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EXISTENCE OF SOLUTIONS TO p-LAPLACIAN EQUATIONS INVOLVING GENERAL SUBCRITICAL GROWTH

YONG-YI LAN

Abstract. In this article, we consider the quasilinear elliptic equation $-\Delta_p u = \mu f(x,u)$ with the Dirichlet boundary coditions, and under suitable growth condition on the nonlinear term f. Existence of solutions is given for all $\mu > 0$ via the variational method and some analysis techniques.

1. Introduction and main results

In this article, we consider the Dirichlet boundary-value problem

$$\begin{gathered} -\Delta_p u = \mu f(x,u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is an open bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $p >1$, $-\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian of u, $f(x,t)$ is continuous on $\overline{\Omega} \times \mathbb{R}$.

We look for the weak solutions of (1.1) which are the same as the critical points of the functional $I_\mu : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$I_\mu(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \mu \int_\Omega F(x,u)\,\mathrm{d}x, \tag{1.2}$$

where $F(x,t) = \int_0^t f(x,s)\,\mathrm{d}s$, and $W_0^{1,p}(\Omega)$ is the Sobolev space with the usual norm:

$$\|u\|^p = \int_\Omega |\nabla u|^p \,\mathrm{d}x.$$

In this article, the hypotheses on the nonlinearity $f(x,t)$ are the following:

(F1) There exist constants $\theta \ge 1$, $\alpha > 0$ such that

$$\theta G(x,t) + \alpha \ge G(x,st) \quad \text{for all } t \in \mathbb{R},\ x \in \overline{\Omega},\ s \in [0,1],$$

where $G(x,t) := tf(x,t) - pF(x,t)$.

(F2)

$$\lim_{|t|\to\infty} \frac{f(x,t)}{t|t|^{p^*-2}} = 0 \quad \text{uniformly a.e. } x \in \Omega,$$

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where $p^* = \frac{Np}{N-p}$ if $1 < p < N$ and $p^* = +\infty$ if $p \geq N$ is the Sobolev critical exponent.

(F3)
$$\lim_{t\to 0} \frac{f(x,t)}{t|t|^{p-2}} = 0 \quad \text{uniformly a.e. } x \in \Omega.$$

(F4)
$$\lim_{|t|\to +\infty} \frac{F(x,t)}{|t|^p} = +\infty \quad \text{uniformly a.e. } x \in \Omega.$$

Problem (1.1) is one of the main quasilinear elliptic problems which have been studied extensively for many years, see, for example [1]–[19], [22]. Since Ambrosetti and Rabinowitz proposed the mountain-pass theorem in 1973 (see [1]), critical point theory has become one of the main tools for finding solutions to elliptic equations of variational type. A standard existence result for (1.1) is that for any $\mu > 0$, (1.1) possesses at least a nontrivial solution if $f(x,t)$ satisfies the following conditions:

(1) p-superlinear at $t = 0$: $\lim_{t\to 0} \frac{f(x,t)}{t|t|^{p-2}} = 0$ uniformly a.e. $x \in \Omega$.
(2) subcritical at $t = \infty$: there are positive constants a and b such that
$$|f(x,t)| \leq a + b|t|^{q-1}, \quad \forall t \in \mathbb{R},\ x \in \Omega.$$
where $1 \leq q < p^*$.
(3) the Ambrosetti–Rabinowitz condition (AR for short): for some $\theta > p$, $C > 0$,
$$0 < \theta F(x,t) \leq f(x,t)t, \quad \forall |t| \geq C,\ x \in \Omega. \tag{1.3}$$

