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EXISTENCE OF SOLUTIONS TO DIRICHLET IMPULSIVE DIFFERENTIAL EQUATIONS THROUGH A LOCAL MINIMIZATION PRINCIPLE

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Abstract. A critical point theorem (local minimum result) for differentiable functionals is used for proving that a Dirichlet impulsive differential equation admits at least one non-trivial solution. Some particular cases and a concrete example are also presented.

1. Introduction

In this article, we study the existence of at least one non-trivial classical solution to the nonlinear Dirichlet boundary-value problem

$$\begin{gathered} -(p(t)u'(t))' + q(t)u(t) = \lambda f(t, u(t)), \quad t \in [0, T],\ t \neq t_j, \\ u(0) = u(T) = 0, \\ \Delta u'(t_j) = \lambda I_j(u(t_j)), \quad j = 1, 2, \dots, n, \end{gathered} \tag{1.1}$$

where $T > 0$, $p \in C^1([0, T],]0, +\infty[)$, $q \in L^\infty([0, T])$, $\lambda \in]0, +\infty[$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $0 = t_0 < t_1 < t_2 < \dots < t_n < t_{n+1} = T$, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = \lim_{t\to t_j^+} u'(t) - \lim_{t\to t_j^-} u'(t)$ and $I_j : \mathbb{R} \to \mathbb{R}$ are continuous for every $j = 1, 2, \dots, n$.

The study of impulsive boundary-value problems is important due to its various applications in which abrupt changes at certain times in the evolution process appear. The dynamics of evolving processes is often subjected to abrupt changes such as shocks, harvesting, and natural disasters. Often these short-term perturbations are treated as having acted instantaneously or in the form of impulses. Such problems arise in physics, population dynamics, biotechnology, pharmacokinetics, industrial robotics.

Recently, many researchers pay their attention to impulsive differential equations by variational method and critical point theory, and we refer the reader to [12, 15, 17, 18, 19, 20] and references cited therein.

Our analysis is mainly based on the critical point theorem by Bonanno [2], contained in Theorem 2.1 below. This theorem has been used in several works

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to obtain existence results for different kinds of problems (see, for instance, [1, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14]).

2. Preliminaries

Our main tool is Ricceri's variational principle [16, Theorem 2.5] as given in [2, Theorem 5.1] which is below recalled (see also [2, Proposition 2.1]).

For a given non-empty set X, and two functionals $\Phi, \Psi : X \to \mathbb{R}$, we define the functions

$$\beta(r_1, r_2) := \inf_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\sup_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}$$

and

$$\rho(r_1, r_2) := \sup_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u)}{\Phi(v) - r_1}$$

for all $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$.

Theorem 2.1 ([2, Theorem 5.1]). *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume that there are $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho(r_1, r_2). \tag{2.1}$$

Then, setting $I_\lambda := \Phi - \lambda\Psi$, for each $\lambda \in]\frac{1}{\rho(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)}[$ there is $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

In the Sobolev space $X := H_0^1(0, T)$, consider the inner product

$$(u, v) := \int_0^T p(t)u'(t)v'(t)\,dt + \int_0^T q(t)u(t)v(t)\,dt\,,$$

which induces the norm

$$\|u\| := \Big(\int_0^T p(t)(u'(t))^2\,dt + \int_0^T q(t)(u(t))^2\,dt\Big)^{1/2}.$$

Then the following Poincaré-type inequality holds:

$$\Big[\int_0^T u^2(t)\,dt\Big]^{1/2} \le \frac{T}{\pi}\Big[\int_0^T (u')^2(t)\,dt\Big]^{1/2}. \tag{2.2}$$

Let us introduce some notation that will be used later. Assume that $\frac{T^2 q^-}{\pi^2} > -p^-$, where

$$p^- := \operatorname{ess\,inf}_{t\in[0,T]} p(t) > 0, \quad q^- := \operatorname{ess\,inf}_{t\in[0,T]} q(t).$$

Moreover, put $\sigma_0 := \min\{T^2 q^-/\pi^2, 0\}$ and $\delta := \sqrt{p^- + \sigma_0}$. Then, we have the following useful proposition.

