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# MULTIPLICITY OF SOLUTIONS FOR ELLIPTIC BOUNDARY VALUE PROBLEMS

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Abstract. In this article, we study the existence of infinitely many solutions for the semilinear elliptic equation $-\Delta u+a(x)u = f(x,u)$ in a bounded domain of $\mathbb{R}^N$ $(N \geq 3)$ with the Dirichlet boundary conditions, where the primitive of the nonlinearity $f$ is either superquadratic at infinity or subquadratic at zero.

## 1. Introduction and main results

Consider the Dirichlet boundary-value problem

$$\begin{gathered} -\Delta u + a(x)u = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega$ is a bounded domain of $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, $a \in L^s(\Omega)$, $s > N/2$. We assume that $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ satisfies:

(F1) There exist $a_1 > 0$ and $p \in (2, 2^*)$ such that

$$|f(x,t)| \leq a_1(1 + |t|^{p-1}), \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where $2^* = 2N/(N-2)$.

The existence of infinitely many solutions of problem (1.1) was first proved in Ambrosetti and Rabinowitz [1] under the superquadratic condition

(AR) There exist $\mu > 2$ and $r > 0$ such that

$$0 < \mu F(x,t) \leq f(x,t)t, \quad \forall x \in \Omega,\ |t| \geq r,$$

where $F(x,t) := \int_0^t f(x,s)ds$ be the primitive of $f$.

Since then, this condition has appeared in most of the studies for superlinear problems, e.g., elliptic equations, Hamiltonian systems and wave equations, see [2, 4, 13, 15, 16, 20, 21] and references therein. Indeed, condition (AR) implies that there exists $C > 0$ such that $F(x,t) \geq C|t|^\mu$ for $|t| \geq 1$ and all $x \in \Omega$. A more natural superquadratic condition is that:

(F2) $F(x,t)/t^2 \to +\infty$ uniformly in $x$ as $|t| \to \infty$.

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Although the condition (AR) is quite natural and important not only to ensure the Euler-Lagrange functional $\varphi$ of problem (1.1) has a mountain pass geometry, but also to guarantee every Palais-Smale sequence of $\varphi$ is bounded, it is somewhat restrictive and eliminates many functionals. For example, the function

$$F(x,t) = t^2 \ln(1+t^2), \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

is superquadratic at infinity, but it does not satisfy condition (AR) for any $\mu > 2$.

For this reason, in recent years, some authors studied the superquadratic problem (1.1) trying to remove the (AR) condition, we refer the readers to [3, 5, 6, 8, 10, 11, 12, 14, 18, 19, 23, 24]. [3, 6, 8, 23] studied problem (1.1) replacing (AR), among other conditions, by

$$\lim_{|t|\to\infty} \frac{tf(x,t) - 2F(x,t)}{|t|^\mu} \ge c > 0 \quad \text{uniformly for a.e. } x \in \Omega,$$

where $\mu > 0$. In [18], to get an existence of nontrivial solution result, Schechter and Zou assumed

$$\text{either} \quad \lim_{t\to-\infty} \frac{F(x,t)}{t^2} = +\infty \quad \text{or} \quad \lim_{t\to+\infty} \frac{F(x,t)}{t^2} = +\infty$$

instead of (AR). While in [3, 5, 6, 8, 10, 12, 14], the authors adapted the monotonicity trick. In particular, under the strictly increasing assumption; i.e.,

(F3') $t \mapsto f(x,t)/|t|$ is strictly increasing on $(-\infty, 0)$ and on $(0, +\infty)$.

Szulkin and Weth [19] proved the following theorem.

**Theorem 1.1** ([19, Theorem 3.2])**.** *Suppose that* (F1), (F2), (F3') *are satisfied and*

(F4) $f(x,-t) = -f(x,t)$ *for all* $(x,t) \in \Omega \times \mathbb{R}$.
(F5) $f(x,t) = o(t)$ *uniformly in* $x$ *as* $t \to 0$.

*Then problem* (1.1)*, where* $a(x) \equiv \lambda$*, has infinitely many solutions.*

Zou [24] considered the global monotonicity condition, i.e.,

(F3") $t \mapsto f(x,t)/|t|$ is increasing on $(-\infty, 0)$ and on $(0, +\infty)$.

By using the special version of fountain theorem established there (see [24, Theorem 2.1]), he obtained the next theorem.

**Theorem 1.2** ([24, Theorem 3.2])**.** . *Suppose that* (F1), (F3"), (F4) *are satisfied and*

(F2') $\liminf_{|t|\to\infty} \frac{f(x,t)t}{|t|^\mu} \ge c > 0$ *uniformly for* $x \in \mathbb{R}^N$*, where* $\mu > 2$.

