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EIGENVALUE PROBLEMS WITH p-LAPLACIAN OPERATORS

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Abstract. In this article, we study eigenvalue problems with the p-Laplacian operator:

$$-(|y'|^{p-2}y')' = (p-1)(\lambda\rho(x) - q(x))|y|^{p-2}y \quad \text{on } (0, \pi_p),$$

where $p > 1$ and $\pi_p \equiv 2\pi/(p\sin(\pi/p))$. We show that if $\rho \equiv 1$ and q is single-well with transition point $a = \pi_p/2$, then the second Neumann eigenvalue is greater than or equal to the first Dirichlet eigenvalue; the equality holds if and only if q is constant. The same result also holds for p-Laplacian problem with single-barrier ρ and $q \equiv 0$. Applying these results, we extend and improve a result by [24] by using finitely many eigenvalues and by generalizing the string equation to p-Laplacian problem. Moreover, our results also extend a result of Huang [14] on the estimate of the first instability interval for Hill equation to single-well function q.

1. Introduction

Recently there are many studies on the p-Laplacian operator:

$$-(|y'|^{p-2}y')' = (p-1)(\lambda\rho(x) - q(x))|y|^{p-2}y \quad \text{on } (0, \pi_p), \tag{1.1}$$

where $p > 1$ and $\pi_p \equiv 2\pi/(p\sin(\pi/p))$. An application for (1.1), the most cited nowadays, is that of a highly viscid fluid flow (cf. Ladyzhenskaya [16], and Lions [19]). This involves partial differential equations, but for symmetric flows, only the ordinary differential operator (perhaps in radial form) is involved (see, e.g., Binding and Drábek [1], del Pino, Elgueta and Manasevich [21], del Pino and Manasevich [22], Rabinowitz [23], and Walter [25]).

In 1979, Elbert [11] showed that the inverse function $S_p(x) \equiv w$ of the integral

$$x = \int_0^w \frac{dt}{(1-t^p)^{1/p}} \qquad \text{for } 0 \le w \le 1\,,$$

satisfies the initial valued problem

$$-(|u'|^{p-2}u')' = (p-1)|u|^{p-2}u, \qquad u(0) = 0,\ u'(0) = 1.$$

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The function $S_p(x)$ is called a generalized sine function and the value $\pi_p \equiv 2\int_0^1 (1-t^p)^{-1/p}dt = 2\pi/(p\sin(\pi/p))$ is the first zero of $S_p(x)$. Continuing $S_p(x)$ symmetrically over $x \in [\pi_p/2, \pi_p]$ and antisymmetrically outside $[0, \pi_p]$ by defining

$$S_p(x) = \begin{cases} S_p(\pi_p - x), & \text{if } \frac{\pi_p}{2} \le x \le \pi_p\,, \\ -S_p(x-\pi_p), & \text{if } \pi_p \le x \le 2\pi_p\,, \end{cases}$$

and $S_p(x) = S_p(x - 2n\pi_p)$ for $n = \pm 1, \pm 2, \dots$, he obtained a sine-like function defined on $\mathbb{R}$. Furthermore, he found the Pythagorean trigonometric identity for p-version:

$$|S_p(x)|^p + |S_p'(x)|^p = 1\,.$$

Similarly, it may be defined an analogue of the hyperbolic sine function (see [18]) $Sh_p(x) \equiv v$ by the inverse function of the integral $x = \int_0^v (1+|t|^p)^{-1/p}dt$. It is clearly that $Sh_p(x) = (-1)^{-1/p}S_p((-1)^{1/p}x)$ and $Sh_p'(x) = S_p'((-1)^{1/p}x)$ where $(-1)^{1/p} = e^{\pi i/p}$. Furthermore, we have $Sh_p''(x) = \frac{|Sh_p(x)|^{p-2}Sh_p(x)}{Sh_p'^{p-2}(x)}$ and

$$Sh_p'^p(x) - |Sh_p(x)|^p = 1.$$

Denote by σ_{2k} (σ_{2k-1}) the set of periodic (anti-periodic) eigenvalues of (1.1) which admit the corresponding eigenfunctions with exactly $2k$ zeros in $[0, \pi_p)$. In 2001, Zhang [26] used a rotation number function to show the existence of the minimal eigenvalue $\underline{\lambda}_n = \min \sigma_n$ and the maximal eigenvalue $\overline{\lambda}_n = \max \sigma_n$, respectively. In particular, Binding and Rynne in a series of papers [2, 3, 4] showed that (1.1) has an infinite sequence of variational and non-variational periodic eigenvalues and the multiplicity of the periodic eigenvalue can be arbitrary. They also showed that the Dirichlet eigenvalues $\{\mu_n\}_{n\ge 1}$ and Neumann eigenvalues $\{\nu_n\}_{n\ge 0}$ for (1.1) acting on $(0, \pi_p)$ satisfy

$$\cdots \le \overline{\lambda}_{2n-2} < \underline{\lambda}_{2n-1} \le \mu_{2n-1},$$
$$\nu_{2n-1} \le \overline{\lambda}_{2n-1} < \underline{\lambda}_{2n} \le \mu_{2n}, \nu_{2n} \le \overline{\lambda}_{2n} < \underline{\lambda}_{2n+1} \le \dots .$$

Note that, for $q \equiv 0$ and $\rho \equiv 1$, we find $\nu_0 = 0$ and $\mu_n = \nu_n = n^p$ for $n \ge 1$.