Proposition 2.2. *Let $u \in X$. Then*

$$\|u'\|_{L^2([0,T])} \le \frac{1}{\delta}\|u\|, \tag{2.3}$$

$$\|u\|_\infty \le \frac{\sqrt{T}}{2\delta}\|u\|. \tag{2.4}$$

Proof. First we prove (2.3). To this end, let $q^- \geq 0$. Then, $\sigma_0 = 0$ and $\delta = \sqrt{p^-}$. Therefore,

$$\begin{aligned} \|u'\|^2_{L^2([0,T])} &\leq \frac{1}{p^-}\int_0^T p(t)|u'(t)|^2\,dt \\ &\leq \frac{1}{p^-}\int_0^T \big(p(t)|u'(t)|^2 + q(t)|u(t)|^2\big)\,dt = \frac{1}{\delta^2}\|u\|^2. \end{aligned}$$

Thus, the desired inequality (2.3) follows. On the other hand, if $q^- < 0$, we have $\sigma_0 = \frac{T^2 q^-}{\pi^2}$ and $\delta = \sqrt{p^- + \frac{T^2 q^-}{\pi^2}}$. Obviously,

$$q^-\|u\|^2_{L^2([0,T])} \leq \int_0^T q(t)|u(t)|^2\,dt.$$

Now, applying inequality (2.2) and bearing in mind that $q^- < 0$, one has

$$\frac{T^2 q^-}{\pi^2}\|u'\|^2_{L^2([0,T])} \leq q^-\|u\|^2_{L^2([0,T])}.$$

By the above inequalities we have

$$\frac{T^2 q^-}{\pi^2}\|u'\|^2_{L^2([0,T])} \leq \int_0^T q(t)|u(t)|^2\,dt.$$

This inequality together with

$$p^-\|u'\|^2_{L^2([0,T])} \leq \int_0^T p(t)|u'(t)|^2\,dt,$$

imply (2.3).

In view of Hölder's inequality and (2.3), one has

$$\|u\|_\infty \leq \frac{\sqrt{T}}{2}\|u'\|_{L^2([0,T])} \leq \frac{\sqrt{T}}{2\delta}\|u\|,$$

which completes and the proof. □

Put $k := \big(\|p\|_\infty + \frac{T^2}{\pi^2}\|q\|_\infty\big)^{1/2}$. Then, from (2.2) we have

$$\|u\| \leq k\|u'\|_{L^2([0,T])}. \tag{2.5}$$

Here and in the sequel $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, namely:

- (a) the mapping $t \mapsto f(t,x)$ is measurable for every $x \in \mathbb{R}$;
- (b) the mapping $x \mapsto f(t,x)$ is continuous for almost every $t \in [0,T]$;
- (c) for every $\rho > 0$ there exists a function $l_\rho \in L^1([0,T])$ such that

$$\sup_{|x|\leq\rho} |f(t,x)| \leq l_\rho(t)$$

for almost every $t \in [0,T]$.

Corresponding to f we introduce the function $F : [0,T]\times\mathbb{R} \to \mathbb{R}$ as follows

$$F(t,x) := \int_0^x f(t,\xi)\,d\xi,$$

for all $(t,x) \in [0,T]\times\mathbb{R}$.

By a *classical solution* of problem (1.1), we mean a function

$$u \in \big\{w \in C([0,T]) : w|_{[t_j,t_{j+1}]} \in H^2([t_j,t_{j+1}])\big\}$$

that satisfies the equation in (1.1) a.e. on $[0,T]\setminus\{t_1,\dots,t_n\}$, the limits $u'(t_j^+)$, $u'(t_j^-)$, $j=1,\dots,n$, exist, satisfy the impulsive conditions $\Delta u'(t_j)=\lambda I_j(u(t_j))$ and the boundary condition $u(0)=u(T)=0$.

We say that a function $u\in X$ is a *weak solution* of problem (1.1), if u satisfies

$$\begin{aligned}&\int_0^T p(t)u'(t)v'(t)\,dt+\int_0^T q(t)u(t)v(t)\,dt\\ &-\lambda\Big(\int_0^T f(t,u(t))v(t)\,dt-\sum_{j=1}^n p(t_j)I_j(u(t_j))v(t_j)\Big)=0,\end{aligned}$$

for any $v\in X$.

Lemma 2.3 ([3, Lemma 2.3]). *The function $u\in X$ is a weak solution of problem* (1.1) *if and only if u is a classical solution of* (1.1).

