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# A VARIATIONAL FORMULATION FOR TRAVELING WAVES AND ITS APPLICATIONS

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Abstract. In this article, we give a variational formulation for traveling wave solutions that decay exponentially at one end of the cylinder for parabolic equations. The variational formulation allows us to obtain the monotone dependence of the velocity on the domain and the nonlinearity, since the velocity is related to the infimum. In particular, we apply this method to Ginzburg-Landau-type problems and a scalar reaction-diffusion-advection equation in infinite cylinders. For the former, we not only obtain the existence, non-existence, boundedness and regularity of the solutions, but also obtain the monotone dependence of the velocity on the nonlinearity and the domain. For the later, we obtain the monotone dependence of the velocity on the nonlinearity and the domain besides the existence, uniqueness, monotonicity and asymptotic behavior at infinity of the solutions. Moreover, we deduce that the influence of the advection on the traveling waves is different from a flow along the cylinder axis considered in many articles.

## 1. Introduction

This article concerns the reaction diffusion equation

$$u_t = \Delta u + f(u) \tag{1.1}$$

in an infinite cylinder $\Sigma$ with either Neumann or Dirichlet boundary condition

$$(\nu \cdot \nabla u)|_{\partial\Sigma} = 0 \quad \text{or} \quad u|_{\partial\Sigma} = 0. \tag{1.2}$$

Here $u = u(x,t) \in \mathbb{R}$, $x = (y,z) \in \Sigma = \Omega \times \mathbb{R}$, $\Omega \subset \mathbb{R}^{n-1}$ $(n \geq 3)$ is a bounded domain with smooth boundary; $\nu$ is the outward normal to $\partial\Sigma$. The nonlinearity $f : \mathbb{R} \mapsto \mathbb{R}$ is in $C^1$, and $f(0) = 0$, then $u = 0$ is the trivial solution of (1.1) and (1.2).

It is known that traveling wave solution is an important class of solutions to investigate the long time behavior of solutions of Cauchy problems; see, for example, [1, 2, 3, 4, 5, 6, 11, 12, 16, 17, 19]. In this article, we use a variational formulation to study the existence of traveling wave solutions which are characterized by a fast exponential decay at one end of the cylinder and properties of obtained traveling wave solutions. And the variational formulation is given to (1.1), the nonlinearity

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$f$ of which is quite general, so the variational formulation can be applied to the study of traveling wave solutions of many other problems.

Heinze [7] first proposed the idea of coverting the existence of traveling wave solutions into the existence of constraint minimizer in two dimensional strip with Dirichlet boundary condition. In [8], Heinze studied a model for the heater in boiling systems and extended the results of [7] to the mixed nonlinear Neumann and Dirichlet boundary problems in infinite cylinder. And [8] obtained the existence of traveling wave solutions of problem

$$\begin{gathered}\partial_t u(y,z,t)=\Delta u(y,z,t)+f(u(y,z,t),y),\quad (y,z,t)\in\Omega\times\mathbb{R}\times\mathbb{R}^+,\\ \partial_\nu u(y,z,t)=g(u(y,z,t),y),\quad (y,z,t)\in\Gamma_1\times\mathbb{R}\times\mathbb{R}^+,\\ u(y,z,t)=0,\quad (y,z,t)\in\Gamma_2\times\mathbb{R}\times\mathbb{R}^+,\end{gathered}\tag{1.3}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^{n-1}$ with $C^1$ boundary. The boundary $\partial\Omega$ consists of two parts $\Gamma_1,\Gamma_2$ corresponding to different boundary conditions. $\Gamma_1$ and $\Gamma_2$ may be empty.

Let $u(y,z,t)=\bar{u}(y,c(z+ct))$ with $c\neq 0$ as the unknown wave velocity, then Equation (1.3) is written as

$$\begin{gathered}\partial_z\bar{u}=\partial_{zz}\bar{u}+\lambda_1(\Delta_y\bar{u}+f(\bar{u},y)),\quad (y,z)\in\Omega\times\mathbb{R},\\ \partial_\nu\bar{u}=g(\bar{u},y),\quad (y,z)\in\Gamma_1\times\mathbb{R},\\ \bar{u}=0,\quad (y,z)\in\Gamma_2\times\mathbb{R}.\end{gathered}$$

Then by defining the following two functionals

$$\begin{gathered}I[u]=\frac{1}{2}\int_\Sigma e^{-z}|\partial_z u|^2\mathrm{d}z\,\mathrm{d}y,\\ J[u]=\int_\Omega e^{-z}\Big(\frac{1}{2}|\nabla_y u|^2-F(u,y)\Big)\mathrm{d}z\,\mathrm{d}y-\int_{\mathbb{R}\times\partial\Gamma_1}e^{-z}G(u,y)\mathrm{d}z\,\mathrm{d}T_y,\end{gathered}$$

where $F(u,y)=\int_0^u f(s,y)\mathrm{d}s$, $G(u,y)=\int_0^u g(s,y)\mathrm{d}s$, Heinze [8] obtained the minimization problem

$$\inf_{\{u\in X\,|\,J[u]=b\}} I[u]\tag{1.4}$$

in the weighted space $X=H^1(\mathbb{R}\times\Omega,e^{-z})$. Moreover, He also obtained $\lambda_1=\frac{1}{c^2}=\inf_{\{u\in X\,|\,J[u]=-1\}}I[u]$ by letting $b=-1$ in (1.4).

For nonlinear Neumann boundary condition, the existence of traveling wave solutions was obtained by Kyed [10] for the problem

$$\begin{gathered}\partial_t u-\Delta u=0,\quad \text{in } \Omega\times\mathbb{R}\times\mathbb{R}^+,\\ \frac{\partial u}{\partial\nu}=f(u),\quad \text{on } \partial\Omega\times\mathbb{R}\times\mathbb{R}^+,\end{gathered}$$

which appears in the study of transient boiling processes by variational methods. Here $\Omega$ is a bounded domain in $\mathbb{R}^{n-1}$ with $C^1$ boundary.

