Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 15, pp. 1–22.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

LACK OF COERCIVITY FOR N-LAPLACE EQUATION WITH CRITICAL EXPONENTIAL NONLINEARITIES IN A BOUNDED DOMAIN

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Abstract. In this article, we study the existence and multiplicity of non-negative solutions of the N-Laplacian equation

$$\begin{gathered} -\Delta_N u + V(x)|u|^{N-2}u = \lambda h(x)|u|^{q-1}u + u|u|^p e^{|u|^\beta} \quad \text{in } \Omega \\ u \geq 0 \quad \text{in } \Omega, \quad u \in W_0^{1,N}(\Omega), \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, $0 < q < N-1 < p+1$, $\beta \in (1, \frac{N}{N-1}]$ and $\lambda > 0$. By minimization on a suitable subset of the Nehari manifold, and using fiber maps, we find conditions on V, h for the existence and multiplicity of solutions, when V and h are sign changing and unbounded functions.

1. Introduction

We consider the following quasilinear elliptic equation in $\Omega \subset \mathbb{R}^N$, $N \geq 2$:

$$\begin{gathered} -\Delta_N u + V(x)|u|^{N-2}u = \lambda h(x)|u|^{q-1}u + u|u|^p e^{|u|^\beta} \quad \text{in } \Omega \\ u \geq 0 \quad \text{in } \Omega, \quad u \in W_0^{1,N}(\Omega), \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

where $\Delta_N u = \nabla \cdot (|\nabla u|^{N-2}\nabla u)$, $0 < q < N-1 < p+1$, $\beta \in (1, \frac{N}{N-1}]$ and $\lambda > 0$. Let $\gamma = \frac{N}{N-q-1}$, $k = \frac{p+2+\beta}{q+1} > 1$ and $k' = \frac{k}{k-1}$. We assume the following:

- (A1) $V \in L^s(\Omega)$, $s > 1$ be an indefinite and unbounded function;
- (A2) $h^+ \not\equiv 0$, h can be indefinite and vanish in some open subset of Ω and moreover $h \in L^\gamma(\Omega)$.

These conditions ensure that $E_V(u) := \int_\Omega (|\nabla u|^N + V(x)|u|^N)dx$ is weakly lower semi-continuous on $W_0^{1,N}(\Omega)$, where as $H(u) := \int_\Omega h(x)|u|^{q+1}dx$ is weakly continuous on $W_0^{1,N}(\Omega)$. Problems of the type (1.1) are motivated by the following Trudinger-Moser inequality.

2000 *Mathematics Subject Classification.* 35J35, 35J60, 35J92.
Key words and phrases. Quasilinear problem; critical exponent; Trudinger-Moser embedding; sign-changing weight function.

Submitted May 23, 2013. Published January 8, 2014.

Theorem 1.1 ([16])**.** *For $N \geq 2$, $u \in W_0^{1,N}(\Omega)$*

$$\sup_{\|u\|\leq 1} \int_\Omega e^{\alpha |u|^{\frac{N}{N-1}}} dx < \infty \tag{1.2}$$

if and only if $\alpha \leq \alpha_N$, where $\alpha_N = N w_{N-1}^{\frac{1}{N-1}}$, $w_{N-1} =$ volume of S^{N-1}.

The embedding $W_0^{1,N}(\Omega) \ni u \longmapsto e^{|u|^\beta} \in L^1(\Omega)$ is compact for all $\beta \in (1, \frac{N}{N-1})$ and is continuous for $\beta = \frac{N}{N-1}$. The non-compactness of the embedding can be shown using a sequence of functions that are truncations of fundamental solution of $-\Delta_N$ on $W_0^{1,N}(\Omega)$. The existence results for quasilinear problems with exponential terms on bounded domains was initiated and studied by Adimurthi [1].

Starting from the pioneering works of Tarantello [21] and Ambrosetti-Brezis-Cerami [5], a lot of work has been done to address the multiplicity of positive solutions for semilinear and quasilinear elliptic problems with positive nonlinearities. Recently, many works are devoted to the study of these multiplicity results with polynomial type nonlinearity with sign-changing weight functions using the Nehari manifold and fibering map analysis (see [2, 3, 8, 10, 11, 12, 21, 22, 23, 24, 25]). Nonhomogeneous elliptic equation with exponential nonlinearity is also dealt in [18]. In [19], Quoirin studied the quasilinear equation

$$-\Delta_p u + V(x)u^{p-1} = \lambda a(x)u^{r-1} + b(x)u(x)^{q-1} \text{ in } \Omega, \quad u(x) = 0 \text{ on } \partial\Omega,$$

where $1 < r < p < q < p^*$, $V(x)$, $a(x)$ and $b(x)$ are indefinite functions. Under suitable condition on V, a, b, he showed the existence of four non-negative solutions when $\lambda_1(V) < 0$, the first eigenvalue of $-\Delta_N + V$. Also In [20], Quoirin and Ubilla showed the existence and multiplicity of non-negative solutions of the following equation:

$$-\Delta u + V(x)u^{p-1} = \lambda a(x)u^{r-1} + b(x)u^{q-1} \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where $1 < r < 2 < q \leq \frac{2N}{N-2}$, $V(x)$, $a(x)$ and $b(x)$ are sign changing. Our work in this paper is motivated by the work of Quoirin [19].

The first and second eigenvalues of the operator $-\Delta_N + V$ on $W_0^{1,N}(\Omega)$ are denoted by $\lambda_1(V) < \lambda_2(V)$ respectively and $\phi_V > 0$ be its L^N-normalized eigenfunction corresponding to $\lambda_1(V)$. Also $\lambda_1(V)$ is characterized as $\min\{E_V(u) : \|u\|_N = 1\}$, is simple and principal so that ϕ_V is unique and positive.

When $\lambda_1(V) \leq 0$, the effect of potential V on (1.1) is relevant as $E_V(u)$ becomes non-coercive. We will see that in this case, (1.1) has existence and multiplicity of non-negative solutions in critical and subcritical case respectively which are distinguished by the sign of V and h. As in [19] and [9], we define,

$$\alpha(V,h) := \min\{E_V(u) : \|u\|_N = 1, H(u) = 0\},$$
$$\beta(V,h) := \min\{E_V(u) : H(u) = 1\}.$$

Then $\alpha(V,h)$ is well-defined by assuming the convention that $\min \emptyset = \infty$. It is clear that $\lambda_1(V) \leq \alpha(V,h)$ and $\lambda_1(V) = \alpha(V,h)$ if and only if $H(\phi_V) = 0$. Also one can easily see that $\beta(V,h)$ is well-defined if $\alpha(V,h) > 0$ (see [19, Lemma 4.3]). Also we introduce some symbols:

$$E^\pm := \{u \in W_0^{1,N}(\Omega) : E_V(u) \gtrless 0\}, \quad E_0 := \{u \in W_0^{1,N}(\Omega) : E_V(u) = 0\},$$
$$H^\pm := \{u \in W_0^{1,N}(\Omega) : H(u) \gtrless 0\}, \quad H_0 := \{u \in W_0^{1,N}(\Omega) : H(u) = 0\},$$

and $H_0^{\pm} := H^{\pm} \cup H_0$, $E_0^{\pm} := E^{\pm} \cup E_0$. We have the following existence result.

Theorem 1.2. *Let $\beta \in (1, \frac{N}{N-1}]$, V and h satisfy* (A1), (A2) *respectively. Then there exists $\lambda_0 > 0$ such that for $\lambda \in (0, \lambda_0)$,* (1.1) *admits a non-negative solution u_λ in each of the following cases:*

(1) $\lambda_1(V) > 0$,
(2) $\lambda_1(V) = 0$ *and* $\phi_V \in H^-$,
(3) $\lambda_1(V) < 0 < \lambda_2(V)$, $\phi_V \in H^-$ *and* $\alpha(V, h) > 0$.

Moreover, u_λ is a local minimum for J_λ on $W_0^{1,N}(\Omega)$.

We have the following multiplicity result in the subcritical case when $h(x)$ changes sign:

Theorem 1.3. *For $\beta \in (1, \frac{N}{N-1})$, V and h satisfy* (A1), (A2) *respectively. Then for $\lambda \in (0, \lambda_0)$,* (1.1) *has a non-negative second solution v_λ in each the cases (1)-(3) above.*

Finally, in the critical case, we obtain the following multiplicity result.

Theorem 1.4. *For $\beta = \frac{N}{N-1}$, $h \geq 0$ and $\lambda_1(V) > 0$ then for $\lambda \in (0, \lambda_0)$,* (1.1) *has at least two non-negative solutions.*

Here λ_0 is the maximum of λ such that for $\lambda < \lambda_0$, the fibering map $t \mapsto J_\lambda(tu)$ has exactly two critical points for each $u \in E^+ \cap H^+$.

The Euler functional associated with the problem (1.1) is $J_\lambda : W_0^{1,N}(\Omega) \to \mathbb{R}$ defined as

$$J_\lambda(u) = \frac{1}{N} E_V(u) - \frac{\lambda}{q+1} H(u) - \int_\Omega G(u) dx, \tag{1.3}$$

where $g(u) = u|u|^p e^{|u|^\beta}$ and $G(u) = \int_0^u g(s) ds$.

Definition 1.5. We say that $u \in W_0^{1,N}(\Omega)$ is a weak solution of (1.1) if for all $\phi \in W_0^{1,N}(\Omega)$, we have

$$\int_\Omega \left(|\nabla u|^{N-2} \nabla u \nabla \phi + V(x)|u|^{N-2} u\phi \right) dx = \int_\Omega g(u)\phi dx + \lambda \int_\Omega h(x)|u|^{q-1} u\phi dx. \tag{1.4}$$

We remark that the similar existence results with some obvious modification can be proved for critical exponent problem for p-Laplacian with $p < N$ and $g(u) = |u|^{p^*-2}u$ where $p^* = \frac{Np}{N-p}$, while subcritical case is studied by Quoirin in [19]. The multiplicity in the critical case can be obtained with some condition on p, q as in [13].

This paper is organized as follows: In section 2, we introduce Nehari manifold and study the behavior of the Nehari manifold using the fibering map analysis for (1.1). Section 3 contains the existence results for critical and subcritical nonlinearities. In section 4, we show the existence of a second solution. In section 5 we study non-existence results.

We shall throughout use the following notation: The norm on $W_0^{1,N}(\Omega)$ and $L^p(\Omega)$ are denoted by $\|\cdot\|$, $\|u\|_p$ respectively. The weak convergence is denoted by $\rightharpoonup$ and $\to$ denotes strong convergence.

