2, there exists a set $E_1 \subset (1, +\infty)$ that has a finite logarithmic measure, such that for point z with $|z| = r \notin [0, 1] \cup E_1$ and $|g(z)| = M(r, g)$, we have

$$\frac{f^{(s)}(z)}{f(z)} = \Big(\frac{v_g(r)}{z}\Big)^s (1 + o(1)). \tag{2.1}$$

Since $\mu(g) = \liminf_{r\to\infty} \frac{\log^+ v_g(r)}{\log r}$, for any given $\varepsilon > 0$, there exists $R > 0$ such that

$$v_g(r) > r^{\mu(g)-\varepsilon} \tag{2.2}$$

holds for $r > R$. If $\mu(g) = \infty$, then we replace $\mu(g) - \varepsilon$ by a sufficiently large positive constant M. Let $E = E_1 \bigcup [1, R]$, then E has a finite logarithmic measure, and by (2.1) and (2.2), we obtain the result in Lemma 2.3. $\square$

Lemma 2.4 ([4, 6]). *Let $g(z)$ be a meromorphic function of finite order. Then for any given $\varepsilon > 0$, there exists a set $E \subset (1, +\infty)$ that has a finite logarithmic measure, such that for all z with $|z| = r \notin [0, 1] \cup E$, we have $|g(z)| \le \exp\{r^{\sigma(g)+\varepsilon}\}$.*

Lemma 2.5 ([3, 15]). *Let $g(z)$ be an entire function with $0 \le \mu(g) < 1$. Then for every $\alpha \in (\mu(g), 1)$, there exists a set $E \subset [0, \infty)$ such that*

$$\overline{\log \operatorname{dens}} E \ge 1 - \frac{\mu(g)}{\alpha},$$

where $E = \{r \in [0, \infty) : m(r) > M(r) \cos \pi\alpha\}$, $m(r) = \inf_{|z|=r} \log |g(z)|$, $M(r) = \sup_{|z|=r} \log |g(z)|$.

Lemma 2.6 ([13]). *Let $g : (0, \infty) \to R$ and $h : (0, \infty) \to R$ be monotone nondecreasing functions such that $g(r) \le h(r)$ outside of an exceptional set H of finite logarithmic measure. Then for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \le h(\alpha r)$ holds for all $r > r_0$.*

Lemma 2.7. *Let $f(z)$ be a meromorphic function such that $\lambda(1/f) < \mu(f) < 1/2$. Then for any given ε $(0 < 2\varepsilon < \mu(f) - \lambda(1/f))$, there exists a set $E \subset (1, +\infty)$ with $\overline{\log \operatorname{dens}} E > 0$, such that for all z satisfying $|z| = r \in E$, we have*

$$|f(z)| \ge \exp\{(1 - o(1)) r^{\mu(f)-\varepsilon}\}.$$

The above result might be known, but we give the proof for the convenience of the readers.

Proof. From the Hadamard factorization theorem, we obtain

$$f(z) = \frac{g(z)}{d(z)}, \tag{2.3}$$

where $g(z)$ is an entire function, $d(z)$ is the canonical product of $f(z)$ formed with its poles such that

$$\lambda(d) = \sigma(d) = \lambda\Big(\frac{1}{f}\Big) < \mu(f). \tag{2.4}$$

By (2.3) we obtain $T(r, g) \le T(r, f) + T(r, d)$. Then combining with (2.4), we obtain $\mu(g) \le \mu(f)$. On the other hand, take a sequence $\{r_n\}$ such that

$$\lim_{r_n \to \infty} \frac{\log T(r_n, g)}{\log r_n} = \mu(g),$$

hence we have

$$\liminf_{r_n \to \infty} \frac{\log T(r_n, f)}{\log r_n} \ge \mu(f). \tag{2.5}$$

By (2.4) and (2.5), for any given $\varepsilon(0 < 2\varepsilon < \mu(f) - \sigma(d))$, there exists a subsequence of $\{r_n\}$, for convenience, we also denote it as $\{r_n\}$, such that for sufficiently large r_n, we have

$$T(r_n, f) > r_n^{\mu(f)-\varepsilon}, \quad T(r_n, d) < r_n^{\sigma(d)+\varepsilon}.$$

