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MULTIPLE SOLUTIONS FOR SCHRÖDINGER-MAXWELL SYSTEMS WITH UNBOUNDED AND DECAYING RADIAL POTENTIALS

FANGFANG LIAO, XIAOPING WANG, ZHIGANG LIU

Abstract. This article concerns the nonlinear Schrödinger-Maxwell system

$$-\Delta u + V(|x|)u + Q(|x|)\phi u = Q(|x|)f(u), \quad \text{in } \mathbb{R}^3$$
$$-\Delta\phi = Q(|x|)u^2, \quad \text{in } \mathbb{R}^3$$

where V and Q are unbounded and decaying radial. Under suitable assumptions on nonlinearity $f(u)$, we establish the existence of nontrivial solutions and a sequence of high energy solutions in weighted Sobolev space via Mountain Pass Theorem and symmetric Mountain Pass Theorem.

1. Introduction

This article concerns the nonlinear Schrödinger-Maxwell system

$$\begin{gathered} -\Delta u + V(|x|)u + Q(|x|)\phi u = Q(|x|)f(u), \quad \text{in } \mathbb{R}^3 \\ -\Delta\phi = Q(|x|)u^2, \quad \text{in } \mathbb{R}^3. \end{gathered} \tag{1.1}$$

Such a system, also known as the nonlinear Schrödinger-Maxwell system, arises in an interesting physical context. Indeed, according to a classical model, the interaction of a charge particle with an electromagnetic field can be described by coupling the nonlinear Schrödinger and the Maxwell equations. For more details on the physical aspects, we refer to [1]. In particular, if we are looking for electrostatic-type solutions, we just have to solve (1.1).

For this problem in a bounded domain, there are some works. Let us recall some recent results. Benci and Fortunato obtained the existence of infinitely many solutions of an eigenvalue problem in [1]. D'Aprile and Wei [2] studied concentration phenomena for the system in the unit ball B_1 of $\mathbb{R}^3$ with Dirichlet boundary conditions. Candela and Salvatore [3] considered the problem with a non-homogeneous term and obtained infinitely many radially symmetric solutions.

Recently, the problem in the whole space $\mathbb{R}^3$ was considered in some works, see for instance $[4-15]$ and the references therein. We recall some of them as follows.

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Ruiz [4] considered the system

$$\begin{gathered} -\Delta u+V(x)u+\lambda\phi u=Q(x)f(u), \quad \text{in } \mathbb{R}^3 \\ -\Delta\phi=u^2, \quad \text{in } \mathbb{R}^3 \end{gathered} \tag{1.2}$$

and obtained the existence and nonexistence of radial solutions for (1.2) with $V(x)=Q(x)=1, f(u)=u^p(1<p<5)$. Later, Ambrosetti and Ruiz in [5] obtained multiplicity results for (1.2) with $V(x)=Q(x)=1$. For the critical growth case, we refer to [6]. Zhao and Zhao established the existence of a positive solution by the concentration compactness principle. Sun, Chen and Nieto [7] obtained the existence of ground state solutions when $V(x)=1, \lambda=K(x)$ and $f(u)$ is asymptotically linear at infinity. When the potential V is not a constant, Wang and Zhou [8] also considered that the case $f(x,u)$ is asymptotically linear and the positive potential V is bounded and non-radial. Mercuri [9] considered the potential V may vanish at infinity and bounded; i.e., $\frac{a}{1+|x|^\alpha}\le V(x)\le A$ for some $\alpha\in(0,2]$, $a,A>0$. By using the classical Mountain Pass Theorem, the author obtained the existence of positive solutions with $\lambda=1$ and $f(u)=u^p(1<p<\frac{N+2}{N-2}, N=3,4,5)$. Soon after, Sun, Chen and Yang [10] considered the asymptotically linear case under the assumptions in [9], the existence and nonexistence of solutions are obtained depending on the parameters λ. When $\lambda=1$ and $Q=1$, Chen and Tang [11] considered the potential $V(x)$ satisfies some coercive condition, i.e.,

- (V0) $V\in C(\mathbb{R}^3,\mathbb{R})$ satisfies $\inf_{x\in\mathbb{R}^3}V(x)>0$ and for each $M>0$, $\operatorname{meas}\{x\in\mathbb{R}^3 | V(x)\le M\}<+\infty$,

and proved that (1.2) has infinitely many high energy solutions under the condition that $f(x,u)$ is superlinear at infinity in u by fountain theorem established in [12]. Soon after, Li, Su and Wei [13] improved their results. For $V(x)$ and $f(x,u)$ are 1-periodic in each x. Zhao and Zhao [14] considered this case and obtained the existence of infinitely many geometrically distinct solutions. For the result of semiclassical solutions, we refer to [17].

In the present paper, we will consider more general radial potential, that is, the potential $V(x)$ may be unbounded, decaying and vanishing. We make the following assumptions:

- (V1) $V(r)\in C((0,+\infty))$, $V(r)\ge 0$ and there exist a_0 and a_1 such that
$$\liminf_{r\to 0}\frac{V(r)}{r^{a_0}}>0, \quad \liminf_{r\to+\infty}\frac{V(r)}{r^{a_1}}>0,$$
- (Q0) $Q(r)\in C((0,+\infty))$, $Q(r)\ge 0$ and there exist b_0 and b_1 such that
$$\limsup_{r\to 0}\frac{Q(r)}{r^{b_0}}>0, \quad \limsup_{r\to+\infty}\frac{Q(r)}{r^{b_1}}>0.$$

Next we introduce notation. Let $C_0^\infty(\mathbb{R}^3)$ denote the collection of smooth functions with compact support and

$$C_{0,r}^\infty(\mathbb{R}^3)=\{u\in C_0^\infty(\mathbb{R}^3): u \text{ is radial}\}.$$