The (AR) condition has appeared in most of the studies for quasilinear problems and plays an important role in studying the existence of nontrivial solutions of many quasilinear elliptic boundary value problems. It is quite natural and important not only to ensure that the Euler-Lagrange functional associated to problem (1.1) has a mountain pass geometry, but also to guarantee that Palais-Smale sequence of the Euler-Lagrange functional is bounded. Since then, the (AR) condition has been used extensively in many literature sources (see [2, 5, 7]). But this condition is very restrictive eliminating many nonlinearities. There are always many functions that do not satisfy the (AR) condition. For example, for the sake of simplicity, we consider the case $p = 2$,
$$f(x,t) = 2t\ln(1+|t|).$$
Many efforts have been made to extend the range of the nonlinearity. For example, Miyagaki and Souto [15] studied (1.1) for when $p = 2$ and replaced the (AR) condition by some monotonicity arguments. They assumed that there is $t_0 > 0$ such that
$$\frac{f(x,t)}{t} \text{ is increasing for } t \geq t_0 \text{ and decreasing for } t \leq -t_0, \text{ for all } x \in \Omega; \tag{1.4}$$
or a weaker condition is that there exist $C > 0$ such that
$$tf(x,t) - 2F(x,t) \leq sf(x,s) - 2F(x,s) + C, \tag{1.5}$$
for all $0 < t < s$ or $s < t < 0$, for all $x \in \Omega$.

There are some other well known solvability conditions (see [4, 8, 10, 19, 22]). Moreover, in the study of critical points of real-valued functionals, with or without constraints, the Palais-Smale condition(the (P.S.) condition for short) and its variants play a essential role.

To ensure the global compactness, one needs to impose the subcritical growth condition on the nonlinearity $f(x,t)$: there exists a constant $C_0 > 0$ such that

$$|f(x,t)| \le C_0(1+|t|^{p-1}), \quad \forall t \in \mathbb{R},\ x \in \Omega,$$

where $1 < p < p^*$. However, in the present paper, we consider a class of elliptic partial differential equations with more general growth condition, that is (F2).

Based on variational methods, Miyagaki and Souto in [15] obtained the following theorem:

Theorem 1.1 ([15, Theorem 1.1])**.** *Under hypotheses* (1.4)*,* (1.2)*,* (F3) *and* (F4)*, problem* (1.1) *for when* $p = 2$ *has a nontrivial weak solution, for all* $\lambda > 0$*.*

Many efforts have been made to extend these results (see [11, 13, 14] and the references therein). Li and Yang extend the results from $p = 2$ to $p > 1$ in [11], they obtained the following theorem:

Theorem 1.2 ([11, Theorem 1.1])**.** *Under hypotheses* (1.4)*,* (1.2)*,* (F3) *and* (F4)*, problem* (1.1) *has a nontrivial weak solution, for all* $\lambda > 0$*.*

The aim of the article is to consider the problem in a different case: based on a variant version of mountain pass theorem, we can prove the same result under more generic conditions, which generalizes Theorems 1.1 and 1.2.

Our main results reads as follows:

Theorem 1.3. *Suppose that* (F1)–(F4) *hold. Then* (1.1) *has a weak nontrivial solution, for all* $\lambda > 0$*.*

Note that (F3) implies that problem (1.1) has a trivial solution $u = 0$ and we are interested in the existence of nontrivial solutions.

Remark 1.4. Theorem 1.3 improves Theorem 1.2 in two aspects. To show this, it suffices to compare condition (F1) with (1.4) and (1.5), and to compare condition (F2) with (1.2).

At first, we can easily prove that (F1) is equivalent to (1.5) when $\theta = 1$, and (F1) gives some general sense of monotony when $\theta > 1$. There are functions satisfying our condition (F1) and not satisfying the condition (1.5). For example, for the sake of simplicity, we consider the case $p = 2$, let

$$F(x,t) = t^2 \ln(1+t^2) + t \sin t,$$

then

$$f(x,t) = 2t\ln(1+t^2) + t^2 \cdot \frac{2t}{1+t^2} + \sin t + t\cos t,$$

it follows that

$$G(x,t) = tf(x,t) - 2F(x,t) = 2(t^2-1) + \frac{2}{1+t^2} + (t^2\cos t - t\sin t).$$

Let $\theta = 1000$, we can prove by some simple computation that G satisfies (F1) but does not satisfy the condition (1.5) any more.