*Then problem* (1.1)*, where* $a(x) \equiv 0$*, has infinitely many solutions.*

In the present paper, base on an approach different to that of the results mentioned above, i.e., the classical Fountain Theorem of Bartsch, we can prove the same result of problem (1.1), where $a(x)$ does not necessarily equal to constant, under more general assumptions, unifying and improving Theorems 1.1 and 1.2.

**Theorem 1.3.** *Assume that* (F1), (F2), (F4) *hold and*

(F3) *There exist* $\theta \ge 1$ *and* $C^* \ge 0$ *such that*

$$\theta \mathcal{F}(x,t) \ge \mathcal{F}(x,st) - C^*, \quad \forall (x,t) \in \Omega \times \mathbb{R},\ s \in [0,1],$$

*where* $\mathcal{F}(x,t) = f(x,t)t - 2F(x,t)$.

*Then problem* (1.1) *possesses infinitely many solutions* $(u_k)$ *such that*

$$\frac{1}{2}\int_\Omega \big(|\nabla u_k|^2 + a(x)u_k^2\big)dx - \int_\Omega F(x,u_k)dx \to +\infty \quad \text{as } k \to \infty.$$

**Remark 1.4.** (i) Condition (F3) with $C^* = 0$ is originally due to Jeanjean [7] for a semilinear problem on $\mathbb{R}^N$. For $p$-Laplacian equations setting on a bounded domain, it was used in [10] to obtain infinitely many solutions and in [5] to compute the critical groups of the energy functional $\varphi$ at infinity and obtain nontrivial solutions via Morse theory.

(ii) It turns out that if for fixed $x \in \Omega$ and some $r > 0$,

$$f(x,t)/|t| \text{ is increasing on } (-\infty,-r) \text{ and on } (r,+\infty),$$

then (F3) holds with $\theta = 1$ and

$$C^* = 1 + \sup_{(x,t)\in\Omega\times[-r,r]} \mathcal{F}(x,t) - \inf_{(x,t)\in\Omega\times[-r,r]} \mathcal{F}(x,t),$$

see [11] for a proof. Thus, (F3) is much weaker than the globally condition (F3') and (F3").

(iii) There are functions $f(x,t)$ satisfying (F3) and not satisfying (F3') and (F3"). For example, let

$$f(x,t) = \begin{cases} t(2\ln|t| + 1), & |t| \ge 1,\\ -|t|t + 2t, & |t| \le 1. \end{cases} \tag{1.2}$$

Simple computation shows that

$$F(x,t) = \begin{cases} t^2\ln|t| + \frac{2}{3}, & |t| \ge 1,\\ -\frac{1}{3}|t|^3 + t^2, & |t| \le 1. \end{cases}$$

Thus it is easy to check that $f$ satisfies (F3) with $\theta = 1$ and $C^* = 1$. But it does not satisfy (F3'), (F3"), since $f(x,t)/t$ is increasing on $(-1,0)$ and decreasing on $(0,1)$.

**Remark 1.5.** Theorem 1.3 unifies and generalizes Theorems 1.1 and 1.2. First, the globally monotonicity conditions (F3') and (F3") respectively in Theorems 1.1 and 1.2 are replaced by the more generic assumption (F3). In addition, condition (F2') in Theorem 1.2 is stronger than (F2) and the condition (F5) in Theorem 1.1 is completely removed. Therefore, our result applies to more general situations. For example, the function listed in (1.2) satisfies our Theorem 1.3. But it does not satisfy Theorems 1.1 and 1.2, and the results in [3, 6, 8, 12, 13, 23].

**Remark 1.6.** Comparing with Theorems 1.1 and 1.2, our approach is much simpler.

- In [19], the difficulty that without (AR) the Palais-Smale sequences of $\varphi$ may be unbounded is solved by minimizing $\varphi$ over the set $M$. Since it is not assumed that $f$ is differentiable, $M$ need not be a $C^1$-submanifold of $E$. Hence, to show that minimizers of $\varphi$ over $M$ are critical points of $\varphi$ is not easy.
- In [24], Zou constructed a variant fountain theorem, and as an application, studied the boundary value problem (1.1) with symmetry. He dealt with a family of perturbed functional. Nevertheless, this approach is not very satisfactory, because working with a family of perturbed functionals makes things unnecessarily complicated.

We shall prove Theorem 1.3 by directly applying the usual variational method to $\varphi$. The main ingredient in our argument is based on the observation that: although there may exist unbounded Palais-Smale sequences, we can prove that all Cerami sequences of $\varphi$ are bounded (see Lemma 2.1 below). Then Theorem 1.3 follows from the Fountain Theorem of Bartsch.

Furthermore, He and Zou [6, Theorem 1.3] considered the asymptotically linear case. They obtained the following theorem via the variant fountain theorem due to Zou [24, Theorem 2.2].