Let $D_r^{1,2}(\mathbb{R}^3)$ be the completion of $C_{0,r}^\infty(\mathbb{R}^3)$ under the norm

$$\|u\|_{D_r^{1,2}}=\Big(\int_{\mathbb{R}^3}|\nabla u|^2dx\Big)^{1/2}.$$

Define

$$L^p(\mathbb{R}^3;Q)=\{u:\mathbb{R}^3\to\mathbb{R}; u \text{ is measurable and } \int_{\mathbb{R}^3}Q(|x|)|u|^p dx<\infty\}.$$

with norm

$$\|u\|_p=\Big(\int_{\mathbb{R}^3}Q(|x|)|u|^p dx\Big)^{1/p}.$$

Set

$$E=H^1_r(\mathbb{R}^3;V)=D^{1,2}_r(\mathbb{R}^3)\cap L^2(\mathbb{R}^3;V),$$

which is a Hilbert space with the norm

$$\|u\|_E=\Big(\int_{\mathbb{R}^3}|\nabla u|^2+V(|x|)u^2 dx\Big)^{1/2}.$$

Corresponding to [15], if (V1) and (Q0) are satisfied, for $N=3$, we define

$$\overline{p}(a_0,b_0)=\begin{cases}6+2b_0, & b_0\geq -2,\ a_0\geq -2,\\ \frac{8+4b_0-2a_0}{4+a_0}, & -2\geq a_0>-4,\ b_0\geq a_0,\\ \infty, & a_0\leq -4,\ b_0>-4,\end{cases}$$

$$\underline{p}(a_1,b_1)=\begin{cases}\frac{8+4b_1-2a_1}{4+a_1}, & b_1\geq a_1>-2,\\ 6+2b_1, & b_1\geq -2,\ a_1\leq -2,\\ 2, & b_1\leq max\{a_1,-2\}.\end{cases}$$

On the other hand, recently, Su, Wang and Willem [15] studied the nonlinear Schrödinger equation

$$\begin{gathered}-\Delta u+V(|x|)u=Q(|x|)f(u),\quad \text{in } \mathbb{R}^N\\ u(x)\to 0,\quad \text{as } |x|\to\infty.\end{gathered}\tag{1.3}$$

and assumed the Ambrosetti-Rabinowitz condition holds; i.e., there exists $\mu>2$ such that

$$0<\mu F(u)\leq uf(u),\quad \forall u\in\mathbb{R},$$

where $F(u)=\int_0^u f(s)ds$. They proved the existence of ground states solutions when V and Q satisfy the assumption (V1) and (Q0).

Motivated by the above facts, as in [15], the purpose of this paper is to extend the existence results of problem (1.3) to Schrödinger-Maxwell system (1.1). Moreover, we assume

(Q1) $Q\in L^{\frac{6p-12}{5p-12}}(\mathbb{R}^3)$ for all $p>12/5$.

To reduce our statement, we first make the following assumption on f.

(F1) $f\in C(\mathbb{R},\mathbb{R})$, and $|f(u)|\leq c(|u|^{p_1-1}+|u|^{p_2-1})$

for some $\underline{p}<p_1\leq p_2<\overline{p}$ ($\underline{p},\overline{p}$ will be defined later), where c is a positive constant.

Throughout this article we denote by c_i, C_i various positive constants, $|\cdot|_p$ denotes the usual $L^p(\mathbb{R}^3)$-norm, and $\|\cdot\|_q$ denotes the $L^q(\mathbb{R}^3,Q)$-norm.

2. Preliminaries

To prove our results, we use the following lemma from [15].

Lemma 2.1. *Assume* (V1) *and* (Q0) *with* $\overline{p} = \overline{p}(a_0, b_0) \geq \underline{p} = \underline{p}(a_1, b_1)$. *Then*

$$H^1_r(\mathbb{R}^3; V) \hookrightarrow L^p(\mathbb{R}^3; Q),$$

for $\underline{p} \leq p \leq \overline{p}$ *when* $\overline{p} < \infty$ *and for* $\underline{p} \leq p < \overline{p}$ *when* $\overline{p} = \infty$. *Furthermore, if* $b_1 \geq \max\{a_1, -2\}$, *the embedding is compact for* $\underline{p} < p < \overline{p}$, *and if* $b_1 < max\{a_1, -2\}$, *the embedding is compact for* $2 \leq p < \overline{p}$.

Remark 2.2. In particular, we can take $\overline{p} = \overline{p}(a_0, b_0) \geq \underline{p} = \max\{\underline{p}(a_1, b_1), 12/5\}$ if we take suitable a_0, b_0. Clearly, $\underline{p} = \max\{\underline{p}(a_1, b_1), \frac{12}{5}\} \geq \underline{p}(a_1, b_1)$. Thus, Lemma 2.1 holds for $\underline{p}$=max$\{\underline{p}(a_1, b_1), 12/5\}$.

It is well known that system (1.1) is the Euler-Lagrange equation of the functional $J : E \times D^{1,2}_r(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$J(u, \phi) = \frac{1}{2}\|u\|^2_E - \frac{1}{4}\int_{\mathbb{R}^3} |\nabla\phi|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} Q(|x|)\phi u^2 dx - \int_{\mathbb{R}^3} Q(|x|)F(u)dx.$$