2. Nehari manifold and fibering map analysis for (1.1)

The energy functional J_λ is not bounded below on the space $W_0^{1,N}(\Omega)$, but is bounded below on an appropriate subset of $W_0^{1,N}(\Omega)$ and a minimizer on subsets of this set gives rise to solutions of (1.1). To obtain the existence results, we introduce the Nehari manifold

$$\mathcal{N}_\lambda = \{u \in W_0^{1,N}(\Omega) : \langle J'_\lambda(u), u\rangle = 0\} = \{u \in W_0^{1,N}(\Omega) : \phi'_u(1) = 0\}$$

where $\langle \cdot,\cdot\rangle$ denotes the duality between $W_0^{1,N}(\Omega)$ and its dual space. Therefore $u \in \mathcal{N}_\lambda$ if and only if

$$E_V(u) - \lambda H(u) - \int_\Omega g(u)u dx = 0. \tag{2.1}$$

We note that $\mathcal{N}_\lambda$ contains every non zero solution of (1.1). Now as we know that the Nehari manifold is closely related to the behaviour of the functions $\phi_u : \mathbb{R}^+ \to \mathbb{R}$ defined as $\phi_u(t) = J_\lambda(tu)$. Such maps are called fiber maps and were introduced by Drabek and Pohozaev in [10]. For $u \in W_0^{1,N}(\Omega)$, we have

$$\begin{aligned}
\phi_u(t) &= \frac{t^N}{N} E_V(u) - \frac{\lambda t^{q+1}}{q+1} H(u) - \int_\Omega G(tu)dx,\\
\phi'_u(t) &= t^{N-1}E_V(u) - \lambda t^q H(u) - \int_\Omega g(tu)u dx,\\
\phi''_u(t) &= (N-1)t^{N-2}E_V(u) - q\lambda t^{q-1}H(u) - \int_\Omega g'(tu)u^2 dx.
\end{aligned}$$

Then it is easy to see that $tu \in \mathcal{N}_\lambda$ if and only if $\phi'_u(t) = 0$ and in particular, $u \in \mathcal{N}_\lambda$ if and only if $\phi'_u(1) = 0$. Thus it is natural to split $\mathcal{N}_\lambda$ into three parts corresponding to local minima, local maxima and points of inflection. For this we set

$$\begin{aligned}
\mathcal{N}^\pm_\lambda &:= \{u \in \mathcal{N}_\lambda : \phi''_u(1) \gtrless 0\} = \{tu \in W_0^{1,N}(\Omega) : \phi'_u(t) = 0,\ \phi''_u(t) \gtrless 0\},\\
\mathcal{N}^0_\lambda &:= \{u \in \mathcal{N}_\lambda : \phi''_u(1) = 0\} = \{tu \in W_0^{1,N}(\Omega) : \phi'_u(t) = 0,\ \phi''_u(t) = 0\}.
\end{aligned}$$

Now we describe the behavior of the fibering map ϕ_u according to the sign of $H(u)$ and $E_V(u)$.
Case 1: $u \in H^+ \cap E^-$. In this case $\phi_u(0) = 0$, $\phi'_u(t) < 0$ for all $t > 0$ which implies that ϕ_u is strictly decreasing and hence no critical point.
Case 2: $u \in H^- \cap E^-$. In this case, firstly we define $m_u : \mathbb{R}^+ \to \mathbb{R}$ by

$$m_u(t) = t^{N-1-q}E_V(u) - t^{-q}\int_\Omega g(tu)u dx.$$

Clearly, for $t > 0$, $tu \in \mathcal{N}_\lambda$ if and only if t is a solution of $m_u(t) = \lambda H(u)$.

$$\begin{aligned}
m'_u(t) &= (N-1-q)t^{(N-2-q)}E_V(u) - t^{-q}\int_\Omega g'(tu)u^2 dx + qt^{-q-1}\int_\Omega g(tu)u dx\\
&= (N-1-q)t^{N-2-q}E_V(u) - (1+p-q)t^{-1-q}\int_\Omega g(tu)u\\
&\quad - \beta t^{-q-1+\beta}\int_\Omega |u|^\beta g(tu)u.
\end{aligned}$$

Therefore $m'_u(t) < 0$ for all $t > 0$, since $u \in E^-$. As $u \in H^-$ so there exists $t_*(u)$ such that $m_u(t_*) = \lambda H(u)$. Thus for $0 < t < t_*$, $\phi'_u(t) = t^q(m_u(t) - \lambda H(u)) > 0$

and for $t > t_*$, $\phi_u'(t) < 0$. Hence ϕ_u is increasing on $(0, t_*)$, decreasing on (t_*, ∞). Since $\phi_u(t) > 0$ for t close to 0 and $\phi_u(t) \to -\infty$ as $t \to \infty$, we obtain ϕ_u has exactly one critical point $t_1(u)$, which is a global maximum point. Hence $t_1(u)u \in \mathcal{N}_\lambda^-$.
Case 3: $u \in E^+ \cap H^-$. In this case, we have

$$
\begin{aligned}
&m_u'(t)\\
&= (N-1-q)t^{(N-2-q)}E_V(u) - t^{-q}\int_\Omega g'(tu)u^2dx + qt^{-q-1}\int_\Omega g(tu)u\\
&= t^{N-2-q}\Big[(N-1-q)E_V(u) - (1+p-q)t^{1-N}\\
&\quad\times \int_\Omega g(tu)u - \beta t^{1-N+\beta}\int_\Omega |u|^\beta g(tu)u\Big].
\end{aligned}
\tag{2.2}
$$

It is easy to see that $\lim_{t\to 0^+} m_u'(t) > 0$ and sum of second and third term in (2.2) is a monotone function in t. Therefore there exists a unique $t_* = t_*(u) > 0$ such that $m_u(t)$ is increasing on $(0, t_*)$, decreasing on (t_*, ∞) and $m_u'(t_*) = 0$. As $m_u(t) \to -\infty$ as $t \to \infty$, $u \in H^-$ so $\exists\ t_1(u)$ such that $m_u(t_1) = \lambda H(u)$. Thus for $0 < t < t_1$, $\phi_u'(t) > 0$ and for $t > t_1$, $\phi_u'(t) < 0$. Thus ϕ_u has exactly one critical point $t_1(u)$, which is a global maximum point. Hence $t_1(u)u \in \mathcal{N}_\lambda^-$
Case 4: $u \in E^+ \cap H^+$. In this case, we claim that there exists $\lambda_0 > 0$ and a unique t_* such that for $\lambda \in (0, \lambda_0)$, ϕ_u has exactly two critical points $t_1(u)$ and $t_2(u)$ such that $t_1(u) < t_*(u) < t_2(u)$, and moreover $t_1(u)$ is a local minimum point and $t_2(u)$ is a local maximum point. Thus $t_1(u)u \in \mathcal{N}_\lambda^+$ and $t_2(u)u \in \mathcal{N}_\lambda^-$.

To show this we need following Lemmas:

Lemma 2.1. *If $\alpha(V, h) > 0$, $\beta(V, h) > 0$, then $H_0^+ \subseteq E^+$ and moreover, there exists a constant $K > 0$ such that $E_V(u) \ge K\|u\|^N$ for all $u \in H_0^+$.*

Proof. Let $u \in H_0^+$ then $H(u) \ge 0$ and $\alpha(V, h) > 0$, $\beta(V, h) > 0$ implies that $E_V(u) > 0$. Next we show that there exists a constant $K > 0$ such that $E_V(u) \ge K\|u\|^N$ for all $u \in H_0^+$. Suppose this is not true, then for each n, there exist $u_n \in H_0^+$ such that $E_V(u_n) < \frac{\|u_n\|^N}{n}$. Let $v_n = \frac{u_n}{\|u_n\|}$. Then v_n is bounded. So there exists a subsequence v_n such that $v_n \rightharpoonup v_0$ weakly in $W_0^{1,N}(\Omega)$. Also $H(v_0) \ge 0$ and $0 \le E_V(v_n) < \frac{1}{n}$ implies $E_V(v_0) \le 0$. Moreover $v_0 \ne 0$ because if $v_0 = 0$ then we obtain $\|v_n\| \le E_V(v_n) + \int_\Omega |V(x)||v_n|^N dx \to 0$, which is a contradiction as $\|v_n\| = 1$. Thus $E_V(v_0) \le 0 \le H(v_0)$ imply $\alpha(V, h) \le 0$, $\beta(V, h) \le 0$ which contradict the given assumptions. $\square$

Next, we define $\Lambda = \{u \in W_0^{1,N}(\Omega) \mid E_V(u) \le \frac{N-q}{(N-1-q)}\int_\Omega g'(u)u^2dx\}$. Then, we prove the following Lemma.

Lemma 2.2. *Let $\alpha(V, h) > 0$, $\beta(V, h) > 0$ then there exists $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$,*

$$
\Lambda_m := \inf_{u\in\Lambda\setminus\{0\}\cap H_0^+}\Big\{\int_\Omega \big(p+2-N+\beta|u|^\beta\big)|u|^{p+2}e^{|u|^\beta} - (N-1-q)\lambda H(u)\Big\} > 0.
\tag{2.3}
$$

Proof. Step 1: $\inf_{u\in\Lambda\setminus\{0\}\cap H_0^+} E_V(u) > 0$. In view of Lemma 2.1 it is sufficient to show that

$$
\inf_{u\in\Lambda\setminus\{0\}\cap H_0^+}\|u\| > 0.
$$

Suppose this is not true. Then we find a sequence $\{u_n\} \subset \Lambda \setminus \{0\} \cap H_0^+$ such that $\|u_n\| \to 0$ and we have

$$E_V(u_n) \leq (\frac{N-q}{N-1-q}) \int_\Omega g'(u_n)u_n^2 \, dx \quad \forall n. \tag{2.4}$$

From $g(u) = u|u|^p e^{|u|^\beta}$, Hölders inequality and Sobolev inequality, we have

$$\begin{aligned}
\int_\Omega g'(u_n)u_n^2 dx &= \int_\Omega \left(p+1+\beta|u_n|^\beta\right)|u_n|^{p+2}e^{|u_n|^\beta}dx \\
&\leq C\int_\Omega |u_n|^{p+2}e^{(1+\delta)|u_n|^\beta}dx \\
&\leq C\Big(\int_\Omega |u_n|^{(p+2)t'}dx\Big)^{1/t'}\Big(\int_\Omega e^{t(1+\delta)|u_n|^\beta}\,dx\Big)^{1/t} \\
&\leq C'\|u_n\|^{p+2}\Big(\sup_{\|w_n\|\leq 1}\int_\Omega e^{t(1+\delta)\|u_n\|^\beta|w_n|^\beta}dx\Big)^{1/t},
\end{aligned}$$

since $\|u_n\| \to 0$ as $n \to \infty$, we can choose $\alpha = t(1+\delta)\|u_n\|^\beta$ such that $\alpha \leq \alpha_N$. Hence by this, (2.4) and Lemma 2.1, we obtain $1 \leq K'\|u_n\|^{p+2-N} \to 0$ as $n \to \infty$, since $p+2 > N$, which gives a contradiction.

Step 2: Let $C_1 = \inf_{u\in\Lambda\setminus\{0\}\cap H_0^+}\int_\Omega \left(p+2-N+\beta|u|^\beta\right)|u|^{p+2}e^{|u|^\beta}dx$. Then $C_1 > 0$. From Step 1 and the definition of Λ, we obtain

$$0 < \inf_{u\in\Lambda\setminus\{0\}\cap H_0^+}\int_\Omega g'(u)u^2dx = \inf_{u\in\Lambda\setminus\{0\}\cap H_0^+}\int_\Omega \left(p+1+\beta|u|^\beta\right)|u|^{p+2}e^{|u|^\beta}dx.$$

Using this it is easy to check that

$$\inf_{u\in\Lambda\setminus\{0\}\cap H_0^+}\int_\Omega \left(p+2-N+\beta|u|^\beta\right)|u|^{p+2}e^{|u|^\beta}dx > 0.$$

This completes step 2.

Step 3: Let $\lambda < \frac{1}{(N-q-1)}(\frac{C_1}{l})^{\frac{(k-1)}{k}}$, where $l = \int_\Omega |h(x)|^{\frac{k}{k-1}}dx$. Then (2.3) holds. Using Hölder's inequality and (A2) we have,

$$\begin{aligned}
H(u) &\leq \Big(\int_\Omega |h(x)|^{\frac{k}{k-1}}dx\Big)^{\frac{k-1}{k}}\Big(\int_\Omega |u|^{(q+1)k}dx\Big)^{1/k} \\
&= l^{\frac{k-1}{k}}\Big(\int_\Omega |u|^{p+2+\beta}dx\Big)^{1/k} \\
&\leq l^{\frac{k-1}{k}}\Big(\int_\Omega (p+2-N+\beta|u|^\beta)|u|^{p+2}e^{|u|^\beta}dx\Big)^{1/k} \\
&\leq (\frac{l}{C_1})^{\frac{k-1}{k}}\int_\Omega \left(p+2-N+\beta|u|^\beta\right)|u|^{p+2}e^{|u|^\beta}\,dx.
\end{aligned}$$

The above inequality combined with step 2 proves the Lemma. □

The following Lemma completes the proof of claim made in case (4) above.

