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EXISTENCE OF POSITIVE SOLUTIONS FOR $p(x)$-LAPLACIAN EQUATIONS WITH A SINGULAR NONLINEAR TERM

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Dedicated to Professor Xianling Fan on his 70th birthday

Abstract. In this article, we study the existence of positive solutions for the $p(x)$-Laplacian Dirichlet problem

$$-\Delta_{p(x)}u = \lambda f(x,u)$$

in a bounded domain $\Omega \subset \mathbb{R}^N$. The singular nonlinearity term f is allowed to be either $f(x,s) \to +\infty$, or $f(x,s) \to +\infty$ as $s \to 0^+$ for each $x \in \Omega$. Our main results generalize the results in [15] from constant exponents to variable exponents. In particular, we give the asymptotic behavior of solutions of a simpler equation which is useful for finding supersolutions of differential equations with variable exponents, which is of independent interest.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with C^2 boundary. We consider the existence of positive solutions for elliptic problems with variable exponent of the form

$$\begin{gathered}
-\Delta_{p(x)}u = \lambda f(x,u), \quad \text{in } \Omega,\\
u(x) > 0, \quad \text{in } \Omega,\\
u(x) = 0, \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.1}$$

where $-\Delta_{p(x)}u = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ with $\nabla u = (\partial_{x_1}u, \partial_{x_2}u, \dots, \partial_{x_N}u)$ which is so-called $p(x)$-Laplacian, $p(\cdot)$ is a function which satisfies some conditions specified below, $f : \Omega \times (0,\infty) \to [0,\infty)$ is a continuous function, and $\lambda > 0$ is a real parameter. Throughout this paper, we will denote $d(x) = d(x, \partial\Omega)$.

In recent years, the study of differential equations and variational problems with nonstandard $p(x)$-growth condition has been an interesting topic. The $p(x)$-Laplacian arises from the study of nonlinear elasticity, electrorheological fluids and image restoration etc. For example, electrorheological fluids have an extensive applications in robotics, aircraft and aerospace. We refer readers to [1, 5, 19, 41, 42, 46] for more detailed background of applications. There are many reference papers related to the study of differential equations and variational problems with variable exponent. Far from being complete, we refer readers to [1, 2, 3, 6, 7, 8, 9, 10, 11,

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12, 14, 20, 21, 22, 23, 24, 25, 26, 30, 31, 32, 33, 34, 35, 36, 39, 42, 43, 44, 45, 46] and references cited therein. For example, the regularity of weak solutions for differential equations with variable exponent was studied in [1, 7], and existence of solutions for variable exponent problems was studied in a series of papers [3, 9, 12, 20, 24, 30, 33, 36, 39, 44, 45]. Recently, the applications of variable exponent analysis in image restoration attracted more and more attention [16, 17, 23, 28]. In this paper, our aim is to study the existence of positive solution for problem (1.1) with singular nonlinear term f.

Clearly, if $p(\cdot) \equiv p$, a constant, the operator is the well-known p-Laplacian, and (1.1) is the usual p-Laplacian equation, but for non-constant $p(\cdot)$, $p(x)$-Laplacian problems are more complicated due to the non-homogeneity of $p(x)$-Laplacian. For example, if Ω is a smooth bounded domain, the Rayleigh quotient

$$\lambda_{p(\cdot)} = \inf_{u \in W_0^{1,p(\cdot)}(\Omega)\setminus\{0\}} \frac{\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx}{\int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx}$$

is zero in general, and $\lambda_{p(\cdot)} > 0$ only under some special conditions (see [13]). It is also possible the first eigenvalue and eigenfunction of $p(x)$-Laplacian do not exist, even though the existence of the first eigenvalue and eigenfunction is very important in the study of elliptic problems related to p-Laplacian problems. For example, in [15], the author use the first eigenfunction and the first eigenvalue to construct subsolutions. Fan [8] considered the eigenvalue problem of $p(x)$-Laplacian equation with the Neumann boundary condition, the existence of infinite many eigenvalues has been established. Benouhiba [2] studied the eigenvalue problem

$$-\Delta_{p(x)}u = \lambda V(x)|u|^{q(x)-2}u, \quad x \in \mathbb{R}^N,$$

where $1 < p(\cdot)$; $q(\cdot) \in C(\mathbb{R}^N)$ and $V(\cdot)$ is an indefinite weight function. The results show that the spectrum of such problems contains a continuous family of eigenvalues.

There are many papers deal with the existence of positive solution for a class p-Laplacian equation with singular nonlinearity (see [15, 18, 37, 38, 40] Mohammed [37], Perera and Silva —citep1, Qing and Yang [40] and Guo et al [18] studied the solvability of (1.1) with $\lambda = 1, p(\cdot) \equiv p \neq 2$ and $f(\cdot,\cdot)$ satisfies various conditions. In [38], the authors considered a boundary condition in a more general sense.

Mohammed [37] considered the existence and uniqueness of weak solutions of the singular boundary value problem with constant exponent as follows.

$$\begin{gathered} -\Delta_p u = f(x,u), \quad \text{in } \Omega, \\ u(x) > 0, \quad \text{in } \Omega, \\ u(x) = 0, \quad \text{on } \partial\Omega, \end{gathered}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $C^{1,\omega}$ boundary for some $0 < \omega < 1$, and singular nonlinearity term $f(x,t)$ could show up when $t \to 0^+$. Mohammed make the following two assumptions:

(1) For each $\theta \in (0,1)$, there is a constant $C_\theta \geq 1$ such that $g(\theta t) \leq C_\theta g(t)$ for all $t > 0$;
(2) $f(x,s) \geq a(x)$ for any $(x,s) \in \Omega \times (0,\infty)$.

In [15], the authors studied the existence of solutions of the nonlinear elliptic problem with constant exponent,

$$\begin{gathered}
-\Delta_p u = \lambda f(x,u), \quad \text{in } \Omega,\\
u(x) > 0, \quad \text{in } \Omega,\\
u(x) = 0, \quad \text{on } \partial\Omega,
\end{gathered}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $1 < p < N$, $f : \Omega \times (0,\infty) \to [0,\infty)$ is a suitable function and $\lambda > 0$ is a real parameter. The nonlinearity term f is allowed to be either $f(x,s) \to +\infty$ or $f(x,s) \to +\infty$ as $s \to 0^+$ for each $x \in \Omega$, and the assumptions (1) and (2) are not assumed.

Results on elliptic problems with singular nonlinearity are rare (see [29, 44]). In [44], by using the sub-supersolution method, we studied the existence and the boundary asymptotic behavior of solutions of the elliptic problem with variable exponent,

$$\begin{gathered}
-\Delta_{p(x)} u = \frac{\lambda}{u^{\gamma(x)}}, \quad \text{in } \Omega,\\
u(x) > 0, \quad \text{in } \Omega,\\
u(x) = 0, \quad \text{on } \partial\Omega,
\end{gathered}$$

where $\Omega \subset \mathbb{R}^N$ is a domain with C^2 boundary, λ is a positive parameter which is large enough.

Liu [29] generalized the results of [37] to $p(x)$-Laplacian by making the similar assumptions. The condition (1) implies that $g(t) \le Ct^{-a}$ when $t \le 1$ for some $a > 0$, which is invalid for $g(t) = e^{1/t}$, and the condition (2) is a bit strong in some sense. Motivated by [15], in this rticle we partly generalized the results to $p(x)$-Laplacian.