For any $u \in E$, consider the linear functional $T_u : D^{1,2}_r(\mathbb{R}^3) \to \mathbb{R}$ defined as

$$T_u(v) = \int_{\mathbb{R}^3} Q(|x|)u^2 v dx.$$

For $\underline{p} < p < \overline{p}$, by (Q1), the Hölder inequality and Lemma 2.1, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} Q(|x|)u^2 v dx \\
&= \int_{\mathbb{R}^3} Q(|x|)^{\frac{p-2}{p}} Q(|x|)^{2/p} u^2 v dx \\
&\leq \Big(\int_{\mathbb{R}^3} Q(|x|)^{\frac{p-2}{p}\cdot\frac{6p}{5p-12}} dx\Big)^{\frac{5p-12}{6p}} \Big(\int_{\mathbb{R}^3} (Q(|x|)^{2/p}u^2)^{p/2} dx\Big)^{2/p} \Big(\int_{\mathbb{R}^3} v^6 dx\Big)^{1/6} \\
&\leq S^{-1}|Q|^{\frac{p-2}{p}}_{\frac{6p-12}{5p-12}} \int_{\mathbb{R}^3} \big(Q(|x|)u^p\big)^{2/p} \|v\|_{D^{1,2}_r} \\
&\leq c_1 S^{-1}|Q|^{\frac{p-2}{p}}_{\frac{6p-12}{5p-12}} \|u\|^2_E \|v\|_{D^{1,2}_r}.
\end{aligned}$$

where S is the best Sobolev embedding constant. Hence, the Lax-Milgram theorem implies that for every $u \in E$, there exists a unique $\phi_u \in D^{1,2}_r(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} Q(|x|)u^2 v = \int_{\mathbb{R}^3} \nabla\phi_u \cdot \nabla v, \quad \text{for any } v \in D^{1,2}_r(\mathbb{R}^3),$$

Using the integration by parts, we obtain

$$\int_{\mathbb{R}^3} \nabla\phi_u \nabla v dx = -\int_{\mathbb{R}^3} v\Delta\phi_u dx, \quad \text{for any } v \in D^{1,2}_r(\mathbb{R}^3);$$

therefore, $-\Delta\phi_u = Q(|x|)u^2$. We can write an integral expression for ϕ_u in the form

$$\phi_u = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{Q(|y|)u^2(y)}{|x-y|} dy, \tag{2.1}$$

for any $u \in C_0^\infty(\mathbb{R}^3)$, by density it can be extended for any $u \in E$. Moreover, the functions ϕ_u possess the following properties:

$$\phi_u \geq 0, \quad \|\phi_u\|_{D_r^{1,2}} \leq c_2\|u\|_p^2 \leq c_3\|u\|_E^2.$$

In fact, clearly, $\phi_u \geq 0$ by (2.1). Using integration by parts, $-\Delta\phi_u = Q(|x|)u^2$, the Hölder inequality and the Sobolev inequality, for any $u \in E$, we obtain

$$\begin{aligned}
\|\phi_u\|_{D_r^{1,2}}^2 &= \int_{\mathbb{R}^3} \nabla\phi_u \cdot \nabla\phi_u dx = -\int_{\mathbb{R}^3} \Delta\phi_u \cdot \phi_u dx \\
&= \int_{\mathbb{R}^3} Q(|x|)\phi_u u^2 dx \\
&\leq c_1 S^{-1} |Q|^{\frac{p-2}{p}}_{\frac{6p-12}{5p-12}} \|u\|_p^2 \|\phi_u\|_{D_r^{1,2}} \\
&\leq c_2 |u|_p^2 \|\phi_u\|_{D_r^{1,2}}.
\end{aligned}$$

It follows that

$$\|\phi_u\|_{D_r^{1,2}} \leq c_2\|u\|_p^2 \leq c_3\|u\|_E^2.$$

Moreover, there exists $c_4 > 0$ such that

$$\int_{\mathbb{R}^3} Q(|x|)\phi_u u^2 dx \leq c_4\|u\|_E^4. \tag{2.2}$$

So, we can consider the functional $I : E \to \mathbb{R}^3$ defined by $I(u) = J(u, \phi_u)$. By (2.1) the reduced functional takes the form

$$I(u) = \frac{1}{2}\|u\|_E^2 + \frac{1}{4}\int_{\mathbb{R}^3} Q(|x|)\phi_u u^2 dx - \int_{\mathbb{R}^3} Q(|x|)F(u)dx. \tag{2.3}$$

It is clear that I is well defined. Moreover, Our hypotheses imply that $I \in C^1(E, \mathbb{R})$ and a standard argument shows that $(u, \phi) \in E \times D_r^{1,2}(\mathbb{R}^3)$ is a critical point of J if and only if u is a critical point of I and $\phi = \phi_u$ (see [22]).

Lemma 2.3. *If assumptions* (V1), (Q0), (Q1), (F1) *hold, then* $I \in C^1(E, \mathbb{R})$ *and*

$$\langle I'(u), v\rangle = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(|x|)uv)dx + \int_{\mathbb{R}^3} Q(|x|)\phi_u uv dx - \langle \Psi'(u), v\rangle, \tag{2.4}$$

where $\Psi(u) = \int_{\mathbb{R}^3} Q(|x|)F(u)dx$.

Proof. First, we prove the existence of the Gateaux derivative of Ψ. From (F1), we have

$$|f(u)| \leq c(|u|^{p_1-1} + |u|^{p_2-1}), \tag{2.5}$$

$$|F(u)| \leq c(\frac{1}{p_1}|u|^{p_1} + \frac{1}{p_2}|u|^{p_2}). \tag{2.6}$$

For any $u, v \in E$ and $0 < |t| < 1$, by the mean value and (2.5), there exists $0 < \theta < 1$ such that

$$\begin{aligned}
&\frac{|Q(|x|)F(u + tv) - Q(|x|)F(u)|}{|t|} \\
&= |Q(|x|)f(u + \theta tv)v| \\
&\leq cQ(|x|)(|u + \theta tv|^{p_1-1} + |u + \theta tv|^{p_2-1})|v| \\
&\leq c_5 Q(|x|)[(|u|^{p_1-1}|v| + |v|^{p_1}) + (|u|^{p_2-1}|v| + |v|^{p_2})]
\end{aligned}$$

The Hölder inequality implies

$$g(x) := cQ(|x|)[(|u|^{p_1-1}|v| + |v|^{p_1}) + (|u|^{p_2-1}|v| + |v|^{p_2})] \in L^1(\mathbb{R}^3).$$

Consequently, by the Lebesgue's dominated convergence theorem, one has

$$\langle \Psi'(u), v\rangle = \int_{\mathbb{R}^3} Q(|x|)f(u)v dx.$$

Next,we show that $\Psi'(\cdot) : E \to E^*$ is continuous. Assume that $u_n \to u$ in E. By Lemma 2.1, we know that $u_n \to u$ in $L^p(\mathbb{R}^3; Q)$, for $\underline{p} \le p \le \overline{p}$ when $\overline{p} < \infty$ and for $\underline{p} \le p < \overline{p}$ when $\overline{p} = \infty$.