Lemma 2.3. *Let $\alpha(V,h) > 0$, $\beta(V,h) > 0$ and λ be such that (2.3) holds. Then for every $u \in H^+ \setminus \{0\}$, there is a unique $t_* = t_*(u) > 0$ and unique $t_1 = t_1(u) < t_* < t_2 = t_2(u)$ such that $t_1u \in \mathcal{N}_\lambda^+$, $t_2u \in \mathcal{N}_\lambda^-$ and $J_\lambda(t_1u) = \min_{0\leq t\leq t_2} J_\lambda(tu)$, $J_\lambda(t_2u) = \max_{t\geq t_*} J_\lambda(tu)$.*

Proof. Fix $0 \neq u \in H^+$, Then by Lemma 2.1, $u \in E^+$. Define $m_u : \mathbb{R}^+ \to \mathbb{R}$ by

$$m_u(t) = t^{N-1-q} E_V(u) - t^{-q} \int_\Omega g(tu)u dx.$$

We note that $m_u(t) \to -\infty$ as $t \to \infty$ and

$$\begin{aligned} m_u'(t) &= (N-1-q)t^{(N-2-q)} E_V(u) - t^{-q} \int_\Omega g'(tu)u^2 dx + qt^{-q-1} \int_\Omega g(tu)u\, dx \qquad (2.5)\\ &= t^{N-2-q}\Big[(N-1-q)E_V(u) - (1+p-q)t^{1-N} \int_\Omega g(tu)u \\ &\quad - \beta t^{1-N+\beta} \int_\Omega |u|^\beta g(tu)u dx\Big]. \qquad (2.6) \end{aligned}$$

It is easy to see that $\lim_{t\to 0^+} m_u'(t) > 0$ and sum of second and third term in (2.5) is a monotone function in t. So there exists a unique $t_* = t_*(u) > 0$ such that $m_u(t)$ is increasing on $(0, t_*)$, decreasing on (t_*, ∞) and $m_u'(t_*) = 0$. Using this and (2.5), we obtain $t_* u \in \Lambda \setminus \{0\} \cap H^+$. From $t_*^{q+2} m_u'(t_*) = 0$ and by definition of m_u, we obtain

$$m_u(t_*) = \frac{1}{t_*^{q+1}(N-1-q)}\Big[\int_\Omega g'(t_*u)(t_*u)^2 dx - (N-1)\int_\Omega g(t_*u)t_*u dx\Big].$$

Using Lemma 2.2 and that $g'(s)s^2 - (N-1)g(s)s = (p+2-N+\beta|s|^\beta)|s|^{p+2}e^{|s|^\beta}$, we have

$$\begin{aligned} m_u(t_*) - \lambda H(u) &= \frac{1}{t_*^{q+1}(N-1-q)}\Big[\int_\Omega \big(g'(t_*u)(t_*u)^2 - (N-1)g(t_*u)t_*u\big)dx \\ &\quad - (N-1-q)\lambda H(t_*(u))\Big] \\ &> \frac{\Lambda_m}{t_*^{q+1}(N-1-q)} > 0. \end{aligned}$$

Since $m_u(0) = 0$, m_u is increasing in $(0, t_*)$ and strictly decreasing in (t_*, ∞), $\lim_{t\to\infty} m_u(t) = -\infty$ and $u \in H^+$. Then there exists a unique $t_1 = t_1(u) < t_*$ and $t_2 = t_2(u) > t_*$ such that $m_u(t_1) = \lambda H(u) = m_u(t_2)$ implies $t_1 u$, $t_2 u \in \mathcal{N}_\lambda$. Also $m_u'(t_1) > 0$ and $m_u'(t_2) < 0$ give $t_1 u \in \mathcal{N}_\lambda^+$ and $t_2 u \in \mathcal{N}_\lambda^-$. Since $\phi_u'(t) = t^q(m_u(t) - \lambda H(u))$. Then $\phi_u'(t) < 0$ for all $t \in [0, t_1)$ and $\phi_u'(t) > 0$ for all $t \in (t_1, t_2)$ so $\phi_u(t_1) = \min_{0\le t\le t_2} \phi_u(t)$. Also $\phi_u'(t) > 0$ for all $t \in [t_*, t_2)$, $\phi_u'(t_2) = 0$ and $\phi_u'(t) < 0$ for all $t \in (t_2, \infty)$ implies that $\phi_u(t_2) = \max_{t\ge t_*} \phi_u(t)$. □

Lemma 2.4. *If $\alpha(V,h) > 0$, $\beta(V,h) > 0$ and λ be such that (2.3) holds. Then $\mathcal{N}_\lambda^0 = \{0\}$.*

Proof. Suppose $u \in \mathcal{N}_\lambda^0$, $u \not\equiv 0$. Then by definition of $\mathcal{N}_\lambda^0$, we have the following two equations

$$(N-1)E_V(u) = \int_\Omega g'(u)u^2 dx + \lambda q H(u), \qquad (2.7)$$

$$E_V(u) = \int_\Omega g(u)u dx + \lambda H(u). \qquad (2.8)$$

Let $u \in H^+ \cap \mathcal{N}_\lambda^0$ and $\lambda \in (0, \lambda_0)$. Then from above equations, we can easily deduce that

$$(N-1-q)E_V(u) \leq \int_\Omega g'(u)u^2 dx,$$

which shows $u \in \Lambda \setminus \{0\}$. Noting that $g'(s)s^2 - (N-1)g(s)s = (p+2-N+\beta|s|^\beta)|s|^{p+2}e^{|s|^\beta}$, from (2.7) and (2.8), we obtain

$$(N-1-q)\lambda H(u) = \int_\Omega \left(p+2-N+\beta|u|^\beta\right)|u|^{p+2}e^{|u|^\beta}dx,$$

which violates Lemma 2.2. Hence $\mathcal{N}_\lambda^0 = \{0\}$. In other cases, $u \in H^- \cap E^- \cap \mathcal{N}_\lambda^0$ and $u \in H^- \cap E^+ \cap \mathcal{N}_\lambda^0$, we have 1 is critical point of ϕ_u and $\phi_u''(1) = 0$ but $u \in H^- \cap E^-$ and $u \in H^- \cap E^+$ implies that ϕ_u has exactly one critical point corresponding to global maxima i.e $\phi_u''(1) \neq 0$ which is a contradiction. Hence $\mathcal{N}_\lambda^0 = \{0\}$. □

3. Existence of solutions

In this section we show that J_λ is bounded below on $\mathcal{N}_\lambda$. Also we show that under suitable condition on V and h, J_λ attains its minimizer on $E^+ \cap H^+ \cap \mathcal{N}_\lambda^+$. We define $\theta_\lambda := \inf\{J_\lambda(u) \mid u \in \mathcal{N}_\lambda\}$ and prove the following lower bound.

Theorem 3.1. *J_λ is bounded below on $\mathcal{N}_\lambda$. Moreover, there exists a constant $C = C(p,q,N) > 0$ such that $\theta_\lambda \geq -C\lambda^{\frac{k}{k-1}}$.*

Proof. Let $u \in \mathcal{N}_\lambda$. Then

$$J_\lambda(u) = \frac{1}{N}\int_\Omega g(u)u\,dx - \int_\Omega G(u)\,dx - \lambda\left(\frac{1}{q+1} - \frac{1}{N}\right)H(u). \tag{3.1}$$

If $u \in H_0^-$, then $J_\lambda(u)$ is bounded below by 0. If $u \in H^+$ then by using Hölder's inequality, we have

$$H(u) \leq l^{\frac{k-1}{k}}\Big(\int_\Omega |u|^{(q+1)k}\,dx\Big)^{1/k},$$

where $l = \int_\Omega |h(x)|^{k/k-1}dx$. Also, It is easy to see that

$$\frac{1}{N}g(u)u - G(u) \geq \left(\frac{1}{N} - \frac{1}{p+2}\right)|u|^{p+2+\beta}, \tag{3.2}$$

From the above inequalities, we obtain

$$J_\lambda(u) \geq \left(\frac{1}{N} - \frac{1}{p+2}\right)\int_\Omega |u|^{(q+1)k}dx - \frac{\lambda(N-q-1)l^{\frac{k-1}{k}}}{N(q+1)}\Big(\int_\Omega |u|^{(q+1)k}dx\Big)^{1/k},$$

where $k = \frac{p+2+\beta}{q+1}$. By considering the global minimum of the function $\rho(x) : \mathbb{R}^+ \to \mathbb{R}$ defines as

$$\rho(x) = \left(\frac{1}{N} - \frac{1}{p+2}\right)x^k - \Big(\frac{\lambda(N-q-1)l^{\frac{k-1}{k}}}{N(q+1)}\Big)x,$$

it can be shown that

$$\inf_{u\in\mathcal{N}_\lambda} J_\lambda(u) \geq \rho\Big[\Big(\frac{\lambda(N-q-1)(p+2)l^{\frac{k-1}{k}}}{k(q+1)(p+2-N)}\Big)^{\frac{1}{k-1}}\Big].$$

From this, it follows that

$$\theta_\lambda \geq -C(p,q,N)\lambda^{\frac{k}{k-1}}, \tag{3.3}$$

where

$$C(p,q,N) = \Big(\frac{1}{k^{\frac{1}{k-1}}} - \frac{1}{k^{\frac{k}{k-1}}}\Big)\frac{l(p+2)^{\frac{1}{k-1}}(N-q-1)^{\frac{k}{k-1}}}{N(p+2-N)^{\frac{1}{k-1}}(q+1)^{\frac{k}{k-1}}} > 0.$$

Hence J_λ is bounded below on $\mathcal{N}_\lambda$. $\square$

The following lemma shows that minimizers for J_λ on any subset of $\mathcal{N}_\lambda$ are usually critical points for J_λ.