**Theorem 1.7** ([6, Theorem 1.3]). *Suppose that $F(x,t)$ satisfies the following conditions:*

- (F6) $F(x,t)=\frac{1}{2}\lambda t^2+H(x,t)$, *where $\lambda\notin\sigma(-\Delta+a)$ a constant; $\sigma$ denotes the spectrum.*
- (F7') *There exist $\delta_i\in(1,2)$, $i=1,2$, and $b_1$, $b_2>0$ such that*
$$b_1|t|^{\delta_1}\le H(x,t),\quad H(x,0)\equiv 0,\quad |H_t(x,t)|\le b_2|t|^{\delta_2-1} \tag{1.3}$$
*for all $(x,t)\in\Omega\times\mathbb{R}$.*
- (F8) $H(x,-t)=H(x,t)$ *for all $(x,t)\in\Omega\times\mathbb{R}$.*
- (F9) *0 is an eigenvalue of $-\Delta+a$ with the Dirichlet boundary condition.*

*Then problem* (1.1) *has infinitely many nontrivial solutions.*

In this article, with the aid of the new version of the symmetric mountain pass lemma developed in Kajikiya [9], we obtain the following theorem, which sharply improves Theorem 1.7.

**Theorem 1.8.** *Assume that* (F1), (F4) *are satisfied and*

- (F7) $\lim_{t\to 0}\frac{F(x,t)}{t^2}=+\infty$ *uniformly for $x\in\Omega$.*

*Then problem* (1.1) *possesses infinitely many nontrivial solutions $(u_k)$ such that*
$$\frac{1}{2}\int_\Omega\big(|\nabla u_k|^2+a(x)u_k^2\big)\,dx-\int_\Omega F(x,u_k)dx\to 0^-\quad \text{as } k\to\infty.$$

**Remark 1.9.** Theorem 1.8 extends Theorem 1.7 in three aspects. First, noting $p>2>\delta_2$, (F6) and the third inequality of (1.3) imply that
$$\begin{aligned}|f(x,t)|&\le\lambda|t|+|H_t(x,t)|\\&\le\lambda|t|+b_2|t|^{\delta_2-1}\\&\le(\lambda+b_2)(1+|t|^{p-1}),\quad\forall(x,t)\in\Omega\times\mathbb{R},\end{aligned}$$
which is just (F1) with $a_1=\lambda+b_2$. Secondly, it follows from (F6) and the first inequality of (1.3) that
$$\frac{F(x,t)}{t^2}\ge\frac{\lambda}{2}+\frac{b_1}{|t|^{2-\delta_1}},\quad\forall(x,t)\in\Omega\times\mathbb{R},$$
which implies that
$$\lim_{t\to 0}\frac{F(x,t)}{t^2}=+\infty\quad\text{uniformly for } x\in\Omega.$$
And finally, the condition (F9) in Theorem 1.7 is completely dropped. There are functionals $F$ satisfying Theorem 1.8 and not satisfying the results in [6]. For

example, let

$$H(x,t) = -|t|^{3/2} \ln\big(\frac{1+t^2}{4}\big), \quad \forall (x,t) \in \Omega \times \mathbb{R},$$
$$F(x,t) = \frac{1}{2}\lambda t^2 + H(x,t), \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where $\lambda \in \sigma(-\Delta + a)$. A straightforward computation shows that $F(x,t)$ satisfies all the assumptions of Theorem 1.8. But it does not satisfy Theorem 1.7, since $\lambda \in \sigma(-\Delta + a)$ and $H(x,t) \le 0$ for all $x \in \Omega$ and $|t| \ge \sqrt{3}$.

The paper is organized as follows. In Section 2 we investigate the superquadratic case and give the proof of Theorem 1.1. In Section 3 we deal with the subquadratic case and prove Theorem 1.8.

## 2. Proof of Theorem 1.1

Let $X := H_0^1(\Omega)$ be the Sobolev space equipped with the norm

$$\|u\| = \Big( \int_\Omega |\nabla u|^2 dx \Big)^{1/2}.$$

Noting $s > N/2$, one has $2s/(s-1) < 2^*$, and then, using the fact that the embedding of $H_0^1(\Omega) \hookrightarrow L^r(\Omega)$ $(1 \le r < 2^*)$ is compact, we obtain

$$|u|_r \le C\|u\|, \quad \forall u \in X, \tag{2.1}$$

for some $C > 0$, where $r = 1$, $2s/(s-1)$, and $|\cdot|_r$ denotes the usual norm of $L^r(\Omega)$. Denote by $\lambda_1 \le \lambda_2 \le \lambda_3 \le \dots$ (counted in their multiplicities) the eigenvalues of $-\Delta + a$ on $H_0^1(\Omega)$ and by $(e_n)_{n=1}^\infty$ the corresponding system of eigenfunctions, which forms an orthonormal basis of $H_0^1(\Omega)$. Assume $\lambda_1, \dots, \lambda_{n^-} < 0$, $\lambda_{n^-+1} = \dots = \lambda_{n^*} = 0$ and let $X^- :=\text{span}\{e_1, \dots, e_{n^-}\}$, $X^0 :=\text{span}\{e_{n^-+1}, \dots, e_{n^*}\}$ and $X^+ := \overline{\text{span}\{e_{n^*+1}, \dots\}}$. Then we have the following decomposition

