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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR $p(x)$-LAPLACIAN EQUATIONS IN $\mathbb{R}^N$

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Abstract. This article concerns the existence and multiplicity of solutions to a class of $p(x)$-Laplacian equations. We introduce a revised Ambrosetti-Rabinowitz condition, and show that the problem has a nontrivial solution and infinitely many solutions.

1. Introduction

The study of various mathematical problems with variable exponent growth condition has received considerable attention in recent years; see e.g. [1, 16, 6, 13, 14, 15]. For background information, we refer the reader to [19, 21]. The aim of this paper is to discuss the existence and multiplicity of solutions of the following $p(x)$-Laplacian equation in $\mathbb{R}^N$:

$$\begin{gathered} -\Delta_{p(x)}u + |u|^{p(x)-2}u = K(x)f(u), \quad \text{in } \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N), \end{gathered} \tag{1.1}$$

where $p(x) = p(|x|) \in C((\mathbb{R}^N))$ with $2 \le N < p^- := \inf_{\mathbb{R}^N} p(x) \le p^+ := \sup_{\mathbb{R}^N} p(x) < +\infty$, $K : \mathbb{R}^N \to \mathbb{R}$ is a measurable function and $f \in C(\mathbb{R}, \mathbb{R})$.

Problem (1.1) has been widely studied. The following equation also has been studied very well

$$\begin{gathered} -\Delta_{p(x)}u + |u|^{p(x)-2}u = f(x,u), \quad \text{in } \mathbb{R}^N, \\ u \in W^{1,p(x)}(\mathbb{R}^N). \end{gathered} \tag{1.2}$$

When $p(x) = p(|x|) \in C(\mathbb{R}^N)$ with $2 \le N < p^- \le p^+ < +\infty$, the authors in [4] proved the existence of infinitely many distinct homoclinic radially symmetric solutions for (1.2), under adequate hypotheses about the nonlinearity at zero (and at infinity).

The case of p Lipschitz continuous with $1 < p^- \le p^+ < N$ was discussed by [7, 12]. Fu-Zhang [12] uses a nonlinearity on the right-hand side of the form $h(x)|u|^{\beta(x)-1}$ where $h \in L_+^\infty(\mathbb{R}^N) \cap L^{q(x)}(\mathbb{R}^N)$, $1 < \beta(x) < p(x)$, $q(x) = \frac{p^*(x)}{p^*x-\beta(x)}$, $p^*(x) = \frac{Np(x)}{N-p(x)}$, and they prove the existence of at least two nontrivial solutions to

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problem (1.2). In [7], through the critical point theory, three main results on the existence of solutions of problem (1.2) obtained, treating separately the three cases; i.e., when the nonlinear term $f(x,u)$ is sublinear, superlinear and concave-convex nonlinearity.

Fan and Han [7] established the existence of nontrivial solutions for problem (1.1) under the case of superlinear, by assuming the following key condition:

(F1') there exist $\theta > p^+$ and $M > 0$ such that

$$0 < \theta F(t) := \theta \int_0^t f(s)ds \le f(t)t, \quad \forall |t| \ge M.$$

This condition is originally due to Ambrosetti and Rabinowitz [2] in the case $p(x) \equiv 2$, and then was used in [3, 5, 8, 9] for $p(x)$-Laplacian equations. Actually, condition (F1') is quite natural and important not only to ensure that the Euler-Lagrange functional associated to problem (1.2) has a mountain pass geometry, but also to guarantee that Palais-Smale sequence of the Euler-Lagrange functional is bounded. But this condition is very restrictive eliminating many nonlinearities. In this paper, we introduce a new condition (F1), below, which is different from the Ambrosetti-Rabinowitz-type condition (F1').

(F1) there exist a constant $M \ge 0$ and a decreasing function τ in the space $C(\mathbb{R} \setminus (-M, M), \mathbb{R})$, such that

$$0 < (p^+ + \tau(t))F(t) := (p^+ + \tau(t)) \int_0^t f(s)ds \le f(t)t, \quad |t| \ge M,$$

where $\tau(t) > 0$, $\lim_{|t|\to+\infty} |t|\tau(t) = +\infty$ and $\lim_{|t|\to+\infty} \int_M^{|t|} \frac{\tau(s)}{s} ds = +\infty$.