On the space $L^{p_1}(\mathbb{R}^3; Q) \cap L^{p_2}(\mathbb{R}^3; Q)$, we define the norm

$$\begin{aligned} \|u\|_{p_1\wedge p_2} &= \|u\|_{p_1} + \|u\|_{p_2} \\ &= \Big(\int_{\mathbb{R}^3} Q(|x|)|u|^{p_1} dx\Big)^{1/p_1} + \Big(\int_{\mathbb{R}^3} Q(|x|)|u|^{p_2} dx\Big)^{1/p_2} \end{aligned}$$

On the space $L^{p_1}(\mathbb{R}^3; Q) + L^{p_2}(\mathbb{R}^3; Q)$, we define the norm

$$\|u\|_{p_1\vee p_2} = \inf\big\{\|v\|_{p_1} + \|w\|_{p_2} : v \in L^{p_1}(\mathbb{R}^3; Q), w \in L^{p_2}(\mathbb{R}^3; Q), u = v + w\big\}.$$

Since $\underline{p} < p_1 \le p_2 < \overline{p}$, one has $u_n \to u$ in $L^{p_1}(\mathbb{R}^3; Q) \cap L^{p_2}(\mathbb{R}^3; Q)$. Similar to [22, Theorem A.4], we have

$$f(u_n) \to f(u) \quad \text{in } L^{p_1'}(\mathbb{R}^3; Q) + L^{p_2'}(\mathbb{R}^3; Q).$$

By the Hölder inequality, we have

$$\begin{aligned} |\langle \Psi'(u_n) - \Psi'(u), v\rangle| &\le \|f(u_n) - f(u)\|_{p_1'\vee p_2'}\|v\|_{p_1\wedge p_2} \\ &\le c_6\|f(u_n) - f(u)\|_{p_1'\vee p_2'}\|v\|_E, \end{aligned}$$

where $p_i' = p_i/(p_i - 1)$, $i = 1, 2$. Hence

$$\|\Psi'(u_n) - \Psi'(u)\| \le c_6\|f(u_n) - f(u)\|_{p_1'\vee p_2'} \to 0 \quad \text{as } n \to \infty.$$

This shows $\Psi'(\cdot) : E \to E^*$ is continuous. This completes the proof. □

Lemma 2.4. *Under the condition* (F1)*, if* $\{u_n\} \subset E$ *is a bounded sequence with* $I'(u_n) \to 0$*, then* $\{u_n\}$ *has a convergent subsequence.*

Proof. Since $\{u_n\} \subset E$ is bounded and the embedding $E \hookrightarrow L^s(\mathbb{R}^3; Q)$ is compact for each $s \in (\underline{p}, \overline{p})$, passing to a subsequence, we can assume that $u_n \rightharpoonup u$ in E, and

$$u_n \to u \quad \text{in } L^s(\mathbb{R}^3; Q),\ s \in (\underline{p}, \overline{p}).$$

Note that

$$\begin{aligned} &\langle I'(u_n) - I'(u), u_n - u\rangle \\ &= \|u_n - u\|_E^2 + \int_{\mathbb{R}^3} \big(Q(|x|)\phi_{u_n} u_n^2 - Q(|x|)\phi_{u_n} u_n u\big)\, dx \\ &\quad + \int_{\mathbb{R}^3} \big(Q(|x|)\phi_u u^2 - Q(|x|)\phi_u u_n u\big)\, dx - \int_{\mathbb{R}^3} Q(|x|)\,(f(u_n) - f(u))\,(u_n - u)dx. \end{aligned}$$

We have

$$\begin{aligned} &\|u_n - u\|_E^2 \\ &= \langle I'(u_n) - I'(u), u_n - u\rangle - \int_{\mathbb{R}^3} \big(Q(|x|)\phi_{u_n} u_n^2 - Q(|x|)\phi_{u_n} u_n u\big)\, dx \end{aligned}$$

$$- \int_{\mathbb{R}^3} \big(Q(|x|)\phi_u u^2 - Q(|x|)\phi_u u_n u\big)\, dx + \int_{\mathbb{R}^3} Q(|x|)\left(f(u_n) - f(u)\right)(u_n - u)dx.$$

Since $u_n \rightharpoonup u$ in E and $I'(u_n) \to 0$, we have

$$\langle I'(u_n) - I'(u), u_n - u\rangle \to 0 \quad \text{as } n \to \infty.$$

On one hand, by (Q1), for $\underline{p} < p < \overline{p}$, we have

$$\begin{aligned}
\int_{\mathbb{R}^3} Q(|x|)(\phi_{u_n} u_n^2 - \phi_{u_n} u_n u)dx &= \int_{\mathbb{R}^3} Q(|x|)\phi_{u_n} u_n(u_n - u)dx \\
&\le |Q|^{\frac{p-2}{p}}_{\frac{6p-12}{5p-12}} \|u_n - u\|_p \|u_n\|_p \|\phi_{u_n}\|_6 \\
&\le c_7 \|u_n - u\|_p \|u_n\|_p \|\phi_{u_n}\|_{D_r^{1,2}}.
\end{aligned}$$