$$X = X^- \oplus X^0 \oplus X^+,$$

and there exists $\delta > 0$ such that

$$\int_\Omega (|\nabla u|^2 + a(x)u^2) dx \ge \delta \|u\|^2, \quad \forall u \in X^+, \tag{2.2}$$
$$\int_\Omega (|\nabla u|^2 + a(x)u^2) dx \le -\delta \|u\|^2, \quad \forall u \in X^-. \tag{2.3}$$

Under assumption (F1), the functional associated to problem (1.1) given by

$$\varphi(u) = \frac{1}{2} \int_\Omega \big( |\nabla u|^2 + a(x)u^2 \big)\, dx - \int_\Omega F(x,u) dx$$

is continuously differentiable on $X$, and

$$\langle \varphi'(u), v \rangle = \int_\Omega (\nabla u \cdot \nabla v + a(x)uv) dx - \int_\Omega f(x,u) v dx$$

for all $u$, $v \in X$. It is well known that the weak solutions of problem (1.1) correspond to the critical points of $\varphi$.

To find critical points of $\varphi$, we shall show that $\varphi$ satisfies the Cerami condition, that is, $(u_n)$ has a convergent subsequence in $X$ whenever $\{\varphi(u_n)\}$ is bounded and $(1 + \|u_n\|)\|\varphi'(u_n)\| \to 0$ as $n \to \infty$.

**Lemma 2.1.** *Assume that assumptions* (F1), (F2), (F3) *hold. Then* $\varphi$ *satisfies the (C) condition.*

*Proof.* We adapt an argument in [10, Lemma 2.2], see also [11, Lemma 2.5]. Let $(u_n)$ be a Cerami sequence of $\varphi$. We claim that $(u_n)$ is bounded. Otherwise, up to a subsequence, we can assume that, for some $c_1 > 0$,

$$\varphi(u_n) \to c_1, \quad (1 + \|u_n\|)\|\varphi'(u_n)\| \to 0 \quad \text{and} \quad \|u_n\| \to \infty \tag{2.4}$$

as $n \to \infty$. Particularly,

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega \big(\frac{1}{2} f(x, u_n)u_n - F(x, u_n)\big)\, dx &= \lim_{n\to\infty} \big(\varphi(u_n) - \frac{1}{2}\langle \varphi'(u_n), u_n\rangle\big) \\ &= c_1. \end{aligned} \tag{2.5}$$

Setting $w_n = u_n/\|u_n\|$, then $\|w_n\| = 1$. Going if necessary to a subsequence, we may assume that

$$\begin{gathered} w_n \rightharpoonup w \quad \text{in } H_0^1(\Omega), \\ w_n \to w \quad \text{in } L^r(\Omega)\ (1 \le r < 2^*), \\ w_n(x) \to w(x) \quad \text{a.e. } x \in \Omega. \end{gathered} \tag{2.6}$$

If $w = 0$, we choose a sequence $(s_n) \subset \mathbb{R}$ such that

$$\varphi(s_n u_n) = \max_{s\in[0,1]} \varphi(s u_n).$$

For any $m > 0$, letting $v_n = \sqrt{2m}\, w_n$, one has

$$v_n \to 0 \quad \text{in } L^r(\Omega)\ (1 \le r < 2^*) \tag{2.7}$$

by (2.6). From (F1), we have

$$|F(x, t)| \le \int_0^1 |f(x, st)t| ds \le a_1(|t| + |t|^p), \quad \forall (x, t) \in \Omega \times \mathbb{R}, \tag{2.8}$$

which, together with (2.7), shows that

$$\int_\Omega F(x, v_n) dx \le a_1 \int_\Omega (|v_n| + |v_n|^p) dx = a_1 \left(|v_n|_1 + |v_n|_p^p\right) \to 0 \tag{2.9}$$

as $n \to \infty$. Taking $s' = 2s/(s-1)$, since $s > N/2$, we have

$$1 \le s' < 2^* \quad \text{and} \quad \frac{1}{s} + \frac{2}{s'} = 1,$$

so that, using Hölder's inequality and (2.7),

$$\begin{aligned} \int_\Omega a(x) v_n^2 dx &\le \Big(\int_\Omega |a(x)|^s dx\Big)^{1/s} \Big(\int_\Omega |v_n|^{s'} dx\Big)^{2/s'} \\ &= |a|_s |v_n|_{s'}^2 \to 0. \end{aligned} \tag{2.10}$$

