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A BOUNDARY PROBLEM WITH INTEGRAL GLUING CONDITION FOR A PARABOLIC-HYPERBOLIC EQUATION INVOLVING THE CAPUTO FRACTIONAL DERIVATIVE

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ABSTRACT. In the present work we investigate the Tricomi problem with an integral gluing condition for a parabolic-hyperbolic equation involving the Caputo fractional differential operator. Using the method of energy integrals, we prove the uniqueness of the solution for the considered problem. The existence of the solution have been proved applying methods of ordinary differential equations and Fredholm integral equations. The solution is represented in an explicit form.

1. FORMULATION OF THE PROBLEM

Fractional analogs of main ODEs and PDEs one motivated by their appearance in real-life processes [1, 8]. They are as well interesting for mathematicians as natural generalizations of integer order ODEs and PDEs. Specialists in the theory of boundary problems for PDEs began to develop it in this direction. There are many works [7, 9, 11, 12] devoted to the investigation of various boundary problems for PDEs.

A distinctive side of this work is the usage of gluing condition of the integral form, containing regular continuous gluing condition as a particular case. We note that for the first time boundary problem with integral gluing condition for a parabolic-hyperbolic type equation was used in the work [5]. Then some generalizations of this work were published in [4, 6]. Special gluing condition of the integral form for parabolic-hyperbolic equation with the Riemann-Liouville fractional differential operator was discussed in [2].

In the present work we use an integral gluing condition with a kernel, which has a more general form than the kernel used in [3]. The uniqueness of the solution requires restrictions on the kernel (see Theorem 1), however, the existence of the solution does not need the conditions required for uniqueness (see Theorem 2).

Consider the equation

$$0 = \begin{cases} u_{xx} - {}_C D_{0y}^{\alpha} u, & y > 0, \\ u_{xx} - u_{yy}, & y < 0 \end{cases} \tag{1.1}$$

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in the domain $\Omega = \Omega^+ \cup \Omega^- \cup AB$, where $0 < \alpha < 1$, $AB = \{(x, y) : 0 < x < 1,\ y = 0\}$, $\Omega^+ = \{(x, y) : 0 < x < 1,\ 0 < y < 1\}$, $\Omega^- = \{(x, y) : -y < x < y + 1,\ -1/2 < y < 0\}$,

$$ {}_C D^{\alpha}_{0y} f = \frac{1}{\Gamma(1-\alpha)} \int_0^y (y-t)^{-\alpha} f'(t)dt $$

is the Caputo fractional differential operator of order α $(0 < \alpha < 1)$, $\Gamma(\cdot)$ is the Euler's gamma-function [10].

Problem. Find a solution of the equation (1.1) belonging to

$$ W = \{u(x, y) : u \in C(\overline{\Omega}) \cap C^2(\Omega^-),\ u_{xx} \in C(\Omega^+),\ {}_C D^{\alpha}_{0y} u \in C(\Omega^+)\} $$

satisfying the boundary conditions

$$ u(0, y) = \varphi_1(y), \quad 0 \le y \le 1, \tag{1.2} $$

$$ u(1, y) = \varphi_2(y), \quad 0 \le y \le 1, \tag{1.3} $$

$$ u(x, -x) = \psi(x), \quad 0 \le x \le 1/2, \tag{1.4} $$

and the gluing condition

$$ \lim_{y\to+0} y^{1-\alpha} u_y(x, y) = \gamma_1 u_y(x, -0) + \gamma_2 \int_0^x u_y(t, -0)Q(x, t)dt, \quad 0 < x < 1. \tag{1.5} $$

Here φ_1, φ_2, ψ, $Q(\cdot, \cdot)$ are given functions, such that $\varphi_1(0) = \psi(0)$, γ_1, γ_2 are constants $\gamma_1^2 + \gamma_2^2 \neq 0$.

2. Uniqueness of the solution

Let us set

$$ u(x, +0) = \tau_1(x),\ 0 \le x \le 1,\ \ u(x, -0) = \tau_2(x),\ 0 \le x \le 1, $$
$$ u_y(x, -0) = \nu_2(x),\ 0 < x < 1,\ \lim_{y\to+0} y^{1-\alpha} u_y(x, y) = \nu_1(x), \quad 0 < x < 1, $$
$$ u_x(x, +0) = {\tau'}_1(x),\ 0 < x < 1, \quad u_x(x, -0) = {\tau'}_2(x),\ 0 < x < 1. $$

It is known that the solution of the Cauchy problem for (1.1) in Ω^- can be represented as

$$ u(x, y) = \frac{1}{2}\Big[\tau_1(x + y) + \tau_2(x - y) - \int_{x-y}^{x+y} \nu_2(t)dt\Big]. \tag{2.1} $$