Lemma 2.4 ([3, Lemma 3.1]). *Assume that*

(A1) *there exist constants $\eta,\theta>0$ and $\sigma\in[0,1[$ such that*
$$|I_j(x)|\le 2\eta|x|+\theta|x|^{\sigma+1}\quad\text{for all } x\in\mathbb{R},\ j=1,2,\dots,n.$$

Then, for any $u\in X$, we have

$$\Big|\sum_{j=1}^n p(t_j)\int_0^{u(t_j)} I_j(x)\,dx\Big|\le\sum_{j=1}^n p(t_j)\Big(\eta\|u\|_\infty^2+\frac{\theta}{\sigma+2}\|u\|_\infty^{\sigma+2}\Big). \tag{2.6}$$

Also put

$$\tilde p:=\sum_{j=1}^n p(t_j),\quad \mu(\tau):=\frac{\sqrt{2}k\tau}{\delta},\quad \Gamma_c:=\frac{\eta}{c}+\Big(\frac{\theta}{\sigma+2}\Big)c^{\sigma-1},$$

where η, θ, σ are given by (A1) and c,τ are positive constants. We assume throughout, and without further mention, that the assumption (A1) holds.

3. Main results

For a given non-negative constant ν and a positive constant τ with $\delta^2\nu^2\neq 2k^2\tau^2$, put

$$a_\tau(\nu):=\frac{\int_0^T\max_{|x|\le\nu}F(t,x)\,dt+\tilde p\nu^3\Gamma_\nu+\tilde p(\mu(\tau))^3\Gamma_{\mu(\tau)}-\int_{T/4}^{3T/4}F(t,\tau)\,dt}{\delta^2\nu^2-2k^2\tau^2}.$$

Theorem 3.1. *Assume that there exist a non-negative constant ν_1 and two positive constants ν_2 and τ, with $\nu_1<\sqrt{2}\tau<\delta\nu_2/k$, such that*

(A2) *$F(t,\xi)\ge 0$ for all $(t,\xi)\in([0,\frac{T}{4}]\cup[\frac{3T}{4},T])\times[0,\tau]$;*

(A3) *$a_\tau(\nu_2)<a_\tau(\nu_1)$.*

Then, for each $\lambda\in]\frac{2}{T\,a_\tau(\nu_1)},\frac{2}{T\,a_\tau(\nu_2)}[$, problem (1.1) *admits at least one non-trivial classical solution $u_0\in X$ such that*

$$\frac{2\delta\nu_1}{\sqrt{T}}<\|u_0\|<\frac{2\delta\nu_2}{\sqrt{T}}.$$

Proof. The aim is to apply Theorem 2.1 to our problem. To this end, we introduce the functionals $\Phi,\Psi:X\to\mathbb{R}$ by setting

$$\Phi(u):=\frac12\|u\|^2,$$

$$\Psi(u) := \int_0^T F(t,u(t))\,dt - \sum_{j=1}^n p(t_j)\int_0^{u(t_j)} I_j(x)\,dx,$$

for every $u \in X$, and put

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u) \quad \forall\ u \in X.$$

Clearly, Φ and Ψ are well defined and continuously Gâteaux differentiable functionals whose Gâteaux derivatives at the point $u \in X$ are the functionals $\Phi'(u), \Psi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_0^T p(t)u'(t)v'(t)\,dt + \int_0^T q(t)u(t)v(t)\,dt,$$
$$\Psi'(u)(v) = \int_0^T f(t,u(t))v(t)\,dt - \sum_{j=1}^n p(t_j)I_j(u(t_j))v(t_j),$$

for every $v \in X$. Moreover, Φ is coercive and sequentially weakly lower semicontinuous and Ψ is sequentially weakly upper semicontinuous. Also, Φ' admits a continuous inverse on X^* and Ψ' is compact. Note that the critical points of the functional I_λ in X are exactly the weak solutions of problem (1.1). We verify condition (2.1) of Theorem 2.1. To this end, we put

$$r_1 := \frac{2\delta^2}{T}\nu_1^2, \quad r_2 := \frac{2\delta^2}{T}\nu_2^2,$$
$$w(t) := \begin{cases} \frac{4\tau}{T}t, & \text{if } t \in [0, T/4[, \\ \tau, & \text{if } t \in [T/4, 3T/4], \\ \frac{4\tau}{T}(T-t), & \text{if } t \in]3T/4, T]. \end{cases}$$

It is easy to verify that $w \in X$ and, in particular, taking (2.3) and (2.5) into account, one has

$$\frac{8\delta^2\tau^2}{T} \le \|w\|^2 \le \frac{8k^2\tau^2}{T}.$$

So, we have

$$\frac{4\delta^2\tau^2}{T} \le \Phi(w) \le \frac{4k^2\tau^2}{T}.$$

From the condition $\nu_1 < \sqrt{2}\tau < \delta\nu_2/k$, we obtain $r_1 < \Phi(w) < r_2$. Moreover, for all $u \in X$ such that $u \in \Phi^{-1}(]-\infty, r_2[)$, from (2.4), one has $|u(t)| < \nu_2$ for all $t \in [0,T]$, from which it follows