Then combining with $T(r, f) \le T(r, g) + T(r, d) + O(1)$, we obtain $\mu(f) \le \mu(g)$. Hence we have

$$\mu(g) = \mu(f) < 1/2.$$

By Lemma 2.5, set $\alpha_0 = \frac{\frac{1}{2}+\mu(g)}{2}$, then there exists a set E_1 with $\overline{\log\operatorname{dens}}E_1 \geq 1 - \frac{\mu(g)}{\alpha_0}$, such that for all z with $|z| = r \in E_1$, we have

$$\log|g(z)| \geq \cos(\pi\alpha_0)\log M(r,g). \tag{2.6}$$

By the definition of $\mu(g)$, for any given $\varepsilon(0 < 2\varepsilon < \mu(f) - \lambda(\frac{1}{f}))$, there exists $r_1 > 0$ such that

$$\log M(r,g) \geq r^{\mu(g)-\frac{\varepsilon}{2}} \tag{2.7}$$

holds for $r > r_1$. Since

$$\frac{\cos(\pi\alpha_0)r^{\mu(g)-\frac{\varepsilon}{2}}}{r^{\mu(g)-\varepsilon}} \to +\infty, (r \to +\infty), \tag{2.8}$$

by (2.6)–(2.8), there exists $r_2(\geq r_1)$ such that for all z with $|z| = r \in E_1 \setminus [0, r_2]$, we have

$$|g(z)| \geq \exp\{\cos(\pi\alpha_0)r^{\mu(g)-\frac{\varepsilon}{2}}\} \geq \exp\{r^{\mu(g)-\varepsilon}\}. \tag{2.9}$$

On the other hand, there exists $R > 0$ such that for $r > R$, we have

$$|d(z)| \leq \exp\{r^{\sigma(d)+\varepsilon}\}. \tag{2.10}$$

Set $E = E_1 \cap [R, +\infty] \cap [r_2, +\infty]$, then $\overline{\log\operatorname{dens}}E > 0$. By (2.3),(2.9) and (2.10), we obtain that

$$|f(z)| \geq \exp\{r^{\mu(g)-\varepsilon} - r^{\sigma(d)+\varepsilon}\} = \exp\{(1 - o(1))r^{\mu(f)-\varepsilon}\}$$

holds for $|z| = r \in E$. $\square$

3. Proofs of main results

In the sequel, we use the symbols E and E_1 to denote any set of finite logarithmic measure and any set of finite linear measure, not necessarily the same at each occurrence. We also use M to denote any positive constant, not necessarily the same at each occurrence.

Proof of Theorem 1.6. Firstly, suppose that $f(z)$ is a transcendental meromorphic solution whose poles are of uniformly bounded multiplicities of (1.2). From (1.2), we know that the poles of $f(z)$ can only occur at the poles of $A_0, A_1, \dots, A_{k-1}$, F. Note that the multiplicities of poles of f are uniformly bounded, and thus we have

$$n(r,f) \leq O\Big\{\sum_{j=0}^{k-1}\overline{n}(r,A_j) + \overline{n}(r,F)\Big\}. \tag{3.1}$$

Then by (1.4) and (3.1) we obtain

$$\lambda\Big(\frac{1}{f}\Big) \leq b. \tag{3.2}$$

From (1.2) we obtain

$$-A_s = \frac{f^{(k)}}{f^{(s)}} + \dots + A_{s+1}\cdot\frac{f^{(s+1)}}{f^{(s)}} + \Big[A_{s-1}\cdot\frac{f^{(s-1)}}{f} + \dots + A_0\Big]\cdot\frac{f}{f^{(s)}} - \frac{F}{f}\frac{f}{f^{(s)}}. \tag{3.3}$$