Before stating our main results, we make the following assumptions throughout this paper:

- (H0) $p(\cdot) \in C^1(\overline{\Omega})$, $1 < p^- := \inf_\Omega p(x) \le p^+ := \sup_\Omega p(x) < \infty$, $p(\cdot) < N$.
- (H1) $f(x,s) \le b(x)g(s)$ for all $(x,s) \in \Omega \times (0,\infty)$, where $g : (0,+\infty) \to (0,+\infty)$ is a continuous function, $sg(s)$ is decreasing for $s \le 1$; and $b : \Omega \to [1,\infty)$, $b(\cdot) \in L^{\alpha(\cdot)}(\Omega)$, $1 < \alpha(\cdot) \in C(\overline{\Omega})$, and $\frac{1}{\alpha(x)} + \frac{1}{p^*(x)} < 1$, for all $x \in \overline{\Omega}$.
- (H2) $f(x,s)$ satisfies
$$\liminf_{s\to 0^+} \frac{f(x,s)}{s^{p^- -1}|\ln s|^{p^-}} = +\infty \quad \text{uniformly for } x \in \Omega. \tag{1.2}$$

Theorem 1.1. *Assume that* (H0), (H1), (H2) *hold. Then problem* (1.1) *has a solution when λ is small enough.*

Theorem 1.2. . *Assume that* (H0), (H1), (H2) *hold. Also assume that*

- (i) *there is a small $\delta > 0$ such that $p(x) \equiv p$ (a constant) for any $x \in \Omega$ with $d(x) \le \delta$;*
- (ii) $\lim_{s\to+\infty} \frac{g(s)}{s^{p^- -1-\varepsilon}} := g_\infty \in [0,+\infty)$, *where $\varepsilon > 0$ is small enough;*
- (iii) $\alpha(\cdot) > N$ *on* $\overline{\Omega}$.

Then problem (1.1) *has a solution for any positive λ.*

Theorem 1.3. *Assume that* (H0), (H1), (H2) *hold. Also assume that*

- (i) $\frac{\partial p(\cdot)}{\partial \nu} < 0$ *on $\partial\Omega$, where ν is the inward unit normal vector of $\partial\Omega$;*

(ii) $\lim_{s\to+\infty}\frac{g(s)}{s^{p^--1-\varepsilon}}:=g_\infty\in[0,+\infty)$, *where* $\varepsilon>0$ *is small enough;*
(iii) $\alpha(\cdot)>N$ *on* $\overline{\Omega}$.

Then problem (1.1) *has a solution for any positive constant* λ.

Theorem 1.4. *Assume that* (H0), (H1), (H2) *hold. Also assume that*

(i) *Equation* (1.1) *is radial;*
(ii) $\lim_{s\to+\infty}\frac{g(s)}{s^{p^--1-\varepsilon}}:=g_\infty\in[0,+\infty)$, *where* $\varepsilon>0$ *is small enough.*

Then problem (1.1) *has a solution for any positive* λ.

This paper is organized as follows. In section 2, we will recall some basic facts about the variable exponent Lebesgue and Sobolev spaces which we will use later, and we will also give a general principle of sub-supersolution method. Proofs of our results will be presented in section 3.

2. Preliminaries

Throughout this paper, the letters c, c_i, C, C_i $(i=1,2,\dots)$, denote positive constants which may vary from line to line, but they are independent of the terms which will take part in any limit process.

To deal with the $p(x)$-Laplacian problem, we need introduce some functional spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$, $W_0^{1,p(\cdot)}(\Omega)$ and properties of the $p(x)$-Laplacian which we will use later. Denote by $S(\Omega)$ be the set of all measurable real-valued functions defined in Ω. Note that two measurable functions are considered as the same element of $S(\Omega)$ when they are equal almost everywhere. Let

$$L^{p(\cdot)}(\Omega)=\big\{u\in S(\Omega):\int_\Omega|u(x)|^{p(x)}dx<\infty\big\},$$

with the norm

$$|u|_{p(\cdot)}=|u|_{L^{p(\cdot)}(\Omega)}=\inf\big\{\lambda>0:\int_\Omega|\frac{u(x)}{\lambda}|^{p(x)}dx\le1\big\}.$$

The space $(L^{p(\cdot)}(\Omega),|\cdot|_{p(\cdot)})$ becomes a Banach space. We call it variable exponent Lebesgue space. Moreover, this space is a separable, reflexive and uniform convex Banach space; see [14, Theorems 1.6, 1.10, 1.14].

The variable exponent Sobolev space

$$W^{1,p(\cdot)}(\Omega)=\big\{u\in L^{p(\cdot)}(\Omega):|\nabla u|\in L^{p(\cdot)}(\Omega)\big\},$$

can be equipped with the norm

$$\|u\|=|u|_{p(\cdot)}+|\nabla u|_{p(\cdot)},\quad\forall u\in W^{1,p(\cdot)}(\Omega).$$

Note that $W_0^{1,p(\cdot)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. The spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable, reflexive and uniform convex Banach spaces (see [14, Theorem 2.1].

For $u,v\in S(\Omega)$, we write $u\le v$ if $u(x)\le v(x)$ for a.e. $x\in\Omega$. Let $\rho(x,s)$ be a Carathéodory function on $\Omega\times\mathbb{R}$ with property that for any $s_0>0$ there exists a constant A such that

$$|\rho(x,s)|\le A\quad\text{for a.e. } x\in\Omega \text{ and all } s\in[-s_0,s_0]. \tag{2.1}$$

Definition 2.1. (i) Let $\underline{u}, \overline{u} \in W^{1,p(\cdot)}_{\rm loc}(\Omega) \cap C_0(\overline{\Omega})$ satisfy $\underline{u}, \overline{u} > 0$ in Ω. We say $\underline{u}$ and $\overline{u}$ are a subsolution and a supersolution of (1.1) respectively, if

$$\begin{gathered}\int_\Omega |\nabla \underline{u}|^{p(x)-2}\nabla \underline{u}\nabla\phi\,dx \le \int_\Omega \lambda f(x,\underline{u})\phi\,dx,\\ \int_\Omega |\nabla \overline{u}|^{p(x)-2}\nabla \overline{u}\nabla\phi\,dx \ge \int_\Omega \lambda f(x,\overline{u})\phi\,dx,\end{gathered}$$

for all $\phi \in C_0^\infty(\Omega)$ with $\phi \ge 0$ and $\operatorname{supp}\phi \subset\subset \Omega$. We say u is a solution of (1.1), if it is both a subsolution and a supersolution of (1.1).

(ii) A function $u \in W^{1,p(\cdot)}(\Omega) \cap C(\overline{\Omega})$ is called a weak solution of the problem

$$\begin{gathered}-\Delta_{p(x)}u = \rho(x,u), \quad \text{in } \Omega,\\ u(x) = \varphi(x), \quad \text{on } \Omega,\end{gathered} \tag{2.2}$$

where $\varphi(\cdot) \in C(\overline{\Omega})$, if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\phi\,dx = \int_\Omega \rho(x,u)\phi\,dx, \forall\phi \in C_0^\infty(\Omega);$$

(iii) $\underline{u}, \overline{u} \in W^{1,p(\cdot)}(\Omega) \cap C(\overline{\Omega})$ are called a weak subsolution and a weak supersolution of the problem (2.2) respectively if $\underline{u} \le \varphi$ and $\overline{u} \ge \varphi$ on $\partial\Omega$ and for all $\phi \in C_0^\infty(\Omega)$, $\phi \ge 0$,

$$\begin{gathered}\int_\Omega |\nabla \underline{u}|^{p(x)-2}\nabla \underline{u}\nabla\phi\,dx \le \int_\Omega \rho(x,\underline{u})\phi\,dx,\\ \int_\Omega |\nabla \overline{u}|^{p(x)-2}\nabla \overline{u}\nabla\phi\,dx \ge \int_\Omega \rho(x,\overline{u})\phi\,dx.\end{gathered}$$