Recently, the eigenvalue gap/ratio are concerned. We say a function f is single-well with transition point a if f is decreasing on $(0, a)$ and increasing on (a, π_p); f is single-barrier if $-f$ is single-well. In 2010, Bognár and Dosly [6] used the Prüfer transformation derived by generalized sine function to show that the Dirichlet eigenvalues for (1.1) with $\rho \equiv 1$ and nonnegative single-well $q(x)$ satisfy $\mu_n/\mu_m \le n^p/m^p$. Furthermore, Chen, Law, Lian and Wang [8] also used the generalized Prüfer transformation to show that $\mu_n/\mu_1 \le n^p$ for (1.1) with $\rho \equiv 1$ and nonnegative continuous $q(x)$. On the other hand, the authors in [9] studied the first two Dirichlet eigenvalues for (1.1) and showed that (i) $\mu_2 - \mu_1 \ge 2^p - 1$ if $\rho \equiv 1$ and $q(x)$ is single-well with transition point at $\pi_p/2$; (ii) $\mu_2/\mu_1 \ge 2^p$ if $q(x) \equiv 0$ and $\rho(x)$ is single-barrier with transition point at $\pi_p/2$.

In this article, we study the gap between the Dirichlet eigenvalues and Neumann eigenvalues. In [24, Theorem 2.5], Shen considered the spectra $\sigma_D = \{\mu_1, \mu_2, \dots\}$, $\sigma_{DN} = \{\tau_1, \tau_2, \dots\}$, $\sigma_{ND} = \{\gamma_1, \gamma_2, \dots\}$, and $\sigma_N = \{\nu_0, \nu_1, \nu_2, \dots\}$ for the following string equations

$$y''(x) + \mu\rho(x)y(x) = 0, \quad y(0) = y(\pi) = 0,$$
$$u''(x) + \tau\rho(x)u(x) = 0, \quad u(0) = u'(\pi) = 0,$$

$$z''(x) + \gamma\rho(x)z(x) = 0, \quad z'(0) = z(\pi) = 0,$$
$$v''(x) + \nu\rho(x)v(x) = 0, \quad v'(0) = v'(\pi) = 0,$$

respectively, where ρ is a positive piecewisely continuous function defined on $[0, \pi]$. He showed that if $\sigma_{DN} = \sigma_{ND}$ and $\sigma_N = \sigma_D \cup \{0\}$, then $\rho(x)$ is a constant function at its points of continuity.

Consider (1.1) and assume q and ρ satisfy (i) $\rho \equiv 1$ and q is single-well with transition point $a = \pi_p/2$, or (ii) $q \equiv 0$ and ρ is single-barrier with transition point $a = \pi_p/2$. In this paper, we show that $\mu_1 = \nu_1$ if and only if (i) q is constant, or (ii) ρ is constant, respectively. Our results extend and improve the result of Shen [24, Theorem 2.5] by using finitely many eigenvalues and by generalizing the string equation to p-Laplacian eigenvalue problem.

Theorem 1.1. *Consider* (1.1) *with* $q(x) \in L^1(0, \pi_p)$ *and* $\rho \equiv 1$. *If* $q(x)$ *is single-well on* $(0, \pi_p)$ *with transition point* $a = \pi_p/2$, *then* $\nu_1 \geq \mu_1$. *Equality holds if and only if* q *is constant. If* $a \neq \pi_p/2$, *then there exist some functions* q *giving* $\nu_1 < \mu_1$.

Theorem 1.2. *Consider* (1.1) *with a positive piecewisely continuous function* ρ *and* $q \equiv 0$. *If* $\rho(x)$ *is single-barrier on* $(0, \pi_p)$ *with transition point* $a = \pi_p/2$, *then* $\nu_1 \geq \mu_1$. *Equality holds if and only if* ρ *is constant. If* $a \neq \pi_p/2$, *then there exist some functions* ρ *giving* $\nu_1 < \mu_1$.

The proof of Theorem 1.1 follows the method developed by Horváth [13]. We first perturb the extremal function q and study the identity for $\frac{d}{dt}(\nu_1(t) - \mu_1(t))$ where t is a parameter. We will show that the optimal function q is a step function with a jump at $\pi_p/2$ and then compel it to be constant. Furthermore, by the principle of duality, the same method also works for (1.1) with $q \equiv 0$ and single-barrier ρ.