Let $u(y,z,t)=\bar{u}(y,z+ct)$, $(y,z,t)\in\Omega\times\mathbb{R}\times\mathbb{R}^+$, then traveling wave equation is

$$\begin{gathered}\Delta\bar{u}-c\partial_z\bar{u}=0,\quad \text{in } \Omega\times\mathbb{R},\\ \frac{\partial\bar{u}}{\partial\nu}=f(\bar{u}),\quad \text{on } \partial\Omega\times\mathbb{R}.\end{gathered}$$

Define

$$\varepsilon[u] = \frac{1}{2}\int_{\Sigma} e^{-z}|Du|^2 \mathrm{d}z\mathrm{d}y,$$
$$J[u] = \int_{\mathbb{R}\times\Gamma} e^{-z} F(u)\mathrm{d}S(y)\mathrm{d}z,$$

where $F(u) = \int_0^u f(s)\mathrm{d}s$, then the variational formulation is $\min_{u\in\mathcal{C}} \varepsilon[u]$, $\mathcal{C} := \{u \in H^1(\mathbb{R}\times\Omega, e^{-z}) | J[u] = 1\}$.

In 2004, Muratov [14] showed that traveling wave solutions with a fast exponential decay at one end of the cylinder are critical points of certain functional. And this type of traveling wave solutions are called variational traveling wave solutions. Furthermore, under certain assumptions on the shape of the solutions, [14] showed that there exists a reference frame in which the solution of the initial value problem converges to the variational traveling wave at least in a sequence of time. Recently, Lucia et al. [13] studied variational traveling wave solutions for Ginzburg-Landau-type problems

$$u_t = \Delta u + f(u), \quad f(u) = -\nabla_u V(u) \tag{1.5}$$

with

$$(\nu\cdot\nabla u)|_{\partial\Sigma} = 0 \quad \text{or} \quad u|_{\partial\Sigma} = 0, \tag{1.6}$$

where $u = u(x,t) \in \mathbb{R}^m$, $V : \mathbb{R}^m \mapsto \mathbb{R}$, $x = (y,z) \in \Sigma = \Omega\times\mathbb{R}$, $\Omega \subset \mathbb{R}^{n-1}$ $(n \geq 3)$ is a bounded domain with boundary of class $C^2$. Let $u(y,z,t) = \bar{u}(y, z-ct)$, then the traveling wave equation is

$$\bar{u}_{zz} + \Delta_y \bar{u} + c\bar{u}_z + f(\bar{u}) = 0$$

with the boundary condition (1.6). By defining

$$\Phi_c[u] = \int_{\Sigma} e^{cz}\Big(\frac{1}{2}\sum_{i=1}^m |\nabla u_i|^2 + V(u)\Big)\mathrm{d}y\,\mathrm{d}z$$

and

$$\Gamma_c[u] = \frac{1}{2}\int_{\Sigma} e^{cz}\sum_{i=1}^m |\frac{\partial u_i}{\partial z}|^2 \mathrm{d}x,$$

the constraint minimization problem is

$$\Phi_c[u_c] = \inf_{\{u\in H_c^1(\Sigma;\mathbb{R}^m)|\Gamma_c[u]=1\}} \Phi_c[u] \leq 0. \tag{1.7}$$

Then under the following three assumptions

- (H1) The function $V : \mathbb{R}^m \mapsto \mathbb{R}$ satisfies $V \in C^0(\mathbb{R}^m)$, $V(0) = \nabla_u V(0) = 0$, $V(u) \geq -C|u|^2$ for some $C \geq 0$;
- (H2) There exists a convex compact set $\mathcal{K} \subset \mathbb{R}^m$ which contains the origin, such that $V \in C^{1,1}(\mathcal{K})$ and for all $u \notin \mathcal{K}$, $V(u) \geq V(\Pi_{\mathcal{K}}(u))$, where $\Pi_{\mathcal{K}} : \mathbb{R}^m \mapsto \mathbb{R}^m$ is the projection on the set $\mathcal{K}$, that is, $\Pi_{\mathcal{K}}(u)$ is the closest point to $u$ which lies in $\mathcal{K}$;
- (H3) There exist $c > 0$ such that $c^2 + 4\upsilon_0 > 0$, and $u \in H_c^1(\Sigma, \mathbb{R}^m)$, $u \not\equiv 0$ such that $\Phi_c[u] \leq 0$, where

$$\upsilon_0 = \mu_0 + \liminf_{|u|\to 0} \frac{2V(u)}{|u|^2},$$

where $\mu_0$ is the smallest eigenvalue of $-\Delta_y$ with the boundary condition (1.6), they obtained the existence, non-existence, and many properties of variational traveling waves.

Furthermore, Muratov and Novaga [15] discussed front propagation problem for a reaction-diffusion-advection equation in infinite cylinder

$$u_t + \mathbf{v} \cdot \nabla u = \Delta u + f(u, y), \quad \mathbf{v} = (-\nabla_y \varphi, 0), \quad \varphi : \overline{\Omega} \mapsto \mathbb{R}, \tag{1.8}$$

$$u|_{\partial\Sigma_\pm} = 0, \quad \nu \cdot \nabla u|_{\partial\Sigma_0} = 0, \tag{1.9}$$

where $u = u(x, t) \in \mathbb{R}$, $x = (y, z) \in \Sigma = \Omega \times \mathbb{R}$, $\Omega \subset \mathbb{R}^{n-1}$ is a bounded domain with $C^2$ boundary; $\mathbf{v}$ is an imposed advection flow; $\partial\Sigma_\pm = \partial\Omega_\pm \times \mathbb{R}$, $\partial\Sigma_0 = \partial\Omega_0 \times \mathbb{R}$, $\partial\Omega_\pm$ and $\partial\Omega_0$ are defined as parts of $\partial\Omega$ by $\nu \cdot \nabla_y \varphi > 0$, $\nu \cdot \nabla_y \varphi < 0$ and $\nu \cdot \nabla_y \varphi = 0$, respectively. They were concerned with a particular situation in which the flow $\mathbf{v}$ is transverse to the axis of the cylinder; i.e., $\mathbf{v}$ does not have a component along $z$.