Lemma 3.2. *Let u be a local minimizer for J_λ in any of the subsets $\mathcal{N}_\lambda^\pm \cap E^+ \cap H^+$ of $\mathcal{N}_\lambda$ such that $u \notin \mathcal{N}_\lambda^0$, then u is a non-negative critical point for J_λ.*

Proof. Let u be a local minimizer for J_λ in any of the subsets of $\mathcal{N}_\lambda$. We can take $u \geq 0$ as $J_\lambda(|u|) = J_\lambda(u)$ for every u. Then, in any case u is a minimizer for J_λ under the constraint $I_\lambda(u) := \langle J'_\lambda(u), u\rangle = 0$. Hence, by the theory of Lagrange multipliers, there exists $\mu \in \mathbb{R}$ such that $J'_\lambda(u) = \mu I'_\lambda(u)$. Thus $\langle J'_\lambda(u), u\rangle = \mu\langle I'_\lambda(u), u\rangle = \mu\phi''_u(1)$=0, but $u \notin \mathcal{N}_\lambda^0$ and so $\phi''_u(1) \neq 0$. Hence $\mu = 0$ completes the proof. $\square$

Lemma 3.3. *If $\alpha(V,h) > 0$, $\beta(V,h) > 0$ and $\{u_n\} \in \mathcal{N}_\lambda \cap H_0^+$ be a sequence such that $J_\lambda(u_n)$ is bounded. Then the sequence $\{u_n\}$ is bounded.*

Proof. On $\mathcal{N}_\lambda$,

$$\begin{aligned} J_\lambda(u_n) &= (\frac{1}{N} - \frac{1}{p+2})E_V(u_n) - \frac{\lambda(p+1-q)}{(q+1)(p+2)}H(u_n) \\ &\quad + \int_\Omega \Big(\frac{1}{p+2}g(u_n)u_n - G(u_n)\Big)\,dx. \end{aligned} \tag{3.4}$$

As $u_n \in H_0^+$, we have

$$\frac{(p+2-N)}{N(p+2)}E_V(u_n) \leq J_\lambda(u_n) + \frac{\lambda(p+1-q)}{(q+1)(p+2)}H(u_n).$$

Then by using Lemma 2.1 and Hölders inequality, we obtain

$$K\|u_n\|^N \leq \frac{(p+2-N)}{N(p+2)}E_V(u_n) \leq J_\lambda(u_n) + \frac{\lambda(p+1-q)}{(q+1)(p+2)}\|h\|_{(\frac{N}{q+1})'}\|u_n\|^{q+1},$$

and hence $\{u_n\}$ is bounded. $\square$

Lemma 3.4. *Let $\alpha(V,h) > 0$, $\beta(V,h) > 0$, and let λ satisfy (2.3). Then given $u \in \mathcal{N}_\lambda \setminus \{0\}$, there exist $\epsilon > 0$ and a differentiable function $\xi : B(0,\epsilon) \subset W_0^{1,N}(\Omega) \to \mathbb{R}$ such that $\xi(0) = 1$, the function $\xi(w)(u-w) \in \mathcal{N}_\lambda$ and for all $w \in W_0^{1,N}(\Omega)$,*

$$\langle \xi'(0), w\rangle = \frac{NR(u,w) - \int_\Omega \big(g(u) + g'(u)u\big)w\,dx - \lambda(q+1)\int_\Omega h(x)|u|^{q-1}uw\,dx}{(N-q-1)E_V(u) - \int_\Omega g'(u)u^2dx + q\int_\Omega g(u)u\,dx}, \tag{3.5}$$

where $R(u,w) = \int_\Omega(|\nabla u|^{N-2}\nabla u\nabla w + V(x)|u|^{N-2}uw)dx$.

Proof. Fix $u \in \mathcal{N}_\lambda \setminus \{0\}$, define a function $G_u : \mathbb{R} \times W_0^{1,N}(\Omega) \to \mathbb{R}$ as follows:

$$G_u(t,v) = t^{N-1-q}E_V(u-v) - t^{-q}\int_\Omega g(t(u-v))(u-v)dx - \lambda H(u-v).$$

Then $G_u \in C^1(\mathbb{R} \times W_0^{1,N}(\Omega);\mathbb{R})$, $G_u(1,0) = \langle J'_\lambda(u), u\rangle = 0$ and

$$\frac{\partial}{\partial t}G_u(1,0) = (N-1-q)E_V(u) - \int_\Omega g'(u)u^2dx + q\int_\Omega g(u)udx \neq 0,$$

since $\mathcal{N}_\lambda^0 = \{0\}$. By the Implicit function theorem, there exist $\epsilon > 0$ and a differentiable function $\xi : B(0,\epsilon) \subset W_0^{1,N}(\Omega) \to \mathbb{R}$ such that $\xi(0) = 1$, and $G_u(\xi(w), w) = 0$ for all $w \in B(0,\epsilon)$ which is equivalent to $\langle J'_\lambda(\xi(w)(u-w)), \xi(w)(u-w)\rangle = 0$ for all $w \in B(0,\epsilon)$ and hence $\xi(w)(u-w) \in \mathcal{N}_\lambda$. Now differentiating $G_u(\xi(w), w) = 0$ with respect to w we obtain (3.5). $\square$

Lemma 3.5. *Let $\alpha(V,h) > 0$, $\beta(V,h) > 0$ then exists a constant $C_2 > 0$ such that $\theta_\lambda \leq -\frac{(p+1-q)}{(q+1)(p+2)N}C_2$.*

Proof. $\alpha(V,h) > 0$, $\beta(V,h) > 0$ implies that $E^+ \cap H_0^+ \neq \emptyset$. Let $v \in E^+ \cap H^+$. Then by the fibering map analysis, we can find $t_1 = t_1(v) > 0$ such that $t_1 v \in \mathcal{N}_\lambda^+$. Thus

$$\begin{aligned} J_\lambda(t_1v) &= \big(\frac{1}{N} - \frac{1}{q+1}\big)E_V(t_1v) - \int_\Omega G(t_1v)dx + \frac{1}{q+1}\int_\Omega g(t_1v)t_1v\,dx \\ &\leq \frac{q+N}{N(q+1)}\int_\Omega g(t_1v)t_1vdx - \int_\Omega G(t_1v)dx - \frac{1}{N(q+1)}\int_\Omega g'(t_1v)(t_1v)^2dx, \end{aligned} \tag{3.6}$$

since $t_1v \in \mathcal{N}_\lambda^+ \cap E^+$. We now consider the function

$$\rho(s) = \frac{q+N}{N(q+1)}g(s)s - G(s) - \frac{1}{N(q+1)}g'(s)s^2.$$

Then

$$\begin{aligned} \rho'(s) &= \frac{(q+N-2)}{N(q+1)}g'(s)s - \frac{q(N-1)}{N(q+1)}g(s) - \frac{1}{(q+1)N}g''(s)s^2 \\ &= \Big(\frac{(q+N-2-p)(p+1) - (N-1)q}{N(q+1)}\Big)g(s) \\ &\quad + \beta\Big(\frac{q-p+N-2-\beta-p-1}{N(q+1)}\Big)g(s)|s|^\beta - \frac{\beta^2}{N(q+1)}g(s)|s|^{2\beta}. \end{aligned}$$

Now it is not difficult to see that coefficients in the first and second term are negative, since $p > N-2$. As $\rho(0) = 0$, it follows that $\rho(s) \leq 0$ for all $s \in \mathbb{R}^+$. Also it can be easily verified that

$$\lim_{s\to 0}\frac{\rho(s)}{|s|^{p+2}} = -\frac{(p+1-q)(p+2-N)}{N(q+1)(p+2)};$$
$$\lim_{s\to\infty}\frac{\rho(s)}{|s|^{p+2+\beta}e^{|s|^\beta}} = -\frac{\beta}{N(q+1)}.$$

From these two estimates, we obtain

$$\rho(s) \leq -\frac{(p+1-q)}{N(q+1)(p+2)}\left(p+2-N+\beta|s|^\beta\right)|s|^{p+2}e^{|s|^\beta}. \tag{3.7}$$

Therefore, using (3.6) and (3.7), we obtain

$$J_\lambda(t_1v) \leq -\frac{(p+1-q)}{N(q+1)(p+2)}\int_\Omega \left(p+2-N+\beta|t_1v|^\beta\right)|t_1v|^{p+2}e^{|t_1v|^\beta}dx$$

$$\leq -\frac{(p+1-q)}{N(q+1)(p+2)} \int_\Omega |t_1 v|^{p+2+\beta} dx$$

Hence $\theta_\lambda \leq \inf_{u\in\mathcal{N}_\lambda^+\cap H^+} J_\lambda(u) \leq -\frac{(p+1-q)}{N(q+1)(p+2)}\, C_2$, where $C_2 = \int_\Omega |t_1 v|^{p+2+\beta} dx$. □

By Lemma 3.1, J_λ is bounded below on $\mathcal{N}_\lambda$. So, by Ekeland's Variational principle, we can find a sequence $\{u_n\} \in \mathcal{N}_\lambda \setminus \{0\}$ such that

$$J_\lambda(u_n) \leq \theta_\lambda + \frac{1}{n}, \tag{3.8}$$

$$J_\lambda(v) \geq J_\lambda(u_n) - \frac{1}{n}\|v-u_n\| \quad \forall\, v \in \mathcal{N}_\lambda. \tag{3.9}$$

We claim that if $\alpha(V,h) > 0$, $\beta(V,h) > 0$ then $u_n \in E^+ \cap H^+$. Now from (3.8) and Lemma 3.5, we have

$$J_\lambda(u_n) \leq -\frac{(p+1-q)}{N(q+1)(p+2)}\, C_3. \tag{3.10}$$

As $u_n \in \mathcal{N}_\lambda$, we have

$$J_\lambda(u_n) = \big(\frac{1}{N} - \frac{1}{q+1}\big) E_V(u_n) + \int_\Omega \big(\frac{1}{q+1} g(u_n)u_n - G(u_n)\big) dx. \tag{3.11}$$

From (3.10) and (3.11), we obtain

$$E_V(u_n) \geq \frac{(p+1-q)}{(N-q+1)(p+2)}\, C_3 > 0 \tag{3.12}$$

Also as $u_n \in \mathcal{N}_\lambda$, we have

$$\begin{aligned} &J_\lambda(u_n)\\ &= \big(\frac{1}{N} - \frac{1}{p+2}\big) E_V(u_n) - \frac{\lambda(p+1-q)}{(q+1)(p+2)} H(u_n) + \int_\Omega \big(\frac{1}{p+2} g(u_n)u_n - G(u_n)\big) dx. \end{aligned}$$

By this equality, (3.12) and (3.10), we obtain

$$H(u_n) \geq \frac{C_3}{\lambda N} > 0 \quad \forall n. \tag{3.13}$$

Thus we have $u_n \in \mathcal{N}_\lambda \cap E^+ \cap H^+$. Now we prove the following result.

Proposition 3.6. *Let $\alpha(V,h) > 0$, $\beta(V,h) > 0$ and λ satisfies* (2.3). *Then $\|J_\lambda'(u_n)\|_* \to 0$ as $n \to \infty$.*

Proof. Step 1: $\liminf_{n\to\infty} E_V(u_n) > 0$. Applying Hölders inequality in (3.13), we have $K'\|u_n\|^{q+1} \geq H(u_n) \geq \frac{C_3}{\lambda N} > 0$ which implies that $\liminf_{n\to\infty} \|u_n\| > 0$. Using this and Lemma 2.1 we obtain $\liminf_{n\to\infty} E_V(u_n) > 0$.

Step 2: We claim that

$$K := \liminf_{n\to\infty} \Big\{ (N-1-q)E_V(u_n) - \int_\Omega g'(u_n)u_n^2 dx + q\int_\Omega g(u_n)u_n dx \Big\} > 0. \tag{3.14}$$

Assume by contradiction that for some subsequence of $\{u_n\}$, still denoted by $\{u_n\}$ we have

$$(N-1-q)E_V(u_n) - \int_\Omega g'(u_n)u_n^2 dx + q\int_\Omega g(u_n)u_n dx = o_n(1).$$

From this and the fact that $E_V(u_n)$ is bounded away from 0, we obtain that $\liminf_{n\to\infty}\int_\Omega g'(u_n)u_n^2dx > 0$. Hence, we obtain $u_n \in \Lambda\setminus\{0\}$ for all n large. Using this and the fact that $u_n \in \mathcal{N}_\lambda\setminus\{0\}$, we have

$$o_n(1) = \lambda(N-q-1)H(u_n) - \int_\Omega (g'(u_n)u_n^2 - (N-1)g(u_n)u_n)dx < -\Lambda_m$$

by (2.3), which is a contradiction.