Using condition (1.4) in (2.1), we find

$$ \tau_2'(x) - 2\psi'(x/2) = \nu_2(x), \quad 0 < x < 1. \tag{2.2} $$

From (1.1) as $y \to +0$ we obtain [13]

$$ \tau_1''(x) - \Gamma(\alpha)\nu_1(x) = 0. \tag{2.3} $$

Below we prove the uniqueness of the solution of the formulated problem. For this aim, first we suppose that the problem has two solutions, then denoting the difference of these solutions by u we will get an appropriate homogeneous problem. If we prove that this homogeneous problem has only the trivial solution, then we can state that the original problem has a unique solution.

We multiply equation (2.3) by $\tau_1(x)$ and integrate from 0 to 1:

$$ \int_0^1 \tau_1''(x)\tau_1(x)dx - \Gamma(\alpha) \int_0^1 \tau_1(x)\nu_1(x)dx = 0. \tag{2.4} $$

We investigate the integral $I=\int_0^1 \tau_1(x)\nu_1(x)dx$. Considering the gluing condition (1.5), we have

$$\nu_1(x)=\gamma_1\nu_2(x)+\gamma_2\int_0^x \nu_2(t)Q(x,t)dt, \quad 0<x<1. \tag{2.5}$$

In the homogeneous case; i.e., $\psi(x)=0$, from (2.2) we obtain $\nu_2(x)=\tau_2'(x)$, hence (2.5) will be written as

$$\nu_1(x)=\gamma_1\tau_2'(x)+\gamma_2\int_0^x \tau_2'(t)Q(x,t)dt, \quad 0<x<1. \tag{2.6}$$

We substitute this expression into the integral I and consider $\tau_1(0)=0$, $\tau_1(1)=0$ (which are deduced from conditions (1.2), (1.3) in the homogeneous case), we have

$$\begin{aligned} I&=\int_0^1 \tau_1(x)\nu_1(x)dx \\ &=\gamma_2\int_0^1 \tau_1^2(x)Q(x,x)dx-\gamma_2\int_0^1 \tau_1(x)dx\int_0^x \tau_1(t)\frac{\partial}{\partial t}Q(x,t)dt. \end{aligned} \tag{2.7}$$

Let $\frac{\partial}{\partial t}Q(x,t)=-Q_1(x)Q_1(t)$. Then

$$I=\gamma_2\int_0^1 \tau_1^2(x)Q(x,x)dx+\frac{\gamma_2\Phi^2(1)}{2}, \tag{2.8}$$

where

$$\Phi(x)=\int_0^x \tau_1(t)Q_1(t)dt, \quad Q(x,t)=Q(x,0)-\int_0^t Q_1(x)Q_1(z)dz.$$

From (2.4) and (2.8), we obtain

$$\int_0^1 {\tau'}_1^2(x)dx+\Gamma(\alpha)\gamma_2\Big[\int_0^1 \tau_1^2(x)Q(x,x)dx+\frac{\Phi^2(1)}{2}\Big]=0. \tag{2.9}$$

Since $\Gamma(\alpha)>0$ for $0<\alpha<1$, then if $\gamma_2\geq 0$, $Q(x,x)>0$ from (2.9) we easily get $\tau_1(x)=0$ for any $x\in[0,1]$.

Based on the solution of the first boundary problem for (1.1) in the domain Ω^+ we obtain $u(x,y)\equiv 0$ in $\overline{\Omega^+}$. Since $u(x,y)\in C(\overline{\Omega})$, we obtain that $u(x,y)\equiv 0$ in $\overline{\Omega}$. Hence, we proved the following result.

Theorem 2.1. *Let $\gamma_2\geq 0$, $\frac{\partial}{\partial t}Q(x,t)=-Q_1(x)Q_1(t)$ and $Q(x,x)>0$. If there exists a solution to problem, then it is unique.*

An example of a function satisfying the conditions of theorem is

$$Q(x,t)=e^{-x}(1+e^{-t}).$$

3. Existence of the solution

From (2.2), (2.3) and (2.5), we have

$${\tau''}_1(x)-A\tau_1(x)=F_1(x), \tag{3.1}$$

where $A=\Gamma(\alpha)\gamma_1$,

$$F_1(x)=\gamma_2\Gamma(\alpha)\int_0^x {\tau'}_1(t)Q(x,t)dt-\Gamma(\alpha)\Big[\gamma_1\psi(\frac{x}{2})+\gamma_2\int_0^x \psi'(\frac{t}{2})Q(x,t)dt\Big]. \tag{3.2}$$

