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NECESSARY AND SUFFICIENT CONDITIONS FOR OSCILLATION OF NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Abstract. In this article, we concerned with oscillation of the neutral delay differential equation $[x(t) - px(t-\tau)]' + qx(t-\sigma) = 0$ with constant coefficients. By constructing several suitable auxiliary functions, we obtained some necessary and sufficient conditions for oscillation of all the solutions of the aforementioned equation for the cases $0 < p < 1$ and $p > 1$.

1. Introduction

Delay differential equations (DDEs) have been applied widely in many fields, such as oscillation theory [1, 2, 3, 4, 5, 6, 7, 8, 9, 13, 14, 19, 21, 28, 32], stability theory [23, 26, 30, 34], periodic solutions [24, 25, 27, 29], population dynamics [10, 18], dynamical behavior of delayed network systems [17, 36] and so on. Theoretical studies on oscillation of solutions of DDEs have fundamental significance [15, 16]. For this reason, many mathematicians have paid a great deal of attention on DDEs in the last few decades.

In this article, we consider the neutral delay differential equation

$$[x(t) - px(t-\tau)]' + qx(t-\sigma) = 0, \quad t \geqslant t_0, \tag{1.1}$$

where t_0 is a positive number and p, q, τ, σ are positive constants. Generally, a solution of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative. It can be seen in the literature that the oscillation theory regarding solutions of (1.1) has been extensively developed in the recent years.

In the research article [33] was derived that if $p \in (0, 1)$ and $q\sigma e > 1 - p$, then all the solutions of (1.1) are oscillatory. The result improves the corresponding result in [20]. Afterward, for $p \in (0, 1)$, many authors have involved to study this problem and have obtained various kinds of better results, see [11, 22, 31, 35, 37]. However, the negative side of all the conclusions reported in literature [11, 20, 22, 31, 33, 35, 37] are limited to sufficient conditions when $0 < p < 1$. Therefore, the main aim of this paper is to establish a set of necessary and sufficient conditions for oscillation of all the solutions of (1.1) for the cases $0 < p < 1$ and $p > 1$. Furthermore, we study deeply on oscillation of solutions under three cases $\tau < \sigma$, $\tau = \sigma$ and $\tau > \sigma$

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when $p > 1$. Besides that, we give some auxiliary criterions of main results (see Remarks 2.6, 2.15, 2.19, 2.24), and obtain several simple sufficient conditions for oscillation of all the solutions of (1.1).

2. Main results

It is well-known from [12] that all the solutions of (1.1) are oscillatory if and only if the characteristic equation (2.1) of (1.1),

$$f(\lambda) \equiv \lambda - p\lambda e^{-\lambda\tau} + qe^{-\lambda\sigma} = 0 \tag{2.1}$$

has no real roots. It is not difficult to show that if $p = 1$, then all the solutions of (1.1) are oscillatory.

Lemma 2.1. *Let $p \in (0, 1)$. Then all the solutions of (1.1) are oscillatory if and only if the equation*

$$g(\mu) = \frac{q}{\mu}e^{\mu\sigma} + pe^{\mu\tau} - 1 = 0 \tag{2.2}$$

has no real roots on $(0, 1/\sigma)$.

Proof. For $\lambda \geqslant 0$, we have that

$$f(\lambda) = \lambda(1 - pe^{-\lambda\tau}) + qe^{-\lambda\sigma} \geqslant qe^{-\lambda\sigma} > 0.$$

Thus, any real root of (2.1) must be negative.

Next, we consider the monotonicity of the function $g(\mu) := \frac{f(-\mu)}{\mu}$. By direct calculation, it has that

$$g'(\mu) = \frac{e^{\mu\sigma}\varphi(\mu)}{\mu^2}, \tag{2.3}$$

where

$$\varphi(\mu) := q(\sigma\mu - 1) + p\tau\mu^2 e^{(\tau-\sigma)\mu}. \tag{2.4}$$

Here, the function $\varphi(\mu)$ satisfies the following properties:

(1) $\varphi(\mu) > 0$ for $\mu \in (\frac{1}{\sigma}, +\infty)$;
(2) $\varphi(\mu)$ is strictly increasing on $(0, 1/\sigma)$ as the function $\mu^2 e^{(\tau-\sigma)\mu}$ is strictly increasing on $(0, 1/\sigma)$.

Further,

$$\varphi(0) = -q < 0 \text{ and } \varphi(\frac{1}{\sigma}) = p\tau\frac{1}{\sigma^2}e^{\frac{\tau-\sigma}{\sigma}} > 0.$$

Thus, we derive that the function $\varphi(\mu)$ has a unique zero θ on $(0, 1/\sigma)$. Hence, $g'(\mu) < 0$ for $\mu \in (0, \theta)$ and $g'(\mu) > 0$ for $\mu \in (\theta, +\infty)$, which imply that $g(\mu)$ is decreasing on $(0, \theta)$ and increasing on $(\theta, +\infty)$. Therefore, $g(\mu) > 0$ for $\mu \in (0, +\infty)$ if and only if (2.2) has no real roots on $(0, 1/\sigma)$. $\square$

By the above proof, it is not difficult to recognize that $g(\theta)$ is the minimum value of $g(\mu)$ on $(0, 1/\sigma)$.