For traveling wave solutions of the form $u(x, t) = \bar{u}(y, z - ct)$, substituting it into (1.8), the traveling wave equation is

$$\bar{u}_{zz} + c\bar{u}_z + \nabla_y \varphi \cdot \nabla_y \bar{u} + \Delta_y \bar{u} + f(\bar{u}, y) = 0$$

with boundary conditions (1.9). By defining the following two functionals

$$\Phi_c[u] = \int_\Sigma e^{cz+\varphi(y)} \Big(\frac{1}{2}|\nabla u|^2 + V(u, y)\Big) \mathrm{d}y \, \mathrm{d}z,$$

$$\Gamma_c[u] = \frac{1}{2} \int_\Sigma e^{cz+\varphi(y)} |\frac{\partial u}{\partial z}|^2 \mathrm{d}x,$$

the constraint minimizer problem was given by

$$\Phi_c[u_c] = \inf_{\{u \in H^1_c(\Sigma) | \Gamma_c[u]=1\}} \Phi_c[u] \le 0. \tag{1.10}$$

Then under the following three assumptions

(A1) The function $f : [0, 1] \times \bar{\Omega} \to \mathbb{R}$ satisfies

$$f(0, y) = 0, \quad f(1, y) \le 0, \quad \forall y \in \Omega;$$

(A2) For some $\alpha \in (0, 1)$

$$f \in C^{0,\alpha}([0, 1] \times \bar{\Omega}), \quad f_u \in C^{0,\alpha}([0, 1] \times \bar{\Omega}), \quad \varphi \in C^{1,\alpha}(\bar{\Omega}),$$

where $f_u = \frac{\partial f}{\partial u}$;

(A3) There exist $c > 0$ satisfying $c^2 + 4\upsilon_0 > 0$, and $u \in H^1_c(\Sigma)$ such that $\Phi_c[u] \le 0$ and $u \not\equiv 0$, where

$$\upsilon_0 = \min_{\{\psi \in H^1(\Omega), \psi|\partial\Omega_\pm=0\}} R(\psi), \;\; R(\psi) = \frac{\int_\Omega e^{\varphi(y)} \left(|\nabla_y \psi|^2 - f_u(0, y)\psi^2\right) \mathrm{d}y}{\int_\Omega e^{\varphi(y)} \psi^2 \mathrm{d}y},$$

they showed only three propagation scenarios are possible: no propagation, a "pulled" front, or a "pushed" front, and the choice of the scenario is completely characterized via a minimization problem (1.10). At the same time, they obtained the uniqueness, monotonicity and the exponential decay behavior besides the existence of the solutions if the functional has non-trivial minimizers. Furthermore, they discussed traveling wave solutions characterized by a certain "minimal speed" if the functional does not have non-trivial minimizers in [15].

However, in both [13] and [15], they did not consider the relations between nonlinear function, domain and wave speed. Furthermore, they did not consider influence of advection on traveling waves in [15] where the advection exists. In this paper, ignited by [8], we give a variational formula (1.4) to investigate the existence of traveling wave solutions. Here, the traveling wave solution with the form of $u(y,z,t) = \bar{u}(y, c(z+ct))$ different from the form $u(x,t) = \bar{u}(y, z-ct)$ considered in [13] and [15]. Due to the differences, our variational formulation relates the wave velocity to the infimum, which enables us to obtain some new results. In the next section, we will give this variational formulation to (1.1). In final, we will apply this variational formulation to a system, i.e., Ginzburg-Landau type problems ([13]) , scalar reaction-diffusion-advection equation in infinite cylinder ([15]). For the former, our variational formulation not only asserts the existence, non-existence, boundedness and regularity of the obtained solutions but also deduces the monotone dependence of the velocity on the nonlinearity and the domain under the same assumptions (i.e. (H1), (H2) and (H3)) in [13]. For the later, we obtain the monotone dependence of the velocity on the nonlinearity and the domain besides the existence, uniqueness, monotonicity and asymptotic behavior at infinity of the obtained solutions under the same assumptions (i.e. (A1), (A2) and (A3)) in [15]. Moreover, we obtain some results about the influence of advection on the traveling waves, which are different from the case of a flow along the cylinder axis considered in many papers (e.g. [3, 4]). The influence of the advection, which transverses to cylinder axis, on traveling waves does not be considered in any other literatures.

**Remark 1.1.** In this article, we only give the variational formulation for Equation (1.1) with boundary (1.2). In fact, by the same analysis to these of [13, 15], this variational formulation can be applied to deduce the existence of traveling wave solutions decaying sufficiently rapidly exponentially at one end of the cylinder under suitable conditions for (1.1) and (1.2). Moreover, we can obtain a variational representation of the wave velocity and the monotone dependence of the wave velocity on the nonlinearity and the domain. For simplicity, we omit the detailed procedures.

Finally, we give the notation used in the paper. Throughout the paper $C^k$, $C_0^\infty$, $C^{k,\alpha}$ denote the usual spaces of continuous functions with $k$ continuous derivatives, smooth functions with compact support, continuously differentially functions with Hölder-continuous derivatives of order $k$ for $\alpha \in (0,1)$, respectively. Unless otherwise specified in the paper, "$\cdot$" denotes a scalar product and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. The symbol $\nabla$ is reserved for the gradient in $\mathbb{R}^n$, while $\nabla_y$ stands for the gradient in $\Omega \subset \mathbb{R}^{n-1}$. Similarly, the symbol $\Delta$ stands for the Laplacian in $\mathbb{R}^n$, and $\Delta_y$ stands for the Laplacian in $\Omega \subset \mathbb{R}^{n-1}$. The numbers $C$, etc., will denote generic positive constants.