Secondly, it is obvious that (1.2) implies (F2). There are functions satisfying our growth condition (F2) and not satisfying the subcritical growth condition (1.2). For example, for the sake of simplicity, we consider the case $p = 2$, let

$$F(t) = \frac{t^{2^*}}{\ln(e+t^2)}$$

then
$$f(t)=\frac{2^*t^{2^*-1}(e+t^2)\ln(e+t^2)-2t^{2^*+1}}{(e+t^2)(\ln(e+t^2))^2}.$$
The proof of Theorem 1.3 is much easier than that of the main results in [11, 15].

Remark 1.5. In assumption (F2), we are dealing with functionals satisfying the so-called non-standard growth conditions. Due to the lack of compactness of the embeddings in $W_0^{1,p}(\Omega)\hookrightarrow L^{p^*}(\Omega)$, we cannot use the standard variational argument directly. We overcome the difficulty by Vitali convergence theorem and some analysis techniques.

This paper is organized as follows. In section 2, we give the proof of the Theorem 1.3. In the following discussion, we denote various positive constants as c or c_i $(i=0,1,2,\dots)$ for convenience.

2. Proof of Theorem 1.3

The proof consists of three steps. We prove Theorem 1.3 only when $\mu=1$. The case of a general $\mu>0$ will follow immediately. In fact, if $\mu>0$ and $\mu\neq 1$, we only let $g(x,t)=\mu f(x,t)$. Then (1.1) becomes
$$\begin{gathered}-\Delta_p u=g(x,u),\quad x\in\Omega,\\ u=0\quad x\in\partial\Omega.\end{gathered}$$
The nonlinear term g also satisfies condition (F1)–(F4), Then the same conclusion as in the case $\mu=1$ holds.

First step: The (C) condition. Let $\{u_n\}$ be any sequence in $W_0^{1,p}(\Omega)$ such that $I(u_n)$ is bounded and $\|I'(u_n)\|(1+\|u_n\|)$ converges to zero; that is,
$$I(u_n)\to c,\quad \|I'(u_n)\|(1+\|u_n\|)\to 0$$
which shows that
$$c=I(u_n)+o(1),\quad \langle I'(u_n),u_n\rangle=o(1) \tag{2.1}$$
where $o(1)\to 0$ as $n\to\infty$.

We now prove that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. By contradiction, we assume $\|u_n\|\to\infty$ as $n\to\infty$. Let $w_n=\frac{u_n}{\|u_n\|}$, then $w_n\in W_0^{1,p}(\Omega)$ with $\|w_n\|=1$. Then there exists a $w\in W_0^{1,p}(\Omega)$ such that
$$\begin{gathered}w_n\rightharpoonup w\quad\text{in } W_0^{1,p}(\Omega),\\ w_n\to w\quad\text{a.e. in }\Omega,\\ w_n\to w\quad\text{in } L^r(\Omega),\ \text{with } 1\le r<p^*,\\ \|w_n\|_{2^*}^{2^*}\le C_1<\infty.\end{gathered}\tag{2.2}$$
Let $\Omega_{\neq}=\{x\in\Omega,\ w(x)\neq 0\}$; then one has
$$\lim_{n\to\infty}w_n(x)=\lim_{n\to\infty}\frac{u_n(x)}{\|u_n\|}=w(x)\neq 0\quad\text{in }\Omega_{\neq}.$$
So we have
$$|u_n(x)|\to+\infty\ \text{ a.e. in }\Omega_{\neq}.\tag{2.3}$$
Using (F4), we have
$$\lim_{n\to\infty}\frac{F(x,u_n(x))}{|u_n(x)|^p}=+\infty,\quad\text{a.e. in }\Omega_{\neq}.\tag{2.4}$$

This means that

$$\lim_{n\to\infty} \frac{F(x,u_n(x))}{|u_n(x)|^p}|w_n(x)|^p = +\infty, \quad \text{a.e. in } \Omega_{\neq}. \tag{2.5}$$