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u) &= \sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Big(\int_0^T F(t,u(t))\,dt - \sum_{j=1}^n p(t_j)\int_0^{u(t_j)} I_j(x)\,dx\Big) \\ &\le \int_0^T \max_{|x|\le\nu_2} F(t,x)\,dt + \tilde{p}\nu_2^3\Gamma_{\nu_2}.\end{aligned}$$

Arguing as before, we obtain

$$\sup_{u\in\Phi^{-1}(]-\infty,r_1])} \Psi(u) \le \int_0^T \max_{|x|\le\nu_1} F(t,x)\,dt + \tilde{p}\nu_1^3\Gamma_{\nu_1}.$$

Since $0 \le w(t) \le \tau$ for each $t \in [0,T]$, assumption (A2) ensures that

$$\int_0^{T/4} F(t,w(t))\,dt + \int_{3T/4}^T F(t,w(t))\,dt \ge 0,$$

and so

$$\begin{aligned}\Psi(w) &\geq \int_{T/4}^{3T/4} F(t,\tau)\,dt - \sum_{j=1}^{n} p(t_j)\int_0^{w(t_j)} I_j(x)\,dx\\ &\geq \int_{T/4}^{3T/4} F(t,\tau)\,dt - \tilde{p}\,(\mu(\tau))^3\Gamma_{\mu(\tau)}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\beta(r_1,r_2) &\leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)-\Psi(w)}{r_2-\Phi(w)}\\ &\leq \frac{\int_0^T \max_{|x|\leq\nu_2} F(t,x)\,dt + \tilde{p}\nu_2^3\Gamma_{\nu_2} + \tilde{p}(\mu(\tau))^3\Gamma_{\mu(\tau)} - \int_{T/4}^{3T/4} F(t,\tau)\,dt}{\frac{2\delta^2\nu_2^2}{T} - \frac{4k^2\tau^2}{T}}\\ &= \frac{T}{2} a_\tau(\nu_2).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\rho(r_1,r_2) &\geq \frac{\Psi(w)-\sup_{u\in\Phi^{-1}(]-\infty,r_1])}\Psi(u)}{\Phi(w)-r_1}\\ &\geq \frac{\int_{T/4}^{3T/4} F(t,\tau)\,dt - \tilde{p}(\mu(\tau))^3\Gamma_{\mu(\tau)} - \tilde{p}\nu_1^3\Gamma_{\nu_1} - \int_0^T \max_{|x|\leq\nu_1} F(t,x)\,dt}{\frac{4k^2\tau^2}{T} - \frac{2\delta^2\nu_1^2}{T}}\\ &= \frac{T}{2} a_\tau(\nu_1).\end{aligned}$$

So, from assumption (A3), it follows that $\beta(r_1,r_2) < \rho(r_1,r_2)$. Therefore, from Theorem 2.1, for each $\lambda \in]\frac{2}{Ta_\tau(\nu_1)}, \frac{2}{Ta_\tau(\nu_2)}[$, the functional I_λ admits at least one critical point u_0 such that $r_1 < \Phi(u_0) < r_2$; that is,

$$\frac{2\delta\nu_1}{\sqrt{T}} < \|u_0\| < \frac{2\delta\nu_2}{\sqrt{T}},$$

and the proof is complete. □

Now, we point out the following consequence of Theorem 3.1.

Theorem 3.2. *Assume that there are two positive constants ν and τ, with $\sqrt{2}k\tau < \delta\nu$, such that assumption* (A2) *in Theorem 3.1 holds. Furthermore, suppose that*

(A4) $\frac{\int_0^T \max_{|x|\leq\nu} F(t,x)\,dt + \tilde{p}\nu^3\Gamma_\nu}{\nu^2} < \frac{\delta^2}{2k^2}\frac{\int_{T/4}^{3T/4} F(t,\tau)\,dt - \tilde{p}\,(\mu(\tau))^3\Gamma_{\mu(\tau)}}{\tau^2}$.