Now, for $n$ large enough, $\sqrt{2m}\|u_n\|^{-1} \in (0, 1)$, we obtain

$$\varphi(s_n u_n) \ge \varphi(v_n) = \frac{1}{2}\|v_n\|^2 + \frac{1}{2}\int_\Omega a(x) v_n^2 dx - \int_\Omega F(x, v_n) dx$$

for all $n$. Combining (2.10) and (2.9), we deduce

$$\liminf_{n\to\infty} \varphi(s_n u_n) \ge m,$$

which implies that

$$\lim_{n\to\infty} \varphi(s_n u_n) = +\infty \tag{2.11}$$

by the arbitrariness of $m$. Noticing $\varphi(0)=0$ and $\varphi(u_n)\to c_1$ $(n\to\infty)$, we see that, for $n$ sufficiently large, $s_n\in(0,1)$ and

$$\begin{aligned}
&\int_\Omega |\nabla(s_nu_n)|^2dx+\int_\Omega a(x)|s_nu_n|^2dx-\int_\Omega f(x,s_nu_n)s_nu_n\,dx\\
&=\langle\varphi'(s_nu_n),s_nu_n\rangle\\
&=s_n\,\frac{d}{ds}\Big|_{s=s_n}\varphi(su_n)=0.
\end{aligned}$$

Therefore, using (2.11) and (F3),

$$\begin{aligned}
&\int_\Omega\Big(\frac12 f(x,u_n)u_n-F(x,u_n)\Big)dx\\
&\geq\frac1\theta\int_\Omega\Big(\frac12 f(x,s_nu_n)s_nu_n-F(x,s_nu_n)\Big)dx-\frac{C^*}{2\theta}|\Omega|\\
&=\frac1\theta\int_\Omega\Big(\frac12|\nabla(s_nu_n)|^2+\frac12a(x)|s_nu_n|^2-F(x,s_nu_n)\Big)dx-\frac{C^*}{2\theta}|\Omega|\\
&=\frac1\theta\varphi(s_nu_n)-\frac{C^*}{2\theta}|\Omega|\to+\infty,
\end{aligned}$$

a contradiction with (2.5).

If $w\neq0$, then the set $\Omega_1=\{x\in\Omega:w(x)\neq0\}$ has positive Lebesgue measure. For $x\in\Omega_1$, we have $|u_n(x)|\to\infty$ as $n\to\infty$, so that, using (F2),

$$\frac{F(x,u_n(x))}{|u_n(x)|^2}|w_n(x)|^2\to+\infty\quad\text{as}\quad n\to\infty.$$

Hence, via Fatou's lemma (see [22]),

$$\int_{w\neq0}\frac{F(x,u_n)}{u_n^2}w_n^2dx\to+\infty\quad\text{as } n\to\infty. \tag{2.12}$$

On the other hand, (F2) implies that there exists $r_1>0$ such that

$$F(x,t)\geq0,\quad\forall x\in\Omega,\quad|t|\geq r_1.$$

From (2.8), one has

$$|F(x,t)|\leq c_2,\quad\forall x\in\Omega,\ |t|\leq r_1,$$

where $c_2=a_1(r_1+r_1^p)$. It follows that $F(x,t)\geq-c_2$ for all $(x,t)\in\Omega\times\mathbb{R}$. Hence we have

$$\int_{w=0}\frac{F(x,u_n)}{\|u_n\|^2}dx\geq-\frac{\int_{w=0}c_2dx}{\|u_n\|^2}\geq-\frac{c_2|\Omega|}{\|u_n\|^2},\quad\forall n\in\mathbb{N},$$

which implies that

$$\liminf_{n\to\infty}\int_{w=0}\frac{F(x,u_n)}{\|u_n\|^2}dx\geq0. \tag{2.13}$$

Notice that

$$\int_\Omega F(x,u_n)dx=\frac12\|u_n\|^2+\frac12\int_\Omega a(x)u_n^2dx-\varphi(u_n),\qquad\forall n\in\mathbb{N}.$$

Dividing both sides by $\|u_n\|^2$ and letting $n\to\infty$, we obtain via (2.13), (2.12) and the first limit of (2.4) that

$$\frac12+\frac12\int_\Omega a(x)w^2dx\geq\limsup_{n\to\infty}\int_\Omega\frac{F(x,u_n)}{\|u_n\|^2}dx$$

$$= \limsup_{n\to\infty} \Big( \int_{w=0} + \int_{w\neq 0} \Big) \frac{F(x,u_n)}{u_n^2} w_n^2 dx = +\infty.$$

This is impossible.