By (F4) again, there is an $C_0 > 0$ such that

$$\frac{F(x,t)}{|t|^p} > 1, \tag{2.6}$$

for any $x \in \Omega$ and $|t| \geq C_0$. Since $F(x,t)$ is continuous on $\overline{\Omega} \times [-C_0, C_0]$, there is an $M > 0$ such that

$$|F(x,t)| \leq M, \quad \text{for all } (x,t) \in \overline{\Omega} \times [-C_0, C_0]. \tag{2.7}$$

From (2.6), (2.7), we see that there is a constant C such that for any $(x,t) \in \overline{\Omega} \times \mathbb{R}$, we have

$$F(x,t) \geq C \tag{2.8}$$

which shows that

$$\frac{F(x,u_n(x)) - C}{\|u_n\|^p} \geq 0.$$

This implies that

$$\frac{F(x,u_n(x))}{|u_n(x)|^p}|w_n(x)|^p - \frac{C}{\|u_n\|^p} \geq 0. \tag{2.9}$$

Using (2.1) we have

$$c = I(u_n) + o(1) = \frac{1}{p}\|u_n\|^p - \int_\Omega F(x,u_n)\,\mathrm{d}x + o(1).$$

So we see that

$$\|u_n\|^p = pc + p\int_\Omega F(x,u_n)\,\mathrm{d}x + o(1). \tag{2.10}$$

By (2.1) and (2.10), we obtain

$$\int_\Omega F(x,u_n)\,\mathrm{d}x \to +\infty. \tag{2.11}$$

We claim that $|\Omega_{\neq}| = 0$. In fact, if $|\Omega_{\neq}| \neq 0$, then combining (2.5) and (2.9) with Fatou's lemma, one has

$$\begin{aligned}
+\infty &= \int_{\Omega_{\neq}} \liminf_{n\to+\infty} \frac{F(x,u_n(x))}{|u_n(x)|^p}|w_n(x)|^p\,\mathrm{d}x - \int_{\Omega_{\neq}} \limsup_{n\to+\infty} \frac{C}{\|u_n\|^p}\,\mathrm{d}x \\
&\leq \int_{\Omega_{\neq}} \liminf_{n\to+\infty} \Big(\frac{F(x,u_n(x))}{|u_n(x)|^p}|w_n(x)|^p - \frac{C}{\|u_n\|^p}\Big)\,\mathrm{d}x \\
&\leq \liminf_{n\to+\infty} \int_{\Omega_{\neq}} \Big(\frac{F(x,u_n(x))}{|u_n(x)|^p}|w_n(x)|^p - \frac{C}{\|u_n\|^p}\Big)\,\mathrm{d}x \\
&\leq \liminf_{n\to+\infty} \int_{\Omega} \Big(\frac{F(x,u_n(x))}{|u_n(x)|^p}|w_n(x)|^p - \frac{C}{\|u_n\|^p}\Big)\,\mathrm{d}x \\
&= \liminf_{n\to+\infty} \int_{\Omega} \frac{F(x,u_n(x))}{\|u_n\|^p}\,\mathrm{d}x \\
&\leq \liminf_{n\to+\infty} \frac{\int_\Omega F(x,u_n(x))\,\mathrm{d}x}{pc + p\int_\Omega F(x,u_n)\,\mathrm{d}x + o(1)}.
\end{aligned} \tag{2.12}$$

So by (2.11) and (2.12) we deduce a contradiction. This shows that $|\Omega_{\neq}| = 0$. Hence $w(x) = 0$ a.e. in Ω.