Lemma 2.2 ([12, Proposition 2.1])**.** *The space $(L^{p(\cdot)}(\Omega), |\cdot|_{p(\cdot)})$ is a separable, uniform convex Banach space, and its conjugate space is $L^{p^0(\cdot)}(\Omega)$, where $p^0(\cdot)$ is the conjugate function of $p(\cdot)$ satisfying $\frac{1}{p(\cdot)} + \frac{1}{p^0(\cdot)} \equiv 1$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p^0(\cdot)}(\Omega)$, we have the following Hölder inequality*

$$|\int_\Omega uv dx| \le \int_\Omega |uv|\,dx \le (\frac{1}{p^-} + \frac{1}{(p^0)^-})|u|_{p(\cdot)}|v|_{p^0(\cdot)} \le 2|u|_{p(\cdot)}|v|_{p^0(\cdot)}.$$

Definition 2.3. Let $u, v \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$. We say that $-\Delta_{p(x)}u + \rho(x,u) \le -\Delta_{p(x)}v + \rho(x,v)$ in Ω if

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\phi\,dx + \int_\Omega \rho(x,u)\phi\,dx \le \int_\Omega |\nabla v|^{p(x)-2}\nabla v\nabla\phi\,dx + \int_\Omega \rho(x,v)\phi\,dx$$

for all $\phi \in C_0^\infty(\Omega)$, $\phi \ge 0$.

Next we give a comparison principle as follows.

Lemma 2.4 ([43, Lemma 2.3])**.** *Let $\rho(x,t)$ be a function satisfying* (2.1) *and non-decreasing in t. Let $u, v \in W^{1,p(\cdot)}(\Omega)$ satisfy*

$$-\Delta_{p(x)}u + \rho(x,u) \le -\Delta_{p(x)}v + \rho(x,v), \quad (x \in \Omega),$$

if $u \le v$ on $\partial\Omega$, then $u \le v$ in Ω.

Lemma 2.5 ([6, Theorem 8.3.1])**.** *For every $u \in W_0^{1,p(\cdot)}(\Omega)$, the inequality*

$$|u|_{p^*(\cdot)} \le C|\nabla u|_{p(\cdot)}$$

holds with a constant C depending only on the dimension N and p^+ and independent of Ω.

Lemma 2.6. *Suppose the domain Ω has finite measure, i.e. $|\Omega| < +\infty$, $p(\cdot), q(\cdot) \in C(\overline{\Omega})$, and $1 < p(x) < q(x) < N$, for all $x \in \overline{\Omega}$. Then for every $u \in L^{q(\cdot)}(\Omega)$, the following inequality holds:*

$$|u|_{p(\cdot)} \le 2|\Omega|^{\frac{1}{p(\xi)} - \frac{1}{q(\xi)}} |u|_{q(\cdot)}, \quad \text{where } \xi \in \overline{\Omega}.$$

Moreover, $|u|_{p(\cdot)} \le 2|\Omega|^{1/N} |u|_{p^*(\cdot)}$ for all $u \in W_0^{1,p(\cdot)}(\Omega)$.

The basic principle of sub-supersolution method for (1.1) can be stated as follows.

Lemma 2.7. *Suppose that* (H0) *holds and $\underline{u}, \overline{u} \in W_{\rm loc}^{1,p(\cdot)}(\Omega) \cap C_0(\overline{\Omega})$. Let $\underline{u}$ and $\overline{u}$ be a subsolution and a supersolution of* (1.1) *respectively satisfying $\underline{u} \le \overline{u}$. If $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, then* (1.1) *has a solution $u \in W_{\rm loc}^{1,p(\cdot)}(\Omega) \cap C_0(\overline{\Omega})$ satisfy $\underline{u} \le u \le \overline{u}$.*

Proof. Denote $\Omega_n = \{x \in \Omega : d(x) > 1/n\}$. Let

$$\widetilde{f}(x,u) = \begin{cases} f(x, \overline{u}), & u \ge \overline{u}, \\ f(x,u), & \underline{u} < u < \overline{u}, \\ f(x, \underline{u}), & u \le \underline{u}. \end{cases}$$

Consider

$$\begin{gathered} -\Delta_{p(x)} u = \lambda \widetilde{f}(x,u), \quad \text{in } \Omega_n, \\ u(x) > 0, \quad \text{in } \Omega_n, \\ u(x) = \underline{u}(x), \quad \text{on } \partial\Omega_n. \end{gathered} \tag{2.3}$$

Since $|\widetilde{f}(x,u)|$ is bounded on $\overline{\Omega_n}$ and $\underline{u} \in C_0(\overline{\Omega})$, it is easy to see that (2.3) has a solution u_n, satisfy $\underline{u} \le u_n \le \overline{u}$. By [11, Theorem 1.2], we can see that $\{u_n\}_{n \ge n_0+1}$ has uniformly bounded $C^{1,\alpha}$ norm on $\overline{\Omega_{n_0}}$. By the diagonal method, we can choose a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$u_{n_k}(x) \to u(x), \quad \nabla u_{n_k}(x) \to \nabla u(x), \quad \forall x \in \Omega,$$

where $u \in C_0(\overline{\Omega}) \cap C^1(\Omega)$. Thus u is a solution of (1.1) and satisfies $\underline{u} \le u \le \overline{u}$. □

3. Proofs of main results

To study the existence of solutions of (1.1), we need to do some preparation work. Note that by [10, Theorem 4.2], the following problem has a weak solution $\omega_b \in W_0^{1,p(\cdot)}(\Omega)$,

$$\begin{gathered} -\Delta_{p(x)} \omega = b(x), \quad \text{in } \Omega, \\ \omega(x) = 0, \quad \text{on } \partial\Omega. \end{gathered} \tag{3.1}$$

Since $b(\cdot)$ is nonnegative, by the comparison principle it follows that ω_b is nonnegative (see [43, Lemma 2.3]) and it is positive in Ω (see [43, Theorem 1.1]]). From [10, Theorem 4.1], we see that ω_b is bounded. Then we have $\omega_b \in C^{1,\alpha}(\overline{\Omega})$ and $\frac{\partial \omega_b}{\partial \nu} > 0$ on $\partial\Omega$ from the following Lemma.

Lemma 3.1. *(i)* [7, Theorem 1.2] *Let ω_b be a bounded solution of* (3.1)*, then $\omega_b \in C^{1,\alpha}(\overline{\Omega})$;*

(ii) [43, Theorem 1.2] *Let ω_b be a solution of* (3.1)*, $x_1 \in \partial\Omega$, $\omega_b \in C^1(\Omega \cup \{x_1\})$, $\omega_b(x_1) = 0$. If Ω satisfies the inward-ball condition at x_1, then $\frac{\partial \omega_b}{\partial \nu}(x_1) > 0$, where ν is the inward unit normal vector of $\partial\Omega$ on x_1.*

We will prove the Theorems 1.1–1.4 stated in section 1 by using Lemma 2.7. Next we will construct a supersolution of (1.1) when λ is small enough. Before we begin the proof of Theorem 1.1, we need some background.

