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INITIAL DATA PROBLEMS FOR THE TWO-COMPONENT CAMASSA-HOLM SYSTEM

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Abstract. This article concerns the study of some properties of the two-component Camassa-Holm system. By constructing two sequences of solutions of the two-component Camassa-Holm system, we prove that the solution map of the Cauchy problem of the two-component Camassa-Holm system is not uniformly continuous in $H^s(\mathbb{R})$, $s > 5/2$.

1. Introduction

Many authors have studied shallow water equations, of which a typical example is Camassa-Holm (CH) equation. This equation has been extended to a two-component integrable system (CH2) by combining its integrability property with compressibility, or free-surface elevation dynamics in its shallow-water interpretation [10, 23]:

$$\begin{gathered} m_t + um_x + 2mu_x + \sigma\rho\rho_x = 0, \quad t > 0,\ x \in \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, \quad t > 0,\ x \in \mathbb{R}, \end{gathered} \tag{1.1}$$ 1.1

where $m = u - u_{xx}$ and $\sigma = \pm 1$. We remark that $\sigma = 1$ is the hydrodynamically relevant choice, see the discussion in [10]. Local well-posedness of (1.1) with $\sigma = 1$ was obtained by [10, 11]. The precise blow-up scenarios and blow-up phenomena of strong solution for (1.1) was established by [10, 11, 13, 15, 19, 17]. Guan-Yin obtained the existence of global weak solution to (1.1). Just recently, Gui and Liu [18] studied (1.1) with $\sigma = 1$ in Besov space and they obtained the local well-posedness. In this paper, we consider the Cauchy problem of (1.1) and study the some properties of it.

If $\rho \equiv 0$, then (1.1) becomes the well-known Camassa-Holm equation [3]. In the past decade, the Camassa-Holm equation has attracted much attention because of its integrability and the existence of multi-peakon solutions, see [1]-[7] and [33]-[35] for the details. The Cauchy problem and initial boundary value problem of the Camassa-Holm equation have been studied extensively [5, 12]. It has been shown that the Camassa-Holm equation is locally well-posedness [5] for initial data $u_0 \in H^s(\mathbb{R})$, $s > 3/2$. Moreover, it has global strong solutions [5] and finite time

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blow-up solutions [5, 6, 8]. On the other hand, it has global weak solution in $H^1(\mathbb{R})$ [1, 2, 3, 7]. The advantage of the Camassa-Holm equation in comparison with the KdV equation lies in the fact that the Camassa-Holm equation has peaked solutions and models wave breaking (i.e. the solution remains bounded while its slope becomes unbounded in finite time [3, 5, 6, 30]). Here peaked solutions are actually peaked traveling waves, similar to the waves of greatest height encountered in classical hydrodynamics, see the discussion in the papers [4, 9, 31]. Moreover, there is a rich geometric structure underlying the Camassa-Holm equation, see the discussion in the papers [25, 26].

Recently, some properties of solutions to the Camassa-Holm equation have been studied by many authors. Himonas et al. [20] studied the persistence properties and unique continuation of solutions of the Camassa-Holm equation. They showed that a strong solution of the Camassa-Holm equation, initially decaying exponentially together with its spacial derivative, must be identically equal to zero if it also decays exponentially at a later time, see [35, 14] for the similar properties of solutions to other shallow water equation. Just recently, Himonas-Kenig [21] and Himonas et al. [22] considered the non-uniform dependence on initial data for the Camassa-Holm equation on the line and on the circle, respectively. Lv et al. [27] obtained the non-uniform dependence on initial data for μ-b equation. Lv-Wang [28] considered the (1.1) with $\rho = \gamma - \gamma_{xx}$ and obtained the non-uniform dependence on initial data. Wang [32] obtained the non-uniform dependence on initial data of periodic Camassa-Holm system. Tang-Wang [29] obtained the Hölder continuous of Camassa-Holm system.

In this paper, we consider the non-uniform dependence on initial data for (1.1). We remark that there is significant difference between (1.1) and (1.1) with $\rho = \gamma - \gamma_{xx}$. It is easy to see that when $\rho = \gamma - \gamma_{xx}$, there are some similar properties between the two equations in (1.1). Thus the proof of non-uniform dependence on initial data to (1.1) with $\rho = \gamma - \gamma_{xx}$ is similar to the single equation, for example, Camassa-Holm equation. But in (1.1), ρ and u have different properties, see Theorem 2.1. This needs construct different asymptotic solution, see section 3. Besides, the results in this paper are different from those in [27] because of the difference of the two operators $1 - \partial_{xx}$ and $\mu - \partial_{xx}$.

This article is organized as follows. In section 2, we recall the well-posedness result of Constantin-Ivanov [10] and Escher et al. [11] and use it to prove the basic energy estimate from which we derive a lower bound for the lifespan of the solution as well as an estimate of the $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ norm of the solution $(u(t,x), \rho(t,x))$ in terms of $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ norm of the initial data (u_0, ρ_0). In section 3, we construct approximate solutions, compute the error and estimate the H^1-norm of this error. In section 4, we estimate the difference between approximate and actual solutions, where the exact solution is a solution to (1.1) with initial data given by the approximate solutions evaluated at time zero. The non-uniform dependence on initial data for (1.1) is established in section 5 by constructing two sequences of solutions to (1.1) in a bounded subset of the Sobolev space $H^s(\mathbb{R})$, whose distance at the initial time is converging to zero while at any later time it is bounded below by a positive constant.

Notation. In the following, we denote by $*$ the spatial convolution. Given a Banach space Z, we denote its norm by $\|\cdot\|_Z$. Since all space of functions are over $\mathbb{R}$, for simplicity, we drop $\mathbb{R}$ in our notations of function spaces if there is no ambiguity. Let $[A, B] = AB - BA$ denotes the commutator of linear operator A and B. Set $\|z\|^2_{H^s\times H^{s-1}} = \|u\|^2_{H^s} + \|\rho\|^2_{H^{s-1}}$, where $z = (u, \rho)$.

2. Local well-posedness

In this section we first recall the known results of Constantin-Ivanov [10] and Escher et al. [11] and give a new estimate of the solution to (1.1).

Let $\Lambda = (1 - \partial_x^2)^{1/2}$. Then the operator Λ^{-2} acting on $L^2(\mathbb{R})$ can be expressed by its associated Green's function $G(x) = \frac{1}{2}e^{-|x|}$ as

$$\Lambda^{-2} f(x) = (G * f)(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|} f(y)\mathrm{d}y, \quad f \in L^2(\mathbb{R}).$$

Hence (1.1) is equivalent to the system

$$\begin{gathered} u_t + uu_x = -\partial_x\Lambda^{-2}\big(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2\big), \quad t > 0,\ x \in \mathbb{R}, \\ \rho_t + u\rho_x = -u_x\rho, \quad t > 0,\ x \in \mathbb{R}, \end{gathered} \tag{2.1}$$ 2.1

with initial data

$$u(0, x) = u_0(x), \quad \rho(0, x) = \rho_0(x), \quad x \in \mathbb{R}. \tag{2.2}$$ 2.1a

The following result is given by Constantin-Ivanov [10] and Escher et al. [11].

t2.1 **Theorem 2.1.** *Given $z_0 = (u_0, \rho_0) \in H^s \times H^{s-1}$, $s \geq 2$. Then there exists a maximal existence time $T = T(\|z_0\|_{H^s\times H^{s-1}}) > 0$ and a unique solution $z = (u, \rho)$ to (2.1) with (2.2) such that*

$$z = z(\cdot, z_0) \in C([0, T); H^s \times H^{s-1}) \cap C^1([0, T); H^{s-1} \times H^{s-2}).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping

$$z_0 \mapsto z(\cdot, z_0) : H^s \times H^{s-1} \to C([0, T); H^s \times H^{s-1}) \cap C^1([0, T); H^{s-1} \times H^{s-2})$$

is continuous.