Hence,

$$\int_{\mathbb{R}^3} Q(|x|)\left(\phi_{u_n} u_n^2 - \phi_{u_n} u_n u\right) dx \to 0, \quad \text{as } n \to \infty.$$

Similarly,

$$\int_{\mathbb{R}^3} Q(|x|)\left(\phi_u u^2 - \phi_u u_n u\right) dx \to 0, \quad \text{as } n \to \infty.$$

On the other hand,

$$\begin{aligned}
&|\int_{\mathbb{R}^3} Q(|x|)\left(f(u_n) - f(u)\right)(u_n - u)dx| \\
&\le \int_{\mathbb{R}^3} Q(|x|)\left(|f(u_n)| + |f(u)|\right)|u_n - u|dx \\
&\le c\int_{\mathbb{R}^3} Q(|x|)\left(|u_n|^{p_1-1} + |u_n|^{p_2-1} + |u|^{p_1-1} + |u|^{p_2-1}\right)|u_n - u|dx \\
&\le c\Big(\int_{\mathbb{R}^3} Q(|x|)|u_n - u|^{p_1} dx\Big)^{1/p_1}\Big(\Big(\int_{\mathbb{R}^3} Q(|x|)|u_n|^{p_1} dx\Big)^{\frac{p_1-1}{p_1}} \\
&\quad + \Big(\int_{\mathbb{R}^3} Q(|x|)|u|^{p_1} dx\Big)^{\frac{p_1-1}{p_1}}\Big) \\
&\quad + c\Big(\int_{\mathbb{R}^3} Q(|x|)|u_n - u|^{p_2} dx\Big)^{1/p_2}\Big(\Big(\int_{\mathbb{R}^3} Q(|x|)|u_n|^{p_2} dx\Big)^{\frac{p_2-1}{p_2}} \\
&\quad + \Big(\int_{\mathbb{R}^3} Q(|x|)|u|^{p_2} dx\Big)^{\frac{p_2-1}{p_2}}\Big).
\end{aligned}$$

Since $u_n \to u$ in $L^s(\mathbb{R}^3; Q), s \in (\underline{p}, \overline{p})$, we have

$$\int_{\mathbb{R}^3} Q(|x|)\left(f(u_n) - f(u)\right)(u_n - u)dx \to 0 \quad \text{as } n \to \infty.$$

So we have $\|u_n - u\|_E \to 0$. This completes the proof. □

3. Main results

Theorem 3.1. *Assume that conditions* (V1), (Q0), (Q1) *hold. If (F1) and the following conidition hold*

(F2) *There exists* μ *and* $r > 0$ *such that* $\max\{\underline{p}, 4\} < \mu \le \overline{p} < \infty$, *and*

$$\mu F(u) \le u f(u),\ \forall u \in \mathbb{R}, \quad \inf_{|u|=r} F(u) := \beta > 0.$$

Then system (1.1) *has a nontrivial solution. Furthermore, if $f(u)$ is odd in u, then system* (1.1) *has a sequence $\{(u_n, \phi_n)\}$ of solutions in $E \times D_r^{1,2}(\mathbb{R}^3)$ with $\|u_n\| \to \infty$ and $I(u_n) \to +\infty$.*

Proof. From (F1), we have

$$|F(u)| \le c(\frac{1}{p_1}|u|^{p_1} + \frac{1}{p_2}|u|^{p_2}).$$

Note that

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|_E^2 + \frac{1}{4}\int_{\mathbb{R}^3} Q(|x|)\phi_u u^2 dx - \int_{\mathbb{R}^3} Q(|x|)F(u)dx \\ &\ge \frac{1}{2}\|u\|_E^2 - \int_{\mathbb{R}^3} Q(|x|)F(u)dx \\ &\ge \frac{1}{2}\|u\|_E^2 - \frac{c}{p_1}\|u\|_{p_1}^{p_1} - \frac{c}{p_2}\|u\|_{p_2}^{p_2} \\ &\ge \frac{1}{2}\|u\|_E^2 - c_8\|u\|_E^{p_1} - c_9\|u\|_E^{p_2}. \end{aligned}$$

Since $p_1, p_2 > 2$, we can take a small ρ such that

$$I|_{\partial B_\rho} \ge \frac{1}{2}\rho^2 - c_8\rho^{p_1} - c_9\rho^{p_2} := \delta > 0,$$

where $B_\rho = \{u \in E : \|u\|_E < \rho\}$.

For $z \in \mathbb{R}$, set

$$h(t) := F(t^{-1}z)t^\mu, \quad \forall t \in [1, +\infty).$$

For $|z| \ge r$ and $t \in [1, |z|/r]$, by (F2), one has

$$\begin{aligned} h'(t) &= f(t^{-1}z)(-\frac{z}{t^2})t^\mu + F(t^{-1}z)\mu t^{\mu-1} \\ &= t^{\mu-1}\big(\mu F(t^{-1}z) - t^{-1}zf(t^{-1}z)\big) \le 0. \end{aligned}$$

So, we have

$$F(z) = h(1) \ge h(\frac{|z|}{r}) \ge \frac{\beta}{r^\mu}|z|^\mu.$$

Since $\mu > 4$, there exists a constant $\max\{\underline{p}, 4\} < \alpha < \overline{p}$ such that $\alpha < \mu$, and hence

$$\lim_{|u|\to\infty} \frac{F(u)}{|u|^\alpha} = +\infty. \tag{3.1}$$

For any finite dimensional space $E_1 \subset E$, by the equivalence of norms in the finite space, there exists a constant $c_{(\alpha)} > 0$, such that

$$\|u\|_\alpha \ge c_\alpha\|u\|_E, \quad \forall u \in E_1 \tag{3.2}$$

where α is the constant appearing in (3.1). For any $\sigma > 0$, by (F1), there is a constant $c_\sigma > 0$ such that

$$|F(u)| \le c_\sigma|u|^{\underline{p}}, \quad \forall |u| < \sigma.$$

Hence, by (3.1), we know that for $M > 0$, there is a constant $C_M > 0$ such that

$$F(u) \ge M|u|^\alpha - C_M|u|^{\underline{p}}, \ \ \forall u \in \mathbb{R}. \tag{3.3}$$