Define

$$g_{\#}(s)=\begin{cases} g(s), & \text{when } s<1,\\ g(s), & \text{when } s\ge 1 \text{ and } \limsup_{s\to+\infty}\frac{g(s)}{s^{p^--1-\varepsilon}}<+\infty,\\ g(1)s^{p^--1-\varepsilon}, & \text{when } s\ge 1 \text{ and } \limsup_{s\to+\infty}\frac{g(s)}{s^{p^--1-\varepsilon}}=+\infty.\end{cases}$$

Without loss of generality, we assume that $g_{\#}(s)=C_*s^{p^--1-\varepsilon}$ for $s\ge 1$. There exists $M_0=M_0(\delta)$ large enough such that

$$g_{\#}(s)<\delta s^{p^--1},\quad \forall s\ge M_0. \tag{3.2}$$

Now we define a continuous function $\hat g:(0,\infty)\to(0,\infty)$ by

$$\hat g(s):=\sup\big\{\frac{g_{\#}(t)}{t^{p^--1}},t>s\big\},\quad s>0.$$

It follows from (3.2) and the definition of $\hat g$ that

(i) $\hat g$ is non-increasing;
(ii) $\hat g(s)\ge \frac{g_{\#}(s)}{s^{p^--1}}$, $s>0$;
(iii) $\hat g(s)<\delta$, for all $s\ge M_0$.

We also define a C^1-function

$$H(s):=\Big(\frac{2}{s}\int_{\frac{s}{2}}^{s}\hat g(t)dt\Big)^{\frac{1}{p^--1}},\quad s>0.$$

Lemma 3.2. *The function H satisfies*

(i) *H is strictly decreasing, and $-H'(s)\ge\frac{2^\varepsilon-1}{p^--1}\frac{H(s)}{s}$;*
(ii) *$\hat g(s)\le[H(s)]^{p^--1}\le\hat g(s/2)$, $s>0$;*
(iii) *$H(s)\to+\infty$ as $s\to0^+$, $H(s)\to0^+$, when $s\to+\infty$.*

Proof. We only need to prove $-H'(s)\ge\frac{2^\varepsilon-1}{p^--1}\frac{H(s)}{s}$, the rest is easy to be verified. By computations

$$-H'(s)=\frac{1}{p^--1}\Big(\frac{2}{s}\int_{\frac{s}{2}}^{s}\hat g(t)dt\Big)^{\frac{1}{p^--1}-1}\Big(\frac{2}{s^2}\int_{\frac{s}{2}}^{s}\hat g(t)dt+\frac{2}{s}\big(\frac12\hat g(\frac{s}{2})-\hat g(s)\big)\Big).$$

By condition (H1), when $s\le1$, $s^{p^-}\hat g(s)$ is decreasing, then we have $\frac12\hat g(\frac{s}{2})-\hat g(s)\ge0$, and then

$$-H'(s)\ge\frac{1}{p^--1}\Big(\frac{2}{s}\int_{\frac{s}{2}}^{s}\hat g(t)dt\Big)^{\frac{1}{p^--1}-1}\frac{2}{s^2}\int_{\frac{s}{2}}^{s}\hat g(t)dt\ge\frac{2^\varepsilon-1}{p^--1}\frac{H(s)}{s}.$$

Here we note that $\hat g(s)=C_*s^{-\varepsilon}$ for $s\ge1$. When $s\ge2$, we have

$$\begin{aligned}\frac{2}{s}\Big(\frac12\hat g(\frac{s}{2})-\hat g(s)\Big)&=C_*\frac{2}{s}\Big(\frac12(\frac{s}{2})^{-\varepsilon}-s^{-\varepsilon}\Big)\\&=C_*\frac{2}{s}(2^{\varepsilon-1}-1)s^{-\varepsilon}\\&=\frac{2}{s}(2^{\varepsilon-1}-1)\hat g(s)\\&\ge\frac{2}{s}(2^{\varepsilon-1}-1)\frac{2}{s}\int_{\frac{s}{2}}^{s}\hat g(t)dt\end{aligned}$$

$$
= (2^\varepsilon - 2)\frac{2}{s^2}\int_{\frac{s}{2}}^{s}\hat{g}(t)dt,
$$

and

$$
\begin{aligned}
-H'(s) &= \frac{1}{p^- - 1}\Big(\frac{2}{s}\int_{\frac{s}{2}}^{s}\hat{g}(t)dt\Big)^{\frac{1}{p^- - 1}-1}\Big(\frac{2}{s^2}\int_{\frac{s}{2}}^{s}\hat{g}(t)dt + \frac{2}{s}(\frac{1}{2}\hat{g}(\frac{s}{2}) - \hat{g}(s))\Big)\\
&\geq \frac{1}{p^- - 1}\Big(\frac{2}{s}\int_{\frac{s}{2}}^{s}\hat{g}(t)dt\Big)^{\frac{1}{p^- - 1}-1}\Big(\frac{2}{s^2}\int_{\frac{s}{2}}^{s}\hat{g}(t)dt + (2^\varepsilon - 2)\frac{2}{s^2}\int_{\frac{s}{2}}^{s}\hat{g}(t)dt\Big)\\
&= \frac{1}{p^- - 1}\Big(\frac{2}{s}\int_{\frac{s}{2}}^{s}\hat{g}(t)dt\Big)^{\frac{1}{p^- - 1}-1}\Big(\frac{2}{s^2}(2^\varepsilon - 1)\int_{\frac{s}{2}}^{s}\hat{g}(t)dt\Big)\\
&= \frac{2^\varepsilon - 1}{p^- - 1}\frac{H(s)}{s}.
\end{aligned}
$$

Note that $s^{p^-}\hat{g}(s)$ is decreasing for $s \leq 1$. When $1 < s < 2$, we have

$$
\begin{aligned}
\frac{2}{s}\big(\frac{1}{2}\hat{g}(\frac{s}{2}) - \hat{g}(s)\big) &= \frac{2}{s}\big(\frac{1}{2}(\frac{s}{2})^{-p^-}(\frac{s}{2})^{p^-}\hat{g}(\frac{s}{2}) - C_* s^{-\varepsilon}\big)\\
&\geq \frac{2}{s}\big(\frac{1}{2}(\frac{s}{2})^{-p^-}(\frac{2}{2})^{p^-}\hat{g}(\frac{2}{2}) - C_* s^{-\varepsilon}\big)\\
&= C_*\frac{2}{s}(\frac{1}{2}(\frac{s}{2})^{-p^-} - s^{-\varepsilon})\\
&= C_*\frac{1}{s}s^{-\varepsilon}(2^{p^-}s^{\varepsilon - p^-} - 2)\\
&\geq C_*\frac{1}{s}s^{-\varepsilon}(2^\varepsilon - 2)\\
&\geq (2^\varepsilon - 2)\frac{2}{s^2}\int_{\frac{s}{2}}^{s}\hat{g}(t)dt.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
-H'(s) &= \frac{1}{p^- - 1}\Big(\frac{2}{s}\int_{\frac{s}{2}}^{s}\hat{g}(t)dt\Big)^{\frac{1}{p^- - 1}-1}\Big(\frac{2}{s^2}\int_{\frac{s}{2}}^{s}\hat{g}(t)dt + \frac{2}{s}(\frac{1}{2}\hat{g}(\frac{s}{2}) - \hat{g}(s))\Big)\\
&\geq \frac{1}{p^- - 1}\Big(\frac{2}{s}\int_{\frac{s}{2}}^{s}\hat{g}(t)dt\Big)^{\frac{1}{p^- - 1}-1}\Big(\frac{2}{s^2}\int_{\frac{s}{2}}^{s}\hat{g}(t)dt + (2^\varepsilon - 2)\frac{2}{s^2}\int_{\frac{s}{2}}^{s}\hat{g}(t)dt\Big)\\
&= \frac{1}{p^- - 1}\Big(\frac{2}{s}\int_{\frac{s}{2}}^{s}\hat{g}(t)dt\Big)^{\frac{1}{p^- - 1}-1}\Big(\frac{2}{s^2}(2^\varepsilon - 1)\int_{\frac{s}{2}}^{s}\hat{g}(t)dt\Big)\\
&= \frac{2^\varepsilon - 1}{p^- - 1}\frac{H(s)}{s}.
\end{aligned}
$$

By summarizing the above discussion, we have

$$
-H'(s) \geq \frac{2^\varepsilon - 1}{p^- - 1}\frac{H(s)}{s}, \quad \forall s > 0.
$$

The proof is complete. □

As a consequence of Lemma 3.2, we can define the function

$$
\eta(s) := \int_0^s \frac{1}{H(t)}dt, s \geq 0, \tag{3.3}
$$

for it is easy to show that $\eta \in C^2(0, \infty)$.