In any case, we deduce a contradiction. Hence $(u_n)$ is bounded in $X$. Next we verify that $(u_n)$ has a convergent subsequence. Without loss of generality, one can suppose that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } X, \\ u_n \to u \quad \text{in } L^r(\Omega)\ (1 \le r < 2^*). \end{gathered} \tag{2.14}$$

By (2.14) and the Hölder inequality, we have

$$\begin{aligned} \int_\Omega a(x)(u_n-u)^2 dx &\le \Big( \int_\Omega |a(x)|^s dx \Big)^{1/s} \Big( \int_\Omega |u_n-u|^{s'} dx \Big)^{2/s'} \\ &= |a|_s |u_n-u|_{s'}^2 \to 0, \end{aligned} \tag{2.15}$$

where $s' = 2s/(s-1)$. It follows from $(f_1)$, (2.14) and Hölder's inequality that

$$\begin{aligned} &\Big| \int_\Omega \big(f(x,u_n)-f(x,u)\big)(u_n-u)dx \Big| \\ &\le \int_\Omega \left(|f(x,u_n)| + |f(x,u)|\right) |u_n-u| dx \\ &\le a_1 \int_\Omega (2 + |u_n|^{p-1} + |u|^{p-1}) |u_n-u| dx \\ &\le 2a_1 |u_n-u|_1 + a_1 \Big( \int_\Omega |u_n|^p dx \Big)^{(p-1)/p} \Big( \int_\Omega |u_n-u|^p dx \Big)^{1/p} \\ &\quad + a_1 \Big( \int_\Omega |u|^p dx \Big)^{(p-1)/p} \Big( \int_\Omega |u_n-u|^p dx \Big)^{1/p} \\ &\le 2a_1 |u_n-u|_1 + a_1 |u_n|_p^{p-1} |u_n-u|_p + a_1 |u|_p^{p-1} |u_n-u|_p \to 0. \end{aligned} \tag{2.16}$$

Moreover, the boundedness of $(u_n)$ and the second limit of (2.4) imply that

$$|\langle \varphi'(u_n), u_n-u\rangle| \le \|\varphi'(u_n)\| (\|u_n\| + \|u\|) \to 0 \quad \text{as } n \to \infty.$$

Combining this with (2.16) and (2.15), we obtain

$$\begin{aligned} \|u_n-u\|^2 = \langle \varphi'(u_n) - \varphi'(u), u_n-u\rangle &- \int_\Omega a(x)(u_n-u)^2 dx \\ &+ \int_\Omega (f(x,u_n)-f(x,u))(u_n-u)dx \to 0. \end{aligned}$$

Thus $u_n \to u$ in $X$ and the proof is complete. □

For convenience to quote, we state the Fountain Theorem of Bartsch (see [2, Theorem 2.5]), which will be used to prove Theorem 1.1.

Let $X$ be a reflexive and separable Banach space, then there are $(e_n)_{n\in\mathbb{N}} \subset X$ and $(e_n^*)_{n\in\mathbb{N}} \subset X^*$ (the dual space of $X$) such that

$$X = \overline{\operatorname{span}\{e_n : n \in \mathbb{N}\}}, \quad X^* = \overline{\operatorname{span}\{e_n^* : n \in \mathbb{N}\}}$$

and

$$\langle e_n, e_m \rangle = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

Let $X_j = \operatorname{span}\{e_j\}$, then $X = \overline{\oplus_{j\geq 1} X_j}$. Now we define

$$Y_k = \oplus_{j=1}^k X_j \quad \text{and} \quad Z_k = \overline{\oplus_{j\geq k} X_j}. \tag{2.17}$$

Then we have the following Fountain Theorem.

**Theorem 2.2** (Fountain Theorem)**.** *Assume that $\varphi \in C^1(X, \mathbb{R})$ satisfies the Cerami condition, $\varphi(-u) = \varphi(u)$. For almost every $k \in N$, there exist $\rho_k > r_k > 0$ such that*

(i) $b_k := \inf_{u\in Z_k, \|u\|=r_k} \varphi(u) \to +\infty$ *as* $k \to \infty$*;*
(ii) $a_k := \max_{u\in Y_k, \|u\|=\rho_k} \varphi(u) \leq 0$.

*Then $\varphi$ has a sequence of critical points $(u_k)$ such that $\varphi(u_k) \to +\infty$.*

**Remark 2.3.** In [2, 21], the Fountain Theorem is established under the Palais-Smale (PS) condition. Since the Deformation Theorem is still valid under the Cerami condition, we see that like many critical point theorems, the Fountain Theorem holds true under the Cerami condition.

*Proof of Theorem 1.3.* For the Hilbert space $X = H_0^1(\Omega)$, define $Y_k$ and $Z_k$ as in (2.17). According to Lemma 2.1 and assumption (F4), we know that $\varphi$ satisfies the Cerami condition and $\varphi(-u) = \varphi(u)$. It remains to verify the conditions (i) and (ii) of Proposition 2.2.