Lemma 2.2. *Let $p \in (0, 1)$ and*

$$h(\mu) := qe^{\mu\sigma}[(\tau - \sigma)\mu + 1] - \tau\mu^2. \tag{2.5}$$

Then all the solutions of (1.1) are oscillatory if and only if

$$h(\theta) = qe^{\theta\sigma}[(\tau - \sigma)\theta + 1] - \tau\theta^2 > 0, \tag{2.6}$$

where θ is a unique zero of the function $\varphi(\mu)$ that is defined by (2.4) on $(0, 1/\sigma)$.

Proof. From the proof of Lemma 2.1, we have that $g(\mu)=0$ has no real roots on $(0,1/\sigma)$ if and only if $g(\theta)>0$. Since

$$g(\theta)=\frac{q}{\theta}e^{\theta\sigma}+pe^{\theta\tau}-1=\frac{h(\theta)}{\tau\theta^2}, \tag{2.7}$$

we obtain Lemma 2.2 immediately. □

Lemma 2.3. *Let $p\in(0,1)$. Then all the solutions of* (1.1) *are oscillatory if and only if one of the following conditions holds.*

- (H1) $q\sigma e\geqslant 1$*;*
- (H2) $\bar\theta>\theta$*, where θ and $\bar\theta$ are the unique zeros of $\varphi(\mu)$ and $h(\mu)$(see* (2.4) *and* (2.5)*) on $(0,1/\sigma)$, respectively.*

Proof. Let

$$y(\mu)=\frac{h(\mu)}{\mu^2}=qe^{\mu\sigma}(\frac{\tau-\sigma}{\mu}+\frac{1}{\mu^2})-\tau,$$

then

$$y'(\mu)=\frac{qe^{\mu\sigma}z(\mu)}{\mu^3}, \tag{2.8}$$

where $z(\mu)=(\tau-\sigma)\sigma\mu^2+(2\sigma-\tau)\mu-2$ which satisfies

$$z(0)=-2<0 \text{ and } z(\frac{1}{\sigma})=-1<0.$$

If $\tau\geqslant\sigma$, we get obviously that $z(\mu)<0$ for all $\mu\in(0,1/\sigma]$; If $\tau<\sigma$, we also get $z(\mu)<0$ for all $\mu\in(0,1/\sigma]$ since $z'(\frac{1}{\sigma})=\tau>0$. Thus, $z(\mu)<0$ for all $\mu\in(0,1/\sigma]$ and from which and (2.8) mean that $y'(\mu)<0$ for all $\mu\in(0,1/\sigma]$. Consequently, $y(\mu)$ is strictly decreasing on $(0,1/\sigma]$. Further,

$$\lim_{\mu\to 0^+}y(\mu)=+\infty \text{ and } y(\frac{1}{\sigma})=(qe\sigma-1)\tau.$$

Therefore, if $q\sigma e\geqslant 1$, then we have $y(\theta)>0$. Hence, $h(\theta)>0$. If $q\sigma e<1$, then we have $y(\frac{1}{\sigma})<0$. Hence, it is easy to find that both functions $y(\mu)$ and $h(\mu)$ have an equal and unique zero $\bar\theta$ on $(0,1/\sigma)$. Consequently, $h(\theta)>0$ is equivalent to $\bar\theta>\theta$.

From Lemma 2.2, all the solutions of (1.1) are oscillatory if and only if one of (H1) or (H2) holds. □

Corollary 2.4. *If $p=0$, then all the solutions of* (1.1) *are oscillatory if and only if $q\sigma e>1$.*

Theorem 2.5. *Assume that $p\in(0,1)$. Then all the solutions of* (1.1) *are oscillatory if and only if there exists a real number $\alpha\in(0,1/\sigma)$ such that*

$$\varphi(\alpha)=p\tau\alpha^2e^{(\tau-\sigma)\alpha}+q(\sigma\alpha-1)>0,$$
$$h(\alpha)=qe^{\sigma\alpha}[(\tau-\sigma)\alpha+1]-\tau\alpha^2>0.$$

Proof. From the proof of Lemma 2.1, the function $\varphi(\mu)$ has a unique zero θ on $(0,1/\sigma)$ and is strictly increasing on $(0,1/\sigma)$. If $q\sigma e\geqslant 1$, then from Lemma 2.3, all the solutions of (1.1) are oscillatory. Now, by the proof of Lemma 2.3, we know that $h(\mu)>0,\ \mu\in(0,1/\sigma)$. So that, conditions of the theorem hold.