The solution of the equation (3.1) together with the conditions

$$\tau_1(0)=\psi(0), \quad \tau_1(1)=\varphi_2(0) \tag{3.3}$$

has the form

$$\tau_1(x)=\frac{1}{1-e^A}[\varphi_2(0)(1-e^{Ax})+\psi(0)(e^{Ax}-e^A)]+\int_0^1 G_0(x,\xi)F_1(\xi)d\xi, \tag{3.4}$$

where

$$G_0(x,\xi)=\frac{1}{A[e^{Ax}-e^{A(x-1)}]}\begin{cases}(1-e^{A\xi})(1-e^{A(x-1)}), & 0\le\xi\le x,\\ (1-e^{A(\xi-1)})(1-e^{Ax}), & x\le\xi\le 1\end{cases} \tag{3.5}$$

is the Green's function of the problem (3.1), (3.3). Considering (3.2) and integrating by parts, we obtain

$$\tau_1(x)-\int_0^1 \tau_1(\xi)K(x,\xi)d\xi=F_2(x), \tag{3.6}$$

where

$$K(x,\xi)=\gamma_2\Gamma(\alpha)\Big[G_0(x,\xi)Q(\xi,\xi)+\int_\xi^1 G_0(\xi,t)\frac{\partial}{\partial\xi}Q(t,\xi)dt\Big], \tag{3.7}$$

$$\begin{aligned}F_2(x)&=\frac{1}{1-e^A}\big[\varphi_2(0)(1-e^{Ax})+\psi(0)(e^{Ax}-e^A)\big]\\ &\quad-\Gamma(\alpha)\int_0^1 G_0(x,\xi)\Big[\gamma_1\psi(\frac{\xi}{2})+\gamma_2\int_0^\xi \psi'(\frac{t}{2})Q(\xi,t)dt\Big]d\xi.\end{aligned} \tag{3.8}$$

Since the kernel $K(x,\xi)$ is continuous and $F_2(x)$ is continuously differentiable, the solution of integral equation (3.6) can be written via the resolvent-kernel:

$$\tau_1(x)=F_2(x)-\int_0^1 F_2(\xi)R(x,\xi)d\xi, \tag{3.9}$$

where $R(x,\xi)$ is the resolvent-kernel of $K(x,\xi)$.

The unknown functions $\nu_1(x)$ and $\nu_2(x)$ will be expressed as

$$\nu_1(x)=\frac{1}{\Gamma(\alpha)}\Big[F_2''(x)-\int_0^1 F_2(\xi)\frac{\partial^2}{\partial x^2}R(x,\xi)d\xi\Big],$$

$$\nu_2(x)=F_2'(x)-\int_0^1 F_2(\xi)\frac{\partial}{\partial x}R(x,\xi)d\xi-\psi'(\frac{x}{2}).$$

The solution of the problem in the domain Ω^+ can be written as

$$\begin{aligned}u(x,y)&=\int_0^y G_\xi(x,y,0,\eta)\varphi_1(\eta)d\eta-\int_0^y G_\xi(x,y,1,\eta)\varphi_2(\eta)d\eta\\ &\quad+\int_0^1 \overline{G}(x-\xi,y)\tau_1(\xi)d\xi,\end{aligned} \tag{3.10}$$

where

$$\overline{G}(x-\xi,y)=\frac{1}{\Gamma(1-\alpha)}\int_0^y \eta^{-\alpha}G(x,y,\xi,\eta)d\eta,$$

$$G(x,y,\xi,\eta)=\frac{(y-\eta)^{\beta-1}}{2}\sum_{n=-\infty}^{\infty}\big[e^{1,\beta}_{1,\beta}\big(-\frac{|x-\xi+2n|}{(y-\eta)^\beta}\big)-e^{1,\beta}_{1,\beta}\big(-\frac{|x+\xi+2n|}{(y-\eta)^\beta}\big)\big]$$

is the Green's function of the first boundary problem for (1.1) in the domain Ω^+ with the Riemann-Liouville fractional differential operator instead of the Caputo ones [13], $\beta = \alpha/2$,

$$e^{1,\beta}_{1,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\beta - \beta n)}$$

is the Wright type function [13].

The solution of the problem in the domain Ω^- will be found by formula (2.1). Hence, we proved the following result.

Theorem 3.1. *If $\varphi_i(y), \psi(x) \in C[0,1] \cap C^1(0,1)$, $Q(x,t) \in C^1([0,1] \times [0,1])$, then there exists a solution of the problem and it can be represented in the domain Ω^+ by formula* (3.10) *and in the domain Ω^- by the formula* (2.1).

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