## 2. Preliminaries and main results

2.1. **Variational formulation.** To derive the variational formulation, we firstly introduce the following exponentially weighted Sobolev spaces in which we will be working.

**Definition 2.1.** Let $H_1^1(\Sigma, \mathbb{R}^m)$ denote the completion of the restrictions of the functions in $\big(C_0^\infty(\mathbb{R}^n)\big)^m$ to $\Sigma$ with respect to the norm

$$\|u\|^2_{H_1^1(\Sigma,\mathbb{R}^m)} = \|u\|^2_{L_1^2(\Sigma,\mathbb{R}^m)} + \|\nabla u\|^2_{L_1^2(\Sigma,\mathbb{R}^m)},$$

$$\|u\|^2_{L^2_1(\Sigma,\mathbb{R}^m)} = \int_\Sigma e^{-z}\sum_{i=1}^m |u_i|^2 \mathrm{d}x.$$

For the Dirichlet boundary condition, replace $C_0^\infty(\mathbb{R}^n)$ with $C_0^\infty(\Sigma)$ above.

**Definition 2.2.** Denote by $H_2^1(\Sigma,\mathbb{R})$ the completion of the restrictions of $C_0^\infty(\mathbb{R}^n)$ to $\Sigma$ with respect to the norm

$$\|u\|^2_{H^1_2(\Sigma,\mathbb{R})} = \|u\|^2_{L^2_2(\Sigma,\mathbb{R})} + \|\nabla u\|^2_{L^2_2(\Sigma,\mathbb{R})},$$
$$\|u\|^2_{L^2_2(\Sigma,\mathbb{R})} = \int_\Sigma e^{-z+\varphi(y)}|u|^2 \mathrm{d}x.$$

For the Dirichlet boundary condition, replace $C_0^\infty(\mathbb{R}^n)$ with $C_0^\infty(\Sigma)$ above.

We are concerned with traveling wave solutions of the form $u(x,t) = u(y,z,t) = \bar{u}(y, c(z+ct))$ with the wave velocity $c \neq 0$. Substituting it into Equation (1.1), one can see that the traveling wave equation becomes

$$\bar{u}_z = \bar{u}_{zz} + \frac{1}{c^2}(\Delta_y \bar{u} + f(\bar{u})) \tag{2.1}$$

with boundary condition (1.2). Moreover, we can always assume $c > 0$ by a possible change of $z$ to $-z$.

Then we define two important functionals as follows:

**Definition 2.3.** Define two functionals in $H_1^1(\Sigma,\mathbb{R})$ by

$$\Gamma[u] = \frac{1}{2}\int_\Sigma e^{-z}\left|\frac{\partial u}{\partial z}\right|^2 \mathrm{d}z\,\mathrm{d}y,$$
$$J[u] = \int_\Sigma e^{-z}\Big(\frac{1}{2}|\nabla_y u|^2 - F(u)\Big)\mathrm{d}z\,\mathrm{d}y,$$

where $F(u) = \int_0^u f(s)\mathrm{d}s$.

Now based on the above preliminaries, we can give the variational formulation (1.4), so that the existence of traveling waves is converted into the existence of constraint minimizers.

**Theorem 2.4.** *We consider the constraint minimization problem*

$$\inf_{\{u\in H_1^1(\Sigma,\mathbb{R})|\Gamma[u]=b\}} J[u], \tag{2.2}$$

*where $b$ is a positive constant. Let $\lambda$ be the Lagrange multiplier, then Equation* (2.1) *is the variational equation corresponding to* (2.2) *and*

$$\lambda\Gamma[u] + J[u] = 0, \tag{2.3}$$

*where $\lambda = c^2$.*

*Proof.* By [18], we can easily obtain that (2.1) is the variational equation corresponding to (2.2) with $\lambda$ as the Lagrange multiplier. In the following, we only need to show (2.3). By multiplying Equation (2.1) by $e^{-z}u_z$ and integrating over $\Sigma$, we obtain

$$\int_\Sigma e^{-z}u_z\big[u_z - u_{zz} - \frac{1}{c^2}(\Delta_y u + f(u))\big]\mathrm{d}z\mathrm{d}y = 0.$$

So (2.3) follows easily by the boundary condition (1.2) and integrating by parts. $\square$

Since $\Gamma[u] = b$, we can always consider $b = 1$ without loss of generality, which can be achieved by a suitable shift in $z$, then the Lagrange multiplier satisfies

$$\lambda = c^2 = - \inf_{\{u \in H_1^1(\Sigma,\mathbb{R}) | \Gamma[u]=1\}} J[u] > 0. \tag{2.4}$$

2.2. **Ginzburg-Landau type problems.** In this subsection, we apply our formulation (2.4) to the Ginzburg-Landau-type problems. Under assumptions (H1), (H2) and (H3), we not only obtain the existence, non-existence, boundedness and regularity of the obtained solutions, but also obtain some new results, i.e. the monotone dependence of the velocity on the nonlinearity and the domain.

For traveling wave solutions of the form $u(y, z, t) = \bar{u}(y, c(z + ct))$, substituting it into (1.5), we obtain traveling wave equation

$$\bar{u}_z = \bar{u}_{zz} + \frac{1}{c^2} \left( \Delta_y \bar{u} - \nabla_{\bar{u}} V \right) \tag{2.5}$$

with boundary condition (1.6). Then by Definition 2.3, we have the following definition.