Next, we will give an explicit estimate for the maximal existence time T. Also, we will show that at any time t in the time interval $[0, T_0]$ the H^s-norm of the solution $z(t, x)$ is dominated by the H^s-norm of the initial data $z_0(x)$. In order to do this, we need the following lemmas.

l2.3 **Lemma 2.2** ([24])**.** *If $r > 0$, then*

$$\|[\Lambda^r, f]g\|_2 \leq C(\|f_x\|_\infty\|\Lambda^{r-1}g\|_2 + \|\Lambda^r f\|_2\|g\|_\infty),$$

where C is a positive constant depending only on r.

t2.2 **Theorem 2.3.** *Let $s > 5/2$. If $z = (u, \rho)$ is a solution of (2.1) with initial data z_0 described in Theorem 2.1, then the maximal existence time T satisfies*

$$T \geq T_0 := \frac{1}{2C_s\|z_0\|_{H^s\times H^{s-1}}}, \tag{2.3}$$ 2.2

where C_s is a constant depending only on s. Also, we have

$$\|z(t)\|_{H^s\times H^{s-1}} \leq 2\|z_0\|_{H^s\times H^{s-1}}, \quad 0 \leq t \leq T_0. \tag{2.4}$$ 2.3

Proof. The derivation of the lower bound for the maximal existence time (2.3) and the solution size estimate (2.4) is based on the following differential inequality for the solution z:

$$\frac{1}{2}\frac{d}{dt}\|z(t)\|^2_{H^s\times H^{s-1}} \le C_s\|z(t)\|^3_{H^s\times H^{s-1}}, \quad 0\le t<T. \tag{2.5}$$

2.4

Suppose that (2.5) holds. Then, integrating (2.5) from 0 to t, we have

$$\|z(t)\|_{H^s\times H^{s-1}} \le \frac{\|z_0\|_{H^s\times H^{s-1}}}{1-C_s\|z_0\|_{H^s\times H^{s-1}}t}.$$

From this inequality it follows that $\|z(t)\|_{H^s\times H^{s-1}}$ is finite if $C_s\|z_0\|_{H^s\times H^{s-1}}t<1$. Let $T_0=\frac{1}{2C_s\|z_0\|_{H^s\times H^{s-1}}}$, then, for $0\le t\le T_0$, we have

$$\|z(t)\|_{H^s\times H^{s-1}} \le \frac{\|z_0\|_{H^s\times H^{s-1}}}{1-C_s\|z_0\|_{H^s\times H^{s-1}}T_0} = 2\|z_0\|_{H^s\times H^{s-1}}.$$

Now we prove the inequality (2.5). Note that the products uu_x and $u\rho_x$ are only in H^{s-1} if $u,\rho\in H^s$. To deal with this problem, we will consider the following modified system

$$\begin{gathered}(J_\varepsilon u)_t + J_\varepsilon(uu_x) = -\partial_x\Lambda^{-2}\Big(J_\varepsilon u^2+\frac{1}{2}J_\varepsilon u_x^2+\frac{1}{2}J_\varepsilon\rho^2\Big), \quad t>0,\ x\in\mathbb{R},\\ (J_\varepsilon\rho)_t + J_\varepsilon(u\rho_x) = -J_\varepsilon(u_x\rho), \quad t>0,\ x\in\mathbb{R},\end{gathered} \tag{2.6}$$

2.5

where for each $\varepsilon\in(0,1]$ the operator J_ε is the Friedrichs mollifier defined by

$$J_\varepsilon f(x) = J_\varepsilon(f)(x) = j_\varepsilon * f.$$

Here $j_\varepsilon(x)=\frac{1}{\varepsilon}j(\frac{x}{\varepsilon})$, and $j(x)$ is a C^∞ function supported in the interval $[-1,1]$ such that $j(x)\ge 0$, $\int_{\mathbb{R}} j(x)\mathrm{d}x=1$. Applying the operator Λ^s and Λ^{s-1} to the first and second equations of (2.6) respectively, then multiplying the resulting equations by $\Lambda^sJ_\varepsilon u$ and $\Lambda^{s-1}J_\varepsilon\rho$, respectively, and integrating them with respect to $x\in\mathbb{R}$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|J_\varepsilon u\|^2_{H^s} &= -\int_{\mathbb{R}}\Lambda^sJ_\varepsilon(uu_x)\Lambda^sJ_\varepsilon u\mathrm{d}x\\ &\quad-\int_{\mathbb{R}}\partial_x\Lambda^{s-2}\partial_x\Lambda^{-2}\Big(J_\varepsilon u^2+\frac{1}{2}J_\varepsilon u_x^2+\frac{1}{2}J_\varepsilon\rho^2\Big)\Lambda^sJ_\varepsilon u\mathrm{d}x,\end{aligned} \tag{2.7}$$

2.6

$$\frac{1}{2}\frac{d}{dt}\|J_\varepsilon\rho\|^2_{H^{s-1}} = -\int_{\mathbb{R}}\Lambda^{s-1}J_\varepsilon(u\rho_x)\Lambda^{s-1}J_\varepsilon\rho\mathrm{d}x - \int_{\mathbb{R}}\Lambda^{s-1}J_\varepsilon(u_x\rho)\Lambda^{s-1}J_\varepsilon\rho\mathrm{d}x. \tag{2.8}$$

2.7

Similar to [32], we can estimate the right-hand sides of (2.7) and (2.8). We obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|J_\varepsilon u\|^2_{H^s} &\le C_s(\|u\|_\infty+\|\rho\|_\infty+\|u_x\|_\infty+\|\rho_x\|_\infty)(\|u\|^2_{H^s}+\|\rho\|^2_{H^{s-1}}),\\ \frac{1}{2}\frac{d}{dt}\|J_\varepsilon\rho\|^2_{H^{s-1}} &\le C_s(\|u\|_\infty+\|\rho\|_\infty+\|u_x\|_\infty+\|\rho_x\|_\infty)(\|u\|^2_{H^s}+\|\rho\|^2_{H^{s-1}}).\end{aligned}$$

Consequently,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|J_\varepsilon u\|^2_{H^s}+\|J_\varepsilon\rho\|^2_{H^{s-1}}\right)\\ &\le C_s(\|u\|_\infty+\|\rho\|_\infty+\|u_x\|_\infty+\|\rho_x\|_\infty)(\|u\|^2_{H^s}+\|\rho\|^2_{H^{s-1}}).\end{aligned}$$

Then, letting ε aproach 0, we have

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{H^s}^2+\|\rho\|_{H^{s-1}}^2\right)\le C_s(\|u\|_\infty+\|\rho\|_\infty+\|u_x\|_\infty+\|\rho_x\|_\infty)(\|u\|_{H^s}^2+\|\rho\|_{H^{s-1}}^2),$$

or

$$\frac{1}{2}\frac{d}{dt}\|z(t)\|_{H^s\times H^{s-1}}^2\le C_s(\|u(t)\|_{C^1}+\|\rho\|_{C^1})\|z(t)\|_{H^s\times H^{s-1}}^2. \tag{2.9}$$ 2.19

Since $s>5/2$, using Sobolev's inequality we have that

$$\|u(t)\|_{C^1}\le C_s\|u(t)\|_{H^s},\quad \|\rho(t)\|_{C^1}\le C_s\|\rho(t)\|_{H^{s-1}}.$$

From (2.9) we obtain the desired inequality (2.5). This completes the proof of Theorem 2.3. □

Recall that $\|z(t)\|_{H^s\times H^{s-1}}^2=\|u(t)\|_{H^s}^2+\|\rho(t)\|_{H^{s-1}}^2$, where $z(t)=(u(t),\rho(t))$. It follows from Theorem 2.3 that

$$\|u(t)\|_{H^s},\ \|\rho(t)\|_{H^{s-1}}\le\|z(t)\|_{H^s\times H^{s-1}}\le 2\|z_0\|_{H^s\times H^{s-1}},\quad 0\le t\le T_0. \tag{2.10}$$ 2.20

r2.1 **Remark 2.4.** Comparing Theorem 2.3 with that in [28], we will see that there exists a significant different between (1.1) and (1.1) with $\rho=\gamma-\gamma_{xx}$. In the other words, we require $s>5/2$ because of the Sobolev embedding Theorem. But in paper [28], since u and γ have the same property, we assume that $s>3/2$.

3. Approximate solutions

In this section we first construct a two-parameter family of approximate solutions by using a similar method to [21], then compute the error and last estimate the H^1-norm of the error.

Following [21], our approximate solutions $u^{\omega,\lambda}=u^{\omega,\lambda}(t,x)$ and $\rho^{\omega,\lambda}=\rho^{\omega,\lambda}(t,x)$ to (2.1) will consist of a low frequency and a high frequency part, i.e.

$$u^{\omega,\lambda}=u_l+u^h,\quad \rho^{\omega,\lambda}=\rho_l+\rho^h,$$

where ω is in a bounded set of $\mathbb{R}$ and $\lambda>0$. The high frequency part is given by

$$\begin{aligned}u^h&=u^{h,\omega,\lambda}(t,x)=\lambda^{-\frac{1}{2}\delta-s}\phi\big(\frac{x}{\lambda^\delta}\big)\cos(\lambda x-\omega t),\\ \rho^h&=\rho^{h,\omega,\lambda}(t,x)=\lambda^{-\frac{1}{2}\delta-s+1}\psi\big(\frac{x}{\lambda^\delta}\big)\cos(\lambda x-\omega t),\end{aligned} \tag{3.1}$$ 3.1

where ϕ and ψ are C^∞ cut-off functions such that

$$\phi(x)=\begin{cases}1 & \text{if } |x|<1,\\ 0 & \text{if } |x|\ge 2,\end{cases}\quad \psi(x)=\begin{cases}1 & \text{if } |x|<1,\\ 0 & \text{if } |x|\ge 2.\end{cases}$$