We shall remark that Theorems 1.1 and 1.2 can be used to solve inverse problems of the instability interval for $p = 2$:

$$-y'' = (\lambda\rho(x) - q(x))y\,. \tag{1.2}$$

Denote by $\{\lambda_n\}_{n\geq 0}$ and $\{\lambda'_n\}_{n\geq 1}$ the eigenvalues of (1.2) with $q(x) = q(x + \pi), \rho(x) = \rho(x + \pi)$ under the periodic ($y(0) = y(\pi)$, $y'(0) = y'(\pi)$), and anti-periodic ($y(0) = -y(\pi)$, $y'(0) = -y'(\pi)$) boundary conditions, respectively. It is known [10] (see also [7, 20]) that $\nu_0 \leq \lambda_0$ and

$$\begin{gathered} \dots \leq \lambda_{2n-2} < \lambda'_{2n-1} \leq \nu_{2n-1}, \\ \mu_{2n-1} \leq \lambda'_{2n} < \lambda_{2n-1} \leq \nu_{2n}, \\ \mu_{2n} \leq \lambda_{2n} < \lambda'_{2n+1} \leq \dots \end{gathered} \tag{1.3}$$

The intervals $(\lambda'_{2n-1}, \lambda'_{2n})$ and $(\lambda_{2n-1}, \lambda_{2n})$ are called the $(2n-1)$-th and $2n$-th instability intervals. The interval $(-\infty, \lambda_0)$ is called the zero-th instability interval.

In 1946, Borg [7] studied an inverse problem for Hill's equation. He showed that the potential $q(x)$ is constant if and only if all instability intervals, except the zero-th, are absent. Later, Hochstadt [12] generalized Borg's result and showed that if q is C^1, then q has period $1/n$ if and only if all those finite instability intervals whose index is not a multiple of n vanish. In 1997, Huang [14] proved that if q is symmetric single-well (or symmetric single-barrier), then q is constant if and only if the first instability interval is absent, i.e. $\lambda'_1 = \lambda'_2$. Thus, for all instability intervals, the first instability gives the most information about the potential q. Using Theorems 1.1 and 1.2, and (1.3), we may eliminate the assumption on the symmetric of q and obtain the following results immediately.

Corollary 1.3. *Consider* (1.2) *with* π*-periodic functions* ρ *and* q*. Then the first instability interval is absent if and only if one of the following conditions holds:*

- (i) $\rho \equiv 1$ *and* q *is single-well with transition point* $a = \pi/2$*.*
- (ii) $q \equiv 0$ *and* ρ *is single-barrier with transition point* $a = \pi/2$*.*

The paper is organized as follows. In section 2, we use a modified Prüfer substitution and comparison theorem to derive properties of eigenfunctions. In section 3, we study two generalized trigonometric equations. The Dirichlet and Neumann eigenvalues are corresponding to the roots of two generalized trigonometric equations, respectively. Finally, in section 4, we give proofs of our main theorems 1.1 and 1.2.

2. Preliminaries

At the beginning of this section, we give two formulas of generalized trigonometric functions. The proof is similar to the classical trigonometric functions, so we omit it here.

Lemma 2.1. *Define the generalized tangent function by* $T_p(x) \equiv S_p(x)/S_p'(x)$ *for* $x \neq (k+1/2)\pi_p$ *and the generalized reciprocal tangent function by* $RT_p(x) \equiv S_p'(x)/S_p(x)$ *for* $x \neq k\pi_p$*. Then we have*

- (i) $T_p'(x) = 1 + |T_p(x)|^p$.
- (ii) $RT_p'(x) = -(RT_p(x))^2(1 + |T_p(x)|^p)$.

Denote by $(\mu_i, \phi_i)_{i\geq 1}$ the normalized Dirichlet eigenpair and $(\nu_i, \psi_i)_{i\geq 0}$ the normalized Neumann eigenpair of (1.1) with $\phi_i(x) > 0$, $\psi_i(x) > 0$ for x near 0^+. The normalized condition means $\int_0^{\pi_p} \rho(x)|\phi_i(x)|^p dx = \int_0^{\pi_p} \rho(x)|\psi_i(x)|^p dx = 1$ for all i.

Definition 2.2. *def1* Let f and g be continuous functions and $g(x) \neq 0$. Define $h(x) \equiv f(x)/g(x)$. We say α_0 is a crossing point of f and g if $h(\alpha_0) = 1$ and h satisfies one of the following conditions

- (i) $h(\alpha_0^+) > 1$ and $h(\alpha_0^-) < 1$.
- (ii) $h(\alpha_0^+) < 1$ and $h(\alpha_0^-) > 1$.