Remark 1.1. Obviously, when $\inf_{|t|\ge M} \tau(t) > 0$, condition (F1) and (F1') are equivalent. However, condition (F1) is weaker than (F1') when $\inf_{|t|\ge M} \tau(t) = 0$. For example, let $|t| \ge M = 2$, and assume that $F(t) = |t|^{p^+} \ln|t|$. Then $f(t) = (p^+ + \tau(t))\mathrm{sgn}(t)|t|^{p^+-1}\ln|t|$ satisfies condition (F1) not (F1'), where $\tau(t) = \frac{1}{\mathrm{lnt}} \in C(\mathbb{R} \setminus (-M, M), \mathbb{R})$.

The aim of this paper is twofold. First, we want to handle the case when $p^- > N$ and the unbounded area $\mathbb{R}^N$. Although important problems can be treated within this framework, only a few works are available in this direction, see [4]. The main difficulty in studying problem (1.1) lies in the fact that no compact embedding is available for $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$. However, the subspace of radially symmetric functions of $W^{1,p(x)}(\mathbb{R}^N)$, denoted further by $W_r^{1,p(x)}(\mathbb{R}^N)$, can be embedded compactly into $L^\infty(\mathbb{R}^N)$ whenever $N < p^- \le p^+ < +\infty$ (cf. [4, Theorem 2.1]). Second, instead of some usual assumption on the nonlinear term f, we assume that it satisfies a modified Ambrosetti-Rabinowitz-type condition (F1).

To state our results, we first introduce the following assumptions:

(H1) $K \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is radial, nonnegative, $K(x) \ge 0$ for any $x \in \mathbb{R}^N$ and $\sup_{d>0} \operatorname{ess\,inf}_{|x|\le d} K(x) > 0$.

(H2) $f(t) = o(t^{p^+-1})$ for t near 0.

Now, we are ready to state the main result of this paper.

Theorem 1.2. *Suppose that* (H1), (H2), (F1) *hold. Then problem* (1.1) *has a nontrivial radially symmetric solution. Furthermore, if* $f(t) = f(-t)$, *then problem* (1.1) *has infinitely many pairs of radially symmetric solutions.*

In the remainder of this section, we recall some definitions and basic properties of variable spaces $L^{p(x)}(\mathbb{R}^N)$ and $W^{1,p(x)}(\mathbb{R}^N)$. For a deeper treatment on these spaces, we refer to [10, 11].

Let $p \in L^\infty(\mathbb{R}^N)$, $p^- > 1$. The variable exponent Lebesgue space $L^{p(x)}(\mathbb{R}^N)$ is defined by

$$L^{p(x)}(\mathbb{R}^N) = \{u : \mathbb{R}^N \to \mathbb{R} : u \text{ is measurable and } \int_{\mathbb{R}^N} |u|^{p(x)} dx < +\infty\}$$

endowed with the norm $|u|_{p(x)} = \{\lambda > 0 : \int_{\mathbb{R}^N} |\frac{u}{\lambda}|^{p(x)} dx \leq 1\}$. Then we define the variable exponent Sobolev space

$$W^{1,p(x)}(\mathbb{R}^N) = \{u \in L^{p(x)}(\mathbb{R}^N) : |\nabla u| \in L^{p(x)}(\mathbb{R}^N)\}$$

with the norm $\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}$.

Proposition 1.3 ([7]). *Set $\psi(u) = \int_{\mathbb{R}^N} (|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)}) dx$. If $u, u_k \in W^{1,p(x)}(\mathbb{R}^N)$, then*

1. $\|u\| < 1 (= 1; > 1) \Leftrightarrow I(u) < 1 (= 1; > 1)$;
2. *If $\|u\| > 1$, then $\|u\|^{p^-} \leq \psi(u) \leq \|u\|^{p^+}$;*
3. *If $\|u\| < 1$, then $\|u\|^{p^+} \leq \psi(u) \leq \|u\|^{p^-}$;*
4. $\lim_{k\to+\infty} \|u_k\| = 0 \Leftrightarrow \lim_{k\to+\infty} \psi(u_k) = 0$;

2. Proof of Theorem 1.2

In this section we prove Theorem 1.2 when $\inf_{|t|\geq M} \tau(t) = 0$. If $\inf_{|t|\geq M} \tau(t) > 0$, then conditions (F1') and (F1) are equivalent, and the proof is rather standard. We may assume that $M \geq 1$, and that there is constant $N_0 > 0$ such that $|\tau(t)| \leq N_0$ for all $t \in \mathbb{R}\backslash(-M, M)$.