The low frequency part $(u_l,\rho_l)=(u_{l,\omega,\lambda}(t,x),\rho_{l,\omega,\lambda}(t,x))$ is the solution to (2.1) with initial data

$$u_l(0,x)=\omega\lambda^{-1}\tilde\phi\big(\frac{x}{\lambda^\delta}\big),\quad \rho_l(0,x)=\omega\lambda^{-1}\tilde\psi\big(\frac{x}{\lambda^\delta}\big),\quad x\in\mathbb{R}, \tag{3.2}$$ 3.2

where $\tilde\phi$ and $\tilde\psi$ are $C_0^\infty(\mathbb{R})$ functions such that

$$\tilde\phi(x)=1\quad \text{if } x\in\operatorname{supp}\phi\cup\operatorname{supp}\psi.$$

We first study the properties of (u_l,ρ_l) and (u^h,ρ^h). The high frequency part (u^h,ρ^h) defined by (3.1) satisfies

$$\|u^h(t)\|_{H^s}\approx O(1),\quad \|\rho^h(t)\|_{H^{s-1}}\approx O(1)\quad \text{for } \lambda\gg 1$$

because of the following result.

l3.1 **Lemma 3.1** ([21]). *Let $\psi \in \mathcal{S}(\mathbb{R})$, $1 < \delta < 2$ and $\alpha \in \mathbb{R}$. Then for any $s \geq 0$ we have that*

$$\lim_{\lambda\to\infty} \lambda^{-\frac{1}{2}\delta - s} \|\psi\big(\frac{x}{\lambda^\delta}\big)\cos(\lambda x - \alpha)\|_{H^s} = \frac{1}{\sqrt{2}}\|\psi\|_2. \tag{3.3}$$ 3.3

Relation (3.3) *is also true if* cos *is replaced by* sin.

For the low frequency part (u_l, ρ_l), we have the following result.

l3.2 **Lemma 3.2.** *Let ω belong to a bounded set of $\mathbb{R}$, $1 < \delta < 2$ and $\lambda \gg 1$. Then the initial-value problem* (2.1)-(3.2) *has a unique solution $(u_l, \rho_l) \in C([0,T); H^s) \times C([0,T); H^{s-1})$, for all $s > 5/2$, satisfying the estimates*

$$\|u_l(t)\|_{H^s} \leq C_s \lambda^{-1+\frac{1}{2}\delta}, \quad \|\rho_l(t)\|_{H^{s-1}} \leq C_{s-1}\lambda^{-1+\frac{1}{2}\delta}.$$

Proof. The existence and uniqueness of local a solution can be derived from Theorem 2.1 for $s > 5/2$.

It follows from [21, Lemma 5] that

$$\|\psi\big(\frac{x}{\lambda^\delta}\big)\|_{H^s} \leq \lambda^{\delta/2}\|\psi\|_{H^s},$$

where $s \geq 0$ and $\psi \in \mathcal{S}(\mathbb{R})$. Using the above inequality, we have that the initial data $(u_l(0,x), \rho_l(0,x))$ satisfies the estimate

$$\|u_l(0)\|_{H^s} \leq |\omega|\lambda^{-1+\frac{1}{2}\delta}\|\tilde{\phi}\|_{H^s}, \quad \|\rho_l(0)\|_{H^{s-1}} \leq |\omega|\lambda^{-1+\frac{1}{2}\delta}\|\tilde{\psi}\|_{H^{s-1}},$$

which decay if $\delta < 2$ and ω is in a bounded set of $\mathbb{R}$. Recall that $\|z_l(t)\|^2_{H^s\times H^{s-1}} = \|u_l(t)\|^2_{H^s} + \|\rho_l(t)\|^2_{H^{s-1}}$, we obtain

$$\|z_l(0)\|_{H^s\times H^{s-1}} = (\|u_l(0)\|^2_{H^s} + \|\rho_l(0)\|^2_{H^{s-1}})^{1/2} \leq |\omega|\lambda^{-1+\frac{1}{2}\delta}(\|\tilde{\phi}\|^2_{H^s} + \|\tilde{\psi}\|^2_{H^{s-1}})^{1/2}.$$

It follows from (3.2) that $z_l(0) \in H^s \times H^{s-1}$ for all $s > 5/2$. If $s > 5/2$, then from estimate (2.3) of Theorem 2.3, we have

$$\|u_l(t)\|_{H^s} \leq C_s\|u_l(0)\|_{H^s} \leq C_s\lambda^{-1+\frac{1}{2}\delta},$$
$$\|\rho_l(t)\|_{H^{s-1}} \leq C_s\|\rho_l(0)\|_{H^{s-1}} \leq C_{s-1}\lambda^{-1+\frac{1}{2}\delta}.$$

The proof is complete. □

Now we compute the error. Substituting the approximate solution $(u^{\omega,\lambda}, \rho^{\omega,\lambda})$ into the first and second equation of (2.1), we obtain the error

$$\begin{aligned}
E &= u^h_t + u_l u^h_x + u^h u_{lx} + u^h u^h_x + \partial_x\Lambda^{-2}\Big((u^h)^2 + k_1 u_l u^h \\
&\quad + \frac{1}{2}(u^h_x)^2 + u_{lx}u^h_x + \frac{1}{2}(\rho^h)^2 + \rho_l\rho^h\Big), \\
F &= \rho^h_t + u_l\rho^h_x + u^h\rho_{lx} + u^h\rho^h_x + \rho^h u_{lx} + \rho_l u^h_x + \rho^h u^h_x,
\end{aligned}$$

where we have used that (u_l, ρ_l) solves (3.2).

Direct calculation shows that

$$u^h_t(t,x) = \omega\lambda^{-\frac{1}{2}\delta - s}\phi\big(\frac{x}{\lambda^\delta}\big)\sin(\lambda x - \omega t),$$
$$\rho^h_t(t,x) = \omega\lambda^{-\frac{1}{2}\delta - s + 1}\psi\big(\frac{x}{\lambda^\delta}\big)\sin(\lambda x - \omega t).$$

Since $\tilde{\phi} = 1$ if $x \in \operatorname{supp}\phi \cup \operatorname{supp}\psi$, we can write u_t^h and ρ_t^h in the form

$$\begin{aligned} u_t^h(t,x) &= \omega\tilde{\phi}\big(\frac{x}{\lambda^\delta}\big)\lambda^{-\frac{1}{2}\delta-s}\phi\big(\frac{x}{\lambda^\delta}\big)\sin(\lambda x-\omega t)\\ &= \lambda u_l(0,x)\lambda^{-\frac{1}{2}\delta-s}\phi\big(\frac{x}{\lambda^\delta}\big)\sin(\lambda x-\omega t),\\ \rho_t^h(t,x) &= \omega\tilde{\phi}\big(\frac{x}{\lambda^\delta}\big)\lambda^{-\frac{1}{2}\delta-s+1}\psi\big(\frac{x}{\lambda^\delta}\big)\sin(\lambda x-\omega t)\\ &= \lambda u_l(0,x)\lambda^{-\frac{1}{2}\delta-s+1}\psi\big(\frac{x}{\lambda^\delta}\big)\sin(\lambda x-\omega t). \end{aligned} \tag{3.4}$$

3.4

Computing the spacial derivatives of u^h and ρ^h, we have

$$\begin{aligned} u_x^h(t,x) &= -\lambda\lambda^{-\frac{1}{2}\delta-s}\phi\big(\frac{x}{\lambda^\delta}\big)\sin(\lambda x-\omega t) + \lambda^{-\frac{3}{2}\delta-s}\phi'\big(\frac{x}{\lambda^\delta}\big)\cos(\lambda x-\omega t),\\ \rho_x^h(t,x) &= -\lambda\lambda^{-\frac{1}{2}\delta-s+1}\psi\big(\frac{x}{\lambda^\delta}\big)\sin(\lambda x-\omega t) + \lambda^{-\frac{3}{2}\delta-s+1}\psi'\big(\frac{x}{\lambda^\delta}\big)\cos(\lambda x-\omega t). \end{aligned} \tag{3.5}$$

3.5

Combining (3.4) with (3.5), we obtain

$$\begin{aligned} u_t^h(t,x) + u_l u_x^h(t,x) &= \lambda[u_l(0,x)-u_l(t,x)]\lambda^{-\frac{1}{2}\delta-s}\phi\left(\frac{x}{\lambda^\delta}\right)\sin(\lambda x-\omega t)\\ &\quad + u_l(t,x)\lambda^{-\frac{3}{2}\delta-s}\phi'\big(\frac{x}{\lambda^\delta}\big)\cos(\lambda x-\omega t),\\ \rho_t^h(t,x) + u_l\rho_x^h(t,x) &= \lambda[u_l(0,x)-u_l(t,x)]\lambda^{-\frac{1}{2}\delta-s+1}\psi\big(\frac{x}{\lambda^\delta}\big)\sin(\lambda x-\omega t)\\ &\quad + u_l(t,x)\lambda^{-\frac{3}{2}\delta-s+1}\psi'\left(\frac{x}{\lambda^\delta}\right)\cos(\lambda x-\omega t). \end{aligned}$$