**Definition 2.5.** Let $u \in H_1^1(\Sigma, \mathbb{R}^m)$, and define functionals by

$$\Gamma_1[u] = \frac{1}{2} \int_\Sigma e^{-z} \sum_{i=1}^m \left| \frac{\partial u_i}{\partial z} \right|^2 \mathrm{d}\, y \mathrm{d}z,$$

$$J_1[u] = \int_\Sigma e^{-z} \Big( \frac{1}{2} \sum_{i=1}^m |\nabla_y u_i|^2 + V(u) \Big) \mathrm{d}y\, \mathrm{d}z.$$

By (2.2)-(2.4), we know that

$$\lambda = - \inf_{u \in \mathcal{B}} J_1[u] > 0, \tag{2.6}$$

where $\lambda = c^2$ is the Lagrange multiplier, $\mathcal{B} = \{u \in H_1^1(\Sigma, \mathbb{R}^m) | \Gamma_1[u] = 1\}$.

Then to show the existence, non-existence, boundedness and regularity analogous to [13] by our variational formulation (2.6), we first show that $u \in H_1^1(\Sigma; \mathbb{R}^m)$ under the assumption $c^2 + 4\upsilon_0 > 0$.

Linearizing Equation (2.5) around $u = 0$ for $z \to -\infty$, then we obtain that the solutions of (2.5) are approximately superposition of functions $u_k(y, z) = e^{-\lambda_k z} v_k(y)$, where $\lambda_k$ satisfies

$$\lambda_k^2 + \lambda_k - \frac{1}{c^2} \upsilon_k = 0, \tag{2.7}$$

$v_k(y)$ and $\upsilon_k \in \mathbb{R}$ are the eigenfunction and the eigenvalue defined by

$$-\Delta_y v_k + H(0) v_k = \upsilon_k v_k, \quad H(u) = (\nabla_u \otimes \nabla_u) V(u),$$

with the boundary condition (1.6) respectively. Where $H(u)$ is the Hessian of the potential $V(u)$ (here we also assume that $V$ is twice differentiable at the origin). From Equation (2.7), we obtain

$$\lambda_k^{\pm}(c) = \frac{-1 \pm \sqrt{1 + \frac{4}{c^2} \upsilon_k}}{2},$$

so by the same discussion to that of [13], we know if $c^2 + 4\upsilon_0 > 0$, $u \in H_1^1(\Sigma, \mathbb{R}^m)$.

Secondly, we can obtain one important inequality that is an analogue of the Poincaré inequality.

**Proposition 2.6.** *Let $u \in H_1^1(\Sigma, \mathbb{R}^m)$, then*

(i)
$$\frac{1}{4}\int_{-\infty}^{-R}\int_{\Omega} e^{-z}\sum_{i=1}^{m}|u_i|^2\mathrm{d}y\mathrm{d}z \le \int_{-\infty}^{-R}\int_{\Omega} e^{-z}\sum_{i=1}^{m}\Big|\frac{\partial u_i}{\partial z}\Big|^2\mathrm{d}y\mathrm{d}z; \tag{2.8}$$
(ii)
$$\int_{\Omega}\sum_{i=1}^{m}u_i^2(y,-R)\mathrm{d}y \le e^{-R}\int_{-\infty}^{-R}\int_{\Omega} e^{-z}\sum_{i=1}^{m}\Big|\frac{\partial u_i}{\partial z}\Big|^2\mathrm{d}y\mathrm{d}z \tag{2.9}$$
*for any* $R \in (-\infty,+\infty)\bigcup\{-\infty,+\infty\}$.

*Proof.* We first prove (i). As
$$\begin{aligned}
&\int_{-\infty}^{-R}\int_{\Omega} e^{-z}|u_i|^2\mathrm{d}y\,\mathrm{d}z\\
&= -e^R\int_{\Omega}u_i^2(y,-R)\mathrm{d}y + 2\int_{-\infty}^{-R}\int_{\Omega}e^{-z}u_i\frac{\partial u_i}{\partial z}\mathrm{d}y\mathrm{d}z\\
&\le 2\Big(\int_{-\infty}^{-R}\int_{\Omega}e^{-z}|u_i|^2\mathrm{d}y\mathrm{d}z\Big)^{1/2}\Big(\int_{-\infty}^{-R}\int_{\Omega}e^{-z}|\frac{\partial u_i}{\partial z}|^2\mathrm{d}y\,\mathrm{d}z\Big)^{1/2},
\end{aligned}$$
Then (2.8) follows.

Now, we give the proof of (ii). Note that
$$\int_{-\infty}^{-R}\int_{\Omega}e^{-z}\big(u_i-\frac{\partial u_i}{\partial z}\big)^2\mathrm{d}y\,\mathrm{d}z \ge 0,$$
we can obtain
$$\begin{aligned}
\int_{-\infty}^{-R}\int_{\Omega}e^{-z}|\frac{\partial u_i}{\partial z}|^2\mathrm{d}y\mathrm{d}z &\ge 2\int_{-\infty}^{-R}\int_{\Omega}e^{-z}u_i\frac{\partial u_i}{\partial z}\mathrm{d}y\mathrm{d}z - \int_{-\infty}^{-R}\int_{\Omega}e^{-z}|u_i|^2\mathrm{d}y\mathrm{d}z\\
&= e^R\int_{\Omega}u_i^2(y,-R)\mathrm{d}y.
\end{aligned}$$
Thus, (2.9) is obtained. □

Let $I_1[u] = c^2\Gamma_1[u] + J_1[u]$, we first note here that there exist $c > 0$ such that $c^2 + 4\upsilon_0 > 0$, and $u \in H_1^1(\Sigma,\mathbb{R}^m)$, $u \not\equiv 0$ such that $I_1[u] \le 0$ by assumption (H3). The functionals $I_1[u]$ and $J_1[u]$ have the same weak lower semicontinuous since $\Gamma_1[u]$ is weak lower semicontinuous. Furthermore, $J_1[u]$ is coercive because of assumption (H1) and Equation (2.8). Hence, $J_1[u]$ has non-trivial constraint minimizers by [13]. If $\bar{u}$ is a minimizer of (2.6), $I_1[\bar{u}] = 0$, then by assumptions (H1) and (H2), our variational formulation yields the non-existence, boundedness and regularity of the obtained solutions by a similar discussion to that of [13].