We introduce the energy function φ associated to problem (1.1) defined by

$$\varphi(u) = \int_{\mathbb{R}^N} \frac{1}{p(x)} (|\nabla u(x)|^{p(x)} + |u(x)|^{p(x)}) dx - \int_{\mathbb{R}^N} K(x) F(u) dx, \quad u \in W_r^{1,p(x)}(\mathbb{R}^N)$$

Due to the principle of symmetric criticality of Palais (see [20]), the critical points of $\varphi|_{W_r^{1,p(x)}(\mathbb{R}^N)}$ are critical points of φ as well, so radially symmetric, weak solutions of problem (1.1).

Claim 2.1. *Let $W = \{w \in W_r^{1,p(x)}(\mathbb{R}^N) : \|w\| = 1\}$. Then, for any $w \in W$, there exist $\delta_w > 0$ and $\lambda_w > 0$, such that*

$$\varphi(\lambda v) < 0, \quad \forall v \in W \cap B(w, \delta_w), \forall |\lambda| \geq \lambda_w,$$

where $B(w, \delta_w) = \{v \in W_r^{1,p(x)}(\mathbb{R}^N) : \|v - w\| < \delta_w\}$.

Proof. Since the embedding $W_r^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ is compact, there is constant $C > 0$ such that $|u|_\infty \leq C\|u\|$. Thus, for all $w \in W$ and a.e. $x \in \mathbb{R}^N$, we have $|w(x)| \leq C$. By the definition of $\tau(t)$, we deduce that there exists $t_\lambda \in \{t \in \mathbb{R} : M \leq |t| \leq |\lambda| C\}$ such that $\tau(t_\lambda) = \min_{M\leq|t|\leq|\lambda|C} \tau(t)$. Then $|\lambda| \geq \frac{t_\lambda}{C}$ and $\lim_{|\lambda|\to+\infty} |t_\lambda| \to +\infty$. From condition (F1), we conclude that $F(t) \geq C_1 |t|^{p^+} H(|t|)$ for all $|t| \geq M$, where $H(t) = \exp(\int_M^{|t|} \frac{\tau(s)}{s} ds)$. Hence, using $\lim_{|t|\to+\infty} \int_M^{|t|} \frac{\tau(s)}{s} ds = +\infty$, it follows that $H(|t|)$ increases when $|t|$ increases, and $\lim_{|t|\to+\infty} H(|t|) = +\infty$.

Fix $w \in W$. By $\|w\| = 1$, we deduce that $\mu(\{x \in \mathbb{R}^N : w(x) \neq 0\}) > 0$, and that there exists a $\bar{t}_w > M$ such that $\mu(\{x \in \mathbb{R}^N : |\bar{t}_w w(x)| \geq M\}) > 0$, where μ is the Lebesgue measure.

Set $\Omega_1 := \{x \in \mathbb{R}^N : |\bar{t}_w w(x)| \geq M\}$ and $\Omega_2 := \mathbb{R}^N \backslash \Omega_1$. Then $\mu(\Omega_1) > 0$. Therefore, for any $x \in \Omega_1$, we have that $|w(x)| \geq \frac{M}{\bar{t}_w}$. Now take $\delta_w = \frac{M}{2C\bar{t}_w}$. Then, for any $v \in W \cap B(w, \delta_w)$, $|v - w|_\infty \leq C\|v - w\| < \frac{M}{2\bar{t}_w}$. Hence, for all $x \in \Omega_1$, we deduce that $|v(x)| \geq \frac{M}{2\bar{t}_w}$ and $|\lambda v(x)| \geq M$ for any $x \in \Omega_1$ and $\lambda \in \mathbb{R}$ with $|\lambda| \geq 2\bar{t}_w$. Thus, for $|\lambda| \geq 2\bar{t}_w$, by the above estimates and $H(|t|)$ increases when $|t|$ increases, we have

$$\begin{aligned} \int_{\Omega_1} K(x)F(\lambda v(x))dx &\geq C_1|\lambda|^{p^+} \int_{\Omega_1} K(x)|v(x)|^{p^+} H(|\lambda v(x)|)dx \\ &\geq C_1|\lambda|^{p^+} (\frac{M}{2\bar{t}_w})^{p^+} H(|\lambda|\frac{M}{2\bar{t}_w}) \int_{\Omega_1} K(x)dx. \end{aligned} \tag{2.1}$$