By the lemma of the logarithmic derivative and (3.3), we obtain

$$\begin{aligned} T(r,A_s) &\le N(r,A_s) + \sum_{j\neq s} m(r,A_j) + m(r,\frac{F}{f}) + 2m(r,\frac{f}{f^{(s)}}) + O(\log rT(r,f)) \\ &\le N(r,A_s) + \sum_{j\neq s} T(r,A_j) + T(r,F) + T(r,\frac{1}{f}) \\ &\quad + 2N(r,f^{(s)}) + 2N(r,\frac{1}{f}) + O(\log rT(r,f)) \\ &\le N(r,A_s) + \sum_{j\neq s} T(r,A_j) + T(r,F) + 2(s+1)N(r,f) \\ &\quad + 3T(r,f) + O(\log rT(r,f)) \\ &\le N(r,A_s) + \sum_{j\neq s} T(r,A_j) + T(r,F) + 2(s+1)N(r,f) \\ &\quad + 4T(r,f), (r\notin E_1). \end{aligned} \tag{3.4}$$

By (1.4), (3.2), (3.4) and Lemma 2.6, we obtain

$$\mu(A_s) \le \mu(f). \tag{3.5}$$

From the Hadamard factorization theorem, we obtain

$$f(z) = \frac{g(z)}{d(z)}, \tag{3.6}$$

where $g(z)$ is an entire function, $d(z)$ is the canonical product of $f(z)$ formed with its poles such that $\lambda(d) = \sigma(d) = \lambda(\frac{1}{f})$. By (1.4),(3.2) and (3.5), we obtain

$$\lambda(d) = \sigma(d) = \lambda(\frac{1}{f}) < \mu(f). \tag{3.7}$$

By the definition of order, for any given $\varepsilon (0 < 2\varepsilon < \sigma(f) - \sigma(d))$, there exists a sequence $\{r_n\}$ such that for sufficiently large r_n, we have

$$T(r_n,f) > r_n^{\sigma(f)-\varepsilon}, \quad T(r_n,d) < r_n^{\sigma(d)+\varepsilon}. \tag{3.8}$$

By (3.6), we obtain

$$T(r,f) \le T(r,g) + T(r,d) + O(1). \tag{3.9}$$

Hence by (3.8) and (3.9), we obtain $\sigma(f) \le \sigma(g)$. On the other hand, by (3.6) we obtain $T(r,g) \le T(r,f) + T(r,d)$. Then combining with (3.7), we obtain $\sigma(g) \le \sigma(f)$. Hence we have

$$\sigma(g) = \sigma(f). \tag{3.10}$$

Using the similar proof to that of Lemma 2.7, we obtain

$$\mu(g) = \mu(f). \tag{3.11}$$

So by (3.7), (3.10),(3.11) and Lemma 2.3, there exists a set $E \subset (1,+\infty)$ of finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0,1]\cup E$ and $|g(z)| = M(r,g)$, we have

$$|\frac{f(z)}{f^{(s)}(z)}| \le r^{2s}. \tag{3.12}$$

By Lemma 2.1, there exists a set $E \subset (1, \infty)$ of finite logarithmic measure and $B > 0$, such that for all z satisfying $|z| = r \notin [0, 1] \cup E$, we have

$$|\frac{f^{(j)}(z)}{f^{(s)}(z)}| \le Br[T(2r, f)]^{j-s+1}, \quad (j = s+1, \dots, k), \tag{3.13}$$

$$|\frac{f^{(j)}(z)}{f(z)}| \le Br[T(2r, f)]^{j+1}, \quad (j = 1, \dots, s-1). \tag{3.14}$$

By (1.4), (3.2), (3.7) and Lemma 2.4, for any given ε $(0 < 2\varepsilon < \mu(A_s) - b)$, there exists a set $E \subset (1, +\infty)$ of finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0, 1] \cup E$, we have

$$|A_j(z)| \le \exp\{r^{b+\varepsilon}\}(j \ne s), \quad |F(z)| \le \exp\{r^{b+\varepsilon/2}\}, \quad |d(z)| \le \exp\{r^{b+\varepsilon/2}\}. \tag{3.15}$$

Hence for all z satisfying $|z| = r \notin [0, 1] \cup E$ and $|g(z)| = M(r, g)$, we have

$$\Big|\frac{F(z)}{f(z)}\Big| = \frac{|F(z)d(z)|}{M(r, g)} \le \exp\{r^{b+\varepsilon}\}. \tag{3.16}$$