Lemma 3.3. *The function η satisfies*

(i) $\eta : (0,\infty) \to (0,\infty)$ *is strictly increasing;*
(ii) *let $\psi = \eta^{-1}$ be the inverse function of η. Then $\psi'(s) = H(\psi(s)), s > 0$.*

Denote $\Omega_\sigma = \{x \in \Omega : \omega_b(x) < \sigma\}$, where $\sigma > 0$ is a small positive constant.

Lemma 3.4. *Assume that* (H0) *and* (H1) *hold. Then there is a supersolution v of* (1.1) *such that $v \in W^{1,p(\cdot)}_{\rm loc}(\Omega) \cap C_0(\overline{\Omega})$ when λ is small enough.*

Proof. Define

$$v(x) := \begin{cases} \psi(k_1\omega_b(x)), & x \in \Omega_\sigma, \\ \omega_b(x) + \psi(k_1\sigma) - \sigma, & x \in \Omega\backslash\Omega_\sigma, \end{cases} \tag{3.4}$$

where ω_b is given by (3.1), and $k_1 > 1$ is a constant. Obviously, $v \in C_0(\overline{\Omega}) \cap W^{1,p(\cdot)}_{\rm loc}(\Omega)$. From the definition of g and $\hat{g}$, Lemma 3.2, and Lemma 3.3, it follows that

$$\psi'(k_1\omega_b(x)) = H(\psi(k_1\omega_b(x))) = H(v(x)), x \in \Omega_\sigma,$$
$$\psi''(s) \le 0, \text{ for all } s \ge 0.$$

We will prove this Lemma in three steps.

Step 1. We will prove that v is a super-solution of (1.1) in Ω_σ; i.e.,

$$\int_\Omega |\nabla v|^{p(x)-2}\nabla v\nabla\phi\,dx \ge \int_\Omega \lambda b g(v)\phi\,dx \ge \int_\Omega \lambda f(x,v)\phi\,dx,$$

for any $\phi \in C_0^\infty(\Omega_\sigma)$ with $\phi \ge 0$ and $\operatorname{supp}\phi \subset\subset \Omega_\sigma$. By computation, we have

$$\begin{aligned}
&\int_\Omega |\nabla v|^{p(x)-2}\nabla v\nabla\phi\,dx \\
&= \int_\Omega [k_1\psi'(k_1\omega_b)]^{p(x)-1}|\nabla\omega_b|^{p(x)-2}\nabla\omega_b\nabla\phi\,dx \\
&= \int_\Omega |\nabla\omega_b|^{p(x)-2}\nabla\omega_b\nabla\left\{\phi[k_1\psi'(k_1\omega_b)]^{p(x)-1}\right\}\,dx \\
&\quad - \int_\Omega (k_1)^{p(x)}|\nabla\omega_b|^{p(x)}\phi(p(x)-1)[\psi'(k_1\omega_b)]^{p(x)-2}\psi''(k_1\omega_b)\,dx \\
&\quad - \int_\Omega (k_1)^{p(x)-1}|\nabla\omega_b|^{p(x)-2}\nabla\omega_b\nabla p[\phi\psi'(k_1\omega_b)^{p(x)-1}]\ln k_1\psi'(k_1\omega_b)\,dx.
\end{aligned}$$

By Lemma 3.2, we have $-H'(v) \ge \frac{2^\varepsilon-1}{p^--1}\frac{H(v)}{v}$ which implies

$$-\psi''(k_1\omega_b) = -H'(v)\psi'(k_1\omega_b) = -H'(v)H(v) \ge \frac{2^\varepsilon-1}{p^--1}\frac{H(v)}{v}H(v).$$

Note that $0 < c_1 \le |\nabla\omega_b| \le c_2$ on $\overline{\Omega_\sigma}$. Let σ be small enough. We can see that v is small enough in Ω_σ, and $H(v)$ is large enough in Ω_σ. Then we have

$$|\nabla\omega_b|\frac{2^\varepsilon-1}{p^--1}k_1\frac{H(v)}{v} \ge c_1\frac{2^\varepsilon-1}{p^--1}k_1\frac{H(v)}{v} \ge |\nabla p|\ln k_1 H(v).$$

By computations, for any $\phi \in C_0^\infty(\Omega)$ with $\phi \ge 0$, we have

$$- \int_\Omega (k_1)^{p(x)}|\nabla\omega_b|^{p(x)}\phi(p(x)-1)[\psi'(k_1\omega_b)]^{p(x)-2}\psi''(k_1\omega_b)\,dx$$

$$
\begin{aligned}
&= -\int_\Omega (k_1)^{p(x)}|\nabla\omega_b|^{p(x)}\phi(p(x)-1)[H(\psi(k_1\omega_b))]^{p(x)-2}\psi''(k_1\omega_b)\,dx\\
&= -\int_\Omega (k_1)^{p(x)}|\nabla\omega_b|^{p(x)}\phi(p(x)-1)[H(v)]^{p(x)-2}H'(v)\psi'(k_1\omega_b)\,dx\\
&\geq \int_\Omega (k_1)^{p(x)}|\nabla\omega_b|^{p(x)}\phi(p(x)-1)[H(v)]^{p(x)-1}\frac{2^\varepsilon-1}{p^- -1}\frac{H(v)}{v}dx\\
&\geq \int_\Omega (k_1)^{p(x)-1}|\nabla\omega_b|^{p(x)-1}|\nabla p|[\phi\psi'(k_1\omega_b)^{p(x)-1}]|\ln k_1 H(v)|\,dx.
\end{aligned}
$$

Here we note that $H(s)\to+\infty$ as $s\to 0^+$ and $v\in C_0(\overline{\Omega})$. Then $v(x)\leq 1$ in Ω_σ and $H(v(x))\geq 1$ for any $x\in\Omega_\sigma$ when σ is small enough. Thus we have

$$
\begin{aligned}
\int_\Omega |\nabla v|^{p(x)-2}\nabla v\nabla\phi\,dx &\geq \int_\Omega |\nabla\omega_b|^{p(x)-2}\nabla\omega_b\nabla\{\phi[k_1\psi'(k_1\omega_b)]^{p(x)-1}\}dx\\
&= \int_\Omega b[k_1\psi'(k_1\omega_b)]^{p(x)-1}\phi\,dx\\
&\geq \int_\Omega b[k_1H(v)]^{p^- -1}\phi\,dx\\
&\geq \int_\Omega bk_1^{p^- -1}\hat{g}(v)\phi\,dx\\
&\geq \int_\Omega bk_1^{p^- -1}\frac{g(v)}{v^{p^- -1}}\phi\,dx\\
&\geq \int_\Omega bg(v)\phi\,dx,
\end{aligned}
$$

for any $\phi\in C_0^\infty(\Omega_\sigma)$ with $\phi\geq 0$ and $\operatorname{supp}\phi\subset\subset\Omega$.