If $q\sigma e<1$, then again from the proof of Lemma 2.3, the function $y(\mu)$ has a unique zero $\bar\theta$ on $(0,1/\sigma)$ and is strictly decreasing on $(0,1/\sigma)$. It can be seen that both functions $h(\mu)$ and $y(\mu)$ have the same sign in the interval $(0,1/\sigma)$. Hence, we obtain Theorem 2.5 from Lemma 2.3. The proof is complete. □

Remark 2.6. The function $\varphi(\mu)$ has a unique zero θ on $(0, 1/\sigma)$, and

$$\varphi(\mu) < 0 \quad \text{for } \mu \in (0, \theta);$$
$$\varphi(\mu) > 0 \quad \text{for } \mu \in (\theta, 1/\sigma).$$

Further, we have that $q\sigma e \geqslant 1$ is equivalent to $h(\mu) > 0, \mu \in (0, 1/\sigma)$ and $q\sigma e < 1$ is equivalent to the fact that $h(\mu)$ has a unique zero $\bar{\theta}$ on $(0, 1/\sigma)$. Here,

$$h(\mu) > 0 \quad \text{for } \mu \in (0, \bar{\theta});$$
$$h(\mu) < 0 \quad \text{for } \mu \in (\bar{\theta}, 1/\sigma).$$

Thus, (1.1) has a non-oscillatory solution if and only if there exists a real number $\beta \in (0, 1/\sigma)$ such that

$$\varphi(\beta) = p\tau\beta^2 e^{(\tau-\sigma)\beta} + q(\sigma\beta - 1) \leqslant 0,$$
$$h(\beta) = qe^{\sigma\beta}[(\tau - \sigma)\beta + 1] - \tau\beta^2 \leqslant 0.$$

Corollary 2.7. *If there exists a real number $\alpha \in [\theta, 1/\sigma)$ such that $qe^{\sigma\alpha} \geqslant \sigma\alpha^2$ where $\theta = \frac{-q\sigma e+\sqrt{(q\sigma e)^2+4eqp\tau}}{2p\tau}$ for $p \in (0, 1)$ and $\tau < \sigma$ hold, then all the solutions of (1.1) are oscillatory.*

Proof. Clearly, we have that $p\tau\alpha^2 + eq(\sigma\alpha - 1) \geqslant 0$ for $\alpha \in [\theta, 1/\sigma)$ where $\theta = \frac{-q\sigma e+\sqrt{(q\sigma e)^2+4eqp\tau}}{2p\tau}$. Therefore, it follows that

$$\varphi(\alpha) = p\tau\alpha^2 e^{(\tau-\sigma)\alpha} + q(\sigma\alpha - 1) > p\tau\alpha^2 e^{-1} + q(\sigma\alpha - 1) \geqslant 0,$$
$$h(\alpha) = qe^{\sigma\alpha}[(\tau - \sigma)\alpha + 1] - \tau\alpha^2 > \frac{\tau qe^{\sigma\alpha}}{\sigma} - \tau\alpha^2 \geqslant 0.$$

Therefore, the conditions of Theorem 2.5 hold. $\square$

Corollary 2.8. *If $p \in (0, 1)$ and $\tau = \sigma$, then all the solutions of (1.1) are oscillatory if and only if there exists a real number $\alpha \in [\theta, 1/\sigma)$ such that*

$$h(\alpha) = qe^{\alpha\tau} - \tau\alpha^2 > 0,$$

where $\theta = \frac{-q\tau+\sqrt{q\tau(q\tau+4p)}}{2p\tau}$.

Corollary 2.9. *If $p \in (0, 1)$, $\tau > \sigma$ and there exists a real number $\alpha \in [\theta, 1/\sigma)$ where $\theta = \frac{-q\sigma+\sqrt{(q\sigma)^2+4qp\tau}}{2p\tau}$ such that $qe^{\sigma\alpha} \geqslant \tau\alpha^2$, then all the solutions of (1.1) are oscillatory.*

Proof. Clearly, we have that $p\tau\alpha^2 + q(\sigma\alpha - 1) \geqslant 0$ for $\alpha \in [\theta, 1/\sigma)$ where $\theta = \frac{-q\sigma+\sqrt{(q\sigma)^2+4qp\tau}}{2p\tau}$. So that it follows that

$$\varphi(\alpha) = p\tau\alpha^2 e^{(\tau-\sigma)\alpha} + q(\sigma\alpha - 1) > p\tau\alpha^2 + q(\sigma\alpha - 1) \geqslant 0,$$
$$h(\alpha) = qe^{\sigma\alpha}[(\tau - \sigma)\alpha + 1] - \tau\alpha^2 > qe^{\sigma\alpha} - \tau\alpha^2 \geqslant 0.$$

Therefore, the conditions of Theorem 2.5 hold. $\square$

So far, we have discussed and have obtained necessary and sufficient conditions for oscillation of all the solutions of (1.1) for $p \in (0, 1)$. Next, we will discuss the behavior of oscillation of solutions of (1.1) for $p > 1$ under three subcases, namely, $\tau < \sigma$, $\tau = \sigma$ and $\tau > \sigma$.

Lemma 2.10. *Let $p > 1$. Then all the solutions of* (1.1) *are oscillatory if and only if the equation*

$$g(\mu) = \frac{q}{\mu}e^{\mu\sigma} + pe^{\mu\tau} - 1 = 0 \tag{2.9}$$

has no real roots on $(\frac{-\ln p}{\tau}, 0)$.