By Lemma 2.7, there exists a set $H_0 \subset (1, +\infty)$ with $\overline{\log\operatorname{dens}} H_0 > 0$, such that for all z satisfying $|z| = r \in H_0$, we have

$$|A_s(z)| \ge \exp\{(1 - o(1))r^{\mu(A_s)-\varepsilon}\}. \tag{3.17}$$

Let $H = H_0 - ([0, 1] \cup E)$, then we have $\overline{\log\operatorname{dens}} H > 0$, and for all z satisfying $|z| = r \in H$ and $|g(z)| = M(r, g)$, by (3.3),(3.12)–(3.17), we have

$$\begin{aligned} \exp\{(1 - o(1))r^{\mu(A_s)-\varepsilon}\} &\le |A_s(z)| \\ &\le (k - s) \cdot \exp\{r^{b+\varepsilon}\}Br \cdot [T(2r, f)]^{k-s+1} + s \cdot \exp\{r^{b+\varepsilon}\} \\ &\quad \times Br \cdot [T(2r, f)]^{s} \cdot r^{2s} + \exp\{r^{b+\varepsilon}\} \cdot r^{2s} \\ &\le (k + 1)Br \cdot \exp\{r^{b+\varepsilon}\} \cdot [T(2r, f)]^{k+1} \cdot r^{2s}. \end{aligned} \tag{3.18}$$

Hence by (3.18), we obtain $\sigma_2(f) \ge \mu(A_s)$. Now we prove that $\sigma_2(f) \le \sigma(A_s)$. By (3.7), (3.10),(3.11) and Lemma 2.2, there exists a set $E \subset (1, +\infty)$ of finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0, 1] \cup E$ and $|g(z)| = M(r, g)$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \Big(\frac{v_g(r)}{z}\Big)^j (1 + o(1)), (j = 1, \dots, k). \tag{3.19}$$

By Lemma 2.4, for any given $\varepsilon > 0$, there exists a set $E \subset (1, +\infty)$ of finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0, 1] \cup E$, we have

$$|A_s(z)| \le \exp\{r^{\sigma(A_s)+\varepsilon}\}. \tag{3.20}$$

Then by (1.2), (3.16), (3.19) and (3.20), for all z satisfying $|z| = r \notin [0, 1] \cup E$ and $|g(z)| = M(r, g)$, we have

$$v_g(r) \le Mr \exp\{r^{\sigma(A_s)+\varepsilon}\}. \tag{3.21}$$

Hence by (3.10), (3.21) and Lemma 2.6, we obtain $\sigma_2(f) \le \sigma(A_s)$.

Let $F \not\equiv 0$. Next we prove that $\overline{\lambda_2}(f) = \lambda_2(f) = \sigma_2(f)$. By (1.2) we obtain

$$\frac{1}{f} = \frac{1}{F}\Big(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \dots + A_0\Big). \tag{3.22}$$

Suppose that z_0 is a zero of f with order $\alpha(>k)$, if z_0 is not a pole of A_j $(j=0,\dots,k-1)$, then z_0 must be a zero of F with order $\alpha-k$. Hence

$$N\Big(r,\frac{1}{f}\Big)\le k\overline{N}\Big(r,\frac{1}{f}\Big)+N\Big(r,\frac{1}{F}\Big)+\sum_{j=0}^{k-1}N(r,A_j). \tag{3.23}$$

By (3.22) we obtain

$$m\Big(r,\frac{1}{f}\Big)\le m\Big(r,\frac{1}{F}\Big)+\sum_{j=0}^{k-1}m(r,A_j)+O(\log rT(r,f)),(r\notin E_1). \tag{3.24}$$

Combining (3.23) and (3.24), we obtain

$$T(r,f)\le k\overline{N}\Big(r,\frac{1}{f}\Big)+T(r,F)+\sum_{j=0}^{k-1}T(r,A_j)+O(\log rT(r,f)),(r\notin E_1). \tag{3.25}$$