Lemma 2.3. *There are exactly two crossing points of* $|\phi_1(x)|$ *and* $|\psi_1(x)|$ *in* $(0, \pi_p)$*.*

Proof. First, we introduce a generalized Prüfer substitution derived by S_p and S_p':

$$\phi_1(x) = r(x)S_p(\theta_D(x)), \quad \phi_1'(x) = r(x)S_p'(\theta_D(x)),$$
$$\psi_1(x) = R(x)S_p(\theta_N(x)), \quad \psi_1'(x) = R(x)S_p'(\theta_N(x)),$$

where $\theta_D(0) = 0$ and $\theta_N(0) = \pi_p/2$. Here, $\theta_D(x)$ and $\theta_N(x)$ are called the Prüfer angles of $\phi_1(x)$ and $\psi_1(x)$, respectively. By direct calculation, we find that

$$\theta_D'(x) = |S_p'(\theta_D(x))|^p + (\mu_1\rho(x) - q(x))|S_p(\theta_D(x))|^p, \tag{2.1}$$
$$\theta_N'(x) = |S_p'(\theta_N(x))|^p + (\nu_1\rho(x) - q(x))|S_p(\theta_N(x))|^p. \tag{2.2}$$

Let x_0 be the unique zero of $\psi_1(x)$ in $(0, \pi_p)$. Since $\phi_1(x) > 0$ on $(0, x_0)$, $0 = \phi_1(0) < \psi_1(0)$ and $\phi_1(x_0) > \psi_1(x_0) = 0$, we find the number of the crossing points of $\phi_1(x)$ and $\psi_1(x)$ in $(0, x_0)$ must be odd. Assume $0 < x_1 < x_2 < x_3 < x_0$ are crossing points of $\phi_1(x)$ and $\psi_1(x)$. Define $v(x) \equiv \frac{\psi_1(x)}{\phi_1(x)}$. Then $v(x_i) = 1$

for $i = 1, 2, 3$. By Rolle's Theorem, there are $z_i \in (x_i, x_{i+1})$, $i = 1, 2$, such that $v'(z_i) = 0$. Note that

$$v'(x) = \frac{\psi_1(x)\phi_1(x)}{\psi_1^2(x)}[\frac{\phi_1'(x)}{\phi_1(x)} - \frac{\psi_1'(x)}{\psi_1(x)}] = \frac{\phi_1(x)}{\psi_1(x)}[RT_p(\theta_D(x)) - RT_p(\theta_N(x))].$$

Hence, we find $\theta_D(z_i) = \theta_N(z_i)$ for $i = 1, 2$. By applying Comparison theorem [5] on (2.1) and (2.2), we obtain $\mu_1 = \nu_1$. This implies that $\theta_D(x) = \theta_N(x)$ for all $x \in (0, x_0)$. But this is a contradiction to $\theta_D(0) = 0$ and $\theta_N(0) = \pi_p/2$. Hence there is exactly one crossing point of $\phi_1(x)$ and $\psi_1(x)$ in $(0, x_0)$.

Similarly, there is also exactly one crossing point of $|\phi_1(x)|$ and $|\psi_1(x)|$ in (x_0, π_p). The proof is complete. □

According to Lemma 2.1, we denote the points $0 < x_- < x_0 < x_+ < \pi_p$ such that $\psi_1(x_0) = 0$ and

$$|\psi_1(x)| - |\phi_1(x)|\{\begin{cases} \geq 0 & \text{on } (0, x_-) \cup (x_+, \pi_p), \\ \leq 0 & \text{on } (x_-, x_+). \end{cases} \tag{2.3}$$

The following lemma is a p-version formula while the similar formulas were derived in [17] and [15] for the Schrödinger equation and string equation, respectively. The argument is similar so we omit here.

Lemma 2.4. *Consider* (1.1) *coupled with Dirichlet or Neumann boundary conditions on* $(0, \pi_p)$. *Let* $q(\cdot, t)$ *be a one-parameter family of continuous functions and* $\rho(\cdot, t)$ *be a one-parameter family of continuous functions such that* $\frac{\partial q}{\partial t}(x, t)$ *and* $\frac{\partial \rho}{\partial t}(x, t)$ *exist. Then*

$$\frac{d}{dt}\lambda(t) = -\lambda(t)\int_0^{\pi_p} \frac{\partial \rho}{\partial t}(x, t)|y(x, t)|^p dx + \int_0^{\pi_p} \frac{\partial q}{\partial t}(x, t)|y(x, t)|^p dx\,. \tag{2.4}$$

The following lemma will be used in the proofs of Theorems 1.1 and 1.2. This lemma makes those proofs simpler.

Lemma 2.5. *Consider* (1.1). *If* $q(x)$ *is increasing and* $\rho(x)$ *satisfies* $\rho(x) \geq \rho(\pi_p - x)$ *for* $x \in (0, \pi_p/2)$, *then* $x_0 \leq \pi_p/2$.