Therefore, we can rewrite the error E and F as

$$E = E_1 + E_2 + \dots + E_8, \quad F = F_1 + F_2 + \dots + F_6,$$

where

$$\begin{gathered} E_1 = -\lambda[u_l(0,x)-u_l(t,x)]\lambda^{-\frac{1}{2}\delta-s}\phi\big(\frac{x}{\lambda^\delta}\big)\sin(\lambda x+\omega t),\\ E_2 = u_l(t,x)\lambda^{-\frac{3}{2}\delta-s}\phi'\big(\frac{x}{\lambda^\delta}\big)\cos(\lambda x+\omega t),\\ E_3 = -u^h u_{lx}, \quad E_4 = -u^h u_x^h,\\ E_5 = -\partial_x\Lambda^{-2}\Big(\frac{k_1}{2}(u^h)^2 + \frac{k_2}{2}(\rho^h)^2\Big), \quad E_6 = -\partial_x\Lambda^{-2}\left(k_1 u_l u^h + k_2\rho_l\rho^h\right),\\ E_7 = -(3-k_1)\partial_x\Lambda^{-2}(u_{lx}u_x^h), \quad E_8 = \frac{3-k_1}{2}\partial_x\Lambda^{-2}\left((u_x^h)^2\right),\\ F_1 = -k_3\lambda[u_l(0,x)-u_l(t,x)]\lambda^{-\frac{1}{2}\delta-s+1}\psi\big(\frac{x}{\lambda^\delta}\big)\sin(\lambda x+\omega t),\\ F_2 = k_3 u_l(t,x)\lambda^{-\frac{3}{2}\delta-s+1}\psi'\big(\frac{x}{\lambda^\delta}\big)\cos(\lambda x+\omega t),\\ F_3 = -k_3 u^h\rho_{lx}, \quad F_4 = -k_3 u^h\rho_x^h,\\ F_5 = -k_3\left(\rho^h u_{lx} + \rho_l u_x^h + \rho^h u_x^h\right). \end{gathered}$$

Now we are ready to estimate the H^1-norm of each error E_i and the L^2-norm of each error F_j ($i = 1,\dots,8, j = 1,\dots,6$). Let C be a generic positive constant. For any positive quantities P and Q, we write $P \lesssim Q$ ($P \gtrsim Q$) means that $P \le CQ$ ($P \ge CQ$) in the following.

Estimates of $\|E_1\|_{H^1}$ and $\|F_1\|_{L^2}$. Note that

$$\|fg\|_{H^1} \le \sqrt{2}\|f\|_{C^1}\|g\|_{H^1}, \quad \forall f \in C^1,\ g \in H^1,$$

and $\|\phi\big(\frac{x}{\lambda^\delta}\big)\sin(\lambda x - \omega t)\|_{C^1} = \lambda\|\phi\|_\infty$, we have

$$\begin{aligned}
\|E_1\|_{H^1} &= \lambda^{1-\frac{1}{2}\delta-s}\|\phi\left(\frac{x}{\lambda^\delta}\right)\sin(\lambda x-\omega t)[u_l(0,x)-u_l(t,x)]\|_{H^1} \\
&\lesssim \lambda^{1-\frac{1}{2}\delta-s}\|\phi\left(\frac{x}{\lambda^\delta}\right)\sin(\lambda x-\omega t)\|_{C^1}\|u_l(0,x)-u_l(t,x)\|_{H^1} \\
&\lesssim \lambda^{2-\frac{1}{2}\delta-s}\|u_l(0,x)-u_l(t,x)\|_{H^1}.
\end{aligned} \tag{3.6}$$

To estimate the H^1-norm of the difference $u_l(0,x)-u_l(t,x)$, we apply the fundamental theorem of calculus in time variable to obtain

$$\|u_l(0,x)-u_l(t,x)\|_{H^1} = \int_0^t \|u_{lt}(\tau)\|_{H^1}\mathrm{d}\tau.$$

It follows from the first equation of (3.2) that

$$\begin{aligned}
\|u_{lt}(t)\|_{H^1} &\le \|u_l u_{lx}\|_{H^1} + \|\partial_x\Lambda^{-2}\big(u_l^2 + \frac{1}{2}u_{lx}^2 + \frac{1}{2}\rho_l^2\big)\|_{H^1} \\
&\le \|u_l\|_{H^1}\|u_l\|_{H^2} + \|u_l^2 + \frac{1}{2}u_{lx}^2 + \frac{1}{2}\rho_l^2\|_2 \\
&\lesssim \|u_l\|_{H^2}^2 + \|u_l\|_\infty\|u_l\|_2 + \|u_{lx}\|_\infty\|u_l\|_{H^1} + \|\rho_l\|_\infty\|\rho_l\|_2 \\
&\lesssim \|u_l\|_{H^2}^2 + \|u_l\|_{H^1}^2 + \|\rho_l\|_{H^2}^2 \\
&\lesssim \|u_l\|_{H^3}^2 + \|\rho_l\|_{H^3}^2 \\
&\lesssim \lambda^{-2+\delta}, \quad \lambda \gg 1,
\end{aligned} \tag{3.7}$$

where we have used Lemma 3.2 and the Sobolev embedding Theorem $H^s \hookrightarrow L^\infty$ for $s > 3/2$.

Combining (3.6) and (3.7), we obtain

$$\|E_1\|_{H^1} \lesssim \lambda^{-s+\frac{1}{2}\delta}, \quad \lambda \gg 1.$$

Similarly,

$$\|F_1\|_{L^2} \lesssim \lambda^{-s+\frac{1}{2}\delta}, \quad \lambda \gg 1.$$

Estimates of $\|E_i\|_{H^1}$ and $\|F_j\|_{H^1}$, $i = 2,\dots,8, j = 2,3$. In [28], the authors obtained the following estimates

$$\begin{gathered}
\|E_2\|_{H^1} \lesssim \lambda^{-s-\delta}, \\
\|E_3\|_{H^1},\ \|E_6\|_{H^1},\ \|E_7\|_{H^1} \lesssim \lambda^{-\frac{1}{2}\delta-s+1}\lambda^{-1+\frac{1}{2}\delta}, \\
\|E_4\|_{H^1},\ \|E_5\|_{H^1},\ \|E_8\|_{H^1} \lesssim \lambda^{-\frac{1}{2}\delta-2s+2}
\end{gathered}$$

Similar to the estimate of $\|E_2\|_{H^1}$, we have

$$\|F_2\|_{L^2} \lesssim \lambda^{-s-\delta}, \quad \lambda \gg 1.$$

Direct calculation shows that

$$\|F_3\|_{L^2} = \|u^h\rho_{lx}\|_{L^2} \lesssim \|u^h\|_{L^\infty}\|\rho_{lx}\|_{H^1} \lesssim \lambda^{-\frac{1}{2}\delta-s}\lambda^{-1+\frac{1}{2}\delta}, \quad \lambda \gg 1.$$

Estimates of $\|F_4\|_{L^2}$. It follows from (3.1) that

$$\|u_x^h(t)\|_\infty \lesssim \lambda^{-\frac{1}{2}\delta-s+1}, \quad \|\rho_x^h(t)\|_\infty \lesssim \lambda^{-\frac{1}{2}\delta-s+2}, \quad \lambda \gg 1. \tag{3.8}$$

By using Lemma 3.1, we have

$$\begin{aligned}\|u^h(t)\|_{H^k} &= \lambda^{-\frac{1}{2}\delta-s}\|\phi(\frac{x}{\lambda^\delta})\cos(\lambda x-\omega t)\|_{H^k}\\ &= \lambda^{-s+k}\lambda^{-\frac{1}{2}\delta-k}\|\phi(\frac{x}{\lambda^\delta})\cos(\lambda x-\omega t)\|_{H^k}\\ &\lesssim \lambda^{-s+k}, \quad \lambda\gg 1.\end{aligned} \tag{3.9}$$ 3.9

The above inequality also holds for $\rho^h(t)$. Combining (3.8) and (3.9), we obtain that, for $\lambda \gg 1$,

$$\|F_4\|_{L^2} = \|u^h\rho^h_x\|_{L^2} \lesssim \|u^h\|_\infty\|\rho^h\|_{H^1} \lesssim \lambda^{-\frac{1}{2}\delta-s}\lambda^{-s+2} \lesssim \lambda^{-\frac{1}{2}\delta-2s+2}.$$

Estimate of $\|F_5\|_{L^2}$. It follows from (3.8) and (3.9) that

$$\begin{aligned}\|F_5\|_{L^2} &= \|\left(\rho^h u_{lx} + \rho_l u^h_x + \rho^h u^h_x\right)\|_{L^2}\\ &\le \left(\|\rho^h\|_\infty\|u_{lx}\|_{H^1} + \|u^h_x\|_\infty\|\rho_l\|_{H^1} + \|\rho^h\|_\infty\|u^h_x\|_{L^2}\right)\\ &\lesssim \|\rho^h\|_\infty\|u_l\|_{H^2} + \|u^h_x\|_\infty\|\rho_l\|_{H^2} + \|\rho^h\|_\infty\|u^h_x\|_{H^1}\\ &\lesssim \lambda^{-\frac{1}{2}\delta-s}\lambda^{-1+\frac{1}{2}\delta} + \lambda^{-\frac{1}{2}\delta-s+1}\lambda^{-1+\frac{1}{2}\delta} + \lambda^{-\frac{1}{2}\delta-s+1}\lambda^{-s+1},\end{aligned}$$

which gives $\|F_5\|_{H^1} \lesssim \lambda^{-\frac{1}{2}\delta-2s+2}$, $\lambda \gg 1$.