Proof. Since

$$g(\mu) = \frac{q}{\mu}e^{\sigma\mu} + pe^{\mu\tau} - 1 > \frac{q}{\mu}(1+\mu\sigma) + p - 1 = \frac{q}{\mu} + q\sigma + p - 1,$$

we know that $g(\mu) > 0$ for $\mu \in (0,\infty)$. It is not difficult to see that $\frac{e^{\sigma\mu}}{\mu}$ is strictly decreasing on $(-\infty, 0)$ while $e^{\mu\tau}$ is strictly increasing on $(-\infty, 0)$. Note that $pe^{\mu\tau} - 1 = 0$ at $\mu = \frac{-\ln p}{\tau}$, we find that

$$g(\mu) < 0 \text{ for } u \in (-\infty, \frac{-\ln p}{\tau}]. \tag{2.10}$$

Clearly, $f(0) = q > 0$. Thus, $f(\lambda)$ has no real roots which is equivalent to $g(\mu)$ has no real roots on $(\frac{-\ln p}{\tau}, 0)$. □

Proposition 2.11. *Suppose that $p > 1$. Then all the solutions of* (1.1) *are oscillatory if and only if*

$$g(\mu) < 0 \quad \textit{for } \mu \in (-\infty, 0). \tag{2.11}$$

Lemma 2.12. *Let $p > 1$ and $\tau < \sigma$. Then all solutions of* (1.1) *are oscillatory if and only if*

$$h(\theta) = qe^{\theta\sigma}[(\tau - \sigma)\theta + 1] - \tau\theta^2 < 0, \tag{2.12}$$

where θ is a unique zero of (2.4) *on* $(-\infty, 0)$.

Proof. Firstly, we prove that $\varphi(\mu)$ has a unique zero θ on $(-\infty, 0)$. In fact,

$$\varphi'(\mu) = p\tau e^{(\tau-\sigma)\mu}[(\tau - \sigma)\mu^2 + 2\mu] + q\sigma. \tag{2.13}$$

It is easy to verify that $\varphi'(\mu)$ is strictly increasing on $(-\infty, 0)$. In addition,

$$\varphi'(\mu) \to -\infty (\mu \to -\infty) \text{ and } \varphi'(0) = q\sigma > 0.$$

Therefore, $\varphi'(\mu)$ has a unique zero ω_0 on $(-\infty, 0)$. Hence, $\varphi(\mu)$ is strictly decreasing on $(-\infty, \omega_0)$ and strictly increasing on $(\omega_0, 0)$. So that, $\varphi(\mu)$ has a unique zero θ on $(-\infty, 0)$ as $\varphi(\mu) \to +\infty(\mu \to -\infty)$ and $\varphi(0) = -q < 0$.

Now, from (2.3), it follows that $g(\theta)$ is the maximum value of $g(\mu)$ on $(-\infty, 0)$. By (2.7), we know that (2.12) is equivalent to $g(\mu) < 0$ for $\mu \in (-\infty, 0)$. □

Proposition 2.13. *Suppose that $p > 1$ and $\tau < \sigma$. Then all the solutions of* (1.1) *are oscillatory if and only if $\theta < \bar{\theta}$, where θ and $\bar{\theta}$ are the unique zeros of $\varphi(\mu)$ and $h(\mu)$ on $(-\infty, 0)$, respectively.*

Proof. Let $y(\mu) = \frac{h(\mu)}{\mu^2} = qe^{\sigma\mu}(\frac{\tau-\sigma}{\mu} + \frac{1}{\mu^2}) - \tau$, then

$$y'(\mu) = \frac{qe^{\sigma\mu}z(\mu)}{-\mu^3}, \tag{2.14}$$

where $z(\mu) = \sigma(\sigma - \tau)\mu^2 - (2\sigma - \tau)\mu + 2$. Further,

$$z'(\mu) = 2\sigma(\sigma - \tau)\mu - 2\sigma + \tau.$$

It is easy to see that $z'(\mu)$ is strictly increasing on $(-\infty, 0)$ and $z'(0) = \tau - 2\sigma < 0$. So that, $z(\mu)$ is strictly decreasing on $(-\infty, 0)$ and $z(\mu) > 0$ on $(-\infty, 0)$ since $z(0) = 2 > 0$. Consequently, $y(\mu)$ is strictly increasing on $(-\infty, 0)$. In addition,

$$y(\mu) \to -\tau \ (\mu \to -\infty) \text{ and } y(\mu) \to +\infty \ (\mu \to 0^-).$$

Therefore, $y(\mu)$ and $h(\mu)$ have an equal and unique zero $\bar{\theta}$ on $(-\infty, 0)$. From Lemma 2.12, it is clear that (2.12) holds if and only if $\theta < \bar{\theta}$. □

Theorem 2.14. *Assume that $p > 1$ and $\tau < \sigma$. Then all the solutions of* (1.1) *are oscillatory if and only if there exists a real number $\alpha \in (-\infty, 0)$ such that*

$$\varphi(\alpha) = p\tau\alpha^2 e^{(\tau-\sigma)\alpha} + q(\sigma\alpha - 1) < 0,$$
$$h(\alpha) = qe^{\sigma\alpha}[(\tau - \sigma)\alpha + 1] - \tau\alpha^2 < 0.$$

Proof. From the proof of Lemma 2.12, $\varphi'(\mu)$ has a unique zero ω_0 on $(-\infty, 0)$, and $\varphi(\mu)$ is strictly decreasing on $(-\infty, \omega_0)$ and strictly increasing on $(\omega_0, 0)$. Further,

$$\varphi(\mu) \to +\infty (\mu \to -\infty) \text{ and } \varphi(0) = -q < 0.$$

Hence, $\varphi(\mu)$ has a unique zero θ on $(-\infty, 0)$.