Then, for each

$$\lambda \in \Big]\frac{\frac{4k^2\tau^2}{T}}{\int_{T/4}^{3T/4} F(t,\tau)\,dt - \tilde{p}\,(\mu(\tau))^3\Gamma_{\mu(\tau)}}, \frac{\frac{2\delta^2\nu^2}{T}}{\int_0^T \max_{|x|\leq\nu} F(t,x)\,dt + \tilde{p}\nu^3\Gamma_\nu}\Big[,$$

problem (1.1) *admits at least one non-trivial classical solution $u_0 \in X$ such that $|u_0(t)| < \nu$ for all $t \in [0,T]$.*

Proof. The conclusion follows from Theorem 3.1, by taking $\nu_1 = 0$ and $\nu_2 = \nu$. Indeed, owing to assumption (A4), one has

$$a_\tau(\nu) < \frac{\big(1-\frac{2k^2\tau^2}{\nu^2\delta^2}\big)\big(\int_0^T \max_{|x|\leq\nu} F(t,x)\,dt + \tilde{p}\nu^3\Gamma_\nu\big)}{\delta^2\nu^2 - 2k^2\tau^2}$$

$$= \frac{\int_0^T \max_{|x|\le\nu} F(t,x)\,dt + \tilde{p}\nu^3\Gamma_\nu}{\delta^2\nu^2}.$$

On the other hand,

$$a_\tau(0) = \frac{\int_{T/4}^{3T/4} F(t,\tau)\,dt - \tilde{p}(\mu(\tau))^3\Gamma_{\mu(\tau)}}{2k^2\tau^2}.$$

Now, owing to assumption (A4) and (2.4), it is sufficient to invoke Theorem 3.1 for concluding the proof. □

The following result gives the existence of at least one non-trivial classical solution in X to problem (1.1) in the autonomous case. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. Put $F(x) := \int_0^x f(\xi)\,d\xi$ for all $x\in\mathbb{R}$. We have the following result as a direct consequence of Theorem 3.1.

Theorem 3.3. *Assume that there exist a non-negative constant ν_1 and two positive constants ν_2 and τ, with $\nu_1 < \sqrt{2}\tau < \delta\nu_2/k$, such that*

- (A5) $f(x) \ge 0$ *for all* $x \in [-\nu_2, \max\{\nu_2,\tau\}]$;
- (A6)

$$\begin{aligned} &\frac{TF(\nu_2) + \tilde{p}{\nu_2}^3\Gamma_{\nu_2} + \tilde{p}\,(\mu(\tau))^3\Gamma_{\mu(\tau)} - \frac{T}{2}F(\tau)}{\delta^2\nu_2^2 - 2k^2\tau^2} \\ &< \frac{TF(\nu_1) + \tilde{p}{\nu_1}^3\Gamma_{\nu_1} + \tilde{p}(\mu(\tau))^3\Gamma_{\mu(\tau)} - \frac{T}{2}F(\tau)}{\delta^2\nu_1^2 - 2k^2\tau^2}. \end{aligned}$$

Then, for each

$$\begin{aligned} \lambda \in \Big] &\frac{2\delta^2\nu_2^2 - 4k^2\tau^2}{T\left(TF(\nu_2) + \tilde{p}\,{\nu_2}^3\Gamma_{\nu_2} + \tilde{p}\,(\mu(\tau))^3\Gamma_{\mu(\tau)} - \frac{T}{2}F(\tau)\right)}, \\ &\frac{2\delta^2\nu_2^2 - 4k^2\tau^2}{T\left(TF(\nu_1) + \tilde{p}\,{\nu_1}^3\Gamma_{\nu_1} + \tilde{p}\,(\mu(\tau))^3\Gamma_{\mu(\tau)} - \frac{T}{2}F(\tau)\right)}\Big[, \end{aligned}$$

the problem

$$\begin{gathered} -(p(t)u'(t))' + q(t)u(t) = \lambda f(u(t)), \quad t\in[0,T],\ t\ne t_j, \\ u(0) = u(T) = 0, \\ \Delta u'(t_j) = \lambda I_j(u(t_j)), \quad j = 1,2,\dots,n, \end{gathered} \tag{3.1}$$

admits at least one non-trivial classical solution $u_0 \in X$ such that

$$\frac{2\delta\nu_1}{\sqrt{T}} < \|u_0\| < \frac{2\delta\nu_2}{\sqrt{T}}.$$

Proof. Since $\delta \le k$, from the condition $\nu_1 < \sqrt{2}\tau < \frac{\delta\nu_2}{k}$ we obtain $\nu_1 < \nu_2$. Therefore, assumption (A5) means $f(x) \ge 0$ for each $x\in[-\nu_1,\nu_1]$ and $f(x)\ge 0$ for each $x\in[-\nu_2,\nu_2]$, which implies

$$\max_{x\in[-\nu_1,\nu_1]} F(x) = F(\nu_1) \quad\text{and}\quad \max_{x\in[-\nu_2,\nu_2]} F(x) = F(\nu_2).$$