Collecting all error estimates together, we have the following theorem.

t3.1 **Theorem 3.3.** *Let $s > 5/2$ and $1 < \delta < 2$. When ω is in a bounded set of $\mathbb{R}$ and $\lambda \gg 1$, we have that*

$$\|E\|_{H^1} \lesssim \lambda^{-r_s}, \quad \|F\|_{L^2} \lesssim \lambda^{-r_s}, \quad \text{for } \lambda\gg 1,\ 0<t<T, \tag{3.10}$$ 3.10

where $r_s = s - \frac{1}{2}\delta > 0$.

4. Difference between approximate and actual solutions

In this section, we estimate the difference between the approximate and actual solutions. Let $(u_{\omega,\lambda}(t,x), \rho_{\omega,\lambda}(t,x))$ be the solution to (2.1) with initial data the value of the approximate solution $(u^{\omega,\lambda}(t,x), \rho^{\omega,\lambda}(t,x))$ at time zero, that is, $(u_{\omega,\lambda}(t,x), \rho_{\omega,\lambda}(t,x))$ satisfies

$$\begin{gathered}\partial_t u_{\omega,\lambda} - u_{\omega,\lambda}\partial_x u_{\omega,\lambda} - \partial_x\Lambda^{-2}(u^2_{\omega,\lambda} + \frac{1}{2}(\partial_x u_{\omega,\lambda})^2 + \frac{1}{2}\rho^2_{\omega,\lambda}) = 0, \quad t>0,\ x\in\mathbb{R},\\ \partial_t\rho_{\omega,\lambda} - u_{\omega,\lambda}\partial_x\rho_{\omega,\lambda} - (\partial_x u_{\omega,\lambda}\rho_{\omega,\lambda} + \partial_x\rho_{\omega,\lambda}u_{\omega,\lambda}) = 0, \quad t>0,\ x\in\mathbb{R},\\ u_{\omega,\lambda}(0,x) = u^{\omega,\lambda}(0,x) = \omega\lambda^{-1}\tilde\phi(\frac{x}{\lambda^\delta}) + \lambda^{-\frac{1}{2}\delta-s}\phi(\frac{x}{\lambda^\delta})\cos(\lambda x), \quad x\in\mathbb{R},\\ \rho_{\omega,\lambda}(0,x) = \rho^{\omega,\lambda}(0,x) = \omega\lambda^{-1}\tilde\psi(\frac{x}{\lambda^\delta}) + \lambda^{-\frac{1}{2}\delta-s+1}\psi(\frac{x}{\lambda^\delta})\cos(\lambda x), \quad x\in\mathbb{R}.\end{gathered} \tag{4.1}$$ 4.1

Note that $(u_{\omega,\lambda}(0,x), \rho_{\omega,\lambda}(0,x)) \in H^s\times H^{s-1}$, $s\ge 2$, it follows from Lemma 3.2 and (3.9) that

$$\begin{gathered}\|u_{\omega,\lambda}(0,x)\|_{H^s} \le \|u_l(0)\|_{H^s} + \|u^h(0)\|_{H^s} \lesssim \lambda^{-1+\frac{1}{2}\delta} + 1, \quad \lambda\gg 1,\\ \|\rho_{\omega,\lambda}(0,x)\|_{H^{s-1}} \le \|\rho_l(0)\|_{H^{s-1}} + \|\rho^h(0)\|_{H^{s-1}} \lesssim \lambda^{-1+\frac{1}{2}\delta} + 1, \quad \lambda\gg 1.\end{gathered}$$

Therefore, if $s > 5/2$, by using Theorem 2.1 and 2.3, we have that for any ω in a bounded set and $\lambda \gg 1$, problem (4.1) has a unique solution $z_{\omega,\lambda} \in C([0,T];H^s)\times$

$C([0,T];H^{s-1})$ with

$$T \gtrsim \frac{1}{\|z_{\omega,\lambda}(0)\|_{H^s\times H^{s-1}}} \gtrsim \frac{1}{1+\lambda^{-1+\frac{1}{2}\delta}} \gtrsim 1. \tag{4.2}$$

a.1

To estimate the difference between the approximate and actual solutions, we let

$$v = u^{\omega,\lambda} - u_{\omega,\lambda}, \quad \sigma = \rho^{\omega,\lambda} - \rho_{\omega,\lambda}.$$

Then (v,σ) satisfies

$$\begin{aligned} &v_t - vv_x + u^{\omega,\lambda}v_x + vu_x^{\omega,\lambda} - \partial_x\Lambda^{-2}\Big[v^2 + \frac{1}{2}v_x^2 \\ &+ \frac{1}{2}\sigma^2 - 2u^{\omega,\lambda}v - u_x^{\omega,\lambda}v_x - \rho^{\omega,\lambda}\sigma\Big] = \tilde{E}, \quad t>0,\ x\in\mathbb{R}, \\ &\sigma_t - v\sigma_x + u^{\omega,\lambda}\sigma_x + v\rho_x^{\omega,\lambda} - \big(\sigma v_x - u^{\omega,\lambda}\sigma - \rho^{\omega,\lambda}v_x\big) = \tilde{F}, \quad t>0,\ x\in\mathbb{R}, \\ &v(0,x) = \sigma(0,x) = 0, \quad x\in\mathbb{R}, \end{aligned} \tag{4.3}$$

4.2

where

$$\begin{gathered} \tilde{E} = u_t^{\omega,\lambda} + u^{\omega,\lambda}u_x^{\omega,\lambda} + \partial_x\Lambda^{-2}\Big((u^{\omega,\lambda})^2 + \frac{1}{2}(u_x^{\omega,\lambda})^2 + \frac{1}{2}(\rho^{\omega,\lambda})^2\Big), \\ \tilde{F} = \rho_t^{\omega,\lambda} + u^{\omega,\lambda}\rho_x^{\omega,\lambda} + +\rho^{\omega,\lambda}u_x^{\omega,\lambda}, \end{gathered}$$

Similar to the prove of Theorem 3.3, $\tilde{E}$ and $\tilde{F}$ satisfy the H^1-norm estimation (3.10). Now we prove that the H^1-norm of difference decays.

t4.1 **Theorem 4.1.** *Let $1<\delta<2$ and $s>5/2$, then*

$$\|v(t)\|_{H^1} \lesssim \lambda^{-r_s}, \quad \|\sigma(t)\|_{L^2} \lesssim \lambda^{-r_s}, \quad 0\le t\le T,\ \lambda\gg 1,$$

where $r_s = s - \frac{1}{2}\delta > 0$.