Verification of (i). For $1 \leq r < 2^*$, taking

$$\beta_k := \sup_{u\in Z_k, \|u\|=1} |u|_r,$$

one has $\beta_k \to 0$ as $k \to \infty$ (see [21, Lemma 3.8]). Set

$$r_k := \big(\frac{\delta}{8a_1\beta_k^p}\big)^{\frac{1}{p-2}}.$$

Since $p > 2$, we get $r_k \to +\infty$ as $k \to \infty$. So choosing $k$ large enough such that $Z_k \subset X^+$ and $r_k > 8a_1C/\delta$, we obtain, for $u \in Z_k$ with $\|u\| = r_k$,

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_\Omega (|\nabla u|^2 + a(x)u^2)dx - \int_\Omega F(x, u)dx \\
&\geq \frac{\delta}{2}\|u\|^2 - a_1\int_\Omega |u|dx - a_1\int_\Omega |u|^p dx \\
&\geq \frac{\delta}{2}\|u\|^2 - a_1C\|u\| - a_1\beta_k^p\|u\|^p \\
&\geq \frac{\delta r_k^2}{4}
\end{aligned}$$

by (2.8) and (2.1), which implies that

$$\inf_{u\in Z_k, \|u\|=r_k} \varphi(u) \geq \frac{\delta r_k^2}{4} \to +\infty \quad \text{as } k \to \infty.$$

Verification of (ii). Since $Y_k$ is finite-dimensional, there exists a constant $C_k > 0$ such that

$$C_k|u|_2 \geq \|u\|, \quad \forall u \in Y_k. \tag{2.18}$$

By (F2), there exists $r_2 > 0$ such that

$$F(x, t) \geq C_k^2(1 + |a|_sC^2)t^2, \quad \forall x \in \Omega,\ |t| \geq r_2.$$

From (2.8), one has

$$|F(x,t)| \le a_1(r_2 + r_2^p), \quad \forall x \in \Omega,\ |t| \le r_2.$$

Thus we obtain

$$F(x,t) \ge C_k^2(1 + |a|_s C^2)t^2 - M_k, \quad \forall (x,t) \in \Omega \times \mathbb{R},$$

where $M_k = a_1(r_2 + r_2^p) + C_k^2(1 + |a|_s C^2)r_2^2$. Combining this with (2.18), (2.1) and the Hölder inequality, we obtain

$$\begin{aligned}
\varphi(u) &= \frac{1}{2}\int_\Omega (|\nabla u|^2 + a(x)u^2)dx - \int_\Omega F(x,u)dx \\
&\le \frac{1}{2}\|u\|^2 + \frac{1}{2}|a|_s|u|_{s'}^2 - C_k^2(1 + |a|_s C^2)|u|_2^2 + M_k|\Omega| \\
&\le \frac{1}{2}(1 + |a|_s C^2)\|u\|^2 - (1 + |a|_s C^2)\|u\|^2 + M_k|\Omega| \\
&\le -\frac{1}{2}(1 + |a|_s C^2)\|u\|^2 + M_k|\Omega|
\end{aligned}$$

for all $u \in Y_k$, where $s' = 2s/(s-1)$. Hence, choosing $\rho_k > \max\{r_k, (\frac{4M_k|\Omega|}{1+|a|_s C^2})^{1/2}\}$, we deduce

$$\max_{u \in Y_k, \|u\| = \rho_k} \varphi(u) \le -\frac{1}{4}(1 + |a|_s C^2)\rho_k^2 < 0.$$

Consequently, by Proposition 2.2, $\varphi$ possesses a sequence of critical points $(u_k)$ such that $\varphi(u_k) \to +\infty$ as $k \to \infty$. $\square$

## 3. Proof of Theorem 1.8

To prove Theorem 1.8, we need the variant symmetric mountain pass lemma established in [9]. Before stating it, we first recall the definition of genus.

Let $X$ be a Banach space and $A$ a subset of $X$. $A$ is said to be symmetric if $u \in A$ implies $-u \in A$. Denote by $\Gamma$ the family of closed symmetric subsets $A$ of $X$ which does not contain the origin, i.e.,

$$\Gamma = \{A \subset X\backslash\{0\} : A \text{ is closed and symmetric with respect to zero}\}.$$

For $A \in \Gamma$, we define

$$\gamma(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ \inf\{k \in N : \exists \text{ an odd } \varphi \in C(A, \mathbb{R}^k\backslash\{0\})\}, & \\ +\infty & \text{if no such odd map,} \end{cases}$$

and $\Gamma_k = \{A \in \Gamma : \gamma(A) \ge k\}$.

For convenience of the readers, we summarize the property of genus which will be used in the proof of Theorem 1.8. We refer the readers to [17, Proposition 7.5] for the proof of the next proposition.