Proof. Denote $Q_1(x) \equiv (p-1)(\nu_1\rho(x) - q(x))$, $Q_2(x) \equiv Q_1(\pi_p - x)$, $z_1(x) \equiv \psi_1(x)$, and $z_2(x) \equiv -\psi_1(\pi_p - x)$. Then $Q_2(x) \leq Q_1(x)$ on $(0, \min\{x_0, \pi_p - x_0\}]$ and we have the following two problems

$$\begin{gathered} (|z_1'|^{p-2}z_1')' + Q_1(x)|z_1|^{p-2}z_1 = 0 \quad \text{on } [0, x_0]\,, \\ z_1'(0) = 0,\ z_1(x_0) = 0\,, \end{gathered}$$

and

$$\begin{gathered} (|z_2'|^{p-2}z_2)' + Q_2(x)|z_2|^{p-2}z_2 = 0 \quad \text{on } [0, \pi_p - x_0]\,, \\ z_2'(0) = 0,\ z_2(\pi_p - x_0) = 0\,. \end{gathered}$$

Let $\theta_1(x)$ and $\theta_2(x)$ be the Prüfer angles of $z_1(x)$ and $z_2(x)$ respectively. Then $\theta_1(x)$ and $\theta_2(x)$ satisfy

$$\begin{gathered} \theta_1'(x) = |S_p'(\theta_1(x))|^p + Q_1(x)|S_p(\theta_1(x))|^p \quad \text{on } [0, x_0]\,, \\ \theta_2'(x) = |S_p'(\theta_2(x))|^p + Q_2(x)|S_p(\theta_2(x))|^p \quad \text{on } [0, \pi_p - x_0]\,, \\ \theta_1(0) = \theta_2(0) = \frac{\pi_p}{2}\,, \end{gathered}$$

$$\theta_1(x_0) = \theta_2(\pi_p - x_0) = \pi_p\,.$$

By comparison theorem, we find $\pi_p - x_0 \geq x_0$ and hence $x_0 \leq \pi_p/2$. □

3. Two generalized triangular equations

In this section, we will study the order of the roots of two generalized triangular equations which are obtained from the proofs of Theorems 1.1 and 1.2 in section 4. Define

$$f(t) = t^{1/p} RT_p(t^{1/p}\frac{\pi_p}{2}), \quad g(t) = t^{1/p} RT_p(t^{1/p}\frac{\pi_p}{2} + \frac{\pi_p}{2}).$$

We have the following results.

Lemma 3.1. *Let $m > 0$. Let t_1 be the first root of $f(t) = -f(t-m)$ and t_2 be the second root of $g(t) = -g(t-m)$. Then $t_2 > t_1$.*

Proof. First, note that $t_1 \in (1, \min\{1+m, 2^p\})$ and $t_2 \in (1, 3^p)$ for $m > 0$.

(i) Assume $t \geq 0$. Then, by Lemma 2.1, we find

$$\begin{aligned}
g'(t) &= \frac{1}{p} t^{\frac{1-p}{p}} RT_p(t^{1/p}\frac{\pi_p}{2} + \frac{\pi_p}{2}) - t^{1/p}\big(1 + |T_p(t^{1/p}\frac{\pi_p}{2} + \frac{\pi_p}{2})|^p\big) \\
&\quad \times RT_p^2(t^{1/p}\frac{\pi_p}{2} + \frac{\pi_p}{2}) \cdot \frac{1}{p} t^{\frac{1-p}{p}} \frac{\pi_p}{2} \\
&= \frac{t^{\frac{1-p}{p}}}{2p|S_p(t^{1/p}\frac{\pi_p}{2} + \frac{\pi_p}{2})|^2} \Big\{ 2S_p(t^{1/p}\frac{\pi_p}{2} + \frac{\pi_p}{2}) S_p'(t^{1/p}\frac{\pi_p}{2} + \frac{\pi_p}{2}) \\
&\quad - t^{1/p}\pi_p |S_p'(t^{1/p}\frac{\pi_p}{2} + \frac{\pi_p}{2})|^{2-p} \Big\} \\
&\equiv \frac{t^{\frac{1-p}{p}}}{2p|S_p(t^{1/p}\frac{\pi_p}{2} + \frac{\pi_p}{2})|^2} \tilde{g}(t).
\end{aligned}$$

If $S_p'(t^{1/p}\frac{\pi_p}{2} + \frac{\pi_p}{2}) > 0$, in this case $t^{1/p} \in (2+4n, 4+4n)$ for $n \geq 0$, then

$$\begin{aligned}
\tilde{g}(t) &= S_p'(t^{1/p}\frac{\pi_p}{2} + \frac{\pi_p}{2})[2S_p(t^{1/p}\frac{\pi_p}{2} + \frac{\pi_p}{2}) - t^{1/p}\pi_p |S_p'(t^{1/p}\frac{\pi_p}{2} + \frac{\pi_p}{2})|^{1-p}] \\
&\leq S_p'(t^{1/p}\frac{\pi_p}{2})[2S_p(t^{1/p}\frac{\pi_p}{2}) - t^{1/p}\pi_p] \\
&\equiv S_p'(t^{1/p}\frac{\pi_p}{2}) h(t).
\end{aligned}$$