By (3.2) and (3.3), we have

$$I(u) \le \frac{1}{2}\|u\|_E^2 + \frac{c_4}{4}\|u\|_E^4 - M\|u\|_\alpha^\alpha + C_M\|u\|_{\underline{p}}^{\underline{p}}$$

$$\leq \frac{1}{2}\|u\|_E^2 + \frac{c_4}{4}\|u\|_E^4 - Mc_\alpha^\alpha\|u\|_E^\alpha + C_M\|u\|_E^{\underline{p}},$$

for all $u \in E_1$. Consequently, there is a large $r_1 > 0$ such that $I < 0$ on $E_1 \backslash B_{r_1}$. Consequently, there is a point $e \in E$ with $\|e\|_E > \rho$ such that $I(e) < 0$.

Now, we prove that I satisfies the Palais-Smale condition. By Lemma 2.4 we know that it is sufficient to prove $\{u_n\}$ is bounded in E. Indeed, if a sequence $\{u_n\} \subset E$ such that $I(u_n)$ is bounded and $I'(u_n) \to 0$, then there is positive constant M_0 such that for large n, one has

$$\begin{aligned} M_0 + \|u_n\|_E &\geq I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle \\ &\geq (\frac{1}{2} - \frac{1}{\mu})\|u_n\|_E^2 + (\frac{1}{4} - \frac{1}{\mu})\int_{\mathbb{R}^3} Q(|x|)\phi_{u_n}u_n^2 dx \\ &\quad + \int_{\mathbb{R}^3} Q(|x|)\Big(\frac{f(u_n)u_n}{\mu} - F(u_n)\Big)dx \\ &\geq (\frac{1}{2} - \frac{1}{\mu})\|u_n\|_E^2. \end{aligned}$$

This implies $\{u_n\}$ is bounded.

Obviously, $I(0) = 0$. Hence I possesses a critical value $\eta \geq \delta$ by [20, Theorem 2.2], thus problem (1.1) has a nontrivial solution. Moreover, obviously, I is bounded on each bounded subset of E and $f(u)$ is odd which implies I is even. Hence the second conclusion follows from [20, Theorem 9.12]. This completes the proof. □

Note that $\mu > 4$ in condition (F2). Now, we consider the weak case $\mu = 4$. At this one, we have the following Theorem.

Lemma 3.2. *Assume that conditions* (V1), (Q0), (Q1), (F1) *and the following conditions hold:*

- (F3) $\frac{F(u)}{|u|^4} \to +\infty$ *as* $|u| \to +\infty$.
- (F4) $uf(u) \geq 4F(u)$ *for all* $u \in \mathbb{R}$.

If $\underline{p} < 4 < \overline{p}$, *then system* (1.1) *has at least one nontrivial solution. Furthermore, if* $f(u)$ *is odd in* u, *then system* (1.1) *has a sequence* $\{(u_n, \phi_n)\}$ *of solutions in* $E \times D_r^{1,2}(\mathbb{R}^3)$ *with* $\|u_n\| \to \infty$ *and* $I(u_n) \to +\infty$.

Proof. From the proofs of the first segment in Theorem 3.1, we know that there exist constants $\rho > 0$ and $\delta > 0$ such that

$$I|_{\partial B_\rho} \geq \delta > 0.$$

Moreover, for any finite dimensional space $E_1 \subset E$, by the equivalence of norms in the finite space, there exists a constant $C > 0$, such that

$$\|u\|_4 \geq C\|u\|_E, \quad \forall u \in E_1. \tag{3.4}$$

Since $\underline{p} < 4$, by (F1) and (F3) we know that for any $M > \frac{c_4}{4C^4}$, there is a constant $C_M > 0$ such that

$$F(u) \geq M|u|^4 - c(M)|u|^{\underline{p}}, \quad \forall u \in \mathbb{R}. \tag{3.5}$$

Hence

$$I(u) \leq \frac{1}{2}\|u\|_E^2 + \frac{1}{4}\int_{\mathbb{R}^3} Q(|x|)\phi_{u_n}u_n^2 dx - M\|u\|_4^4 + C_M\|u\|_{\underline{p}}^{\underline{p}}.$$

By (3.4) and (3.5), we know

$$I(u) \le \frac{1}{2}\|u\|_E^2 + \frac{c_4}{4}\|u\|_E^4 - MC^4\|u\|_E^4 + C_M\|u\|_E^{\underline{p}},$$

for all $u \in E_1$. Consequently, there is a large $r_1 > 0$ such that $I < 0$ on $E_1 \backslash B_{r_1}$. Consequently, there is a point $e \in E$ with $\|e\|_E > \rho$, such that $I(e) < 0$.