Since $I(tu_n)$ is continuous in $t \in [0,1]$, there exists $t_n \in [0,1]$ such that

$$I(t_n u_n) = \max_{t\in[0,1]} I(tu_n).$$

Clearly, $t_n > 0$ and $I(t_n u_n) \geq 0 = I(0)$. If $t_n < 1$ we have that $\frac{\mathrm{d}}{\mathrm{d}t} I(tu_n)|_{t=t_n} = 0$, which gives $\langle I'(t_n u_n), t_n u_n\rangle = 0$. If $t_n = 1$, then (2.1) gives that $\langle I'(u_n), u_n\rangle = o(1)$. So we always have

$$\langle I'(t_n u_n), t_n u_n\rangle = o(1).$$

From (F1), for $t \in [0,1]$ we have

$$\begin{aligned}
pI(tu_n) &\leq pI(t_n u_n)\\
&= pI(t_n u_n) - \langle I'(t_n u_n), t_n u_n\rangle + o(1)\\
&= \int_\Omega [t_n u_n f(x, t_n u_n) - pF(x, t_n u_n)]\,\mathrm{d}x + o(1)\\
&\leq \int_\Omega [\theta(u_n f(x, u_n) - pF(x, u_n)) + \alpha]\,\mathrm{d}x + o(1)\\
&\leq \theta(\|u_n\|^p + pc - \|u_n\|^p + o(1)) + \alpha|\Omega| + o(1)\\
&\leq p\theta c + \alpha|\Omega| + o(1).
\end{aligned} \tag{2.13}$$

where we used (2.1) and (2.10), θ and α as in (F1).

Furthermore, by (F2), for every $\varepsilon > 0$, there exists $a(\varepsilon) > 0$, such that

$$|F(x,t)| \leq \frac{1}{2C_1}\varepsilon|t|^{p^*} + a(\varepsilon), \quad \text{for } t \in \mathbb{R} \text{ a.e. } x \in \Omega.$$

Let $\delta = \varepsilon/(2a(\varepsilon)) > 0$, $E \subseteq \Omega$, $\operatorname{meas} E < \delta$, we have

$$\begin{aligned}
\Big|\int_E F(x, w_n)\,\mathrm{d}x\Big| &\leq \int_E |F(x, w_n)|\,\mathrm{d}x\\
&\leq \int_E a(\varepsilon)\,\mathrm{d}x + \frac{1}{2C_1}\varepsilon \int_E |w_n|^{p^*}\,\mathrm{d}x\\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

hence $\{\int_\Omega F(x, w_n)\,\mathrm{d}x,\ n \in N\}$ is equi-absolutely-continuous. It follows easily from Vitali Convergence Theorem that

$$\int_\Omega F(x, w_n)\,\mathrm{d}x \to \int_\Omega F(x, 0)\,\mathrm{d}x = 0;$$

So, for any $R_0 > 0$,

$$pI(R_0 w_n) = \|R_0 w_n\|^p - p\int_\Omega F(x, R_0 w_n)\,\mathrm{d}x = R_0^p + o(1). \tag{2.14}$$

From (2.13), we obtain

$$pI(tu_n) \leq p\theta c + \alpha|\Omega| + o(1), \tag{2.15}$$

for $t \in [0,1]$. So combining (2.14) with (2.15),

$$R_0^p + o(1) = pI(R_0 w_n) \leq p\theta c + \alpha|\Omega| + o(1).$$

Letting $n \to \infty$ we obtain

$$R_0^p \leq p\theta c + \alpha|\Omega| + o(1).$$

Letting $R_0 \to \infty$, we obtain a contradiction. Hence $\|u_n\|$ is bounded.

By the continuity of the embedding, we have $\|u_n\|_{2^*}^{2^*} \le C_2 < \infty$ for all n. If necessary going to a subsequence, one obtains

$$u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega), \quad \text{and} \quad u_n \to u \text{ in } L^r(\Omega), \quad \text{where } 1 \le r < p^*.$$

Using (F2), for every $\varepsilon > 0$, there exists $a(\varepsilon) > 0$, such that

$$|f(x,t)t| \le \frac{1}{2C_2}\varepsilon|t|^{p^*} + a(\varepsilon), \quad \text{for } t \in \mathbb{R}, \text{ a.e. } x \in \Omega.$$