Finally, we show that $\|J'_\lambda(u_n)\|_* \to 0$ as $n\to\infty$. By Lemma 3.4, we obtain a sequence of functions $\xi_n : B(0,\epsilon_n)\to\mathbb{R}$ for some $\epsilon_n > 0$ such that $\xi_n(0) = 1$ and $\xi_n(w)(u_n - w) \in \mathcal{N}_\lambda$ for all $w \in B(0,\epsilon_n)$. Choose $0 < \rho < \epsilon_n$ and $f \in W_0^{1,N}(\Omega)$ such that $\|f\| = 1$. Let $w_\rho = \rho f$. Then $\|w_\rho\| = \rho < \epsilon_n$ and $\eta_\rho = \xi_n(w_\rho)(u_n - w_\rho) \in \mathcal{N}_\lambda$ for all n. Since $\eta_\rho \in \mathcal{N}_\lambda$, we deduce from (3.9) and Taylor's expansion,

$$\begin{aligned}\frac{1}{n}\|\eta_\rho - u_n\| \geq J_\lambda(u_n) - J_\lambda(\eta_\rho) &= \langle J'_\lambda(\eta_\rho), u_n - \eta_\rho\rangle + o(\|u_n - \eta_\rho\|)\\ &= (1-\xi_n(w_\rho))\langle J'_\lambda(\eta_\rho), u_n\rangle + \rho\xi_n(w_\rho)\langle J'_\lambda(\eta_\rho), f\rangle + o(\|u_n - \eta_\rho\|).\end{aligned} \tag{3.15}$$

We note that as $\rho\to 0$, we have $\frac{1}{\rho}\|\eta_\rho - u_n\| = \|u_n\langle\xi'_n(0), f\rangle - f\|$. Now dividing (3.15) by ρ and taking the limit $\rho\to 0$, and using $u_n \in \mathcal{N}_\lambda$, we obtain

$$\langle J'_\lambda(u_n), f\rangle \leq \frac{1}{n}\left(\|u_n\|\|\xi'_n(0)\|_* + 1\right) \leq \frac{1}{n}\frac{C_4\|f\|}{K}, \tag{3.16}$$

by Lemma 3.4 and (3.14). This completes the proof. □

We can now prove the following result.

Lemma 3.7. *Let $\alpha(V,h) > 0$, $\beta(V,h) > 0$ and let λ satisfy* (2.3). *Then there exists a function $u_\lambda \in \mathcal{N}_\lambda^+ \cap H^+ \cap E^+$ such that $J_\lambda(u_\lambda) = \inf_{u\in\mathcal{N}_\lambda\setminus\{0\}} J_\lambda(u)$.*

Proof. Let u_n be a minimizing sequence for J_λ on $\mathcal{N}_\lambda\setminus\{0\}$ satisfying (3.8) and (3.9). Then $\{u_n\}$ is bounded in $W_0^{1,N}(\Omega)$ by Lemma 3.3. Also there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and a function u_λ such that $u_n \rightharpoonup u_\lambda$ weakly in $W_0^{1,N}(\Omega)$, $u_n \to u_\lambda$ strongly in $L^\alpha(\Omega)$ for all $\alpha \geq 1$ and $u_n(x) \to u_\lambda(x)$ a.e in Ω. Also $H(u_n) \to H(u_\lambda)$. By Proposition 3.6, $\|J'_\lambda(u_n)\|_* \to 0$. Then we have

$$\begin{gathered}\nabla u_n(x) \to \nabla u_\lambda(x) \quad \text{a.e. in } \Omega,\\ g(u_n) \to g(u_\lambda) \quad \text{strongly in } L^1(\Omega),\\ |\nabla u_n|^{N-2}\nabla u_n \rightharpoonup |\nabla u_\lambda|^{N-2}\nabla u_\lambda \quad \text{weakly in } (L^{\frac{N}{N-1}}(\Omega))^N.\end{gathered}$$

In particular, it follows that u_λ solves (1.1) and hence $u_\lambda \in \mathcal{N}_\lambda$. Moreover, $\theta_\lambda \leq J_\lambda(u_\lambda) \leq \liminf_{n\to\infty} J_\lambda(u_n) = \theta_\lambda$. Hence u_λ is a minimizer for J_λ on $\mathcal{N}_\lambda$.

Using (3.13), we have $H(u_\lambda) > 0$ and $E_V(u_\lambda) > 0$, since $\beta(V,h) > 0$. Therefore there exists $t_1(u_\lambda)$ such that $t_1(u_\lambda)u_\lambda \in \mathcal{N}_\lambda^+$. We now claim that $t_1(u_\lambda) = 1$ (*i.e.* $u_\lambda \in \mathcal{N}_\lambda^+$). Suppose $t_1(u_\lambda) < 1$. Then $t_2(u_\lambda) = 1$ and hence $u_\lambda \in \mathcal{N}_\lambda^-$. Now $J_\lambda(t_1(u_\lambda)u_\lambda) \leq J_\lambda(u_\lambda) = \theta_\lambda$ which is impossible, as $t_1(u_\lambda)u_\lambda \in \mathcal{N}_\lambda$. □

Theorem 3.8. *Let $\alpha(V,h) > 0$, $\beta(V,h) > 0$ and λ be such that* (2.3) *holds. Then $u_\lambda \in \mathcal{N}_\lambda^+ \cap E^+ \cap H^+$ is also a non-negative local minimum for J_λ in $W_0^{1,N}(\Omega)$.*

Proof. Since $u_\lambda \in \mathcal{N}_\lambda^+$, we have $t_1(u_\lambda) = 1 < t_*(u_\lambda)$. Hence by continuity of $u \mapsto t_*(u)$, given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $1+\epsilon < t_*(u_\lambda - w)$ for all $\|w\| < \delta$. Also, from Lemma 3.5 we have, for $\delta > 0$ small enough, we obtain a

C^1 map $t : \mathrm{B}(0,\delta) \to \mathbb{R}^+$ such that $t(w)(u_\lambda - w) \in \mathcal{N}_\lambda$, $t(0) = 1$. Therefore, for $\delta > 0$ small enough we have $t_1(u_\lambda - w) = t(w) < 1 + \epsilon < t_*(u_\lambda - w)$ for all $\|w\| < \delta$. Since $t_*(u_\lambda - w) > 1$, we obtain $J_\lambda(u_\lambda) < J_\lambda(t_1(u_\lambda - w)(u_\lambda - w)) < J_\lambda(u_\lambda - w)$ for all $\|w\| < \delta$. This shows that u_λ is a local minimizer for J_λ. We can take $u_\lambda \geq 0$ as $J_\lambda(|u_\lambda|) = J_\lambda(u_\lambda)$. □

The following Lemma is taken from [19].

Lemma 3.9. *(i) If either $\lambda_1(V) > 0$ or $\lambda_1(V) = 0$ and $\phi_V \in H^-$ then $\beta(V,h) > 0$.*
(ii) If $\phi_V \in H^-$, $\lambda_1(V) < 0 < \lambda_2(V)$ and $\alpha(V,h) > 0$ then $\beta(V,h) > 0$.

Proof. (i) follows from $\beta(V,h) \geq \lambda_1(V)$ and $\beta(V,h) = \lambda_1(V)$ if and only if $\phi_V \in H^+$.

To prove (ii), define $\tilde{\beta}(V,h) := \min\{E_V(u); H(u) \geq 0, \|u\|_N = 1\}$. This minimum is achieved say u_0 and positive. Also $\tilde{\beta}(V,h) > 0$ implies $\beta(V,h) > 0$. If $H(u_0) = 0$ then $\tilde{\beta}(V,h) \geq \alpha(V,h) > 0$. If $H(u_0) > 0$ then $\tilde{\beta}(V,h)$ is actually an eigenvalue of $-\Delta_N + V$. Since $\phi_V \in H^-$ we see that $\tilde{\beta}(V,h) \geq \lambda_2(V) > 0$. □

Now the proof of Theorem 1.2 follows from Lemma 3.9 and Theorem 3.8.

4. Multiplicity results

4.1. Existence of a second solution in the subcritical case ($1 < \beta < \frac{N}{N-1}$).

Lemma 4.1. *For $\beta \in (1, \frac{N}{N-1})$, $u_n \rightharpoonup u$ implies $G(tu_n) \to G(tu)$ in $L^1(\Omega)$ for all $t \in \mathbb{R}$.*

Proof. Let $u_n \rightharpoonup u$ then $tu_n \rightharpoonup tu$ for every $t \in \mathbb{R}$. By compactness of embedding $u \longmapsto \int_\Omega e^{|u|^\beta}$, we have $\int_\Omega e^{|tu_n|^\beta} \leq C$, for some $C > 0$. From this one can easily show that $g(tu_n) \to g(tu)$ in $L^1(\Omega)$. Also there exists $M > 0$ such that $G(s) \leq (1 + g(s))M$ for every $s \in \mathbb{R}$. Then using this and by applying Lebesgue dominated convergence theorem, we obtain $G(tu_n) \to G(tu)$ in $L^1(\Omega)$. □

Lemma 4.2. *If $\alpha(V,h) > 0$, $\beta(V,h) > 0$. Then J_λ achieve its minimizers on $\mathcal{N}_\lambda^- \cap H_0^+ \cap E^+$.*

Proof. Note that $\alpha(V,h) > 0$ and $\beta(V,h) > 0$ imply $H_0^+ \cap E^+ \neq \emptyset$. Let $u_n \in \mathcal{N}_\lambda^- \cap H_0^+ \cap E^+$ be a minimizing sequence for J_λ. Then $J_\lambda(u_n)$ is bounded. By Lemma 3.3, $\{u_n\}$ is a bounded sequence. Therefore $u_n \rightharpoonup u_0$ weakly in $W_0^{1,N}(\Omega)$ and $H(u_n) \to H(u_0)$. Also $H(u_0) \geq 0$ as $u_n \in H_0^+$. We claim that $u_0 \not\equiv 0$. Let us assume this for a moment, if $H(u_0) = 0$, $\alpha(V,h) > 0$ then $E_V(u_0) > 0$ and if $H(u_0) > 0$, $\beta(V,h) > 0$ then $E_V(u_0) > 0$. Thus $u_0 \in H_0^+ \cap E^+$ and we have ϕ_{u_0} has a global maximum at some t_0 so that $t_0u_0 \in \mathcal{N}_\lambda^- \cap H_0^+ \cap E^+$. Next we claim that $u_n \to u_0$. Suppose this is not true then by using Lemma 4.1, we have

$$J_\lambda(t_0u_0) < \lim_{n\to\infty} J_\lambda(t_0u_n).$$

On the other hand, $u_n \in \mathcal{N}_\lambda^-$ implies that 1 is a global maximum point for ϕ_{u_n}; i.e., $\phi_{u_n}(t_0) \leq \phi_{u_n}(1)$. Thus we have

$$\lim_{n\to\infty} J_\lambda(t_0u_n) \leq \lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in\mathcal{N}_\lambda^-\cap H_0^+\cap E^+} J_\lambda(u),$$

which is a contradiction. Hence $u_n \to u_0$ and moreover $u_0 \in \mathcal{N}_\lambda^+ \cap H_0^+ \cap E^+$, since $\mathcal{N}_\lambda^0 = \emptyset$.