Finally, we give the results of monotone dependence by our variational formulation.

**Theorem 2.7.** *We assume that the following functions* $V$, $\tilde{V}$ *and* $\bar{V}$ *satisfy assumptions* (H1)-(H3)*, then we have*

(i) *If* $\tilde{V} \ge \bar{V}$*, then* $\tilde{\lambda} \le \bar{\lambda}$*, that is,* $\tilde{c} \le \bar{c}$*;*
(ii) *If* $\tilde{\Omega}$, $\bar{\Omega} \subset \mathbb{R}^{n-1}$ $(n \ge 3)$ *are bounded domain with boundary of class* $C^2$ *and* $\tilde{\Omega} \subset \bar{\Omega}$*, then* $\tilde{\lambda} \ge \bar{\lambda}$*; that is,* $\tilde{c} \ge \bar{c}$*;*
(iii) *Let boundary condition* (1.6) *only be Dirichlet boundary condition, then* $\lambda$ *is the most for the ball compared to all domains* $\Omega$ *with the same volume.*

*Proof.* (i) By assumption $\tilde{V} \geq \bar{V}$, we obtain corresponding functionals satisfying $\tilde{J}_1[\bar{u}] \geq \bar{J}_1[\bar{u}]$, and by Equation (2.6), we have $\tilde{\lambda} \leq \bar{\lambda}$; that is, $\tilde{c} \leq \bar{c}$.

(ii) Let $\bar{u}$ be a non-trivial minimizer of (2.6) corresponding to $\tilde{\Omega}$. Then $1 = \tilde{\Gamma}_1[\bar{u}] \leq \bar{\Gamma}_1[\bar{u}]$, therefore, there exists a shift $\bar{u}_a = \bar{u}(z+a, y),\ a \leq 0$ such that

$$\bar{\Gamma}_1[\bar{u}_a] = e^a \bar{\Gamma}_1[\bar{u}] = 1.$$

Let us extend the minimizer in $\tilde{\Omega}$ by 0 to $\bar{\Omega}$, then since $a \leq 0,\ \bar{J}_1[\bar{u}_a] = e^a \bar{J}_1[\bar{u}] \geq -\tilde{\lambda}$, we have $\tilde{\lambda} \geq \bar{\lambda}$; that is, $\tilde{c} \geq \bar{c}$. Where $f = -\nabla_u V(u)$.

(iii) By spherical rearrangement in the coordinate $y$, we can obtain this process decreases the functional $J_1[u]$ and preserves $\Gamma_1[u]$ by [9], so $\lambda$ is the most for the ball in all other domains $\Omega$ with the same volume by (2.6). Where $f = -\nabla_u V(u)$. $\square$

**Remark 2.8.** From the above discussion, we can obtain the boundedness of the obtained solutions analogous to [13, Theorem 3.3], that is, we have $|\bar{u}(y,z)| \leq Ce^{-\lambda z}$ for some $C > 0$ and $\lambda < 0$. Then $\bar{u}(z,.) \to 0$ as $z \to -\infty$ in $C^1(\bar{\Omega})$ is obtained.

2.3. **Scalar reaction-diffusion-advection equations.** In this subsection under the assumptions (A1), (A2) and (A3), we apply our variational formulation to the scalar reaction-diffusion equation (1.8) with boundary conditions (1.9).

For traveling wave solutions of the form $u(y,z,t) = \bar{u}(y, c(z+ct))$, substituting it into (1.8), we obtain the following traveling wave equation

$$\bar{u}_z = \bar{u}_{zz} + \frac{1}{c^2}\left(\Delta_y \bar{u} + \nabla_y \varphi \cdot \nabla_y \bar{u} + f(\bar{u}, y)\right) \tag{2.10}$$

with boundary conditions (1.9).

**Definition 2.9.** Let $u \in H_2^1(\Sigma)$, define functionals

$$\Gamma_2[u] = \frac{1}{2}\int_\Sigma e^{-z+\varphi(y)} |\frac{\partial u}{\partial z}|^2 \mathrm{d}y\,\mathrm{d}z,$$
$$J_2[u] = \int_\Sigma e^{-z+\varphi(y)} \Big(\frac{1}{2}|\nabla_y u|^2 + V(u,y)\Big)\mathrm{d}y\,\mathrm{d}z,$$

where

$$V(u,y) = \begin{cases} 0, & u < 0, \\ -\int_0^u f(s,y)\mathrm{d}s, & 0 \leq u \leq 1, \\ -\int_0^1 f(s,y)\mathrm{d}s, & u > 1. \end{cases}$$

By (2.2)-(2.4) and a similar discussion in Section 2, we have the following:

$$\lambda \Gamma_2[u] + J_2[u] = 0, \tag{2.11}$$
$$\lambda = c^2 = -\inf_{\{u \in H_2^1(\Sigma) | \Gamma_2[u]=1\}} J_2[u] > 0\,. \tag{2.12}$$

Equation (2.10) is the variational equation corresponding to (2.12).

**Theorem 2.10.** *Assume that hypotheses* (A1)-(A3) *hold, then there exists a unique value of $c^\star \geq c$, where $c$ is defined by hypothesis* (A3)*, and a unique function $\bar{u} \in C^2(\Sigma) \bigcap C^1(\bar{\Sigma})$, $\bar{u} \not\equiv 0$, such that $(c^\star, \bar{u})$ is the solution of* (2.10) *with boundary conditions* (1.9)*. Moreover, $\bar{u} \in H^2(\Sigma) \bigcap W^{1,\infty}(\Sigma)$, $\bar{u}_z > 0$ in $\Sigma$, and*

$$\lim_{z \to +\infty}(.,z) = v, \quad \lim_{z \to -\infty}(.,z) = 0 \tag{2.13}$$

*in* $C^1(\bar{\Omega})$.

$$\bar{u}(y,z) = a_0\psi_0(y)e^{-\lambda_-(c^\star,\upsilon_0)} + 0(e^{-\lambda z}) \tag{2.14}$$