So, from assumptions (A5) and (A6), we arrive at assumptions (A2) and (A3), respectively. Hence, Theorem 3.1 yields the conclusion. □

Theorem 3.4. *Let $f:\mathbb{R}\to\mathbb{R}$ be a non-negative continuous function such that*

- (A7) $\lim_{\xi\to 0^+} \frac{f(\xi)}{\xi} = +\infty$.

Then, for each $\lambda \in]0, \frac{2\delta^2}{T} \sup_{\nu>0} \frac{\nu^2}{TF(\nu)+\tilde{p}\nu^3\Gamma_\nu}[$*, problem* (3.1) *admits at least one non-trivial classical solution* $u_0 \in X$*.*

Proof. For fixed λ as in the conclusion, there exists a positive constant ν such that

$$\lambda(TF(\nu) + \tilde{p}\nu^3\Gamma_\nu) < \frac{2\delta^2\nu^2}{T}. \tag{3.2}$$

Moreover, assumption (A7) implies that $\lim_{t\to 0^+} \frac{F(\xi)}{\xi^2} = +\infty$. On the other hand,

$$\lim_{\xi\to 0^+} \frac{(\mu(\xi))^3\Gamma_{\mu(\xi)}}{\xi^2} = \begin{cases} \eta\big(\frac{\sqrt{2}\,k}{\delta}\big)^2, & \text{if } 0 < \sigma < 1, \\ \Gamma_1\big(\frac{\sqrt{2}\,k}{\delta}\big)^2, & \text{if } \sigma = 0. \end{cases}$$

Therefore,

$$\lim_{\xi\to 0^+} \frac{F(\xi) - (\mu(\xi))^3\Gamma_{\mu(\xi)}}{\xi^2} = +\infty.$$

So, a positive constant τ satisfying $\sqrt{2}k\tau < \delta\nu$ can be chosen such that

$$\lambda\Big(\frac{\frac{T}{2}F(\tau) - \tilde{p}\,(\mu(\tau))^3\Gamma_{\mu(\tau)}}{\tau^2}\Big) > \frac{4k^2}{T}. \tag{3.3}$$

Hence, taking (3.2) and (3.3) into account, Theorem 3.1 ensures the conclusion. □

Remark 3.5. Taking (A7) into account, fix $\rho > 0$ such that $f(\xi) > 0$ for all $\xi \in]0, \rho[$. Then, put

$$\lambda_\rho := \frac{2\delta^2}{T} \sup_{\nu\in]0,\rho[} \frac{\nu^2}{TF(\nu) + \tilde{p}\nu^3\Gamma_\nu}.$$

The result of Theorem 3.4 for every $\lambda \in]0, \lambda_\rho[$ holds with $|u_0(t)| < \rho$ for all $t \in [0, T]$, where u_0 is the ensured non-trivial classical solution in X.

Example 3.6. Let $I(u(t_1)) = u(t_1)$ for some $t_1 \in (0, 1)$. Then $I : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the sublinear growth condition (A1) with $\eta = \theta = \frac{1}{3}$ and $\sigma = 0$. Now, put $p(t) = 1$, $q(t) = \frac{-\pi^2}{2}$ for all $t \in [0, 1]$ and $f(\xi) = (1 + \xi)e^\xi$ for every $\xi \in \mathbb{R}$. Clearly, one has $\delta = \frac{1}{\sqrt{2}}$. Hence, since

$$\sup_{\nu\in]0,1[} \frac{\nu^2}{\int_0^\nu f(\xi)\,d\xi + \nu^3\Gamma_\nu} = \sup_{\nu\in]0,1[} \frac{\nu^2}{\nu e^\nu + \nu^3\Gamma_\nu} = \frac{2}{2e + 1},$$

from Remark 3.5, for every $\lambda \in \big]0, \frac{2}{2e+1}\big[$ the problem

$$\begin{gathered} -u''(t) - \frac{\pi^2}{2}u(t) = \lambda(1 + u(t))e^{u(t)}, \quad \text{a.e. in } [0, 1], \\ u(0) = u(1) = 0, \\ \Delta u'(t_1) = \lambda u(t_1), \end{gathered}$$

has at least one non-trivial classical solution $u_0 \in H_0^1(0, 1)$ such that $|u_0(t)| < 1$ for all $t \in [0, 1]$.