Let $\delta = \varepsilon/(2a(\varepsilon)) > 0$, $E \subseteq \Omega$, $\operatorname{meas} E < \delta$, we have

$$\begin{aligned}
\Big|\int_E f(x,u_n)u_n \,\mathrm{d}x\Big| &\le \int_E |f(x,u_n)u_n| \,\mathrm{d}x \\
&\le \int_E a(\varepsilon)\,\mathrm{d}x + \frac{1}{2C_2}\varepsilon \int_E |u_n|^{p^*}\,\mathrm{d}x \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

hence $\{\int_\Omega f(x,u_n)u_n \,\mathrm{d}x,\ n \in N\}$ is equi-absolutely-continuous. It follows easily from Vitali Convergence Theorem that

$$\int_\Omega f(x,u_n)u_n \,\mathrm{d}x \to \int_\Omega f(x,u)u \,\mathrm{d}x. \tag{2.16}$$

From (F2), for any $\varepsilon > 0$ there exists $a(\varepsilon) > 0$ such that

$$|f(x,t)| \le \frac{1}{2c_1c_2}\varepsilon|t|^{p^*-1} + a(\varepsilon) \quad \text{for } t \in \mathbb{R},\ x \in \Omega.$$

where

$$c_1 \ge \Big(\int_\Omega |u_n|^{p^*}\,\mathrm{d}x\Big)^{\frac{p^*-1}{p^*}} \quad \forall n; \quad c_2 := \Big(\int_\Omega |u|^{p^*}\,\mathrm{d}x\Big)^{\frac{1}{p^*}}.$$

From Hölder's inequality, for every $E \subseteq \Omega$, we have

$$\int_E a(\varepsilon)|u|\,\mathrm{d}x \le a(\varepsilon)(\operatorname{meas} E)^{\frac{p^*-1}{p^*}}\Big(\int_E |u|^{p^*}\,\mathrm{d}x\Big)^{\frac{1}{p^*}} \le a(\varepsilon)(\operatorname{meas} E)^{\frac{p^*-1}{p^*}}c_1;$$

$$\int_E |u_n|^{p^*-1}|u|\,\mathrm{d}x \le \Big(\int_E |u_n|^{p^*}\,\mathrm{d}x\Big)^{\frac{p^*-1}{p^*}}\Big(\int_E |u|^{p^*}\,\mathrm{d}x\Big)^{\frac{1}{p^*}} \le c_1c_2.$$

Let $\delta = \big(\frac{\varepsilon}{2c_1a(\varepsilon)}\big)^{\frac{p^*}{p^*-1}} > 0$, $E \subseteq \Omega$, $\operatorname{meas} E < \delta$, we have

$$\begin{aligned}
\Big|\int_E f(x,u_n)u \,\mathrm{d}x\Big| &\le \int_E |f(x,u_n)u| \,\mathrm{d}x \\
&\le \int_E a(\varepsilon)|u|\,\mathrm{d}x + \frac{1}{2c_1c_2}\varepsilon \int_E |u_n|^{p^*-1}|u|\,\mathrm{d}x \\
&\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

hence $\{\int_\Omega f(x,u_n)u \,\mathrm{d}x,\ n \in N\}$ is also equi-absolutely-continuous. It follows from Vitali Convergence Theorem that

$$\int_\Omega f(x,u_n)u \,\mathrm{d}x \to \int_\Omega f(x,u)u \,\mathrm{d}x. \tag{2.17}$$

Since

$$\langle I'(u_n), u\rangle = \int_\Omega \big(|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla u - f(x,u_n)u\big)\,\mathrm{d}x \to 0; \tag{2.18}$$

$$\langle I'(u_n), u_n\rangle = \int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla u_n - f(x, u_n)u_n\right) \mathrm{d}x \to 0. \tag{2.19}$$