On the other hand, by continuity, we deduce that there exists a $C_2 > 0$ such that $F(t) \geq -C_2$ when $|t| \leq M$. Note that $F(t) > 0$ if $|t| \geq M$. Hence,

$$\begin{aligned} \int_{\Omega_2} K(x)F(\lambda v(x))dx &= \int_{\Omega_2 \cup \{x \in \mathbb{R}^N : |\lambda v(x)| \geq M\}} K(x)F(\lambda v(x))dx \\ &\quad + \int_{\Omega_2 \cup \{x \in \mathbb{R}^N : |\lambda v(x)| \leq M\}} K(x)F(\lambda v(x))dx \\ &\geq \int_{\Omega_2 \cup \{x \in \mathbb{R}^N : |\lambda v(x)| \leq M\}} K(x)F(\lambda v(x))dx \\ &\geq -C_2|K|_1. \end{aligned} \tag{2.2}$$

Hence, for $v \in W \cap B(w, \delta_w)$ and $|\lambda| > 1$, from (2.1) and (2.2), we have

$$\begin{aligned} \varphi(\lambda v) &= \int_{\mathbb{R}^N} \frac{|\lambda|^{p(x)}}{p(x)} (|\nabla v|^{p(x)} + |v|^{p(x)})dx - \int_{\mathbb{R}^N} K(x)F(\lambda v(x))dx \\ &\leq |\lambda|^{p^+} - C_1|\lambda|^{p^+} (\frac{M}{2\bar{t}_w})^{p^+} H(|\lambda|\frac{M}{2\bar{t}_w}) \int_{\Omega_1} K(x)dx + C_2|K|_1 \\ &= |\lambda|^{p^+} \Big[1 - C_1(\frac{M}{2\bar{t}_w})^{p^+} H(|\lambda|\frac{M}{2\bar{t}_w}) \int_{\Omega_1} K(x)dx\Big] + C_2|K|_1 \\ &\to -\infty, \end{aligned}$$

as $|\lambda| \to +\infty$, because $\lim_{|t| \to +\infty} H(|t|) = +\infty$. □

Claim 2.2. *There exist $\nu > 0$ and $\rho > 0$ such that* $\inf_{\|u\|=\nu} \varphi(u) \geq \rho > 0$.

Proof. Note that $|u|_\infty \to 0$ if $\|u\| \to 0$. Then, by hypothesis (H2), we have

$$\int_{\mathbb{R}^N} K(x)F(u)dx = |K|_1 o(|u|_\infty^{p^+}) = |K|_1 o(\|u\|^{p^+}),$$

which implies

$$\begin{aligned} \varphi(u) &= \int_{\mathbb{R}^N} \frac{1}{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)})dx - \int_{\mathbb{R}^N} K(x)F(u)dx \\ &\geq \frac{1}{p^+}\|u\|^{p^+} - |K|_1 o(\|u\|^{p^+}). \end{aligned}$$

Therefore, there exist $1 > \nu > 0$ and $\rho > 0$ such that $\inf_{\|u\|=\nu} \varphi(u) \geq \rho > 0$. □

Claim 2.3. *The functional φ satisfies the (PS) condition.*

Proof. Let $\{u_n\} \subset W_r^{1,p(x)}(\mathbb{R}^N)$ be a (PS) sequence of the functional φ; that is, $|\varphi(u_n)| \le c$ and $|\langle \varphi'(u_n), h\rangle| \le \varepsilon_n \|h\|$ with $\varepsilon_n \to 0$, for all $h \in W_r^{1,p(x)}(\mathbb{R}^N)$. We will prove that the sequence $\{u_n\}$ is bounded in $W_r^{1,p(x)}(\mathbb{R}^N)$. Indeed, if $\{u_n\}$ is unbounded in $W_r^{1,p(x)}(\mathbb{R}^N)$, we may assume that $\|u_n\| \to \infty$ as $n \to \infty$. Let $u_n = \lambda_n w_n$, where $\lambda_n \in \mathbb{R}$, $w_n \in W$. It follows that $|\lambda_n| \to \infty$.