Here, we point out a special situation of our main result when the nonlinear term has separable variables. To be precise, let $\alpha \in L^1([0, T])$ such that $\alpha(t) \geq 0$

a.e. $t \in [0,T]$, $\alpha \not\equiv 0$, and let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function. Consider the following Dirichlet boundary-value problem

$$\begin{gathered} -(p(t)u'(t))' + q(t)u(t) = \lambda\alpha(t)g(u(t)), \quad t \in [0,T],\ t \neq t_j, \\ u(0) = u(T) = 0, \\ \Delta u'(t_j) = \lambda I_j(u(t_j)), \quad j = 1,2,\dots,n. \end{gathered} \tag{3.4}$$

Put $G(x) := \int_0^x g(\xi)\,d\xi$ for all $x \in \mathbb{R}$, and set $\|\alpha\|_1 := \int_0^T \alpha(t)\,dt$ and $\alpha_0 := \int_{T/4}^{3T/4} \alpha(t)\,dt$.

Corollary 3.7. *Let $I_j(x) \le 0$ for all $x \in \mathbb{R}$, $j = 1,\dots,n$. Assume that there exist a non-negative constant ν_1 and two positive constants ν_2 and τ, with $\nu_1 < \sqrt{2}\tau < \delta\nu_2/k$, such that*

(A8)
$$\begin{aligned} &\frac{G(\nu_2)\|\alpha\|_1 + \tilde{p}\nu_2^3\Gamma_{\nu_2} + \tilde{p}(\mu(\tau))^3\Gamma_{\mu(\tau)} - G(\tau)\alpha_0}{\delta^2\nu_2^2 - 2k^2\tau^2} \\ &< \frac{G(\nu_1)\|\alpha\|_1 + \tilde{p}\nu_1^3\Gamma_{\nu_1} + \tilde{p}(\mu(\tau))^3\Gamma_{\mu(\tau)} - G(\tau)\alpha_0}{\delta^2\nu_1^2 - 2k^2\tau^2}. \end{aligned}$$

Then, for each

$$\begin{aligned} \lambda \in \Big]&\frac{2\delta^2\nu_1^2 - 4k^2\tau^2}{T\big(G(\nu_1)\|\alpha\|_1 + \tilde{p}\nu_1^3\Gamma_{\nu_1} + \tilde{p}(\mu(\tau))^3\Gamma_{\mu(\tau)} - G(\tau)\alpha_0\big)}, \\ &\frac{2\delta^2\nu_2^2 - 4k^2\tau^2}{T\big(G(\nu_2)\|\alpha\|_1 + \tilde{p}\nu_2^3\Gamma_{\nu_2} + \tilde{p}(\mu(\tau))^3\Gamma_{\mu(\tau)} - G(\tau)\alpha_0\big)}\Big[, \end{aligned}$$

problem (3.4) *admits at least one positive classical solution $u_0 \in X$, such that*

$$\frac{2\delta\nu_1}{\sqrt{T}} < \|u_0\| < \frac{2\delta\nu_2}{\sqrt{T}}.$$

Proof. Put $f(t,\xi) := \alpha(t)g(\xi)$ for all $(t,\xi) \in [0,T] \times \mathbb{R}$. Clearly, $F(t,x) = \alpha(t)G(x)$ for all $(t,x) \in [0,T] \times \mathbb{R}$. Therefore, taking into account that G is a non-decreasing function, Theorem 3.1 and [3, Lemma 3.6] ensure the conclusion. $\square$

An immediate consequence of Corollary 3.7 is the following.

Corollary 3.8. *Let $I_j(x) \le 0$ for all $x \in \mathbb{R}$, $j = 1,\dots,n$. Assume that there exist two positive constants ν and τ, with $\sqrt{2}k\tau < \delta\nu$, such that*

(A9) $\frac{G(\nu)\|\alpha\|_1 + \tilde{p}\nu^3\Gamma_\nu}{\nu^2} < \frac{\delta^2}{2k^2}\frac{G(\tau)\alpha_0 - \tilde{p}(\mu(\tau))^3\Gamma_{\mu(\tau)}}{\tau^2}$.