Now, from the proof of Proposition 2.13, $y(\mu)$ has a unique zero $\bar{\theta}$ on $(-\infty, 0)$ and is strictly increasing on $(-\infty, 0)$. Note that both functions $h(\mu)$ and $y(\mu)$ have the same sign in the interval $(-\infty, 0)$. Therefore, from Proposition 2.13, we obtain Theorem 2.14. The proof is complete. □

Remark 2.15. The equation (1.1) has a non-oscillatory solution if and only if there exists a real number $\beta \in (-\infty, 0)$ such that

$$\varphi(\beta) = p\tau\beta^2 e^{(\tau-\sigma)\beta} + q(\sigma\beta - 1) \geqslant 0,$$
$$h(\beta) = qe^{\sigma\beta}[(\tau - \sigma)\beta + 1] - \tau\beta^2 \geqslant 0.$$

Furthermore, the function $\varphi(\mu)$ has a unique zero θ on $(-\infty, 0)$ and

$$\varphi(\mu) > 0 \quad \text{for } \mu \in (-\infty, \theta);$$
$$\varphi(\mu) < 0 \quad \text{for } \mu \in (\theta, 0).$$

The function $h(\mu)$ has a unique zero $\bar{\theta}$ on $(-\infty, 0)$ and

$$h(\mu) < 0 \quad \text{for } \mu \in (-\infty, \bar{\theta});$$
$$h(\mu) > 0 \quad \text{for } \mu \in (\bar{\theta}, 0).$$

Lemma 2.16. *Let $p > 1$ and $\tau = \sigma$. Then all the solutions of* (1.1) *are oscillatory if and only if*

$$qe^{\theta\tau} < \tau\theta^2, \tag{2.15}$$

where $\theta = \frac{-q\tau - \sqrt{q\tau(q\tau+4p)}}{2p\tau}$.

By using similar procedure which used to prove Lemma 2.2, we can obtain the result. Therefore, we omit the proof.

Proposition 2.17. *Suppose that $p > 1$ and $\tau = \sigma$. Then all the solutions of* (1.1) *are oscillatory if and only if $\theta < \bar{\theta}$, where $\theta = \frac{-q\tau - \sqrt{q\tau(q\tau+4p)}}{2p\tau}$ and $\bar{\theta}$ is a unique zero of $h(\mu) = qe^{\tau\mu} - \tau\mu^2$ on $(-\infty, 0)$.*

Proof. From (2.5), we have that $h(\mu) = qe^{\tau\mu} - \tau\mu^2$. Let $y(\mu) = \frac{h(\mu)}{\mu^2} = \frac{qe^{\tau\mu}}{\mu^2} - \tau$, then

$$y'(\mu) = \frac{qe^{\tau\mu}}{-\mu^3} z(\mu),$$

where $z(\mu) = -\tau\mu + 2$. Since $z(0) = 2 > 0$, we have that $z(\mu) > 0$ for $\mu \in (-\infty, 0)$. This means that $y'(\mu) > 0$ for $\mu \in (-\infty, 0)$. Hence, $y(\mu)$ is strictly increasing on $(-\infty, 0)$. In addition,

$$y(\mu) \to -\tau \ (\mu \to -\infty) \text{ and } y(\mu) \to +\infty \ (\mu \to 0^-).$$

In consequence, the function $y(\mu)$ has a unique zero $\bar{\theta}$ on $(-\infty, 0)$, which implies together with Lemma 2.16 that $qe^{\tau\theta} < \tau\theta^2$ if and only if $\theta < \bar{\theta}$. $\square$

Theorem 2.18. *Assume that $p > 1$ and $\tau = \sigma$. Then all the solutions of* (1.1) *are oscillatory if and only if there exists a real number $\alpha \in [\theta, 0)$ such that*

$$h(\alpha) = qe^{\tau\alpha} - \tau\alpha^2 < 0,$$

where $\theta = \frac{-q\tau - \sqrt{q\tau(q\tau + 4p)}}{2p\tau}$.