*for some* $a_0 > 0$ *and* $\lambda < \lambda_-(c^\star,\upsilon_0)$, *uniformly in* $C^1(\bar{\Omega}\times[-\infty,R])$, *as* $R\to-\infty$.

$$\bar{u}(y,z) = v(y) + \tilde{a}_0\tilde{\psi}_0(y)e^{-\lambda_+(c^\star,\tilde{\upsilon}_0)} + 0(e^{-\lambda z})$$

*for some* $\tilde{a}_0 > 0$ *and* $\lambda > \lambda_+(c^\star,\tilde{\upsilon}_0)$, *uniformly in* $C^1(\bar{\Omega}\times[R,+\infty])$, *as* $R\to+\infty$. *Where* $v:\Omega\mapsto\mathbb{R}$ *is a local minimizer of* $E(v)=\int_\Omega e^{\varphi(y)}(\frac{1}{2}|\nabla_y v|^2 + V(v(y),y))\mathrm{d}y$ *with* $E(v)<0$, $\tilde{\upsilon}_0$, $\tilde{\psi}_0$ *and* $\lambda_+(c^\star,\tilde{\upsilon}_0)$ *are obtained by linearizing* (2.10) *around* $u=v$ *at large* $z$.

*Proof.* Due to the discussion in Section 3.1 and the maximum principle, we only need to show $u\in H_2^1(\Sigma)$ under the assumption $c^2+4\upsilon_0>0$ in (A3) by [15].

Linearizing (2.10) around $u=0$ at large $(-z)$, then we obtain

$$u(y,z)\sim\Sigma_k a_k\psi_k(y)e^{-\lambda_k z},\quad (k=0,1,2,\dots)$$

with $(\lambda_k,\ \upsilon_k)$ satisfying

$$\lambda_k^2+\lambda_k-\frac{1}{c^2}\upsilon_k=0, \tag{2.15}$$

and $\upsilon_k\in\mathbb{R}$ is the eigenvalue defined by

$$\Delta_y\psi_k+\nabla_y\varphi\cdot\nabla_y\psi_k+f_u(0,y)\psi_k=-\upsilon_k\psi_k$$

with boundary conditions (1.9). Then by (2.15), we obtain

$$\lambda_k^\pm(c)=\frac{-1\pm\sqrt{1+\frac{4}{c^2}\upsilon_k}}{2}.$$

Hence, we obtain $u\in H_2^1(\Sigma)$ under the assumption $c^2+4\upsilon_0>0$ by the same discussion to that of [15]. Then, the existence, uniqueness, monotonicity and asymptotic behavior at infinity of the obtained traveling wave solutions are deduced by [15]. ☐

Finally, we give the new results deduced by our variational formulation.

**Theorem 2.11.** *We assume that all the nonlinearities* $f,\tilde{f},\bar{f}$ *satisfy assumptions* (A1)–(A3). *Then we have*

(i) *If* $\tilde{V}(u,y)\le\bar{V}(u,y)$, *then* $\tilde{\lambda}\ge\bar{\lambda}$; *that is,* $\tilde{c}\ge\bar{c}$. *Where*

$$\tilde{V}(u,y)\ (\bar{V}(u,y))=\begin{cases}0, & u<0,\\ -\int_0^u\tilde{f}(s,y)\mathrm{d}s\ (-\int_0^u\bar{f}(s,y)\mathrm{d}s), & 0\le u\le1,\\ -\int_0^1\tilde{f}(s,y)\mathrm{d}s\ (-\int_0^1\bar{f}(s,y)\mathrm{d}s), & u>1.\end{cases}$$

(ii) *If* $\tilde{\Omega},\bar{\Omega}\subset\mathbb{R}^{n-1}$ *are bounded domain with boundary of class* $C^2$, $f_y(u,y)=0$ *and* $\tilde{\Omega}\subset\bar{\Omega}$, *then* $\tilde{\lambda}\ge\bar{\lambda}$; *that is,* $\tilde{c}\ge\bar{c}$;

(iii) *Let* $\Omega_0=\emptyset$ *and* $f_y(u,y)=0$, *then* $\lambda$ *is the most for the ball in all other domains* $\Omega$ *with the same volume.*

The proof of the above theorem is basically the same proof as the one of Theorem 2.7. In the following, we consider the influence of the advection on traveling wave.

**Theorem 2.12.** *We assume $\varphi = \tilde{\varphi}$, $\bar{\varphi}$ and nonlinearity $f$ satisfy assumptions* (A1)-(A3)*, then if*

$$e^{\tilde{\varphi}(y)}V(e^{-\frac{\tilde{\varphi}(y)}{2}}w,y) \geq e^{\bar{\varphi}(y)}V(e^{-\frac{\bar{\varphi}(y)}{2}}w,y) \tag{2.16}$$

*for*

$$|\nabla_y\tilde{\varphi}|^2 + 2\Delta_y\tilde{\varphi} \geq |\nabla_y\bar{\varphi}|^2 + 2\Delta_y\bar{\varphi}, \tag{2.17}$$

*then we have $\tilde{\lambda} \leq \bar{\lambda}$; that is $\tilde{c} \leq \bar{c}$: If*

$$e^{\tilde{\varphi}(y)}V(e^{-\frac{\tilde{\varphi}(y)}{2}}w,y) \leq e^{\bar{\varphi}(y)}V(e^{-\frac{\bar{\varphi}(y)}{2}}w,y) \tag{2.18}$$

*for*

$$|\nabla_y\tilde{\varphi}|^2 + 2\triangle_y\tilde{\varphi} \leq |\nabla_y\bar{\varphi}|^2 + 2\triangle_y\bar{\varphi}, \tag{2.19}$$

*then we have $\tilde{\lambda} \geq \bar{\lambda}$, that is $\tilde{c} \geq \bar{c}$, where $\tilde{c}$ and $\bar{c}$ are wave speeds corresponding to $\tilde{\varphi}$ and $\bar{\varphi}$ respectively.*