Now we show that $u_0 \not\equiv 0$. Suppose $u_0 \equiv 0$. Then $E_V(u_n) = \lambda H(u_n) + \int_\Omega g(u_n)u_n dx \to \lambda H(u_0) + \int_\Omega g(u_0)u_0 dx = 0$ as $n \to \infty$. Therefore we have $u_n \to 0$ strongly in $W_0^{1,N}(\Omega)$ as $n \to \infty$, since $\|u_n\|^N \le E_V(u_n) + \int_\Omega |V(x)||u_n|^N dx \to 0$ as $n \to \infty$. Now let $v_n = u_n/|u_n\|$. Then $v_n \rightharpoonup v_0$ weakly in $W_0^{1,N}(\Omega)$ and $H(v_n) \to H(v_0)$. Also $E_V(u_n) \le \frac{1}{N-1-q}\int_\Omega g'(u_n)u_n^2 dx$, since $u_n \in \mathcal{N}_\lambda^-$. Thus

$$\begin{aligned} E_V(v_n) \le & \frac{(p+1)\|u_n\|^{p+2-N}}{(N-1-q)} \int_\Omega |v_n|^{p+2} e^{\|u_n\|^\beta |v_n|^\beta} dx \\ & + \frac{\beta \|u_n\|^{p+2-N+\beta}}{(N-1-q)} \int_\Omega |v_n|^{p+2+\beta} e^{\|u_n\|^\beta |v_n|^\beta} dx \end{aligned}$$

By using Hölders inequality, Sobolev embeddings, Moser Trudinger inequality and $u_n \to 0$ strongly in $W_0^{1,N}(\Omega)$, one can easily show that $\int_\Omega |v_n|^{p+2} e^{\|u_n\|^\beta |v_n|^\beta}$ and $\int_\Omega |v_n|^{p+2+\beta} e^{\|u_n\|^\beta |v_n|^\beta}$ are bounded. Thus $\limsup E_V(v_n) \to 0$ as $n \to \infty$. So $E_V(v_0) \le 0$. Also

$$\begin{aligned} &\lambda H(v_n) \\ &= \|u_n\|^{N-1-q} E_V(v_n) - \|u_n\|^{-1-q} \int_\Omega g(\|u_n\|v_n)\|u_n\|v_n \, dx \\ &\le \|u_n\|^{N-1-q} E_V(v_n) + \|u_n\|^{p+1-q} \int_\Omega |v_n|^{p+2} e^{\|u_n\|^\beta |v_n|^\beta} \, dx \\ &\le K\|u_n\|^{N-1-q} + \|u_n\|^{p+1-q} \Big(\int_\Omega |v_n|^{(p+2)t'} \Big)^{1/t'} \Big(\sup_{\|w_n\| \le 1} \int_\Omega e^{t\|u_n\|^\beta |w_n|^\beta} \Big)^{1/t} \\ &\le K_1 \|u_n\|^{N-1-q} + K_2 \|u_n\|^{p+1-q} \|v_n\|^{p+2} \Big(\sup_{\|w_n\| \le 1} \int_\Omega e^{\alpha |w_n|^\beta} \Big)^{1/t} \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

since $\|u_n\| \to 0$ as $n \to \infty$ we can choose $\alpha \le \alpha_N$. Thus $H(v_0) = 0$. It is easy to see that $v_0 \ne 0$. Hence $\alpha(V, h) \le 0$, which is a contradiction because $\alpha(V, h) > 0$. Hence $u_0 \not\equiv 0$. □

The proof of Theorem 1.3 follows from Lemmas 3.9 and 4.2.

4.2. **Existence of a second solution in the critical case** ($\beta = \frac{N}{N-1}$)**.** For showing the existence of a second solutions we assume $h \ge 0$. Then $u \in H_0^+$ for every $u \in W_0^{1,N}(\Omega)$. In this subsection, We show that the minimizing sequence in $\mathcal{N}_\lambda^-$ is a Palais-Smale sequence below the critical level. We analyze the critical level and show that the weak limit of minimizing sequence is the required second solution of (1.1). To proceed further, we cut-off the nonlinearity from u_λ. For this we define

$$\tilde g(x,s) = \begin{cases} g(x, u_\lambda) & s \le u_\lambda(x) \\ g(x,s) & s \ge u_\lambda(x) \end{cases} \quad \text{and} \quad \tilde k(x,s) = \begin{cases} h(x)u_\lambda^q(x) & s \le u_\lambda(x) \\ h(x)s^q & s \ge u_\lambda(x), \end{cases}$$

where $\tilde G(x,s) = \int_0^s \tilde g(x,t)dt$ and $\tilde K(x,s) = \int_0^s \tilde k(x,t)dt$. Define $\tilde J_\lambda : W_0^{1,N}(\Omega) \to \mathbb{R}$ as

$$\tilde J_\lambda(u) = \frac{1}{N} E_V(u) - \int_\Omega \tilde G(x,u)dx - \lambda \int_\Omega \tilde K(x,u)dx, \tag{4.1}$$

where $\tilde{G}(x,u) = G(u) + g(u_\lambda)u_\lambda - G(u_\lambda)$ and $\tilde{K}(x,u) = K(u) + \frac{q}{q+1}h(x)u_\lambda^{q+1}$. So

$$\tilde{J}_\lambda(u) = J_\lambda(u) - \int_\Omega (g(u_\lambda)u_\lambda - G(u_\lambda))dx - \frac{\lambda q}{q+1}H(u_\lambda). \tag{4.2}$$

Then we have $\tilde{J}'_\lambda(u) = J'_\lambda(u)$, $\tilde{\mathcal{N}_\lambda} = \mathcal{N}_\lambda$, $\tilde{\mathcal{N}}^+_\lambda = \mathcal{N}^+_\lambda$, $\tilde{\mathcal{N}}^-_\lambda = \mathcal{N}^-_\lambda$, $\tilde{\mathcal{N}}^0_\lambda = \mathcal{N}^0_\lambda$. Define

$$\tilde{c}_1 = \inf_{u\in\mathcal{N}^-_\lambda} \tilde{J}_\lambda(u), \quad c_1 = \inf_{u\in\mathcal{N}^-_\lambda} J_\lambda(u).$$

Then $\tilde{J}_\lambda(u) \le J_\lambda(u)$ for every u and $\tilde{c}_1 \le c_1$. It is easy to see that u_λ is also a local minimum of $\tilde{J}_\lambda$. Next we define

$$U_1 := \Big\{u = 0 \text{ or } u \in W^{1,N}_0(\Omega) : \|u\| < t^-\big(\frac{u}{\|u\|}\big)\Big\}$$

and

$$U_2 := \Big\{u \in W^{1,N}_0(\Omega) : \|u\| > t^-\big(\frac{u}{\|u\|}\big)\Big\}.$$

Then we claim that $\mathcal{N}^-_\lambda = \{u \in W^{1,N}_0 \setminus \{0\} : \|u\| = t^-(\frac{u}{\|u\|})\}$. Indeed, for $u \in \mathcal{N}^-_\lambda$ and $u \in H^+_0$ as $h \ge 0$. Let $v = \frac{u}{\|u\|} \in W^{1,N}_0(\Omega)$. Then by Lemma 2.3, there exists a unique $t^-(v)$ such that $t^-(v)v \in \mathcal{N}^-_\lambda$. Using this and the fact that $u \in \mathcal{N}^-_\lambda$ we have $t^-(v) = \|u\|$. For other side, let $u \in W^{1,N}_0(\Omega) \setminus \{0\}$ such that $t^-(v) = \|u\|$. Then $t^-(v)v \in \mathcal{N}^-_\lambda$ which implies $u \in \mathcal{N}^-_\lambda$.
So, combining above discussion, we have $W^{1,N}_0(\Omega) \setminus \mathcal{N}^-_\lambda = U_1 \cup U_2$. Also, it is easy to see that $\mathcal{N}^+_\lambda \subset U_1$. In particular, $u_\lambda \in U_1$. Fix $n_0 \in \mathbb{N}$ and for any $M > 0$, we define

$$\rho_M = \min_{\gamma\in\mathcal{F}_M} \max_{t\in[0,1]} J_\lambda(\gamma(t)) \tag{4.3}$$

where $\mathcal{F}_M = \{\gamma \in C([0,1] : W^{1,N}_0(\Omega)) : \gamma(0) = u_\lambda,\ \gamma(1) = u_\lambda + M\phi_{n_0}\}$. Then we have the lemma:

Lemma 4.3. *There exist $M > 0$ such that $u_\lambda + M\phi_{n_0} \in U_2$ and moreover, $\rho_M \ge c_1$.*

Proof. Firstly, we can easily choose a suitable constant $S > 0$ such that $0 < t^-(u) < S$ for all $u : \|u\| = 1$. Recall the inequality: For any $p \ge 2$, there exists a constant $C(p) > 0$ such that

$$|\xi_2|^p > |\xi_1|^p + p|\xi_1|^{p-2}\langle \xi_1, \xi_2 - \xi_1\rangle + \frac{C(p)}{2^p - 1}|\xi_2 - \xi_1|^p, \quad \text{for every } \xi_1, \xi_2 \in \Omega.$$

Let $M > 0$ be such that $\frac{C(N)}{2^N-1}M^N \ge S$. Then we show that $w_{n_0} := u_\lambda + M\phi_{n_0} \in U_2$. By using u_λ is a solution of (1.1), $h \ge 0$ and support$(\phi_{n_0}) \subset B_{\delta_n}(0)$, we have

$$\begin{aligned}
\|w_{n_0}\|^N &= \int_\Omega |\nabla(u_\lambda + M\phi_{n_0})|^N dx \\
&\ge \int_\Omega (|\nabla u_\lambda|^N + MN|\nabla u_\lambda|^{N-2}\nabla u_\lambda \cdot \nabla\phi_{n_0} + \frac{C(N)}{2^N - 1}|M\nabla\phi_{n_0}(x)|^N)dx \\
&\ge \int_\Omega \big(|\nabla u_\lambda|^N + MN\big(g(u_\lambda) + \lambda h(x)|u_\lambda|^{q-1}u_\lambda - V(x)|u_\lambda|^{N-2}u_\lambda\big)\phi_{n_0}\big) \\
&\quad + \frac{C(N)M^N}{2^N - 1} \\
&\ge \|u_\lambda\|^N + \frac{C(N)M^N}{2^N - 1}
\end{aligned}$$

$$> \frac{C(N)}{2^N - 1} M^N \geq S > t^-\big(\frac{w_{n_0}}{\|w_{n_0}\|}\big).$$

Next, we show that $\rho_M \geq c_1$. For this, it is sufficient to show that every path starting from u_λ to $u_\lambda + M\phi_{n_0}$ intersects $\mathcal{N}_\lambda^-$. As $u_\lambda \in U_1$, $u_\lambda + M\phi_{n_0} \in U_2$ and γ is a continuous, so there exists some t_0 such that

$$\|u_\lambda + t_0\phi_{n_0}\| = t^-\Big(\frac{u_\lambda + t_0\phi_{n_0}}{\|u_\lambda + t_0\phi_{n_0}\|}\Big).$$

Hence $u_\lambda + t_0\phi_{n_0} \in \mathcal{N}_\lambda^-$. □

Also, we note that $J_\lambda(u_\lambda + tv) \to -\infty$ as $t \to \infty$ for any $v \in W_0^{1,N}(\Omega) \setminus \{0\}$. We obtain an upper bound on ρ_M in the following Lemma. Proof here is adopted from [14].