Since $h((2+4n)^p) < 0$ for $n \geq 0$, and $h'(t) = \frac{t^{\frac{1-p}{p}}\pi_p}{p}(S_p'(t^{1/p}\frac{\pi_p}{2} + \frac{\pi_p}{2}) - 1) < 0$ for $t^{1/p} \in (2+4n, 4+4n)$ and $n \geq 1$, we have $h(t) < 0$ for $t^{1/p} \in (2+4n, 4+4n)$ and $n \geq 0$. Hence $g'(t) < 0$ for $t^{1/p} \in (2+4n, 3+4n) \cup (3+4n, 4+4n)$ and $n \geq 0$.

Similarly, if $S_p'(t^{1/p}\frac{\pi_p}{2} + \frac{\pi_p}{2}) < 0$, in this case $t^{1/p} \in (4n, 4n+2)$ for $n \geq 0$, then

$$\begin{aligned}
\tilde{g}(t) &= S_p'(t^{1/p}\frac{\pi_p}{2} + \frac{\pi_p}{2})[2S_p(t^{1/p}\frac{\pi_p}{2} + \frac{\pi_p}{2}) + t^{1/p}\pi_p |S_p'(t^{1/p}\frac{\pi_p}{2} + \frac{\pi_p}{2})|^{1-p}] \\
&\leq S_p'(t^{1/p}\frac{\pi_p}{2} + \frac{\pi_p}{2})[2S_p(t^{1/p}\frac{\pi_p}{2} + \frac{\pi_p}{2}) + t^{1/p}\pi_p] \\
&\equiv S_p'(t^{1/p}\frac{\pi_p}{2} + \frac{\pi_p}{2}) \tilde{h}(t)\,.
\end{aligned}$$

Since $\tilde{h}((4n)^p) > 0$ for $n \geq 0$ and $\tilde{h}'(t) = \frac{t^{\frac{1-p}{p}}\pi_p}{p}(S_p'(t^{1/p}\frac{\pi_p}{2} + \frac{\pi_p}{2}) + 1) > 0$ for $t^{1/p} \in (4n, 4n+2)$ and $n \geq 0$, we have $\tilde{h}(t) > 0$ for $t^{1/p} \in (4n, 4n+2)$ and $n \geq 0$ and hence $g'(t) < 0$ for $t^{1/p} \in (4n, 4n+1) \cup (4n+1, 4n+2)$ and $n \geq 0$.

(ii) Assume $t < 0$. Let $\hat{t} = -t$ and $\tilde{t} = \hat{t}^{1/p}\frac{\pi_p}{2} + (-1)^{-1/p}\frac{\pi_p}{2}$. Since

$$g(t) = t^{1/p} RT_p(t^{1/p}\frac{\pi_p}{2} + \frac{\pi_p}{2}) = (-1)^{1/p}\hat{t}^{1/p}\frac{S_p'((-1)^{1/p}\tilde{t})}{S_p((-1)^{1/p}\tilde{t})} = \hat{t}^{1/p}\frac{Sh_p'(\tilde{t})}{Sh_p(\tilde{t})}\,,$$

we have

$$\begin{aligned} g'(t) &= -\frac{1}{p}\hat{t}^{\frac{1-p}{p}}\frac{Sh_p'(\tilde{t})}{Sh_p(\tilde{t})} + \hat{t}^{1/p}(-\frac{1}{p}\hat{t}^{\frac{1-p}{p}})\frac{\pi_p}{2}\Big(\frac{Sh_p''(\tilde{t})}{Sh_p(\tilde{t})} - \frac{Sh_p'^2(\tilde{t})}{Sh_p^2(\tilde{t})}\Big) \\ &= \frac{-\frac{1}{p}\hat{t}^{\frac{1-p}{p}}}{Sh_p^2(\tilde{t})}\Big[Sh_p'(\tilde{t})Sh_p(\tilde{t}) + \frac{\pi_p}{2}\hat{t}^{1/p}\Big(\frac{|Sh_p(\tilde{t})|^p}{Sh_p'^{p-2}(\tilde{t})} - Sh_p'^2(\tilde{t})\Big)\Big] \\ &= \frac{-\frac{1}{p}\hat{t}^{\frac{1-p}{p}}}{Sh_p^2(\tilde{t})}[Sh_p'(\tilde{t})Sh_p(\tilde{t}) - \frac{\pi_p}{2}\hat{t}^{1/p}Sh_p'^{2-p}(\tilde{t})] \\ &\equiv \frac{-\frac{1}{p}\hat{t}^{\frac{1-p}{p}}}{Sh_p^2(\tilde{t})}\hat{g}(t)\,. \end{aligned}$$

Using similar argument as step (i), we can show $\hat{g}(t) > 0$ and hence $g'(t) < 0$ for all $t < 0$.