Proof. From the proof of Proposition 2.17, the function $y(\mu)$ has a unique zero $\bar{\theta}$ on $(-\infty, 0)$ and is strictly increasing on $(-\infty, 0)$. Now, it is easy to find that the signs of the functions $h(\mu)$ and $y(\mu)$ are the same in the interval $(-\infty, 0)$. So that, from Proposition 2.17, we obtain Theorem 2.18 immediately. The proof is complete. $\square$

Remark 2.19. The function $h(\mu)$ has a unique zero $\bar{\theta}$ on $(-\infty, 0)$ and

$$h(\mu) < 0 \quad \text{for } \mu \in (-\infty, \bar{\theta});$$
$$h(\mu) > 0 \quad \text{for } \mu \in (\bar{\theta}, 0).$$

Corollary 2.20. *If $p > 1$, $\tau = \sigma$ and $qe^{\frac{q}{p}\tau} \geqslant \frac{p^2}{\tau}$, then all the solutions of* (1.1) *are oscillatory.*

Proof. Let $\alpha = \frac{-q}{p}$, then $\theta = \frac{-q\tau - \sqrt{q\tau(q\tau + 4p)}}{2p\tau} < \alpha < 0$. It follows that

$$h(\alpha) = qe^{\tau\alpha} - \tau\alpha^2 = \frac{q(p^2 e^{-\frac{q}{p}\tau} - q\tau)}{p^2} \leqslant 0.$$

From Theorem 2.18, it can be concluded that all the solutions of (1.1) are oscillatory. $\square$

Proposition 2.21. *Suppose that $p > 1$, $\tau > \sigma$ and $\xi = \frac{\tau - 2\sigma - \sqrt{(2\sigma - \tau)^2 + 8\sigma(\tau - \sigma)}}{2\sigma(\tau - \sigma)}$. Then all the solutions of* (1.1) *are oscillatory if and only if one of the following conditions holds.*

- (H1) $q \geqslant \frac{p\tau\xi^2 e^{(\tau-\sigma)\xi}}{1 - \sigma\xi}$;
- (H2) $h(\theta) = qe^{\sigma\theta}[(\tau - \sigma)\theta + 1] - \tau\theta^2 < 0$, *where θ is a unique zero of* (2.4) *on* $(\xi, 0)$.

Proof. Let $\phi(\mu) = \frac{\varphi(\mu)}{1 - \sigma\mu} = \frac{p\tau\mu^2 e^{(\tau-\sigma)\mu}}{1 - \sigma\mu} - q$, then by (2.3) and (2.4), we have that

$$\varphi(\mu) = (1 - \sigma\mu)\phi(\mu),$$
$$g'(\mu) = \frac{e^{\sigma\mu}(1 - \sigma\mu)\phi(\mu)}{\mu^2}.$$

Differentiation yields that

$$\phi'(\mu) = \frac{-p\tau\mu e^{(\tau-\sigma)\mu}}{(1-\sigma\mu)^2} y(\mu),$$

where $y(\mu) = \sigma(\tau-\sigma)\mu^2 + (2\sigma-\tau)\mu - 2$ and $y(\mu)$ satisfies that

$$\lim_{\mu\to-\infty} y(\mu) = +\infty \quad \text{and} \quad y(0) = -2 < 0.$$

Thus, $y(\mu)$ has a unique zero $\xi = \frac{\tau-2\sigma-\sqrt{(2\sigma-\tau)^2+8\sigma(\tau-\sigma)}}{2\sigma(\tau-\sigma)}$ on $(-\infty,0)$. Consequently, $\phi'(\mu) > 0$ and $\phi'(\mu) < 0$ for $\mu \in (-\infty,\xi)$ and $\mu \in (\xi,0)$, respectively. As a result, the function $\phi(\mu)$ is strictly increasing on $(-\infty,\xi)$ and strictly decreasing on $(\xi,0)$. This implies that $\phi(\xi)$ is the maximum value of $\phi(\mu)$ for $\mu \in (-\infty,0)$. In addition,

$$\lim_{\mu\to-\infty} \phi(\mu) = \phi(0) = -q.$$

So that, if $q \geqslant \frac{p\tau\xi^2 e^{(\tau-\sigma)\xi}}{1-\sigma\xi}$ and $\phi(\xi) \leqslant 0$, then we have that $g'(\mu) \leqslant 0$, $\mu \in (-\infty,0)$. Now, it easy to find that $\lim_{\mu\to-\infty} g(\mu) = -1$. Hence, (2.11) holds.

If $q < \frac{p\tau\xi^2 e^{(\tau-\sigma)\xi}}{1-\sigma\xi}$, i.e. $\phi(\xi) > 0$, Consequently, $\phi(\mu)$ has a unique zero ω on $(-\infty,\xi)$ and a unique zero θ on $(\xi,0)$. In consequence, we have that $g'(\mu) < 0$ on $(-\infty,\omega)$, $g'(\mu) > 0$ on (ω,θ) and $g'(\mu) < 0$ on $(\theta,0)$, which means that the function $g(\mu)$ is strictly decreasing, strictly increasing and strictly decreasing on $(-\infty,\omega)$, (ω,θ) and $(\theta,0)$, respectively. Further,

$$\lim_{\mu\to-\infty} g(\mu) = -1 \text{ and } \lim_{\mu\to 0^-} g(\mu) = -\infty,$$

and $\phi(\theta) = 0$, i.e. $\varphi(\theta) = 0$. Therefore, (2.11) holds if and only if $g(\theta) < 0$ (the condition (H2) holds). □