Lemma 4.4. *Let ρ_M be defined as in* (4.3). *Then $\rho_M < J_\lambda(u_\lambda) + \frac{1}{N}\alpha_N^{N-1}$.*

Proof. Let $\delta_n > 0$ be such that $\delta_n \to 0$ as $n \to \infty$. Then we define a sequence of Moser functions in Ω as

$$\phi_n(x) = \frac{1}{w_{N-1}^{1/N}} \begin{cases} (\log n)^{\frac{N-1}{N}} & 0 \leq \frac{|x|}{\delta_n} \leq \frac{1}{n}; \\ \frac{\log \frac{\delta_n}{|x|}}{(\log n)^{1/N}} & \frac{1}{n} \leq \frac{|x|}{\delta_n} \leq 1; \\ 0 & \frac{|x|}{\delta_n} \geq 1, \end{cases}$$

with support$(\phi_n) \subset B_{\delta_n}(0)$. We choose this ball in such a way that $V \leq 0$ on $B_{\delta_n}(0)$. It can be easily seen that $\|\nabla\phi_n\|_N = 1$ for all n. We prove the Lemma by contradiction argument. Suppose $\rho_M \geq J_\lambda(u_\lambda) + \frac{1}{N}\alpha_N^{N-1}$. Then for each n, there exist t_n such that

$$\sup_{t>0} J_\lambda(u_\lambda + t\phi_n) = J_\lambda(u_\lambda + t_n\phi_n) \geq J_\lambda(u_\lambda) + \frac{1}{N}\alpha_N^{N-1}. \tag{4.4}$$

Then from (4.4), we obtain $\{t_n\}$ is a bounded sequence, otherwise $J_\lambda(u_\lambda + t_n\phi_n) \to -\infty$.

Now using the one dimensional inequality: $(1 + t^2 + 2t\cos\alpha)^{N/2} \leq 1 + t^N + Nt\cos\alpha + O(t^2 + t^{N-1})$ for $t \geq 0$ to estimate $|\nabla(u_\lambda + t\phi_n)|^N$ in $J_\lambda(u_\lambda + t_n\phi_n)$ as in [14], we obtain

$$J_\lambda(u_\lambda + t_n\phi_n) \leq \frac{t_n^N}{N} + J_\lambda(u_\lambda) + t_n^2 O\Big(\delta_n^{N-2}(\log n)^{\frac{-2}{N}}\Big) + t_n^{N-1} O\Big(\delta_n(\log n)^{\frac{(1-N)}{N}}\Big)$$

Using (4.4), and choosing $\delta_n = (\log n)^{-1/N}$, we obtain

$$t_n^N \geq \alpha_N^{N-1}, \tag{4.5}$$

since t_n is bounded. Now t_n is a point of maximum for one dimensional map $t \to J_\lambda(u_\lambda + t_n\phi_n)$ and hence, $\frac{d}{dt}J_\lambda(u_\lambda + t\phi_n)|_{t=t_n} = 0$. So,

$$\begin{aligned} &\int_\Omega (|\nabla(u_\lambda + t_n\phi_n)|^{N-2}\nabla(u_\lambda + t_n\phi_n)\nabla\phi_n + V(x)|u_\lambda + t_n\phi_n|^{N-2}(u_\lambda + t_n\phi_n)\phi_n)dx \\ &= \int_\Omega g(u_\lambda + t_n\phi_n)\phi_n\,dx + \lambda\int_\Omega h(x)|u_\lambda + t_n\phi_n|^{q-1}(u_\lambda + t_n\phi_n)\phi_n\,dx \end{aligned} \tag{4.6}$$

Let $c_n = \min_{|x|\le \frac{\delta_n}{n}} u_\lambda(x)$. Then

$$\begin{aligned}\int_\Omega g(u_\lambda + t_n\phi_n)\phi_n\,dx &\ge \int_{|x|\le\frac{\delta_n}{n}} g(u_\lambda+t_n\phi_n)e^{|u_\lambda+t_n\phi_n|^{\frac{N}{N-1}}}\phi_n\,dx\\ &\ge \frac{w_{N-1}}{N}\big(\frac{\delta_n}{n}\big)^N\phi_n(0)|t_n\phi_n(0)|^{p+1}e^{(c_n+t_n\phi_n(0))^{\frac{N}{N-1}}}.\end{aligned} \tag{4.7}$$

Now using Taylor's expansion and (4.5), for some $K_0>0$, we obtain

$$\begin{aligned}(c_n+t_n\phi_n(0))^{\frac{N}{N-1}} &\ge (t_n\phi_n(0))^{\frac{N}{N-1}} + \frac{Nc_n}{N-1}(t_n\phi_n(0))^{\frac{1}{N-1}}\\ &\ge N\log n + K_0(\log n)^{1/N}t_n^{\frac{1}{N-1}},\end{aligned} \tag{4.8}$$

since t_n is bounded away from zero. Using (4.7) and (4.8), we obtain

$$\begin{aligned}\int_\Omega g(u_\lambda+t_n\phi_n)\phi_n &\ge \frac{w_{N-1}}{N}\big(\frac{\delta_n}{n}\big)^N\phi_n(0)|t_n\phi_n(0)|^{p+1}e^{(c_n+t_n\phi_n(0))^{\frac{N}{N-1}}}\\ &\ge \frac{w_{N-1}}{N}\big(\frac{\delta_n}{n}\big)^N\phi_n(0)|t_n\phi_n(0)|^{p+1}e^{(N\log n+K_0(\log n)^{1/N})}\\ &\ge \frac{w_{N-1}}{N}|t_n\phi_n(0)|^{p+2}(\log n)^{-1}e^{\frac{K_0}{2}(\log n)^{1/N}}\\ &\to\infty \quad \text{as } n\to\infty.\end{aligned} \tag{4.9}$$

So, the right hand side of (4.6) tends to ∞ as $n\to\infty$ but the left hand side is bounded, which is a contradiction. Hence $\rho_M < J_\lambda(u_\lambda) + \frac{1}{N}\alpha_N^{N-1}$. □

Lemma 4.5. *Let $\alpha(V,h)>0$, $\beta(V,h)>0$, then given $u\in\mathcal{N}_\lambda^-\cap H^+$, there exist $\epsilon>0$ and a differentiable function $\xi^- : B(0,\epsilon)\subset W_0^{1,N}(\Omega)\to\mathbb{R}$ such that $\xi^-(0)=1$ and the function $\xi^-(w)(u-w)\in\mathcal{N}_\lambda^-$ and for all $w\in W_0^{1,N}(\Omega)$,*

$$\begin{aligned}&\langle(\xi^-)'(0),w\rangle\\ &= \frac{NR(u,w) - \int_\Omega (g(u)w+g'(u)uw)\,dx - \lambda(q+1)\int_\Omega h(x)|u|^{q-1}uwdx}{(N-q-1)E_V(u) - \int_\Omega g'(u)u^2dx + q\int_\Omega g(u)udx}.\end{aligned}$$

Proof. First, we note that if $u\in\mathcal{N}_\lambda^-$, then $u\in\Lambda\setminus\{0\}$, satisfies (2.3). Then Lemma 3.4, there exist $\epsilon>0$ and a differentiable function $\xi^- : B(0,\epsilon)\subset W_0^{1,N}(\Omega)\to\mathbb{R}$ such that $\xi^-(0)=1$ and the function $\xi^-(w)(u-w)\in\mathcal{N}_\lambda$ for all $w\in B(0,\epsilon)$. Since $u\in\mathcal{N}_\lambda^-$, we have

$$(N-1-q)E_V(u) + q\int_\Omega g(u)u\,dx - \int_\Omega g'(u)u^2\,dx < 0.$$

Thus by continuity of J_λ' and ξ^-, we have

$$\begin{aligned}&\phi''_{(\xi^-(w)(u-w))}(1)\\ &= (N-1-q)E_V(\xi^-(w)(u-w)) + q\int_\Omega g(\xi^-(w)(u-w))\xi^-(w)(u-w)\\ &\quad - \int_\Omega g'(\xi^-(w)(u-w))(\xi^-(w)(u-w))^2 < 0,\end{aligned}$$

if ϵ is sufficiently small. This concludes the proof. □

We recall the following results which will be used later.

Proposition 4.6 ([15])**.** *Let $\{v_n : \|v_n\| = 1\}$ be a sequence in $W_0^{1,N}(\Omega)$ converging weakly to a non-zero v. Then for every $p < (1 - \|v\|^N)^{\frac{-1}{N-1}}$,*

$$\sup_n \int_\Omega exp(p\alpha_N |v_n|^{\frac{N}{N-1}}) < \infty.$$

Lemma 4.7 ([17, 14])**.** *Let $\{v_n\} \subset W_0^{1,N}(\Omega)$ be Palais-Smale sequence; that is, $J(v_n) \to c$, $J'(v_n) \to 0$ as $n \to \infty$. Then there exists a subsequence $\{v_n\}$ of $\{v_n\}$ and $v \in W_0^{1,N}(\Omega)$ such that $G(v_n) \to G(v)$, $g(v_n) \to g(v)$ strongly in $L^1(\Omega)$.*

Now we show the existence of a second non-trivial solution which is different from u_λ.

Theorem 4.8. *Let $\alpha(V,h) > 0$, $\beta(V,h) > 0$ and λ satisfies (2.3). Then there exist a minimizing sequence $\{v_n\}$ in $\mathcal{N}_\lambda^-$ and v_λ such that $v_n \rightharpoonup v_\lambda$ weakly in $W_0^{1,N}(\Omega)$, v_λ is a non-negative solution for (1.1) and moreover it is different from u_λ.*

Proof. We note that $\mathcal{N}_\lambda^-$ is a closed set, as $t^-(u)$ is a continuous function of u and $\tilde{J}_\lambda$ is bounded below on $\mathcal{N}_\lambda^-$. Therefore, by Ekeland's Variational principle, we can find a sequence $\{v_n\} \in \mathcal{N}_\lambda^-$ such that

$$\begin{aligned} \tilde{J}_\lambda(v_n) &\leq \inf_{u \in \mathcal{N}_\lambda^-} \tilde{J}_\lambda(u) + \frac{1}{n} \\ \tilde{J}_\lambda(v) &\geq \tilde{J}_\lambda(v_n) - \frac{1}{n}\|v - v_n\| \quad \forall v \in \mathcal{N}_\lambda^-. \end{aligned} \tag{4.10}$$

Now $v_n \in \mathcal{N}_\lambda \cap H_0^+$ then by Lemma 3.3, we have $\{v_n\}$ is a bounded sequence in $W_0^{1,N}(\Omega)$. From (4.10), we have $J_\lambda(v) \geq J_\lambda(v_n) - \frac{1}{n}\|v - v_n\|$ for all $v \in \mathcal{N}_\lambda^-$. It is easy to see that $v_n \in \mathcal{N}_\lambda^-$ implies $v_n \in \Lambda \setminus \{0\}$. Then $\liminf_{n\to\infty} E_V(v_n) > 0$, follows from the Step 1 of Lemma 2.2. Thus by Lemma 4.5 and following the proof of Proposition 3.6, we obtain $\|\tilde{J}'_\lambda(v_n)\|_* \to 0$ as $n \to \infty$. Thus following the proof as in Lemma 3.7, we have v_λ, weak limit of sequence $\{v_n\}$, is a solution of (1.1). Taking $\phi = v_\lambda^-$ as a test function in (1.4), we have

$$\begin{aligned} &\int_\Omega \left(|\nabla v_\lambda|^{N-2}\nabla v_\lambda \nabla v_\lambda^- + V(x)|v_\lambda|^{N-2}v_\lambda v_\lambda^-\right) dx \\ &= \int_\Omega g(v_\lambda)v_\lambda^- dx + \lambda \int_\Omega h(x)|v_\lambda|^{q-1}v_\lambda v_\lambda^- dx. \end{aligned}$$

Using this we obtain

$$\begin{aligned} E_V(v_\lambda^-) &= \int_\Omega (|\nabla v_\lambda^-|^N + V(x)|v_\lambda^-|^N) dx \\ &= -\int_\Omega g(v_\lambda)v_\lambda^- dx - \lambda \int_\Omega h(x)|v_\lambda|^{q-1}v_\lambda v_\lambda^- dx \leq 0. \end{aligned}$$

since $\tilde{g}(x,s) \geq 0$ and $\tilde{k}(x,s) \geq 0$ for all $s \in \mathbb{R}$. As $H(v_\lambda^-) \geq 0$, $\alpha(V,h) > 0$ and $\beta(V,h) > 0$ so by using Lemma 2.1 we obtain that $\|v_\lambda^-\| = 0$. Hence $v_\lambda \geq 0$ in Ω.