Then, for each

$$\lambda \in \Big]\frac{4k^2\tau^2}{T\big(G(\tau)\alpha_0 - \tilde{p}(\mu(\tau))^3\Gamma_{\mu(\tau)}\big)}, \frac{2\delta^2\nu^2}{T\big(G(\nu)\|\alpha\|_1 + \tilde{p}\nu^3\Gamma_\nu\big)}\Big[,$$

problem (3.4) *admits at least one positive classical solution $u_0 \in X$, such that $|u_0(t)| < \nu$ for all $t \in [0,T]$.*

The above corollary follows directly from Theorem 3.2 and [3, Lemma 3.6].

Now, consider the nonlinear Dirichlet boundary-value problem

$$\begin{gathered} -u''(t)+a(t)u'(t)+b(t)u(t)=\lambda h(t,u(t)), \quad t\in[0,T],\ t\neq t_j,\\ u(0)=u(T)=0,\\ \Delta u'(t_j)=\lambda I_j(u(t_j)), \quad j=1,2,\dots,n, \end{gathered} \tag{3.5}$$

where $h:[0,T]\times\mathbb{R}\to\mathbb{R}$ is an L^1-Carathéodory function and $a,b\in L^\infty([0,T])$ satisfy the following conditions

$$\operatorname{ess\,inf}_{t\in[0,T]} a(t)\geq 0, \quad \operatorname{ess\,inf}_{t\in[0,T]} \big\{b(t)e^{-A(t)}\big\} > -\frac{\pi^2}{T^2}e^{-A(T)},$$

where $A(t)$ be a primitive of $a(t)$.

It is easy to see that the solutions of (1.1) are solutions of (3.5) if

$$p(t)=e^{-\int_0^t a(\xi)\,d\xi}, \quad q(t)=b(t)e^{-\int_0^t a(\xi)\,d\xi}, \quad f(t,u)=h(t,u)e^{-\int_0^t a(\xi)\,d\xi}.$$

Let $H(t,x):=\int_0^x h(t,\xi)\,d\xi$. Then, by a simple computation, we obtain

$$F(t,x)=e^{-A(t)}H(t,x), \quad \forall (t,x)\in[0,T]\times\mathbb{R}.$$

Set

$$\tilde{k}:=\Big(1+\frac{T^2}{\pi^2}\|be^{-A}\|_\infty\Big)^{1/2}, \quad \tilde{\sigma}_0:=\min\Big\{\frac{T^2}{\pi^2}\operatorname{ess\,inf}_{t\in[0,T]}\big(b(t)e^{-A(t)}\big),0\Big\},$$
$$\tilde{\delta}:=\sqrt{e^{-A(T)}+\tilde{\sigma}_0}.$$

For a given non-negative constant ν and a positive constant τ with $\tilde{\delta}^2\nu^2\neq 2\tilde{k}^2\tau^2$, put $\tilde{\mu}(\tau):=\sqrt{2}\tilde{k}\tau/\tilde{\delta}$ and

$$\begin{aligned} \tilde{a}_\tau(v):=\Big(&\int_0^T e^{-A(t)}\max_{|x|\leq\nu}H(t,x)\,dt+\tilde{p}\nu^3\Gamma_\nu+\tilde{p}\,(\tilde{\mu}(\tau))^3\Gamma_{\tilde{\mu}(\tau)}\\ &-\int_{T/4}^{3T/4}e^{-A(t)}H(t,\tau)\,dt\Big)\Big/\big(\tilde{\delta}^2\nu^2-2\tilde{k}^2\tau^2\big). \end{aligned}$$

With the above notation and by Theorem 3.1, we obtain the following existence property for problem (3.5).

Theorem 3.9. *Assume that there exist a non-negative constant ν_1 and two positive constants ν_2 and τ, with $\nu_1<\sqrt{2}\tau<\tilde{\delta}\nu_2/\tilde{k}$, such that*

(A10) *$H(t,\xi)\geq 0$ for all $(t,\xi)\in([0,\frac{T}{4}]\cup[\frac{3T}{4},T])\times[0,\tau]$;*
(A11) *$\tilde{a}_\tau(\nu_2)<\tilde{a}_\tau(\nu_1)$.*

Then, for each $\lambda\in]\frac{2}{T\,\tilde{a}_\tau(\nu_1)},\frac{2}{T\,\tilde{a}_\tau(\nu_2)}[$, problem (3.5) admits at least one non-trivial classical solution $u_0\in X$ such that

$$\frac{2\tilde{\delta}\nu_1}{\sqrt{T}}<\|u_0\|<\frac{2\tilde{\delta}\nu_2}{\sqrt{T}}.$$

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