It follows from (2.16)-(2.19) that $\|u_n\| \to \|u\|$. By Kadec-Klee property, we have

$$u_n \to u \quad \text{in } W_0^{1,p}(\Omega).$$

Second step: Mountain-pass geometric structure. I has a mountain pass geometry; i.e., there exist $u_1 \in W_0^{1,p}(\Omega)$ and constants $r, \rho > 0$ such that $I(u_1) < 0$, $\|u_1\| > r$ and

$$I(u) \geq \rho, \quad \text{when } \|u\| = r. \tag{2.20}$$

Indeed, By (F3), we have $t_0 > 0$ and $\lambda \in (0, \lambda_1)$ such that

$$\frac{f(x,t)}{t|t|^{p-2}} < \lambda, \quad \text{for } |t| < t_0,$$

where

$$\lambda_1 = \inf_{u \in W_0^{1,p}(\Omega),\, u \neq 0} \frac{\|u\|^p}{\|u\|_p^p} > 0$$

is the first eigenvalue of the operator $-\Delta_p$ with the Dirichlet boundary value in Ω. This implies that

$$F(x,t) \leq \frac{\lambda}{p}|t|^p, \quad \text{for } |t| \leq t_0.$$

This inequality with (F2) shows that

$$F(x,t) \leq \frac{\lambda}{p}|t|^p + C|t|^{p^*}, \quad \text{for } t \in \mathbb{R}$$

with some $C > 0$. Since $\lambda_1 > 0$ denotes the first eigenvalue of the operator $-\Delta_p$ with the Dirichlet boundary value in Ω, it follows that $\|u\|^p \geq \lambda_1\|u\|_p^p$ for $u \in W_0^{1,p}(\Omega)$. Then I is estimated as

$$I(u) \geq \frac{1}{p}\|u\|^p - \frac{\lambda}{p}\|u\|_p^p - C\|u\|_{p^*}^{p^*} \geq \frac{\lambda_1 - \lambda}{p\lambda_1}\|u\|^p - C'\|u\|^{p^*}.$$

This shows the existence of r and ρ satisfying:

$$I(u) \geq \rho, \quad \text{when } \|u\| = r.$$

From (F4) follows that, for all $M > 0$ there exists $C_M > 0$, such that

$$F(x,t) \geq M|t|^p - C_M, \quad \forall\, x \in \Omega,\ t > 0. \tag{2.21}$$

Let ϕ be a function such that $\phi \in W_0^{1,p}(\Omega)$, $\phi \geq 0$, $\phi \not\equiv 0$. From (2.21) we obtain

$$\begin{aligned} I(t\phi) &= \frac{|t|^p}{p}\|\phi\|^p - \int_\Omega F(x, t\phi)\,\mathrm{d}x \\ &\leq \frac{|t|^p}{p}\|\phi\|^p - t^p\int_\Omega M\phi^p\,\mathrm{d}x + c|\Omega| \to -\infty \quad \text{as } t \to \infty. \end{aligned}$$

We fix $t > 0$ large so that $I(t\phi) < 0$ and $t\|\phi\| > r$. Let $u_1 := t\phi \in W_0^{1,p}(\Omega)$ and then constants r, $\rho > 0$ such that $I(u_1) < 0$, $\|u_1\| > r$ and satisfies (2.20), i.e. I has a mountain pass geometry.

Third step: Critical value of I. For u_1 in second step, we define

$$\Gamma := \{\gamma : C[0,1] \to W_0^{1,p}(\Omega) : \gamma(0) = 0,\ \gamma(1) = u_1\},$$
$$c_0 := \inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1} I(\gamma(t))$$

As shown in [18], a deformation lemma can be proved with the (C) condition, replacing the usual Palais-Smale condition, and it turns out that the Mountain Pass Theorem still holds. Then c_0 is a critical value of I. For the proof, we refer the reader to [17, 20, 21].

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Yong-Yi Lan

School of Sciences, Jimei University, Xiamen 361021, China

E-mail address: `lanyongyi@jmu.edu.cn`