Next we prove that I satisfies the Palais-Smale condition. Indeed, if a sequence $\{u_n\} \subset E$ is such that $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$, then there is a positive constant M_1 such that for large n, one has

$$\begin{aligned} M_1 + \|u_n\|_E &\ge I(u_n) - \frac{1}{4}\langle I'(u_n), u_n\rangle \\ &= \frac{1}{4}\|u_n\|_E^2 + \int_{\mathbb{R}^3} Q(|x|)\big(\frac{1}{4}f(u_n)u_n - F(u_n)\big)dx \\ &\ge \frac{1}{4}\|u_n\|_E^2. \end{aligned}$$

This implies $\{u_n\}$ is bounded. Hence $\{u_n\} \subset E$ has a convergent subsequence by Lemma 2.4. This shows that I satisfies the Palais-Smale condition. Finally, the conclusions follows from [20, Theorem 2.2 and 9.12]. □

Corollary 3.3. *Assume that conditions* (V1), (Q0), (Q1), (F1), (F3) *and the following conditions hold:*

(F4') *$u \to f(u)/|u|^3$ is increasing on $(-\infty, 0)$ and on $(0, +\infty)$.*

If $\underline{p} < 4 < \overline{p}$, then system (1.1) *has at least one nontrivial solution. Furthermore, if $f(u)$ is odd in u, then system* (1.1) *has a sequence $\{(u_n, \phi_n)\}$ of solutions in $E \times D_r^{1,2}(\mathbb{R}^3)$ with $\|u_n\| \to \infty$ and $I(u_n) \to +\infty$.*

Proof. It is sufficient to prove that (F4') implies (F4). In fact, whenever $u > 0$,

$$F(u) = \int_0^1 f(ut)u\,dt = \int_0^1 \frac{f(ut)}{(ut)^3}u^4t^3dt \le \int_0^1 \frac{f(u)}{(u)^3}u^4t^3dt = \frac{1}{4}f(u)u.$$

Whenever $u < 0$,

$$F(u) = \int_0^1 f(ut)u\,dt = -\int_0^1 \frac{f(ut)}{(-ut)^3}u^4t^3dt \le \int_0^1 \frac{f(u)}{(u)^3}u^4t^3dt = \frac{1}{4}f(u)u.$$

This shows (F4) holds. □

Theorem 3.4. *Assume that condition* (V1), (Q1), (F1), (F3) *and the following condition hold:*

(F5) *$F(u) \ge 0$ for all $u \in \mathbb{R}$ and $G(s) \le G(t)$ whenever $(s,t) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $s \le t$, where $G(u) = f(u)u - 4F(u)$.*

If $\underline{p} < 4 < \overline{p}$, then system (1.1) *has at least one nontrivial solution. Furthermore, if $f(u)$ is odd in u, then system* (1.1) *has a sequence $\{(u_n, \phi_n)\}$ of solutions in $E \times D_r^{1,2}(\mathbb{R}^3)$ with $\|u_n\| \to \infty$ and $I(u_n) \to +\infty$.*

Proof. Similar to the proof of Lemma 3.2, we know that there exist $\rho > 0$, $\delta > 0$ such that

$$I|_{\partial B_\rho} \ge \delta > 0.$$

Moreover, for any finite dimensional subspace $E_1 \subset E$, there is a large $r_1 > 0$ such that $I < 0$ on $E_1 \backslash B_{r_1}$.

Now, we prove that I satisfies the Cerami condition. Indeed, if a sequence $\{u_n\} \subset E$ is such that $\{I(u_n)\}$ is bounded and $(1 + \|u_n\|)I'(u_n) \to 0$, then we claim that $\{u_n\}$ is bounded. If this is false, then we can assume $\|u_n\| \to +\infty$. Set $v_n = \frac{u_n}{\|u_n\|_E}$, then $\|v_n\|_E = 1$. By virtue of Lemma 2.1, passing to a subsequence, we may assume

$$v_n \rightharpoonup v \quad \text{in } E,$$
$$u_n \to u \quad \text{in } L^s(\mathbb{R}^3; Q), s \in (\underline{p}, \overline{p}).$$

Since $\{I(u_n)\}$ is bounded, there exists a constant $C_1 > 0$ such that

$$\int_{\mathbb{R}^3} \frac{Q(|x|)F(u_n)}{\|u_n\|_E^4} dx \le C_1 < \infty.$$

Set $\Omega = \{x \in \mathbb{R}^3 : v(x) \neq 0\}$. Then $|u_n(x)| \to +\infty$ for a.e. $x \in \Omega$. If $\operatorname{meas}(\Omega) > 0$, then, by (F4)

$$\frac{F(u_n)}{\|u_n\|_E^4} = \frac{F(u_n)}{|u_n|^4}|v_n(x)|^4 \to \infty, \quad \text{as } n \to \infty.$$

Since $Q(|x|) > 0$, using Fatou's lemma, we obtain

$$\int_{\mathbb{R}^3} \frac{Q(|x|)F(u_n)}{\|u_n\|_E^4} dx \to \infty.$$

A contradiction, so $\operatorname{meas}(\Omega) = 0$. Therefore, $v(x) = 0$ a.e. $x \in \mathbb{R}^3$. Next, as in [19], we define

$$I(t_n u_n) = \max_{t \in [0,1]} I(t u_n).$$

For any $M > 0$, set $\tilde{v}_n = \sqrt{4M}\frac{u_n}{\|u_n\|_E} = \sqrt{4M} v_n$. Since $|F(u)| \le c(\frac{1}{p_1}|u|^{p_1} + \frac{1}{p_2}|u|^{p_2})$ for $u \in \mathbb{R}$,

$$|\int_{\mathbb{R}^3} Q(|x|)F(\tilde{v}_n)dx| \le \frac{c}{p_1}\int_{\mathbb{R}^3} Q(|x|)|\tilde{v}_n|^{p_1} dx + \frac{c}{p_2}\int_{\mathbb{R}^3} Q(|x|)|\tilde{v}_n|^{p_2} dx \to 0,$$

as $n \to \infty$. Consequently, for large n, one has

$$\begin{aligned} I(t_n u_n) &\ge I(\tilde{v}_n) \\ &\ge \frac{1}{2}\|\tilde{v}_n\|_E^2 + \frac{1}{4}\int_{\mathbb{R}^3} Q(|x|)\phi_{\tilde{v}_n}\tilde{v}_n^2 dx - \int_{\mathbb{R}^3} Q(|x|)F(\tilde{v}_n)dx \\ &\ge M. \end{aligned}$$