Then for any $\phi\in C_0^\infty(\Omega_\sigma)$ with $\phi\geq 0$ and $\operatorname{supp}\phi\subset\subset\Omega_\sigma$, we have

$$
\int_\Omega |\nabla v|^{p(x)-2}\nabla v\nabla\phi\,dx \geq \int_\Omega \lambda b(x)g(v)\phi\,dx \geq \int_\Omega \lambda f(x,v)\phi\,dx. \tag{3.5}
$$

Step 2. We will prove that v is a supersolution of (1.1) in $\Omega\backslash\overline{\Omega_\sigma}$; i.e.,

$$
\int_\Omega |\nabla v|^{p(x)-2}\nabla v\nabla\phi\,dx \geq \int_\Omega \lambda f(x,v)\phi\,dx, \forall\phi\in C_0^\infty(\Omega\backslash\overline{\Omega_\sigma}), \phi\geq 0.
$$

Let λ be small enough such that $\lambda g(v(x))\leq 1$, for all $x\in\Omega\backslash\overline{\Omega_\sigma}$. For any $\phi\in C_0^\infty(\Omega\backslash\overline{\Omega_\sigma})$ with $\phi\geq 0$, we have

$$
\int_\Omega |\nabla v|^{p(x)-2}\nabla v\nabla\phi\,dx = \int_\Omega b\phi\,dx \geq \int_\Omega \lambda bg(v)\phi\,dx \geq \int_\Omega \lambda f(x,v)\phi\,dx. \tag{3.6}
$$

Step 3. We will prove that v is a super-solution of (1.1) in Ω; i.e.,

$$
\int_\Omega |\nabla v|^{p(x)-2}\nabla v\nabla\phi\,dx \geq \int_\Omega \lambda f(x,v)\phi\,dx,
$$

for all $\phi\in C_0^\infty(\Omega)$ with $\phi\geq 0$ and $\operatorname{supp}\phi\subset\subset\Omega$. Denote $d_1(x)=d(x,\partial(\Omega\backslash\Omega_\sigma))$, and

$$
\xi_n(x)=\begin{cases} 2n(\frac{1}{n}-d_1(x)), & \frac{1}{2n}\leq d_1(x)\leq\frac{1}{n},\\ 1, & 0\leq d_1(x)<\frac{1}{2n},\ n\in\mathbb{N}\\ 0, & \text{other wise.}\end{cases}
$$

For $\phi \in C_0^\infty(\Omega_\sigma)$, with $\phi \geq 0$ and $\operatorname{supp}\phi \subset\subset \Omega$, we have $\phi = \phi_{1,n} + \phi_{2,n} + \phi_{3,n}$, where $\phi_{1,n} = \phi\chi_{\overline{\Omega_\sigma}}(1-\xi_n) \in C_0^\infty(\Omega)$ satisfies $\operatorname{supp}\phi_{1,n} \Subset \Omega_\sigma$, $\phi_{2,n} = \phi\xi_n \in C_0^\infty(\Omega)$, and $\phi_{3,n} = \phi\chi_{\Omega\setminus\overline{\Omega_\sigma}}(1-\xi_n) \in C_0^\infty(\Omega\setminus\overline{\Omega_\sigma})$. Therefore,

$$
\begin{aligned}
&\int_\Omega |\nabla v|^{p(x)-2}\nabla v\nabla\phi\,dx \\
&= \lim_{n\to\infty}\Big\{\int_\Omega |\nabla v|^{p(x)-2}\nabla v\nabla\phi_{1,n}dx + \int_\Omega |\nabla v|^{p(x)-2}\nabla v\nabla\phi_{2,n}dx \\
&\quad + \int_\Omega |\nabla v|^{p(x)-2}\nabla v\nabla\phi_{3,n}dx\Big\} \\
&\geq \lim_{n\to\infty}\Big\{\int_\Omega \lambda f(x,v)\phi_{1,n}dx + \int_\Omega \lambda f(x,v)\phi_{3,n}dx + \int_\Omega |\nabla v|^{p(x)-2}\nabla v\nabla\phi_{2,n}dx\Big\} \\
&= \int_{\Omega_\sigma} \lambda f(x,v)\phi\,dx + \int_{\Omega\setminus\Omega_\sigma} \lambda f(x,v)\phi\,dx + \lim_{n\to\infty}\int_\Omega |\nabla v|^{p(x)-2}\nabla v\nabla\phi_{2,n}dx \\
&= \int_\Omega \lambda f(x,v)\phi\,dx + \lim_{n\to\infty}\int_\Omega \phi|\nabla v|^{p(x)-2}\nabla v\nabla\xi_n dx \\
&= \int_\Omega \lambda f(x,v)\phi\,dx + \int_{\partial(\Omega\setminus\Omega_\sigma)} \phi[(\psi'(k_1\sigma)|\nabla k_1\omega_b|)^{p(x)-1} - |\nabla\omega_b|^{p(x)-1}]\,dS.
\end{aligned}
$$

Here we note that $\psi'(k_1\sigma) = H(k_1\sigma) \to +\infty$ as $\sigma \to 0^+$. Thus

$$
\int_{\partial(\Omega\setminus\Omega_\sigma)} \phi[(\psi'(k_1\sigma)|\nabla k_1\omega_b|)^{p(x)-1} - |\nabla\omega_b|^{p(x)-1}]\,dS \geq 0,
$$

and then

$$
\int_\Omega |\nabla v|^{p(x)-2}\nabla v\nabla\phi\,dx \geq \int_\Omega \lambda f(x,v)\phi\,dx,
$$

for all $\phi \in C_0^\infty(\Omega)$ with $\phi \geq 0$ and $\operatorname{supp}\phi \subset\subset \Omega$. It means that v is a super-solution of (1.1). The proof is complete. $\square$

Proof of Theorem 1.1. At first, we construct a subsolution for problem (1.1). Since $\partial\Omega$ is C^2 smooth, there exists a positive constant ℓ such that $d(\cdot) \in C^2(\overline{\partial\Omega_{3\ell}})$, and $|\nabla d(\cdot)| \equiv 1$, where $\overline{\partial\Omega_{3\ell}} = \{x \in \overline{\Omega} : d(x) \leq 3\ell\}$.

Let $\sigma \in (0,\ell)$ be small enough. Denote

$$
\phi(x) = \begin{cases}
e^{kd(x)} - 1, & d(x) < \sigma,\\
e^{k\sigma} - 1 + \int_\sigma^{d(x)} ke^{k\sigma}(\frac{2\ell-t}{2\ell-\sigma})^{\frac{2}{p^--1}}dt, & \sigma \leq d(x) < 2\ell,\\
e^{k\sigma} - 1 + \int_\sigma^{2\ell} ke^{k\sigma}(\frac{2\ell-t}{2\ell-\sigma})^{\frac{2}{p^--1}}dt, & 2\ell \leq d(x),
\end{cases}
$$

where $k > 0$ is a parameter. It is easy to see that $\phi \in C_0^1(\overline{\Omega})$. By computations it follows that

$$
-\Delta_{p(x)}\mu\phi = \begin{cases}
-k(k\mu e^{kd(x)})^{p(x)-1}\big[(p(x)-1) + (d(x) + \frac{\ln k\mu}{k})\nabla p(x)\nabla d(x)\\
+\frac{\Delta d(x)}{k}\big], \quad \text{if } d(x) < \sigma,\\
\Big\{\frac{1}{2\ell-\sigma}\frac{2(p(x)-1)}{p^--1} - (\frac{2\ell-d(x)}{2\ell-\sigma})[(\ln k\mu e^{k\sigma}(\frac{2\ell-d(x)}{2\ell-\sigma})^{\frac{2}{p^--1}})\nabla p(x)\nabla d(x)\\
+\Delta d(x)]\Big\}(k\mu e^{k\sigma})^{p(x)-1}(\frac{2\ell-d(x)}{2\ell-\sigma})^{\frac{2(p(x)-1)}{p^--1}-1},\\
\quad \text{if } \sigma < d(x) < 2\ell,\\
0 \quad \text{if } 2\ell < d(x).
\end{cases}
$$