Proof. Note that

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{H^1}^2 = \int_{\mathbb{R}} (vv_t + v_xv_{xt})\mathrm{d}x, \tag{4.4}$$

4.3

$$\frac{1}{2}\frac{d}{dt}\|\sigma(t)\|_{L^2}^2 = \int_{\mathbb{R}} \sigma\sigma_t\mathrm{d}x. \tag{4.5}$$

4.4

Applying the operator $1-\partial_x^2 = \Lambda^2$ to both sides of the first equations of (4.3), we have

$$\begin{aligned} v_t &= \Lambda^2\tilde{E} - \Lambda^2(u^{\omega,\lambda}v_x - vu_x^{\omega,\lambda}) - (2u^{\omega,\lambda}v + u_x^{\omega,\lambda}v_x + \rho^{\omega,\lambda}\sigma)_x \\ &\quad + \frac{1}{2}(\sigma^2)_x + 3vv_x - 2v_xv_{xx} - vv_{xxx} + v_{xxt}, \end{aligned} \tag{4.6}$$

4.5

$$\sigma_t = \tilde{F} - (u^{\omega,\lambda}\sigma_x + v\rho_x^{\omega,\lambda}) - (u_x^{\omega,\lambda}\sigma + \rho^{\omega,\lambda}v_x) + (v\sigma)_x. \tag{4.7}$$

4.6

Substituting (4.6) and (4.7) into (4.4) and (4.5), respectively, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v(t)\|_{H^1}^2 &= \int_{\mathbb{R}} v\Lambda^2\tilde{E}\mathrm{d}x - \int_{\mathbb{R}} v\Lambda^2(u^{\omega,\lambda}v_x + vu_x^{\omega,\lambda})\mathrm{d}x \\ &\quad - \int_{\mathbb{R}} v(2u^{\omega,\lambda}v + u_x^{\omega,\lambda}v_x + \rho^{\omega,\lambda}\sigma)_x\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} v(\sigma^2)_x\mathrm{d}x \\ &\quad + \int_{\mathbb{R}} (v(3vv_x - 2v_xv_{xx} - vv_{xxx} + v_{xxt}) + v_xv_{xt})\mathrm{d}x, \end{aligned} \tag{4.8}$$

4.7

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\sigma(t)\|_{L^2}^2 = &\int_{\mathbb{R}} \sigma\tilde{F}\mathrm{d}x - \int_{\mathbb{R}} \sigma(u^{\omega,\lambda}\sigma_x + v\rho_x^{\omega,\lambda})\mathrm{d}x \\
&- \int_{\mathbb{R}} \sigma(\rho^{\omega,\lambda}v_x + \sigma u_x^{\omega,\lambda})\mathrm{d}x + \int_{\mathbb{R}} \sigma(v\sigma)_x\mathrm{d}x.
\end{aligned} \tag{4.9}
$$

4.8

A direct calculation yields

$$
\begin{aligned}
&\int_{\mathbb{R}} (v(3vv_x - 2v_xv_{xx} - vv_{xxx} + v_{xxt}) + v_xv_{xt})\mathrm{d}x \\
&= \int_{\mathbb{R}} [(v^3)_x - (v^2v_{xx})_x + (vv_{xt})_x]\mathrm{d}x = 0.
\end{aligned}
$$

Substituting the above equalities in (4.8), and adding the resulting equations, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|v(t)\|_{H^1}^2 + \|\sigma(t)\|_{L^2}^2\right) \\
&= \int_{\mathbb{R}} v\Lambda^2\tilde{E}\mathrm{d}x + \int_{\mathbb{R}} \sigma\tilde{F}\mathrm{d}x - \int_{\mathbb{R}} v\Lambda^2(u^{\omega,\lambda}v_x + vu_x^{\omega,\lambda})\mathrm{d}x \\
&\quad - \int_{\mathbb{R}} \sigma(u^{\omega,\lambda}\sigma_x + v\rho_x^{\omega,\lambda})\mathrm{d}x - \int_{\mathbb{R}} v(2u^{\omega,\lambda}v + u_x^{\omega,\lambda}v_x + \rho^{\omega,\lambda}\sigma)_x\mathrm{d}x \\
&\quad - \int_{\mathbb{R}} \sigma(\rho^{\omega,\lambda}v_x + \sigma u_x^{\omega,\lambda})\mathrm{d}x + \int_{\mathbb{R}} \left[\frac{1}{2}v(\sigma^2)_x + \sigma(v\sigma)_x\right]\mathrm{d}x \\
&:= I_1 + I_2 + \dots + I_7.
\end{aligned}
$$

We first look at the last term I_7. Integrating by parts gives

$$
I_7 = \int_{\mathbb{R}} \left[\frac{1}{2}v(\sigma^2)_x + \sigma(v\sigma)_x\right]\mathrm{d}x = 0.
$$

Estimates of integrals I_1 and I_2. Integrating by parts and applying the Cauchy-Schwarz inequality, we have

$$
\Big|\int_{\mathbb{R}} v\Lambda^2\tilde{E}\mathrm{d}x\Big| = \Big|\int_{\mathbb{R}} (v\tilde{E} - v_x\tilde{E}_x)\mathrm{d}x\Big| \le \|\tilde{E}\|_{H^1}\|v(t)\|_{H^1},
$$

$$
\Big|\int_{\mathbb{R}} \sigma\tilde{F}\mathrm{d}x\Big| \le \|\tilde{F}\|_{L^2}\|\sigma(t)\|_{L^2}.
$$

Estimates of integrals I_3-I_6. Similar to that in [28], we obtain

$$
\begin{aligned}
\sum_{i=3}^{6} I_i \lesssim &(\|u^{\omega,\lambda}(t)\|_\infty + \|u_x^{\omega,\lambda}(t)\|_\infty + \|u_{xx}^{\omega,\lambda}(t)\|_\infty + \|\rho^{\omega,\lambda}(t)\|_\infty) \\
&\times (\|v(t)\|_{H^1}^2 + \|\sigma(t)\|_{L^2}^2).
\end{aligned}
$$

Combining the estimations for I_1–I_7, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|v(t)\|_{H^1}^2 + \|\sigma(t)\|_{L^2}^2) \\
&\lesssim (\|\tilde{E}\|_{H^1} + \|\tilde{F}\|_{H^1})(\|v(t)\|_{H^1} + \|\sigma(t)\|_{L^2}) \\
&\quad + (\|u^{\omega,\lambda}(t)\|_\infty + \|u_x^{\omega,\lambda}(t)\|_\infty + \|u_{xx}^{\omega,\lambda}(t)\|_\infty + \|\rho^{\omega,\lambda}(t)\|_\infty + \|\rho_x^{\omega,\lambda}(t)\|_\infty) \\
&\quad \times (\|v(t)\|_{H^1}^2 + \|\sigma(t)\|_{H^1}^2).
\end{aligned} \tag{4.10}
$$

4.9

It follows from (3.1) that

$$
u_x^h = -\lambda^{-\frac{3}{2}\delta - s}\phi'\left(\frac{x}{\lambda^\delta}\right)\cos(\lambda x - \omega t) - \lambda^{-\frac{\delta}{2} - s + 1}\phi\left(\frac{x}{\lambda^\delta}\right)\sin(\lambda x - \omega t),
$$

$$
\begin{aligned}
u^h_{xx} &= \lambda^{-\frac{5}{2}\delta-s}\phi''\big(\frac{x}{\lambda^\delta}\big)\cos(\lambda x-\omega t) - 2\lambda^{-\frac{3}{2}\delta-s+1}\phi'\big(\frac{x}{\lambda^\delta}\big)\sin(\lambda x-\omega t)\\
&\quad - 2\lambda^{-\frac{1}{2}\delta-s+2}\phi\big(\frac{x}{\lambda^\delta}\big)\cos(\lambda x-\omega t).
\end{aligned}
$$

Hence

$$
\|u^h(t)\|_\infty + \|u^h_x(t)\|_\infty + \|u^h_{xx}(t)\|_\infty \lesssim \lambda^{-(\frac{1}{2}\delta+s-2)}, \quad \lambda \gg 1.
$$

By using Lemma 3.2, we have

$$
\|u_l(t)\|_\infty + \|u_{lx}(t)\|_\infty + \|u_{lxx}(t)\|_\infty \lesssim \lambda^{-(1-\frac{1}{2}\delta)}, \quad \lambda \gg 1.
$$

Therefore,

$$
\|u^{\omega,\lambda}(t)\|_\infty + \|u^{\omega,\lambda}_x(t)\|_\infty + \|u^{\omega,\lambda}_{xx}(t)\|_\infty \lesssim \lambda^{-\rho_s}, \quad \lambda \gg 1, \tag{4.11}
$$
4.10

where $\rho_s = \min\{\frac{1}{2}\delta + s - 2, 1-\frac{1}{2}\delta\} > 0$ for any $s > 1$ if δ is chosen appropriately in the interval $(1,2)$. Similarly, we can prove that

$$
\|\rho^{\omega,\lambda}(t)\|_\infty \lesssim \lambda^{-s}, \quad \|\rho^{\omega,\lambda}_x(t)\|_\infty \lesssim \lambda^{-\rho_s} \quad \lambda \gg 1. \tag{4.12}
$$
4.11

Let $\tilde{z}(t,x) = (v(t,x), \sigma(t,x))$ and $\|\tilde{z}(t)\|^2_{H^1\times L^2} = \|v(t)\|^2_{H^1} + \|\sigma(t)\|^2_{L^2}$, then by (4.10)-(4.12), we obtain that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\tilde{z}(t)\|^2_{H^1\times L^2} &\lesssim (\|\tilde{E}\|_{H^1} + \|\tilde{F}\|_{L^2})\|\tilde{z}(t)\|_{H^1\times L^2} + \lambda^{-\rho_s}\|\tilde{z}(t)\|^2_{H^1}\\
&\lesssim \lambda^{-r_s}\|\tilde{z}(t)\|_{H^1\times L^2} + \lambda^{-\rho_s}\|\tilde{z}(t)\|^2_{H^1\times L^2}, \quad \lambda \gg 1,
\end{aligned}
$$

where we have used Theorem 3.3. Consequently,

$$
\frac{d}{dt}\|\tilde{z}(t)\|_{H^1\times L^2} \lesssim \lambda^{-\rho_s}\|\tilde{z}(t)\|_{H^1\times L^2} + \lambda^{-r_s}, \quad \lambda \gg 1. \tag{4.13}
$$
4.12