This means that $\lim_{n\to\infty} I(t_n u_n) = \infty$. In view of the choice of t_n we know that $\langle I'(t_n u_n), t_n u_n\rangle = 0$ or $\to 0$. Hence, by (F5) and the oddness of f, one has

$$\begin{aligned} &\infty \leftarrow 4I(t_n u_n) - \langle I'(t_n u_n), t_n u_n\rangle \\ &= t_n^2 \int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + V(|x|)|u_n|^2\right) dx + \int_{\mathbb{R}^3} Q(|x|)\left(f(t_n u_n)t_n u_n - 4F(t_n u_n)\right) dx \\ &\le \|u_n\|_E^2 + \int_{\mathbb{R}^3} Q(|x|)\left(f(u_n)u_n - 4F(u_n)\right) dx \\ &= 4I(u_n) - \langle I'(u_n), u_n\rangle. \end{aligned}$$

This is a contradiction, so $\{u_n\}$ is bounded. Consequently, $\{u_n\} \subset E$ has a convergent subsequence by Lemma 2.4. This shows that I satisfies the Cerami condition. Note that if we use Cerami condition in place of the Palais-Smale condition, then

[20, Theorems 2.2 and 9.12] are still true. Therefore, the conclusion follows from [20, Theorems 2.2 and 9.12]. This completes the proof. □

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References

[1] V. Benci, D. Fortunato; *An eigenvalue problem for the Schröinger-Maxwell equations*, Topol. Methods Nonlinear Anal. 11 (1998) 283-293.
[2] T. D'Aprile, J. Wei; *On bound states concentrating on spheres for the Schröinger-Maxwell equation*, SIAM J. Math. Anal. 37 (2005) 321-342.
[3] A. Candela, A. Salvatore; *Multiple solitary waves for non-homogeneous Schröinger-Maxwell equations*, Mediterr. J. Math. 3 (2006) 483-493.
[4] D. Ruiz; *The Schröinger-Maxwell equation under the effect of a nonlinear local term*, J. Funct. Anal. 237 (2006) 655-674.
[5] A. Ambrosetti, D. Ruiz; *Multiple bound states for the Schröinger-Maxwell problem*, Commun. Contemp. Math. 10 (2008) 391-304.
[6] L. G. Zhao, F. K. Zhao; *Positive solutions for Schröinger-Maxwell equations with a critical exponent*, Nonlinear Anal. 70 (2009) 2150-2164.
[7] J. T. Sun, H. B. Chen, J.J. Nieto, On ground state solutions for some non-autonomous Schröinger-Maxwell systems, J. Differential Equations 252 (2012) 3365-3380.
[8] Z. P. Wang, H. S. Zhou; *Positive solution for a nonlinear stationary Schröinger-Maxwell system in* $\mathbb{R}^3$, Discrete Contin. Dyn. Syst. 18 (2007) 809-816.
[9] C. Mercuri; *Positive solutions of nonlinear Schröinger-Maxwell systems with radial potentials vanishing at infinity*, Rend. Lincei-Matematicae Applicazioni 19 (2008) 211-227.
[10] J. T. Sun, H. B. Chen, L. Yang; *Positive solutions of asymptotically linear Schröinger-Maxwell systems with a radial potential vanishing at infinity*, Nonlinear Anal. 74 (2011) 413-423.
[11] S. J. Chen, C. L. Tang; *High energy solutions for the superlinear Schröinger-Maxwell equations*, Nonlinear Anal. 71 (2009) 4927-4934.
[12] W. Zou; *Variant fountain theorems and their applications*, Manuscripta Math. 104 (2001) 343-358.
[13] Q. D. Li, H. Su, Z. L. Wei; *Existence of infinitely many large solutions for the nonlinear Schröinger-Maxwell equations*, Nonlinear Anal. 72 (2010) 4264-4270.
[14] L. G. Zhao, F. K. Zhao; *On the existence of solutions for the Schröinger-Maxwell equations*, J. Math. Anal. Appl. 346 (2008) 155-169.
[15] J. B. Su, Z. Q. Wang, M. Willem; *Weighted Sobolev embedding with unbounded and decaying radial potentials*, J. Differential Equations 238 (2007) 201-219.
[16] A. Azzollini, A. Pomponio; *Ground state solutions for the nonlinear Schröinger-Maxwell equations*, J. Math. Anal. Appl. 345 (2008) 90-108.
[17] M. B. Yang, Z. F. Shen, Y. H. Ding; *Multiple semiclassical solutions for the nonlinear Schröinger-Maxwell system*, Nonlinear Analysis 71 (2009) 730-739.
[18] M. B. Yang, B. R. Li; *Solitary waves for non-homogeneous Schröinger-Maxwell system*, Applied Mathematics and Computation 215 (2009) 66-70.
[19] L. Jeanjean; *On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer type problem set on* $\mathbb{R}^N$, Proc. Roy. Soc. Edinburgh A 129 (1999) 787-809.
[20] P. H. Rabinowitz; *Minimax Methods in Critical Point Theory with Application to Differential Equations*, in: CBMS Regional Conf. Ser. in Math., vol. 65, American Mathematical Society, Providence, RI, 1986.
[21] W. M. Zou, M. Schechter; *Critical Point Theory and its Applications*, Springer, New York, 2006.
[22] M. Willem; *Minimax Theorems*, Birkhäuser, Berlin, 1996.

Fangfang Liao
School of Mathematics and Statistics, Central South University, Changsha, 410083, Hunan, China.
Department of Mathematics, Xiangnan University, Chenzhou, 423000, Hunan, China
E-mail address: liaofangfang1981@126.com

Xiaoping Wang
Department of Mathematics, Xiangnan University, Chenzhou, 423000, Hunan, China
E-mail address: wxp31415@163.com

Zhigang Liu
Department of Mathematics, Xiangnan University, Chenzhou, 423000, Hunan, China
E-mail address: liuzg22@sina.com