Denote $\sigma = \frac{1}{k}\ln 2^{\frac{1}{p^+}}$. Then

$$e^{k\sigma} = 2^{\frac{1}{p^+}}. \tag{3.7}$$

Let

$$\mu = e^{-ak}, \quad \text{where } a = \frac{p^- - 1}{\max_{x\in\overline{\Omega}}|\nabla p| + 1}. \tag{3.8}$$

Then $k\mu \le 1$ when k is large enough. If k is sufficiently large, it is easy to see that

$$-\Delta_{p(x)}\phi \le 0 \le \lambda f(x,\phi), d(x) < \sigma. \tag{3.9}$$

Since $d(\cdot) \in C^2(\overline{\Omega_{3\ell}})$, there exists a positive constant C_3 such that $|\Delta d| \le C_3$ on $\overline{\Omega_{3\ell}}$. Note that the definition of σ means (3.7); i.e., $e^{k\sigma} = 2^{\frac{1}{p^+}}$. When k is large enough, we have

$$\begin{aligned}
&-\Delta_{p(x)}\mu\phi \\
&\le (k\mu e^{k\sigma})^{p(x)-1}\big(\frac{2\ell - d(x)}{2\ell-\sigma}\big)^{\frac{2(p(x)-1)}{p^- -1}-1}\Big|\frac{2(p(x)-1)}{(2\ell-\sigma)(p^- -1)} \\
&- \big(\frac{2\ell-d(x)}{2\ell-\sigma}\big)\big[(\ln k\mu e^{k\sigma}\big(\frac{2\ell-d(x)}{2\ell-\sigma}\big)^{\frac{2}{p^- -1}})\nabla p(x)\nabla d(x) + \Delta d(x)\big]\Big| \\
&\le C_1(k\mu)^{p(x)-1}|\ln k + \ln\mu|, \quad \sigma < d(x) < 2\ell.
\end{aligned} \tag{3.10}$$

When k is large enough, we can see that μ is small enough and moreover, $k\mu \le 1$. Combining (1.2), (3.8) and (3.10) together, we have

$$\begin{aligned}
-\Delta_{p(x)}\mu\phi &\le C_1(k\mu)^{p(x)-1}|\ln\mu| \\
&\le C_1(k\mu)^{p^- -1}|\ln\mu| \\
&\le C_1 a^{1-p^-}\mu^{p^- -1}|\ln\mu|^{p^-} \\
&\le C_2(\mu\phi)^{p^- -1}|\ln\mu\phi|^{p^-} \\
&\le \lambda f(x,\mu\phi), \sigma < d(x) < 2\ell.
\end{aligned} \tag{3.11}$$

Obviously

$$-\Delta_{p(x)}\mu\phi = 0 \le \lambda f(x,\mu\phi), 2\ell < d(x). \tag{3.12}$$

Combining (3.9), (3.11) and (3.12), we can conclude that

$$-\Delta_{p(x)}\mu\phi \le \lambda f(x,\mu\phi), \quad \text{a.e. in } \Omega. \tag{3.13}$$

Here we note that $\mu\phi = v$ on $\partial\Omega$, and $\mu\nabla\phi \le \nabla v$ near $\partial\Omega$, then $\mu\phi \le v$ on Ω when μ is small enough. By Lemma 2.7 and Lemma 3.4, (1.1) has a solution. The proof is complete. □

To prove Theorem 1.2, we need the following Lemma, which is useful for finding supersolutions of (1.1). We denote by C_0 the best embedding constant of $W_0^{1,1}(\Omega) \subset L^{\frac{N}{N-1}}(\Omega)$ (see [4, Lemma 5.2]); i. e.,

$$|u|_{L^{N/(N-1)}(\Omega)} \le C_0|\nabla u|_{L^1(\Omega)} \quad \text{for } u \in W_0^{1,1}(\Omega). \tag{3.14}$$

Lemma 3.5. *Suppose $0 \le b(\cdot) \in L^{\alpha(\cdot)}(\Omega)$, $N < \alpha(\cdot) \in C(\overline{\Omega})$. Let $M > 0$ and u be the unique solution of the problem*

$$\begin{gathered}
-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = Mb(x), \quad \text{in } \Omega, \\
u = 0, \quad \text{on } \partial\Omega.
\end{gathered} \tag{3.15}$$

Set

$$\rho_0 = \frac{p^-}{2|b(\cdot)|_{L^{\alpha^-}(\Omega)}|\Omega|^{\frac{1}{N}-\frac{1}{\alpha^-}}C_0}.$$

Then $|u|_\infty \le C^* M^{1/(p^--1)}$ *when* $M \ge \rho_0$ *and* $|u|_\infty \le C_* M^{1/(p^+-1)}$ *when* $M < \rho_0$, *where* C^* *and* C_* *are positive constants depending only on* $p^+, p^-, N, |b(\cdot)|_{L^{\alpha^-}(\Omega)}, |\Omega|$ *and* C_0.

Proof. Let u be the solution of (3.15). Then $u \ge 0$. For $k \ge 0$, set $A_k = \{x \in \Omega : u(x) > k\}$. By taking $(u-k)^+$ as a test function of (3.15), it follows from (3.14) and Young inequality that

$$\begin{aligned}
\int_{A_k} |\nabla u|^{p(x)}dx &= M\int_{A_k} b(u-k)dx \\
&\le M|b(\cdot)|_{L^N(A_k)}|(u-k)^+|_{L^{N/(N-1)}(A_k)} \\
&\le M|b(\cdot)|_{L^{\alpha^-}(A_k)}|A_k|^{\frac{1}{N}-\frac{1}{\alpha^-}}|(u-k)^+|_{L^{N/(N-1)}(A_k)} \\
&\le M|b(\cdot)|_{L^{\alpha^-}(\Omega)}|A_k|^{\frac{1}{N}-\frac{1}{\alpha^-}}C_0\int_{A_k}\varepsilon|\nabla u|\varepsilon^{-1}dx \\
&\le M|b(\cdot)|_{L^{\alpha^-}(\Omega)}|A_k|^{\frac{1}{N}-\frac{1}{\alpha^-}}C_0\int_{A_k}\Big(\frac{(\varepsilon|\nabla u|)^{p(x)}}{p(x)} + \frac{(\varepsilon^{-1})^{p^0(x)}}{p^0(x)}\Big)dx \\
&\le \frac{M|b(\cdot)|_{L^{\alpha^-}(\Omega)}|\Omega|^{\frac{1}{N}-\frac{1}{\alpha^-}}C_0}{p^-}\int_{A_k}\varepsilon^{p(x)}|\nabla u|^{p(x)}dx \\
&\quad + \frac{M|b(\cdot)|_{L^{\alpha^-}(\Omega)}|A_k|^{\frac{1}{N}-\frac{1}{\alpha^-}}C_0}{(p^+)^0}\int_{A_k}\varepsilon^{-p^0(x)}dx.
\end{aligned} \tag{3.16}$$