Since $\|\tilde{z}(0)\|_{H^1\times L^2} = (\|v(0)\|^2_{H^1} + \|\sigma(0)\|^2_{L^2})^{1/2} = 0$ and for $s > 1$, we can choose $\delta \in (1,2)$ such that $\rho_s \geq 0$, by (4.13) and Gronwall's inequality, we obtain

$$
\|\tilde{z}(t)\|_{H^1\times L^2} \lesssim \lambda^{-r_s}, \quad 0 \leq t \leq T, \quad \lambda \gg 1.
$$

Note that

$$
\|v(t)\|_{H^1},\ \|\sigma(t)\|_{L^2} \leq \|\tilde{z}(t)\|_{H^1\times L^2},
$$

we see that

$$
\|v(t)\|_{H^1},\ \|\sigma(t)\|_{L^2} \lesssim \lambda^{-r_s}, \quad 0 \leq t \leq T,\ \lambda \gg 1.
$$

This completes the proof. □

5. Non-uniform dependence

In this section, we prove non-uniform dependence for (2.1) by taking advantage of the information provided by Theorem 2.1-2.3, Theorem 3.3 and Theorem 4.1. Our main result is the following.

t5.1 **Theorem 5.1.** *If $s > 5/2$, then the data-to-solution $z(0) \to z(t)$ for (2.1) is not uniformly continuous from any bounded subset of $H^s \times H^{s-1}$ into $C([-T,T]; H^s) \times C([-T,T]; H^{s-1})$, where $z(0) = (u_0(x), \rho_0(x))$ and $z(t) = (u(t,x), \rho(t,x))$. More precisely, there exist two sequences of solutions $(u_\lambda(t), \rho_\lambda(t))$ and $(\tilde{u}_\lambda(t), \tilde{\rho}_\lambda(t))$ to the differential equations of (2.1) in $C([-T,T]; H^s) \times C([-T,T]; H^{s-1})$ such that*

$$
\|u_\lambda(t)\|_{H^s} + \|\tilde{u}_\lambda(t)\|_{H^s} + \|\rho_\lambda(t)\|_{H^{s-1}} + \|\tilde{\rho}_\lambda(t)\|_{H^{s-1}} \lesssim 1, \tag{5.1}
$$

$$
\lim_{\lambda\to\infty}\|u_\lambda(0) - \tilde{u}_\lambda(0)\|_{H^s} = \lim_{\lambda\to\infty}\|\rho_\lambda(0) - \tilde{\rho}_\lambda(0)\|_{H^{s-1}} = 0, \tag{5.2}
$$
5.2

$$
\liminf_{\lambda\to\infty}(\|u_\lambda(t) - \tilde{u}_\lambda(t)\|_{H^s} + \|\rho_\lambda(t) - \tilde{\rho}_\lambda(t)\|_{H^{s-1}}) \gtrsim \sin t, \quad |t| < T \leq 1. \tag{5.3}
$$
5.3

Proof. Let $(u_\lambda(t),\rho_\lambda(t))=(u_{1,\lambda}(t,x),\rho_{1,\lambda}(t,x))$ and let $(\tilde{u}_\lambda(t),\tilde{\rho}_\lambda(t))=(u_{-1,\lambda}(t,x),\rho_{-1,\lambda}(t,x))$, where $(u_{1,\lambda}(t,x),\rho_{1,\lambda}(t,x))$ and $(u_{-1,\lambda}(t,x),\rho_{-1,\lambda}(t,x))$ be the unique solution to problem (4.1) with initial data $(u^{1,\lambda}(0,x),\rho^{1,\lambda}(0,x))$ and $(u^{-1,\lambda}(0,x),\rho^{-1,\lambda}(0,x))$, respectively. From Theorem 2.1 these solutions belong to $C([0,T];H^s)\times C([0,T];H^{s-1})$. By (4.2) and the assumptions after Theorem 2.1, we see that T is independent of $\lambda\gg 1$. Letting $k=[s]+2$ and using estimate (2.10), we have

$$\|u_{\pm1,\lambda}(t)\|_{H^k},\|\rho_{\pm1,\lambda}(t)\|_{H^{k-1}}\lesssim\|z^{\pm1,\lambda}(0)\|_{H^k\times H^{k-1}},\qquad(5.4)$$ 5.4

where $z^{\pm1,\lambda}(0)=(u^{\pm1,\lambda}(0),\rho^{\pm1,\lambda}(0))$ and $\|z^{\pm1,\lambda}(0)\|^2_{H^k\times H^{k-1}}=\|u^{\pm1,\lambda}(0)\|^2_{H^k}+\|\rho^{\pm1,\lambda}(0)\|^2_{H^{k-1}}$. If λ is large enough, then from Lemma 3.1 we have

$$\begin{aligned}\|u^{\pm1,\lambda}(t)\|_{H^k}&\le\|u_{\pm1,\lambda}(t)\|_{H^k}+\lambda^{-\frac12\delta-s}\|\phi(\frac{x}{\lambda^\delta})\cos(\lambda x-\omega t)\|_{H^k}\\&\lesssim\lambda^{-1+\frac12\delta}+\lambda^{k-s}\|\phi\|_2,\end{aligned}$$

which gives

$$\|u^{\pm1,\lambda}(t)\|_{H^k}\lesssim\lambda^{k-s}.\qquad(5.5)$$ 5.5

Combining (5.4) with (5.5), we obtain

$$\|u_{\pm1,\lambda}(t)\|_{H^k}\lesssim\lambda^{k-s},\quad\lambda\gg1.\qquad(5.6)$$ 5.6

Estimates (5.5) and (5.6) yield

$$\|u^{\pm1,\lambda}(t)-u_{\pm1,\lambda}(t)\|_{H^k}\lesssim\lambda^{k-s},\quad\lambda\gg1.\qquad(5.7)$$ 5.7

Theorem 4.1 implies

$$\|u^{\pm1,\lambda}(t)-u_{\pm1,\lambda}(t)\|_{H^1}\lesssim\lambda^{-r_s},\quad\lambda\gg1.\qquad(5.8)$$ 5.8

Now, applying the interpolation inequality

$$\|\varphi\|_{H^s}\le\|\varphi\|_{H^{s_1}}^{(s_2-s)/(s_2-s_1)}\|\varphi\|_{H^{s_2}}^{(s-s_1)/(s_2-s_1)}$$

with $s_1=1$ and $s_2=[s]+2=k$, and using estimates (5.7) and (5.8), we obtain

$$\begin{aligned}&\|u^{\pm1,\lambda}(t)-u_{\pm1,\lambda}(t)\|_{H^s}\\&\le\|u^{\pm1,\lambda}(t)-u_{\pm1,\lambda}(t)\|_{H^1}^{(k-s)/(k-1)}\|u^{\pm1,\lambda}(t)-u_{\pm1,\lambda}(t)\|_{H^k}^{(s-1)/(k-1)}\\&\lesssim\lambda^{-r_s(k-s)/(k-1)}\lambda^{(k-s)(s-1)/(k-1)}\\&\lesssim\lambda^{-(r_s-s+1)(k-s)/(k-1)},\quad\lambda\gg1.\end{aligned}$$

Hence

$$\|u^{\pm1,\lambda}(t)-u_{\pm1,\lambda}(t)\|_{H^s}\lesssim\lambda^{-\varepsilon_s},\quad\lambda\gg1,\qquad(5.9)$$ 5.9

where $\varepsilon_s=(1-\frac12\delta)/(s+2)$.