Let $\Omega_1^n := \{x \in \mathbb{R}^N : |\lambda_n w_n(x)| \ge M\}$ and $\Omega_2^n := \mathbb{R}^N \backslash \Omega_1^n$. Then

$$\begin{aligned}
-\varepsilon_n|\lambda_n| &= -\varepsilon_n\|u_n\| \\
&\le \langle \varphi'(u_n), u_n\rangle \\
&= \int_{\mathbb{R}^N} \Big(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\Big)\,dx - \int_{\mathbb{R}^N} K(x) f(u_n) u_n dx \\
&\le \int_{\mathbb{R}^N} |\lambda_n|^{p(x)} \Big(|\nabla w_n|^{p(x)} + |w_n|^{p(x)}\Big)\,dx - \int_{\Omega_1^n} K(x) f(\lambda_n w_n)\lambda_n w_n dx \\
&\quad - \int_{\Omega_2^n} K(x) f(\lambda_n w_n)\lambda_n w_n dx,
\end{aligned}$$

which implies that

$$\begin{aligned}
\int_{\Omega_1^n} K(x) f(\lambda_n w_n)\lambda_n w_n\,dx &\le \int_{\mathbb{R}^N} |\lambda_n|^{p(x)} \Big(|\nabla w_n|^{p(x)} + |w_n|^{p(x)}\Big)\,dx \\
&\quad + \varepsilon_n|\lambda_n| - \int_{\Omega_2^n} K(x) f(\lambda_n w_n)\lambda_n w_n dx.
\end{aligned}$$

Note that $0 < (p^+ + \tau(t_{\lambda_n}))F(\lambda_n w_n) \le f(\lambda_n w_n)\lambda_n w_n$ in Ω_1^n. So,

$$\int_{\Omega_1^n} K(x) F(\lambda_n w_n) dx \le \frac{1}{p^+ + \tau(t_{\lambda_n})} \int_{\Omega_1^n} K(x) f(\lambda_n w_n)\lambda_n w_n dx.$$

Then it follows that

$$\begin{aligned}
\varphi(u_n) &= \varphi(\lambda_n w_n) \\
&= \int_{\mathbb{R}^N} \frac{|\lambda_n|^{p(x)}}{p(x)} (|\nabla w_n|^{p(x)} + |w_n|^{p(x)}) dx - \int_{\mathbb{R}^N} K(x) F(\lambda_n w_n) dx \\
&= \int_{\mathbb{R}^N} \frac{|\lambda_n|^{p(x)}}{p(x)} \Big(|\nabla w_n|^{p(x)} + |w_n|^{p(x)}\Big)\,dx - \int_{\Omega_1^n} K(x) F(\lambda_n w_n) dx \\
&\quad - \int_{\Omega_2^n} K(x) F(\lambda_n w_n) dx \\
&\ge \frac{1}{p^+} \int_{\mathbb{R}^N} |\lambda_n|^{p(x)} \Big(|\nabla w_n|^{p(x)} + |w_n|^{p(x)}\Big)\,dx \\
&\quad - \frac{1}{p^+ + \tau(t_{\lambda_n})} \int_{\Omega_1^n} K(x) f(\lambda_n w_n)\lambda_n w_n dx - \int_{\Omega_2^n} K(x) F(\lambda_n w_n) dx \\
&\ge \frac{1}{p^+} \int_{\mathbb{R}^N} |\lambda_n|^{p(x)} \Big(|\nabla w_n|^{p(x)} + |w_n|^{p(x)}\Big)\,dx \\
&\quad - \frac{1}{p^+ + \tau(t_{\lambda_n})} \Big[\int_{\mathbb{R}^N} |\lambda_n|^{p(x)} \Big(|\nabla w_n|^{p(x)} + |w_n|^{p(x)}\Big)\,dx + \varepsilon_n|\lambda_n|\Big]
\end{aligned}$$