*Proof.* Let $w = e^{\varphi(y)/2}u$; i.e. $u = e^{-\varphi(y)/2}w$ and replace $u$ by $w$ in the functional $\Gamma_2[u]$ and $J_2[u]$, we obtain

$$\Gamma_2[w] = \frac{1}{2}\int_\Sigma e^{-z}|\frac{\partial w}{\partial z}|^2 \mathrm{d}y\,\mathrm{d}z \tag{2.20}$$

and

$$\begin{aligned} J_2[w] = \frac{1}{2}\int_\Sigma e^{-z}\Big(|\nabla_y w|^2 + \Big(\frac{1}{4}(|\nabla_y\varphi|^2 + 2\triangle_y\varphi)\Big)w^2\Big)\mathrm{d}y\,\mathrm{d}z \\ + \int_\Sigma e^{-z+\varphi(y)}V(e^{-\frac{\varphi(y)}{2}}w,y)\mathrm{d}y\,\mathrm{d}z. \end{aligned} \tag{2.21}$$

Let $\tilde{J}_2, \tilde{\Gamma}_2; \bar{J}_2, \bar{\Gamma}_2$ be corresponding functionals to $\tilde{\varphi}$ and $\bar{\varphi}$ respectively. Thus, by virtue of (2.20) and (2.21), we have $\tilde{J}_2 \geq \bar{J}_2$ if (2.16) and (2.17) hold. And by (2.12), then $\tilde{\lambda} \leq \bar{\lambda}$; that is, $\tilde{c} \leq \bar{c}$.

Similarly, we can obtain the remaining results. □

**Theorem 2.13.** *Assume* (A1)–(A3) *hold. Then if*

$$0 \geq e^{\varphi(y)}V(e^{-\frac{\varphi(y)}{2}}w,y) \geq V(w,y) \tag{2.22}$$

*for*

$$|\nabla_y\varphi|^2 + 2\triangle_y\varphi \geq 0, \tag{2.23}$$

*then we have $\tilde{\lambda} \leq \bar{\lambda}$; that is, $\tilde{c} \leq \bar{c}$. If*

$$e^{\varphi(y)}V(e^{-\frac{\varphi(y)}{2}}w,y) \leq V(w,y) \tag{2.24}$$

*for*

$$|\nabla_y\varphi|^2 + 2\triangle_y\varphi \leq 0, \tag{2.25}$$

*then $\tilde{\lambda} \geq \bar{\lambda}$, that is $\tilde{c} \geq \bar{c}$, where the wave velocity of the advection equation* (1.8) *and equation without advection (that is, $\varphi = 0$ in Equation* (1.8)*) be $\tilde{c}$ and $\bar{c}$ respectively.*

*Proof.* Let $\tilde{J}_2$, $\tilde{\Gamma}_2$ and $\bar{J}_2$, $\bar{\Gamma}_2$ be the corresponding functionals to the advection equation (1.8) and equation without advection (that is, $\varphi = 0$ in Equation (1.8)) respectively. By (2.20) and (2.21), if (2.22) and (2.23) hold, we have $\tilde{J}_2 \geq \bar{J}_2$, and by (2.12), then $\tilde{\lambda} \leq \bar{\lambda}$, that is $\tilde{c} \leq \bar{c}$. Similarly, we also can obtain the remaining results. □

**Remark 2.14.** We can obtain some specific conditions such that (2.16) ((2.22)) and (2.18) ((2.24)) hold in Theorem 2.12 (Theorem 2.13) respectively. For example, if

$$\begin{cases} V(u,y)\geq 0 \text{ and } V(e^{-\frac{\tilde{\varphi}(y)}{2}}w,y)\geq V(e^{-\frac{\bar{\varphi}(y)}{2}}w,y) \text{ (or } V_u(u,y)\leq 0) \text{ for } \tilde{\varphi}\geq\bar{\varphi} \\ \text{or} \\ V(u,y)\leq 0 \text{ and } V(e^{-\frac{\tilde{\varphi}(y)}{2}}w,y)\geq V(e^{-\frac{\bar{\varphi}(y)}{2}}w,y) \text{ (or } V_u(u,y)\geq 0) \text{ for } \tilde{\varphi}\leq\bar{\varphi}, \end{cases}$$

$$\begin{cases} V(u,y)\geq 0 \text{ and } V(e^{-\frac{\tilde{\varphi}(y)}{2}}w,y)\leq V(e^{-\frac{\bar{\varphi}(y)}{2}}w,y) \text{ (or } V_u(u,y)\leq 0) \text{ for } \tilde{\varphi}\leq\bar{\varphi} \\ \text{or} \\ V(u,y)\leq 0 \text{ and } V(e^{-\frac{\tilde{\varphi}(y)}{2}}w,y)\leq V(e^{-\frac{\bar{\varphi}(y)}{2}}w,y) \text{ (or } V_u(u,y)\geq 0) \text{ for } \tilde{\varphi}\geq\bar{\varphi}, \end{cases}$$

$$V(u,y)\leq 0 \text{ and } V(e^{-\frac{\varphi(y)}{2}}w,y)\geq V(w,y) \text{ (or } V_u(u,y)\geq 0) \text{ for } \varphi(y)\leq 0$$

and

$$\begin{cases} V(u,y)\leq 0 \text{ and } V(e^{-\frac{\varphi(y)}{2}}w,y)\leq V(w,y) \text{ (or } V_u(u,y)\geq 0) \text{ for } \varphi(y)\geq 0 \\ \text{or} \\ V(u,y)\geq 0 \text{ and } V(e^{-\frac{\varphi(y)}{2}}w,y)\leq V(w,y) \text{ (or } V_u(u,y)\leq 0) \text{ for } \varphi(y)\leq 0 \end{cases}$$

hold, then (2.16), (2.18), (2.22) and (2.24) corresponding hold.

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