Next, we prove (5.2) and (5.3). Note that $0<\delta<2$, we have

$$\begin{gathered}\|u_{1,\lambda}(0)-u_{-1,\lambda}(0)\|_{H^s}=2\lambda^{-1}\|\tilde{\phi}(\frac{x}{\lambda^\delta})\|_{H^s}\le2\lambda^{-1+\frac12\delta}\|\tilde{\phi}\|_{H^s}\to0,\\\|\rho_{1,\lambda}(0)-\rho_{-1,\lambda}(0)\|_{H^{s-1}}=2\lambda^{-1}\|\tilde{\psi}(\frac{x}{\lambda^\delta})\|_{H^{s-1}}\le2\lambda^{-1+\frac12\delta}\|\tilde{\psi}\|_{H^{s-1}}\to0\end{gathered}$$

as $\lambda\to\infty$, which implies that (5.2) holds. Now, we prove (5.3). It is easy to see that

$$\liminf_{\lambda\to\infty}\left(\|u_\lambda(t)-\tilde{u}_\lambda(t)\|_{H^s}+\|\rho_\lambda(t)-\tilde{\rho}_\lambda(t)\|_{H^{s-1}}\right)\ge\liminf_{\lambda\to\infty}\|u_\lambda(t)-\tilde{u}_\lambda(t)\|_{H^s}.$$

Thus we only prove that

$$\liminf_{\lambda\to\infty} \|u_\lambda(t) - \tilde{u}_\lambda(t)\|_{H^s} \gtrsim \sin t, \quad |t| < T \le 1.$$

Obviously,

$$\begin{aligned} &\|u_{1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^s} \\ &\ge \|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^s} - \|u^{1,\lambda}(t) - u_{1,\lambda}(t)\|_{H^s} - \|u^{-1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^s}. \end{aligned}$$

It follows from (5.9) that

$$\|u_{1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^s} \ge \|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^s} - c\lambda^{-\varepsilon_s}, \quad \lambda \gg 1,$$

which implies that

$$\liminf_{\lambda\to\infty} \|u_{1,\lambda}(t) - u_{-1,\lambda}(t)\|_{H^s} \ge \liminf_{\lambda\to\infty} \|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^s}. \tag{5.10}$$ 5.10

The identity $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$ gives

$$u^{1,\lambda}(t) - u^{-1,\lambda}(t) = u_{l,1,\lambda}(t) - u_{l,-1,\lambda}(t) + 2\lambda^{-\frac{1}{2}\delta-s}\phi\big(\frac{x}{\lambda^\delta}\big)\sin\lambda x\sin t.$$

Thus,

$$\begin{aligned} &\|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^s} \\ &\ge 2\lambda^{-\frac{1}{2}\delta-s}\|\phi\big(\frac{x}{\lambda^\delta}\big)\sin\lambda x\|_{H^s}|\sin t| - \|u_{l,1,\lambda}(t)\|_{H^s} - \|u_{l,-1,\lambda}(t)\|_{H^s} \\ &\gtrsim \lambda^{-\frac{1}{2}\delta-s}\|\phi\big(\frac{x}{\lambda^\delta}\big)\sin\lambda x\|_{H^s}|\sin t| - \lambda^{-1+\frac{1}{2}\delta}, \quad \lambda \gg 1. \end{aligned}$$

Letting $\lambda\to\infty$ in the above inequality, we have

$$\liminf_{\lambda\to\infty} \|u^{1,\lambda}(t) - u^{-1,\lambda}(t)\|_{H^s} \gtrsim |\sin t|. \tag{5.11}$$ 5.11

Summing inequalities (5.10) and (5.11) up, it yields inequality (5.3). This completes the proof. □

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References

BC1 [1] A. Bressan, A. Constantin, *Global conservation solutions of the Camassa-Holm equation*, Arch. Rat. Mech. Anal. **183** (2007) 215-239.

BC2 [2] A. Bressan, A. Constantin, *Global dissipative solutions of the Camassa-Holm equation*, Anal. Appl. **5** (2007) 1-27.

CH [3] R. Camassa, D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Rev. Lett. **71**(1993), 1661–1664.

C2006 [4] A. Constantin, *The trajectories of particles in Stokes waves*, Invent. Math. **f166** (2006) 523-535.

CE1 [5] A. Constantin, J. Escher, *Well-posedness, global existence and blowup phenomena for a periodic quasi-linear hyperbolic equation*, Comm. Pure Appl. Math. **51** (1998) 475-504.

CE2 [6] A. Constantin, J. Escher, *Wave breaking for nonlinear nonlocal shallow water equations*, Acta Math. (1998) 229-243.

CE4 [7] A. Constantin, J. Escher, *Global weak solutions for a shallow water equation*, Indiana Univ. Math. J. **47** (1998) 1527-1245.

CE3 [8] A. Constantin, J. Escher, *On the blow-up rate and the blow-up of breaking waves for a shallow water equation*, Math. Z. **233** (2000) 75-91.

CE2007 [9] A. Constantin, J. Escher, *Particle trajectories in solitary water waves*, Bull. Amer. Math. Soc. **44** (2007) 423-431.

CI [10] A. Constantin, R. Ivanov, *On an integrable two-component Camassa-Holm shallow water system*, Phys. Lett. A **372** (2008) 7129-7132.

ELY [11] J. Escher, O. Lechtenfeld, Z. Yin, *Well-posedness and blow-up phenomena for the 2-component Camassa-Holm equation*, Discrete Contin. Dyn. Syst. **19** (2007) 493-513.

EY1 [12] J. Escher, Z. Yin, *Initial boundary value problems of the Camassa-Holm equation*, Commun. Partial Differential Equations **33** (2008) 377-395.

FQ [13] Y. Fu, C. Qu, *Well-posedness and blow-up solution for a new coupled Camassa-Holm system with peakons*, J. Math. Phys. 50, 012906 (2009).

FL [14] Y. G. Fu, Z. R. Liu, *Non-uniform dependence on initial data for the periodic modified Camassa-Holm equation*, Nonlinear Differential Equations and Applications NoDEA **20** (2013) 741-755.

GY [15] C. Guan, Z. Yin, *Global existence and blow-up phenomena for an integrable two-component Camassa-Holm shallow water system*, J. Differential Equations **248** (2010) 2003-2014.

GY1 [16] C. Guan, Z. Yin, *Global weak solutions for a two-component Camassa-Holm shallow water system*, J. Funct. Anal. **260** (2011) 1132-1154.

GL1 [17] G. Gui, Y. Liu, *On the global existence and wave-breaking criteria for the two-component Camassa-Holm system*, J. Funct. Anal. **258** (2010) 4251-4278.

GL2 [18] G. Gui, Y. Liu, *On the Cauchy problem for the two-component Camassa-Holm system*, Math. Z. **268** (2011) 45-66.

GZ [19] Z. G. Guo, Y. Zhou, *On solutions to a two-component generalized Camassa-Holm equation*, Stud. Appl. Math. **124** (2010) 307C322.

HMPZ [20] A. Himonas, G. Misiolek, G. Ponce, Y. Zhou, *Persistence properties and unique continuous of solutions of the Camassa-Holm equation*, Comm. Math. Phys. **271** (2007) 511-522.

HK [21] A. Himonas, C. Kenig, *Non-uniform dependence on initial data for the CH equation on the line*, Differential Integral Equations **22** (2009) 201-224.

HKM [22] A. Himonas, C. Kenig, G. Misiolek, *Non-uniform dependence for the periodic CH equation*, Commun. Partial Differential Equations **35** (2010) 1145-1162.

HNT [23] D. D. Holm, L. Naraigh, C. Tronci, *Singular solution of a modified two component Camassa-Holm equation*, Phy. Rev. E **79** (2009) 1-13.

IM [24] R. Iorio, de Magãlhaes Iorio, *Fourier Analysis and partial differential equation.* Cambridge University Press, Cambridge (2001).

K2007 [25] B. Kolev, *Bi-Hamiltonian systems on the dual of the Lie algebra of vector fields of the circle and periodic shallow water equations*, Phil. Trans. Roy. Soc. London A **365** (2007) 2333-2357.

K2008 [26] B. Kolev, *Geometric differences between the Burgers and the Camassa-Holm equations*, J. Nonl. Math. Phys. **15** (2008) 116-132.

LPW [27] G. Y. Lv, Peter Y. H. Pang . M. X. Wang, *Non-uniform Dependence on Initial Data for the μ-b Equation*, Z. Angew. Math. Phys. **64** (2013) 1543-1554.

LWjmp [28] G. Y. Lv, M. X. Wang, *Non-uniform dependence for a modified Camassa-Holm system*, J. Math. Physics **53** (2012) 013101.

TW [29] P. Tang, X. Wang, *Hölder continuity for a modified Camassa-Holm system*, Applied Mathematics and Computation **234** (2014) 63-68.

TSD [30] L. X. Tian, C. Y. Shen, D. P. Ding, *Optimal control of the viscous Camassa-Holm equation*, Nonlinear Anal. RWA **10** (2009) 519-530.

T1996 [31] J. F. Toland, *Stokes waves*, Topol. Meth. Nonl. Anal. **7** (1996) 1-48.

Wxh [32] X. Wang, *Non-uniform dependence on initial data of a periodic Camassa-Holm system*, Abstract and Applied Analysis (2014), Article ID 823126, 10 pp.

YLF [33] J. L. Yin, L. X. Tian, X. H. Fan, *Orbital stability of floating periodic peakons for the Camassa-Holm equation*, Nonlinear Anal. RWA **11** (2010) 4021-4026.

Y2 [34] Z. Yin, *Global existence for a new periodic integrable equation*, J. Math. Anal. Appl. **283** (2003) 129-139.

ZC [35] Y. Zhou, H. P. Chen, *Wave breaking and propagation speed for the Camassa-Holm equation with $k \neq 0$*, Nonlinear Anal. Real World Appl. **12** (2011) 1355-1358.

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