Finally, we show $u_\lambda \not\equiv v_\lambda$. By (4.2), we see that $\tilde{J}_\lambda(v_n) \to \tilde{c_1}$ which is equivalent to $J_\lambda(v_n) \to c_1$ as $n \to \infty$.

Case 1: Suppose $u_\lambda \equiv v_\lambda$, $c_0 = c_1$, then

$$J_\lambda(u_\lambda) = c_0 = c_1 = \lim_{n\to\infty} J_\lambda(v_n)$$

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{1}{N}\|v_n\|^N - \int_\Omega G(v_n)dx - \frac{\lambda}{q+1}H(v_n)\\
&= \lim_{n\to\infty} \frac{1}{N}\|v_n\|^N - \int_\Omega G(u_\lambda)dx - \frac{\lambda}{q+1}H(u_\lambda)\\
&= \lim_{n\to\infty} \frac{1}{N}\|v_n\|^N + J_\lambda(u_\lambda) - \frac{1}{N}\|u_\lambda\|^N.
\end{aligned}$$

Thus $\lim_{n\to\infty}\|v_n\| = \|u_\lambda\|$. Since $\nabla v_n(x) \to \nabla u_\lambda(x)$ a.e. pointwise then we have $v_n \to u_\lambda$ strongly in $W_0^{1,N}(\Omega)$ and $u_\lambda \in \mathcal{N}_\lambda^-$, as $\mathcal{N}_\lambda^-$ is a closed set. This is a contradiction as $u_\lambda \in \mathcal{N}_\lambda^+$.

Case 2: Suppose $u_\lambda \equiv v_\lambda$, $c_0 < c_1$. For proving this case, we use the same idea as in [14]. Using $J_\lambda(v_n) \to c_1$ and Lemma 4.7, we have

$$\lim_{n\to\infty}\|v_n\|^N = c_1 N - \int_\Omega V(x)|u_\lambda|^N + N\int_\Omega G(u_\lambda) + \frac{N\lambda}{q+1}H(u_\lambda). \tag{4.11}$$

Setting

$$w_n = \frac{v_n}{\|v_n\|_N} \quad \text{and} \quad w_\lambda = \frac{u_\lambda}{\lim_{n\to\infty}\|v_n\|},$$

we have $\|w_\lambda\| \le 1$ and $w_n \rightharpoonup w_\lambda$ weakly in $W_0^{1,N}(\Omega)$. Now, the following two possibilities occurs:

(i) $\|w_\lambda\| = 1$, in this case we have $\lim_{n\to\infty}\|v_n\| = \|u_\lambda\|$. Then we have $v_n \to u_\lambda$ in $W_0^{1,N}(\Omega)$ and $u_\lambda \in \mathcal{N}_\lambda^-$, which gives a contradiction as $u_\lambda \in \mathcal{N}_\lambda^+$.

(ii) $\|w_\lambda\| < 1$ and we have $c_1 < c_0 + \frac{1}{N}\alpha_N^{N-1}$. Then there exists $\epsilon > 0$ small enough such that

$$(1+\epsilon) < \frac{\alpha_N}{[N(c_1 - J_\lambda(u_\lambda))]^{\frac{1}{N-1}}}.$$

Set

$$\beta_0 := -\frac{1}{N}\int_\Omega V(x)|u_\lambda|^N dx + \int_\Omega G(u_\lambda)dx + \frac{\lambda}{q+1}H(u_\lambda).$$

Then from (4.11) we have $\lim_{n\to\infty}\|v_n\|^N = N(c_1+\beta_0)$. Also for sufficiently large n,

$$(1+\epsilon)\|v_n\|^{\frac{N}{N-1}} < \frac{\alpha_N\|v_n\|^{\frac{N}{N-1}}}{[N(c_1 - J_\lambda(u_\lambda))]^{\frac{1}{N-1}}} = \frac{\alpha_N}{(1-\|w_\lambda\|^N)^{1/N}},$$

since $w_\lambda = \frac{u_\lambda}{(N(c_1+\beta_0))^{1/N}}$. Choosing p such that $(1+\epsilon)\|v_n\|^{\frac{N}{N-1}} < p < \frac{\alpha_N}{(1-\|w_\lambda\|^N)^{1/N}}$. Now,

$$\begin{aligned}
\int_\Omega f(v_n)(v_n - u_\lambda)dx &= \int_\Omega |v_n|^{p+1}e^{|v_n|^{\frac{N}{N-1}}}(v_n - u_\lambda)dx\\
&\le \Big(\int_\Omega |v_n - u_\lambda|^t\Big)^{1/t}\Big(\int_\Omega e^{(1+\epsilon)t'\|v_n\|^{\frac{N}{N-1}}|w_n|^{\frac{N}{N-1}}}\Big)^{1/t'}
\end{aligned}$$

choose $t > 1$ sufficiently close to 1 such that $(1+\epsilon)t'\|v_n\|^{\frac{N}{N-1}} \le p < \frac{\alpha_N}{(1-\|w_\lambda\|^N)^{1/N}}$. Then by Proposition 4.6, we have

$$\int_\Omega f(v_n)(v_n - u_\lambda)dx \to 0 \quad \text{as } n \to \infty.$$

From this and $J_\lambda'(u_n)(v_n - u_\lambda) \to 0$ as $n \to \infty$ we obtain

$$\int_\Omega |\nabla v_n|^{N-2}\nabla v_n(\nabla v_n - \nabla u_\lambda)dx \to 0 \quad \text{as } n \to \infty$$

Moreover, since $v_n \rightharpoonup u_\lambda$ we have

$$\int_\Omega |\nabla u_\lambda|^{N-2}\nabla u_\lambda(\nabla v_n - \nabla u_\lambda)dx \to 0 \quad \text{as } n\to\infty$$

Hence, by using $|a-b|^N \leq 2^{N-2}(|a|^{N-2}a - |b|^{N-2}b)(a-b)$ for all $a, b\in\mathbb{R}^N$ we have

$$\int_\Omega |\nabla v_n - \nabla u_\lambda|^N \to 0 \quad \text{as } n\to\infty.$$

Thus $v_n \to u_\lambda$ strongly in $W_0^{1,N}(\Omega)$, which is again a contradiction as $u_\lambda \in \mathcal{N}_\lambda^+$. Thus u_λ and v_λ are distinct. $\square$

The proof of Theorem 1.4 follows from Theorems 3.8 and 4.8 .

5. Non-existence of solutions

We derive non-existence results for (1.1). Let Ω_h^+ be the largest domain where $h>0$ and $\phi(h^+)$ be the eigenfunction corresponding to the first eigenvalue $\lambda_1(\Omega_h^+)$ of $\Delta_N + V$ in $W_0^{1,N}(\Omega_h^+)$. We also assume that $\Omega_h^+ \neq \emptyset$. First, we recall the Picone's identity (see [4]),

Theorem 5.1. *Let $v>0$, $u\geq 0$ in $W_0^{1,N}(\Omega)$. Then*

$$|\nabla u|^N - \nabla\Big(\frac{u^N}{v^{N-1}}\Big)|\nabla v|^{N-2}\nabla v \geq 0 \quad \text{a.e. in } \Omega;$$

moreover, equality holds if and only if u is a multiple of v.

In the following Lemma, we only show the non-existence of solutions that are positive in Ω_h^+.

Lemma 5.2. *If $\lambda_1(V,\Omega_h^+) < 0$, then for every $\lambda>0$,* (1.1) *has no solution such that $u>0$ in Ω_h^+.*

Proof. We extend $\phi(h^+)$ by zero outside Ω_h^+ so that $\phi(h^+)\in W_0^{1,N}(\Omega)$. Also we have

$$\int_{\Omega_h^+}(|\nabla\phi(h^+)|^N + V|\phi(h^+)|^N)dx = \lambda_1(\Omega_h^+)\int_{\Omega_h^+}\phi(h^+)^N dx.$$

If u is a solution of (1.1). Then for $\epsilon>0$, consider $\frac{\phi(h^+)^N}{(u+\epsilon)^{N-1}}$ as a test function and we obtain

$$\begin{aligned}&\int_{\Omega_h^+}\Big(|\nabla u|^{N-2}\nabla u\cdot\nabla\Big(\frac{\phi(h^+)^N}{(u+\epsilon)^{N-1}}\Big) + V|u|^{N-2}u\frac{\phi(h^+)^N}{(u+\epsilon)^{N-1}}\Big)dx\\ &= \lambda\int_{\Omega_h^+}h(x)u^q\frac{\phi(h^+)^N}{(u+\epsilon)^{N-1}}dx + \int_{\Omega_h^+}g(u)\frac{\phi(h^+)^N}{(u+\epsilon)^{N-1}}dx.\end{aligned}$$

After subtracting the above two equations and taking limit as $\epsilon\to 0$, we see that the left-hand side is non-negative by Theorem 5.1 and the right hand side is negative which is a contradiction. $\square$

Lemma 5.3. *Let $h>0$ and $\lambda_1(V)>0$. Then there exists $\lambda^0>0$ such that* (1.1) *has no solution for $\lambda>\lambda^0$.*

Proof. Assume by contradiction that for all $\lambda > 0$, (P_λ) has a solution u_λ. Then

$$-\Delta_N u_\lambda + V(x)|u_\lambda|^{N-2}u_\lambda = u_\lambda|u_\lambda|^p e^{|u_\lambda|^{\frac{N}{N-1}}} + \lambda h u_\lambda^q, \quad \text{in } \Omega. \tag{5.1}$$

We can choose $\lambda > 0$ large such that

$$\lambda_1(V) - \lambda h t^{q+1-N} - g(t)t^{1-N} < 0 \tag{5.2}$$

for all $t > 0$, and for almost every $x \in \mathbb{R}^N$. Also we have

$$\int_\Omega (|\nabla\phi_1|^N + V|\phi_1|^N)dx = \lambda_1(V)\int_\Omega \phi_1^N dx,$$

where $\lambda_1(V)$ is an eigenvalue of $-\Delta_N + V$ corresponding to ϕ_1. Now multiply (5.1) by $\frac{\phi_1^N}{(u_\lambda+\epsilon)^{N-1}}$ and integrating by parts we obtain

$$\begin{aligned}
&\int_\Omega \Big(|\nabla u_\lambda|^{N-2}\nabla u_\lambda\cdot\nabla\Big(\frac{\phi_1^N}{(u_\lambda+\epsilon)^{N-1}}\Big) + V|u_\lambda|^{N-2}u_\lambda\Big(\frac{\phi_1^N}{(u_\lambda+\epsilon)^{N-1}}\Big)\Big)dx \\
&= \lambda\int_\Omega h(x)u_\lambda^q\Big(\frac{\phi_1^N}{(u_\lambda+\epsilon)^{N-1}}\Big)dx + \int_\Omega g(u_\lambda)\Big(\frac{\phi_1^N}{(u_\lambda+\epsilon)^{N-1}}\Big)dx.
\end{aligned}$$

After subtracting the above two equations and letting $\epsilon \to 0$ we obtain that the left hand side is non-negative by Theorem 5.1, and the right hand side

$$\int_\Omega \Big(\lambda_1(V) - \lambda h(x)u_\lambda^{q+1-N} - g(u_\lambda)u_\lambda^{1-N}\Big)\phi_1^N dx < 0,$$

by (5.2), a contradiction and hence the result follows. □

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