(iii) Let $t = t(m)$. If $g(t) = -g(t-m)$, then $g'(t)\frac{dt}{dm} = -g'(t-m)(\frac{dt}{dm} - 1)$ and hence

$$0 < \frac{dt}{dm} = \frac{g'(t-m)}{g'(t) + g'(t-m)} < 1\,.$$

This implies $t_2(m)$ is strictly increasing for $m > 0$. On the other hand, when $m = 2^p$, we have $t_2 = 2^p$ and

$$t_2 - t_1 > 0 \quad \text{for } m \geq 2^p\,.$$

Theqrefore, we only need to consider $0 < m < 2^p$. In this case, $t_1 \in (1, \min\{1 + m, 2^p\})$ and $t_2 \in (\max\{1, m\}, \min\{2^p, 1 + m\})$.

(iv) Assume $t_1 \geq t_2$ for some $0 < m < 2^p$. By similar arguments as steps (i) and (ii), it can be shown that $f(t)$ is decreasing on $(-\infty, 2^p)$ and $((2n)^p, (2n+2)^p)$ for $n \geq 1$, and $f(1) = 0$. Then

$$-f(t_2 - m) \leq -f(t_1 - m) = f(t_1) \leq f(t_2) \leq f(t_2 - m)\,.$$

This implies $f(t_2 - m) = 0$ and then $t_2 - m = 1$. But $t_2 < 1 + m$. Hence $t_1 < t_2$ for $m > 0$. □

Lemma 3.2. *Let $m > 1$. Let s_1 be the first root of $f(s) = -f(sm)$ and s_2 be the second root of $g(s) = -g(sm)$. Then $s_2 > s_1$.*

Proof. Note that $s_1, s_2 \in (\frac{1}{m}, \min\{1, \frac{2^p}{m}\})$. If $s_1 \geq s_2$ for some $m > 1$, then $\frac{1}{m} \leq s_2 < s_2 m < 2^p$ and

$$f(s_2) \geq f(s_1) = -f(s_1 m) \geq -f(s_2 m) > -f(s_2)\,.$$

This implies $s_2 = 1$. Hence $s_1 \leq s_2$ for $m > 1$. □

4. Proof of Main Theorems

Proof of Theorem 1.1. For $M > 0$, denote

$$A_M = \{0 \le q(x) \le M : q \text{ is single-well with transition point } \frac{\pi_p}{2}\}.$$

Then A_M is closed and $E(q) \equiv (\nu_1 - \mu_1)(q)$ is bounded on A_M. Hence there exists an optimal function q_0 giving the minimal eigenvalue gap $\nu_1 - \mu_1$.

Recall the definitions of x_- and x_+ in (2.3). We shall define $q(x,t) = (1-t)q_0(x) + tq_1(x)$ for $t \in [0, \pi_p]$ for some appropriated function q_1.

First, assume $x_- \le \pi_p/2 \le x_+$. Let

$$q_1(x) = \begin{cases} q_0(x_-) & \text{on } (0, \frac{\pi_p}{2}), \\ q_0(x_+) & \text{on } (\frac{\pi_p}{2}, \pi_p). \end{cases}$$

By the optimality of q_0 and Lemma 2.4, we have

$$\begin{aligned} 0 \le\ & fracddt(\nu_1(t) - \mu_1(t))|_{t=0} \\ &= \int_0^{\pi_p} (q_1(x) - q_0(x))(|\psi_1(x,0)|^p - |\phi_1(x,0)|^p) dt\,, \end{aligned}$$

which is nonpositive. Hence, $q_0(x) = q_1(x)$ a.e. on $[0, \pi_p]$.

If $x_- > \pi_p/2$, we let

$$q_1(x) = \{\begin{cases} 0 & \text{on } (0, x_-), \\ M & \text{on } (x_-, \pi_p). \end{cases}$$

By the normality of ϕ_1 and ψ_1, we have

$$\int_0^{x_-} (|\psi_1(x,0)|^p - |\phi_1(x,0)|^p) dx > 0 > \int_{x_-}^{\pi_p} (|\psi_1(x,0)|^p - |\phi_1(x,0)|^p) dx.$$