Proposition 2.22. *Suppose that $p > 1$, $\tau > \sigma$ and $q < \frac{p\tau\xi^2 e^{(\tau-\sigma)\xi}}{1-\sigma\xi}$. Then all the solutions of* (1.1) *are oscillatory if and only if $\theta < \bar{\theta}$, where θ and $\bar{\theta}$ are the unique zeros of* (2.4) *and* (2.5) *on $(\xi,0)$, respectively and*

$$\xi = \frac{\tau-2\sigma-\sqrt{(2\sigma-\tau)^2+8\sigma(\tau-\sigma)}}{2\sigma(\tau-\sigma)}.$$

Proof. Let $y(\mu) = \frac{h(\mu)}{\mu^2} = qe^{\sigma\mu}(\frac{\tau-\sigma}{\mu} + \frac{1}{\mu^2}) - \tau$, then

$$y'(\mu) = \frac{qe^{\sigma\mu}}{-\mu^3} z(\mu),$$

where $z(\mu) = -\sigma(\tau-\sigma)\mu^2 - (2\sigma-\tau)\mu + 2$ and $z(\mu)$ satisfies that

$$\lim_{\mu\to-\infty} z(\mu) = -\infty \quad \text{and} \quad z(0) = 2 > 0.$$

Therefore, $z(\mu)$ has a unique zero $\omega = \frac{\tau-2\sigma-\sqrt{(2\sigma-\tau)^2+8\sigma(\tau-\sigma)}}{2\sigma(\tau-\sigma)} (\omega < 0)$ on $(-\infty,0)$. This means that $z(\mu) < 0$ for $\mu \in (-\infty,\omega)$ and $z(\mu) > 0$ for $\mu \in (\omega,0)$. Consequently, $y'(\mu) < 0$ and $y'(\mu) > 0$ for $\mu \in (-\infty,\omega)$ and $\mu \in (\omega,0)$, respectively. In consequence, $y(\mu)$ is strictly decreasing on $(-\infty,\omega)$ and strictly increasing on $(\omega,0)$. In addition,

$$\lim_{\mu\to-\infty} y(\mu) = -\tau \quad \text{and} \quad \lim_{\mu\to 0^-} y(\mu) = +\infty.$$

Thus, $y(\mu)$ has a unique zero $\bar{\theta}$ on $(\omega, 0)$. It is easy to find that $\omega = \xi$. This, together with Proposition 2.21 imply that $h(\theta) < 0$ if and only if $\theta < \bar{\theta}$. ☐

Theorem 2.23. *Assume that $p > 1$, $\tau > \sigma$ and $\xi = \frac{\tau-2\sigma-\sqrt{(2\sigma-\tau)^2+8\sigma(\tau-\sigma)}}{2\sigma(\tau-\sigma)}$. Then all the solutions of* (1.1) *are oscillatory if and only if there exists a real number $\alpha \in (\xi, 0)$ such that*

$$\begin{gathered}\varphi(\alpha) = p\tau\alpha^2 e^{(\tau-\sigma)\alpha} + q(\sigma\alpha - 1) < 0,\\ h(\alpha) = qe^{\sigma\alpha}[(\tau-\sigma)\alpha + 1] - \tau\alpha^2 < 0.\end{gathered}$$

Proof. From the proof of Proposition 2.22, $y(\mu)$ has a unique zero $\bar{\theta}$ on $(\xi, 0)$ and $y(\mu)$ is strictly increasing on $(\xi, 0)$. If $q \geqslant \frac{p\tau\xi^2 e^{(\tau-\sigma)\xi}}{1-\sigma\xi}$, then by Proposition 2.21, all the solutions of (1.1) are oscillatory. Now, again from the proof of Proposition 2.21, it follows that $\varphi(\mu) < 0, \mu \in (0, \xi)$. As a result, the conditions of Theorem 2.23 hold.

If $q < \frac{p\tau\xi^2 e^{(\tau-\sigma)\xi}}{1-\sigma\xi}$, then from the proof of Proposition 2.21, the function $\phi(\mu)$ has a unique zero θ on $(\xi, 0)$ and is strictly decreasing on $(\xi, 0)$. Moreover, the function values of $\varphi(\mu)$ and $\phi(\mu)$ have the same sign in the interval $(\xi, 0)$. And also both functions $h(\mu)$ and $y(\mu)$ have the same sign in the interval $(\xi, 0)$. Therefore, from Proposition 2.22, it can be seen that $\theta < \bar{\theta}$ if and only if the conditions of Theorem 2.23 hold. The proof is complete. ☐

Remark 2.24. The function $h(\mu)$ has a unique zero $\bar{\theta}$ on $(\xi, 0)$ and

$$\begin{gathered}h(\mu) < 0 \quad \text{for } \mu \in (\xi, \bar{\theta});\\ h(\mu) > 0 \quad \text{for } \mu \in (\bar{\theta}, 0).\end{gathered}$$

Further, we have that $q \geqslant \frac{p\tau\xi^2 e^{(\tau-\sigma)\xi}}{1-\sigma\xi}$ is equivalent to $\varphi(\mu) < 0$ for $\mu \in (\xi, 0)$, $q < \frac{p\tau\xi^2 e^{(\tau-\sigma)\xi}}{1-\sigma\xi}$ is equivalent to the fact that $\varphi(\mu)$ has a unique zero θ on $(\xi, 0)$. Here,

$$\begin{gathered}\varphi(\mu) > 0 \quad \text{for } \mu \in (\xi, \theta);\\ \varphi(\mu) < 0 \quad \text{for } \mu \in (\theta, 0).\end{gathered}$$