$$\begin{aligned}
&+\frac{1}{p^+ + \tau(t_{\lambda_n})}\int_{\Omega_2^n} K(x)f(\lambda_n w_n)\lambda_n w_n dx - \int_{\Omega_2^n} K(x)F(\lambda_n w_n)dx \\
&= \frac{\tau(t_{\lambda_n})}{p^+(p^+ + \tau(t_{\lambda_n}))}\int_{\mathbb{R}^N} |\lambda_n|^{p(x)}\left(|\nabla w_n|^{p(x)} + |w_n|^{p(x)}\right)dx \\
&\quad - \frac{1}{p^+ + \tau(t_{\lambda_n})}\varepsilon_n|\lambda_n| + T(\lambda_n w_n) \\
&\geq \frac{\tau(t_{\lambda_n})}{p^+(p^+ + N_0)}|\lambda_n|^{p^-} - \frac{1}{p^+}\varepsilon_n|\lambda_n| + T(\lambda_n w_n) \\
&= |\lambda_n|\Big[\frac{|\lambda_n|^{p^- -1}\tau(t_{\lambda_n})}{p^+(p^+ + N_0)} - \frac{\varepsilon_n}{p^+}\Big] + T(\lambda_n w_n) \\
&\geq |\lambda_n|\Big[\frac{|\lambda_n|^{p^- -1}\tau(t_{\lambda_n})}{p^+(p^+ + N_0)} - \frac{\varepsilon_n}{p^+}\Big] - C_2,
\end{aligned}$$

where

$$T(\lambda_n w_n) = \frac{1}{p^+ + \tau(t_{\lambda_n})}\int_{\Omega_2^n} K(x)f(\lambda_n w_n)\lambda_n w_n\, dx - \int_{\Omega_2^n} K(x)F(\lambda_n w_n)\, dx$$

is bounded from below. We know that $|\lambda_n| \to +\infty$, and so $|t_{\lambda_n}| \to +\infty$, as $n \to +\infty$. It follows from (F1) and $p^- > N \geq 2$ that

$$\lim_{n\to+\infty} |\lambda_n|^{p^- -1}\tau(t_{\lambda_n}) \geq \lim_{n\to+\infty} \frac{|t_{\lambda_n}|\tau(t_{\lambda_n})}{M} = +\infty.$$

This means that $\lim_{n\to+\infty}\varphi(u_n) \to +\infty$. This is a contradiction. So, the sequence $\{u_n\}$ is bounded in $W_r^{1,p(x)}(\mathbb{R}^N)$. Note that the embedding $W_r^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ is compact, there exists a $u \in W_r^{1,p(x)}(\mathbb{R}^N)$ such that passing to subsequence, still denoted by $\{u_n\}$, it converges strongly to u in $L^\infty(\mathbb{R}^N)$, and in the same way as the proof of [17, Proposition 3.1] we can conclude that u_n converges strongly also in $W_r^{1,p(x)}(\mathbb{R}^N)$. Thus, φ satisfies the (PS) condition. □

Proof of Theorem 1.2. Due to Claims 2.1, 2.2 and 2.3, we know that φ satisfies the conditions of the classical mountain pass theorem due to Ambrosetti and Rabinowitz [2]. Hence, we obtain a nontrivial critical point, which gives rise to a nontrivial radially symmetric solution to problem (1.1).

Furthermore, if $f(t) = f(-t)$, then φ is even. We will use the following $\mathbb{Z}_2$ version of the mountain pass theorem in [18]. □

Theorem 2.4. *Let E be an infinite-dimensional Banach space, and $\varphi \in C(E, \mathbb{R})$ be even, satisfying the (PS) condition, and having $\varphi(0) = 0$. Assume that $E = V \oplus X$, where V is finite dimensional. Suppose that the following hold.*

- (a) *there are constants $\nu, \rho > 0$ such that $\inf_{\partial B_\nu \cup X}\varphi \geq \rho$.*
- (b) *for each finite-dimensional subspace $\overline{E} \subset E$, there is an $\sigma = \sigma(\overline{E})$ such that $\varphi \leq 0$ on $\overline{E}\backslash B_\sigma$.*

Then φ possesses an unbounded sequence of critical values.

From Claims 2.1 and 2.2, φ satisfies (a) and the (PS) condition. For any finite-dimensional subspace $\overline{E} \subset E$, $S \cap \overline{E} = \{w \in \overline{E} : \|w\| = 1\}$ is compact. By Claim 2.1 and the finite covering theorem, it is easy to verify that φ satisfies condition (b). Hence, by the $\mathbb{Z}_2$ version of the mountain pass theorem, φ has a sequence of critical points $\{u_n\}_{n=1}^{\infty}$. That is, problem (1.1) has infinitely many pairs of radially symmetric solutions.

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