Hence, by the optimality of q_0, we have

$$\begin{aligned} 0 &\le \frac{d}{dt}(\nu_1(t) - \mu_1(t))|_{t=0} \\ &= \int_0^{\pi_p} (q_1(x) - q_0(x))(|\psi_1(x,0)|^p - |\phi_1(x,0)|^p) dx \\ &= \int_0^{x_-} (-q_0(x))(|\psi_1(x,0)|^p - |\phi_1(x,0)|^p) dx \\ &\quad + \int_{x_-}^{\pi_p} (M - q_0(x))(|\psi_1(x,0)|^p - |\phi_1(x,0)|^p) dx \\ &\le -q_0(\frac{\pi_p}{2}) \int_0^{x_-} (|\psi_1(x,0)|^p - |\phi_1(x,0)|^p) dt \\ &\quad + (M - q_0(x_+)) \int_{x_-}^{\pi_p} (|\psi_1(x,0)|^p - |\phi_1(x,0)|^p) dt\,, \end{aligned}$$

which is non-positive. This implies that $q_0 = 0$ on $(0, x_-)$ and $= M$ on (x_-, π_p). But this makes a contradiction to Lemma 2.5. Hence this case is refuted. The case $x_+ \le \pi_p/2$ is similar.

After simplification, the optimal function q_0 is a 1-step function. Without loss of generality, let

$$q_0(x) = \begin{cases} 0 & \text{on } (0, \frac{\pi_p}{2}), \\ m & \text{on } (\frac{\pi_p}{2}, \pi_p). \end{cases}$$

By equating the corresponding ratio by y'/y at $\pi_p/2$, ν_1 is the second root of the functional equation $\lambda^{1/p} RT_p(\frac{\pi_p}{2}\lambda^{1/p} + \frac{\pi_p}{2}) = -(\lambda - m)^{1/p} RT_p(\frac{\pi_p}{2}(\lambda - m)^{1/p} + \frac{\pi_p}{2})$, and, similarly, μ_1 is the first root of $\lambda^{1/p} RT_p(\frac{\pi_p}{2}\lambda^{1/p}) = -(\lambda - m)^{1/p} RT_p(\frac{\pi_p}{2}(\lambda - m)^{1/p})$. Using Lemma 3.1, we obtain $\nu_1 - \mu_1 > 0$.

Finally, if the transition point a is not $\pi_p/2$, we let

$$q(x,t) = \begin{cases} t & \text{on } [0, a], \\ 0 & \text{on } [a, \pi_p]. \end{cases}$$

Since $\phi_1(x,0) = (p/\pi_p)^{1/p} S_p(x)$, $\psi_1(x,0) = (p/\pi_p)^{1/p} S_p(x + \pi_p/2)$, and

$$\int_0^{\pi_p/2} (|\psi_1(x,0)|^p - |\phi_1(x,0)|^p) dx = 0,$$

we have

$$\frac{d}{dt}(\nu_1(t) - \mu_1(t))|_{t=0} = \int_0^a (|\psi_1(x,0)|^p - |\phi_1(x,0)|^p) dx < 0\,,$$

when $0 < a - \frac{\pi_p}{2} << 1$. Hence for small $t > 0$, $\nu_1(t) - \mu_1(t) < 0$ when $0 < a - \pi_p/2 << 1$. □

Proof of Theorem 1.2. For $M > 1$, denote

$$A_M = \{\frac{1}{M} \le \rho(x) \le M : \rho \text{ is single-barrier with transition point } \frac{\pi_p}{2}\}.$$

Then there exists an optimal function ρ_0 giving the minimal eigenvalue ratio ν_1/μ_1.

Similar to the proof of Theorem 1.1 and by Lemma 2.5, the cases $x_+ < \pi_p/2$ and $x_- > \pi_p/2$ are refuted by using suitable ρ_0's. Hence we have $x_- \le \pi_p/2 \le x_+$ and

$$\rho_0(x) = \begin{cases} \rho_0(x_-) & \text{on } (0, \frac{\pi_p}{2}), \\ \rho_0(x_+) & \text{on } (\frac{\pi_p}{2}, \pi_p). \end{cases}$$

That is the optimal function ρ_0 is a 1-step function. Without loss of generality, let

$$\rho_0(x) = \begin{cases} 1 & \text{on } (0, \frac{\pi_p}{2}), \\ m & \text{on } (\frac{\pi_p}{2}, \pi_p), \end{cases}$$

for some $m > 1$. Then ν_1 is the second root of

$$RT_p(\frac{\pi_p}{2}\lambda^{1/p} + \frac{\pi_p}{2}) = -m^{1/p} RT_p(\frac{\pi_p}{2}(m\lambda)^{1/p} + \frac{\pi_p}{2}),$$

and μ_1 is the first root of

$$RT_p(\frac{\pi_p}{2}\lambda^{1/p}) = -m^{1/p} RT_p(\frac{\pi_p}{2}(m\lambda)^{1/p}).$$

Hence, by Lemma 3.2, $\nu_1/\mu_1 > 1$.

Finally, we let

$$\rho(x,t) = \begin{cases} t & \text{on } [0, a], \\ 1 & \text{on } [a, \pi_p]. \end{cases}$$

Then it can be shown that, if $0 < \pi_p/2 - a << 1$, the function $\rho(x,t)$ gives $\nu_1/\mu_1 < 1$ for small $t > 1$. □

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