**Theorem 3.1.** *Let $A, B \in \Gamma$. Then (i)-(iv) below hold.*

(i) *If there is an odd continuous mapping from $A$ to $B$, then $\gamma(A) \le \gamma(B)$.*
(ii) *If $A \subset B$, then $\gamma(A) \le \gamma(B)$.*
(iii) *If $A$ is compact, then $\gamma(A) < +\infty$ and $\gamma(N_\delta(A)) = \gamma(A)$ for $\delta > 0$ small enough, where $N_\delta(A) = \{x \in X : \|x - A\| \le \delta\}$.*
(iv) *The $n$-dimensional sphere $S^n$ has a genus of $n + 1$ by the Borsuk-Ulam theorem.*

Now we state the variant symmetric mountain pass lemma.

**Theorem 3.2** ([9, Theorem 1.1])**.** *Let $X$ be an infinite dimensional Banach space and $I \in C^1(X, \mathbb{R})$ satisfies the following conditions:*

(1) *$I(u)$ is even, bounded from below, $I(0) = 0$ and $I$ satisfies the Palais-Smale condition (PS), i.e., $(u_n) \subset X$ has a convergent subsequence whenever $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$ as $n \to \infty$.*

(2) *For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u\in A_k} I(u) < 0$.*

*Then $I$ possesses a sequence of critical points $(u_k)$ such that $I(u_k) \le 0$, $u_k \neq 0$ and $\lim_{k\to\infty} u_k = 0$.*

*Proof of Theorem 1.8.* We consider the truncated functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + h(\|u\|)\Big(\frac{1}{2}\int_\Omega a(x)u^2 dx - \int_\Omega F(x,u)dx\Big)$$

for all $u \in X$, where $h \in C^1([0,+\infty), \mathbb{R})$ such that $0 \le h \le 1$, $h(t) = 1$ for $0 \le t \le 1$ and $h(t) = 0$ for $t \ge 2$. Obviously, $I \in C^1(X, \mathbb{R})$ and $I(0) = 0$. If we can prove that $I$ admits a sequence of critical points $(u_k)$ such that $I(u_k) \le 0$, $u_k \neq 0$ and $u_k \to 0$ as $k \to \infty$, then the critical points of $I$ satisfying $\|u_k\| \le 1$ are just critical points of $\varphi$, since $I(u) = \varphi(u)$ when $\|u\| \le 1$, and hence Theorem 1.8 follows. Applying Proposition 3.2, we shall verify that $I$ possesses a sequence of nontrivial critical points which converges to the origin.

By the oddness of $f$, we see that $I(-u) = I(u)$. For $\|u\| \ge 2$, one has

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx = \frac{1}{2}\|u\|^2,$$

which implies that $I(u) \to +\infty$ as $\|u\| \to \infty$. Thus $I$ is bounded from below and satisfies the (PS) condition.

Given any $k \in N$, let $E_k = \oplus_{j=1}^k X_j$, where $X_j = \text{pan}\{e_j\}$. Since on the finite-dimensional space $E_k$ all norms are equivalent, there exists $d_k > 0$ such that

$$d_k|u|_2 \ge \|u\| \quad \text{and} \quad d_k\|u\| \ge \|u\|_\infty \tag{3.1}$$

for all $u \in E_k$, where $\|u\|_\infty = \max_{x\in\Omega} |u(x)|$. By (F7), there is $r_3 > 0$ such that

$$F(x,t) \ge d_k^2(1 + |a|_s C^2)t^2, \quad \forall x \in \Omega,\ |t| \le r_3. \tag{3.2}$$

Therefore, for $u \in E_k$ with $\|u\| = l_k := \min\{1/2, r_3/d_k\}$, we obtain

$$\begin{aligned}
I(u) &= \frac{1}{2}\int_\Omega (|\nabla u|^2 + a(x)u^2)dx - \int_\Omega F(x,u)dx \\
&\le \frac{1}{2}\|u\|^2 + \frac{1}{2}|a|_s|u|_{s'}^2 - d_k^2(1 + |a|_s C^2)\int_\Omega u^2 dx \\
&\le \frac{1}{2}(1 + |a|_s C^2)\|u\|^2 - (1 + |a|_s C^2)\|u\|^2 \\
&\le -\frac{1}{2}(1 + |a|_s C^2)l_k^2
\end{aligned}$$

by (3.2), (3.1), (2.1) and Hölder's inequality, where $s' = 2s/(s-1)$. This implies that

$$\{u \in E_k : \|u\| = l_k\} \subset \{u \in X : I(u) \le -\frac{1}{2}(1 + |a|_s C^2)l_k^2\}.$$

So, taking $A_k = \{u \in X : I(u) \le -(1 + |a|_s C^2)l_k^2/2\}$, by Proposition 3.1, we obtain

$$\gamma(A_k) \ge \gamma(\{u \in E_k : \|u\| = l_k\}) \ge k,$$

and hence $A_k \in \Gamma_k$ and

$$\sup_{u\in A_k} I(u) \leq -\frac{1}{2}(1+|a|_s C^2)l_k^2 < 0.$$

Thus, Theorem 1.8 follows from Proposition 3.2 and the proof is complete. $\square$

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