Take a sequence $\{r_n'\}$ satisfying $\lim_{r_n'\to\infty}\frac{\log\log T(r_n',f)}{\log r_n'}=\sigma_2(f)$, set $\operatorname{meas}E_1=\delta$, then there exists $r_n\in[r_n',r_n'+\delta+1]$ such that

$$\liminf_{r_n\to\infty}\frac{\log\log T(r_n,f)}{\log r_n}\ge\lim_{r_n'\to\infty}\frac{\log\log T(r_n',f)}{\log(r_n'+\delta+1)}=\sigma_2(f). \tag{3.26}$$

Hence by (1.4), (3.26) and $\sigma_2(f)\ge\mu(A_s)$, for sufficiently large r_n, we have

$$T(r_n,F)=o(T(r_n,f)),\ T(r_n,A_j)=o(T(r_n,f)),\quad(0\le j\le k-1). \tag{3.27}$$

Then by (3.25) and (3.27), we obtain $\sigma_2(f)\le\overline{\lambda_2}(f)$. Since $\overline{\lambda_2}(f)\le\sigma_2(f)$, we obtain $\mu(A_s)\le\overline{\lambda_2}(f)=\lambda_2(f)=\sigma_2(f)\le\sigma(A_s)$.

Secondly, suppose that f is a nonconstant rational solution of (1.2). When $s\ge2$, if z_0 is a pole of f with order $m(\ge1)$, or f is a polynomial with degree more than $s-1$, then $f^{(s)}\not\equiv0$. Hence by (1.2), (3.15) and (3.17), for all z satisfying $|z|=r\in H_0-([0,1]\cup E)$, we have

$$\exp\{(1-o(1))r^{\mu(A_s)-\varepsilon}\}\le|A_s(z)|\le r^M\exp\{r^{b+\varepsilon}\}.$$

This is impossible. So every nontranscendental solution f of (1.2) is a polynomial with degree $\deg f\le s-1$. By the similar argument, we obtain that every nonconstant solution of (1.2) is transcendental when $s=0$ or 1. □

Proof of Theorem 1.10. From the hypothesis we know that every meromorphic solution of (1.6) is of infinite order. So every meromorphic solution of (1.6) is transcendental. Suppose that $f(z)$ is a transcendental meromorphic solution whose poles are of uniformly bounded multiplicities. Set $f=ge^P$, then we have

$$\overline{\lambda_2}(g)=\overline{\lambda_2}(f),\quad\lambda_2(g)=\lambda_2(f). \tag{3.28}$$

Substituting $f=ge^P$ into (1.6), we obtain

$$g^{(k)}+B_{k-1}g^{(k-1)}+\dots+B_0g=Q, \tag{3.29}$$

where

$$B_{k-1}=A_{k-1}+kP', \tag{3.30}$$

$$
\begin{aligned}
B_{k-j} = A_{k-j} + (k-j+1)A_{k-j+1}P' + \sum_{m=2}^{j} A_{k-j+m}\Big[\binom{m}{k-j+m}(P')^m \\
+ D_{m-1}(P')\Big], \quad j = 2,3,\dots,k;\ A_k \equiv 1.
\end{aligned} \tag{3.31}
$$

Here $D_{m-1}(P')$ is a differential polynomial in P' of degree $m-1$, its coefficients are constants. By (1.5),(3.30) and (3.31), we obtain

$$
\mu(B_0) = \mu(A_0), \quad \lambda\Big(\frac{1}{B_0}\Big) < \mu(A_0), \quad \sigma(B_j) < \mu(A_0), (1 \le j \le k-1). \tag{3.32}
$$

Hence by (1.5), (3.29),(3.32) and Theorem 1.6, we obtain

$$
\mu(A_0) \le \overline{\lambda_2}(g) = \lambda_2(g) = \sigma_2(g) \le \sigma(A_0). \tag{3.33}
$$

Since $\sigma_2(e^P) = \sigma(P) < \mu(A_0) \le \sigma_2(g)$, we obtain $\sigma_2(f) = \sigma_2(g)$. Then combining (3.28) and (3.33), we obtain $\mu(A_0) \le \overline{\lambda_2}(f) = \lambda_2(f) = \sigma_2(f) \le \sigma(A_0)$. $\square$

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