Thus, (1.1) has a non-oscillatory solution if and only if there exists a real number $\beta \in (\xi, 0)$ such that

$$\begin{gathered}\varphi(\beta) = p\tau\beta^2 e^{(\tau-\sigma)\beta} + q(\sigma\beta - 1) \geqslant 0,\\ h(\beta) = qe^{\sigma\beta}[(\tau-\sigma)\beta + 1] - \tau\beta^2 \geqslant 0.\end{gathered}$$

For $\tau > \sigma$, it easy to verify that $\frac{-2}{\tau-\sigma} < \xi = \frac{\tau-2\sigma-\sqrt{(2\sigma-\tau)^2+8\sigma(\tau-\sigma)}}{2\sigma(\tau-\sigma)} < \frac{-1}{\tau-\sigma}$. From the condition (H1) of Proposition 2.21, we obtain the following corollary at once.

Corollary 2.25. *If $p > 1$, $\tau > \sigma$ and $q \geqslant \frac{4p}{\tau-\sigma}$, then all the solutions of* (1.1) *are oscillatory.*

In fact, Corollary 2.25 can be still improved and extended, we have given the following corollary to Corollary 2.25 with some improvements.

Corollary 2.26. *If $p > 1$, $\tau > \sigma$ and $q \geqslant \frac{p}{e(\tau-\sigma)}$, then all the solutions of* (1.1) *are oscillatory.*

Proof. It is easy to see that

$$\xi = \frac{\tau - 2\sigma - \sqrt{(2\sigma-\tau)^2 + 8\sigma(\tau-\sigma)}}{2\sigma(\tau-\sigma)} < \alpha < 0,$$

where $\alpha = \frac{-1}{\tau-\sigma}$. In consequence,

$$\varphi(\alpha) = p\tau\alpha^2 e^{(\tau-\sigma)\alpha} + q(\sigma\alpha - 1) = \frac{\tau[pe^{-1} - q(\tau-\sigma)]}{(\tau-\sigma)^2} \leqslant 0,$$

$$h(\alpha) = qe^{\sigma\alpha}[(\tau-\sigma)\alpha + 1] - \tau\alpha^2 = \frac{-\tau}{(\tau-\sigma)^2} < 0.$$

From Theorem 2.23, all the solutions of (1.1) are oscillatory. □

3. Examples with numerical simulation

In this section, we enumerate some specific examples together with numerical simulation to verify the results that we obtained and to show the simplicity of results.

Example 3.1. Consider the neutral delay differential equation

$$[x(t) - 3x(t-9)]' + 10x(t-9) = 0. \tag{3.1}$$

Then, it is easy to see that $p = 3$, $q = 10$, $\tau = 9$ and $\sigma = 9$. Consequently, $\tau = \sigma$ and $q > p^2/\tau$. Corollary 2.20 shows that all the solutions of (3.1) are oscillatory.

Example 3.2. Again, consider the neutral delay differential equation

$$[x(t) - 8x(t-15)]' + 10x(t-14) = 0. \tag{3.2}$$

It is not difficult to find the values $p = 8$, $q = 10$, $\tau = 15$ and $\sigma = 14$. Consequently, $\tau > \sigma$ and $q > \frac{p}{\tau-\sigma}$. One can see from Corollary 2.26 that all the solutions of (3.2) are oscillatory.

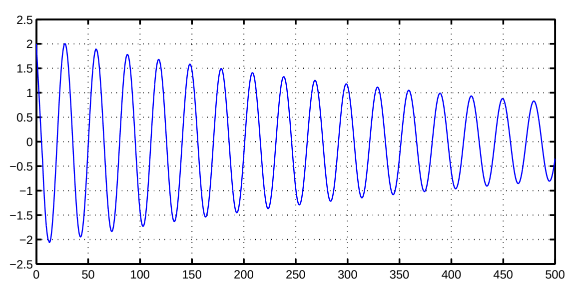

Figure 1. Time evolution of (3.3) with initial value 2

Example 3.3. Consider the neutral delay differential equation

$$[x(t) - \frac{1}{2}x(t-6)]' + \frac{1}{5}x(t-5) = 0. \tag{3.3}$$

Then, it is easy to see that $p = 1/2$, $q = 1/5$, $\tau = 6$ and $\sigma = 5$. If we take $\alpha = 3/16$, then $\alpha \in (0, 1/\sigma)$. Consequently, $\tau > \sigma$ and

$$\theta = \frac{-q\sigma + \sqrt{(q\sigma)^2 + 4qp\tau}}{2p\tau} = \frac{-1 + \sqrt{3.4}}{6} < \frac{3}{16},$$
$$qe^{\sigma\alpha} - \tau\alpha^2 = \frac{1}{5}(5\alpha + 1) - 6\alpha^2 > \frac{1}{5} + \alpha - \frac{6\alpha}{3} = \frac{1}{80} > 0.$$

From Corollary 2.9, it is easy to conclude that all the solutions of (3.3) are oscillatory (see Figure 1).

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