When $M \ge \rho_0$ we can take

$$\varepsilon = \Big(\frac{p^-}{2M|b(\cdot)|_{L^{\alpha^-}(\Omega)}|\Omega|^{\frac{1}{N}-\frac{1}{\alpha^-}}C_0}\Big)^{1/p^-} = (\frac{\rho_0}{M})^{1/p^-}, \tag{3.17}$$

then $\varepsilon \le 1$ and

$$\begin{aligned}
&\frac{M|b(\cdot)|_{L^{\alpha^-}(\Omega)}|\Omega|^{\frac{1}{N}-\frac{1}{\alpha^-}}C_0}{p^-}\int_{A_k}\varepsilon^{p(x)}|\nabla u|^{p(x)}dx \\
&\le \frac{M|b(\cdot)|_{L^{\alpha^-}(\Omega)}|\Omega|^{\frac{1}{N}-\frac{1}{\alpha^-}}C_0}{p^-}\varepsilon^{p^-}\int_{A_k}|\nabla u|^{p(x)}dx \\
&= \frac{1}{2}\int_{A_k}|\nabla u|^{p(x)}dx.
\end{aligned}$$

Consequently, from the inequality above and (3.16) it follows that

$$\begin{aligned}
\int_{A_k}|\nabla u|^{p(x)}dx &\le \frac{2M|b(\cdot)|_{L^{\alpha^-}(\Omega)}|A_k|^{\frac{1}{N}-\frac{1}{\alpha^-}}C_0}{(p^+)^0}\int_{A_k}\varepsilon^{-p^0(x)}dx \\
&\le \frac{2M|b(\cdot)|_{L^{\alpha^-}(\Omega)}C_0\varepsilon^{-(p^-)^0}}{(p^+)^0}|A_k|^{1+\frac{1}{N}-\frac{1}{\alpha^-}}.
\end{aligned} \tag{3.18}$$

Note that $b(\cdot) \geq 1$. From (3.15) and (3.18), we have

$$\int_{A_k} (u-k)dx \leq \int_{A_k} b(x)(u-k)dx = \frac{1}{M}\int_{A_k} |\nabla u|^{p(x)}dx \leq \gamma |A_k|^{1+\frac{1}{N}-\frac{1}{\alpha^-}}, \tag{3.19}$$

where

$$\gamma = \frac{2|b(\cdot)|_{L^{\alpha^-}(\Omega)} C_0 \varepsilon^{-(p^-)^0}}{(p^+)^0}. \tag{3.20}$$

By the [27, Lemma 5.1, Chapter 2], (3.19) implies

$$|u|_\infty \leq \gamma\Big(\frac{\alpha^- N}{\alpha^- - N}+1\Big)^2 |\Omega|^{\frac{1}{N}-\frac{1}{\alpha^-}}. \tag{3.21}$$

From (3.17), (3.20) and (3.21), we see that

$$|u|_\infty \leq C^* M^{1/(p^- -1)}, \tag{3.22}$$

where

$$C^* = \frac{\big(\frac{\alpha^- N}{\alpha^- - N}+1\big)^2 \big(2C_0|b(\cdot)|_{L^{\alpha^-}(\Omega)}|\Omega|^{\frac{1}{N}-\frac{1}{\alpha^-}}\big)^{(p^-)^0}}{(p^+)^0 (p^-)^{(p^-)^0/p^-}}. \tag{3.23}$$

When $M < \rho_0$, take

$$\varepsilon = \Big(\frac{p^-}{2M|b(\cdot)|_{L^{\alpha^-}(\Omega)}|\Omega|^{\frac{1}{N}-\frac{1}{\alpha^-}} C_0}\Big)^{1/p^+} = \Big(\frac{\rho_0}{M}\Big)^{1/p^+}.$$

Note that in this case $\varepsilon > 1$. Using similar arguments as above we obtain

$$|u|_\infty \leq C_* M^{1/(p^+ -1)},$$

where

$$C_* = \frac{\big(\frac{\alpha^- N}{\alpha^- - N}+1\big)^2 \big(2C_0|b(\cdot)|_{L^{\alpha^-}(\Omega)}|\Omega|^{\frac{1}{N}-\frac{1}{\alpha^-}}\big)^{(p^+)^0}}{(p^+)^0 (p^-)^{(p^+)^0/p^+}}.$$

The proof is complete. □

Lemma 3.6. *Suppose there is a small $\delta > 0$ such that $p(x) \equiv p$ (a constant) for any $x \in \Omega$ with $d(x) \leq \delta$ and $N < \alpha(\cdot) \in C(\overline{\Omega})$. Let $M > 1$ and u be the unique solution of the problem*

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = Mb(x), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{3.24}$$

where $0 \leq b(\cdot) \in L^{\alpha(\cdot)}(\Omega)$. Then $|\nabla u(\cdot)| \leq CM^{\frac{1}{p^- -1}}$ on $\partial\Omega$.

Proof. By Lemma 3.5, we have $u(x) \leq C_\# M^{\frac{1}{p^- -1}}$ for all $x \in \Omega$. Let u_2 be the solution of the following p-Laplacian equation (with constant exponent)

$$\begin{gathered} -\operatorname{div}(|\nabla u_2|^{p-2}\nabla u_2) = \varkappa b(x), \quad \text{in } \Omega, \\ u_2 = 0, \quad \text{on } \partial\Omega, \end{gathered}$$

where $\varkappa$ is a positive parameter.

It is easy to see that $u_2 \in C^{1,\alpha}(\overline{\Omega})$. Then $\frac{\partial u_2}{\partial \nu} > 0$ on $\partial\Omega$, where ν is the inward unit normal vector. We can also see that $u_2 > 0$ on $\partial(\Omega \backslash \overline{\Omega_\varepsilon})$ when $\varepsilon \in (0, \delta)$ is small enough. Let $\varkappa$ be large enough, we have $u_2 \geq 2C_\#$ on $\partial(\Omega \backslash \overline{\Omega_\varepsilon})$. It means that $u_2M^{\frac{1}{p^--1}} \geq u$ on $\partial\Omega_\varepsilon$. Define $u_3 = u_2M^{\frac{1}{p^--1}}$. Since $p(x) \equiv p$ for any $x \in \Omega$ with $d(x) \leq \delta$, we have

$$-\operatorname{div}(|\nabla u_3|^{p(x)-2}\nabla u_3) = -\operatorname{div}(|\nabla u_3|^{p-2}\nabla u_3) = M^{\frac{p-1}{p^--1}}\varkappa b(x) \geq Mb(x) \quad \text{in } \Omega_\varepsilon.$$

Therefore, $u_3 = u_2M^{\frac{1}{p^--1}} \geq u$ on $\overline{\Omega_\varepsilon}$ and $|\nabla u| \leq |\nabla u_3| \leq CM^{\frac{1}{p^--1}}$ on $\partial\Omega$. The proof is complete. $\square$

Proof of Theorem 1.2. At first, we construct a supersolution of (1.1). Denote $k_2 = \psi(k_1\sigma)$. Let ω_2 be the solution of the problem

$$\begin{gathered}
-\Delta_{p(x)}\omega_2 = b(x)k_2^{p^--1-\frac{\varepsilon}{2}}, \quad \text{in } \Omega\backslash\overline{\Omega_\sigma}, \\
\omega_2 > 0, \quad \text{in } \Omega\backslash\overline{\Omega_\sigma}, \\
\omega_2 = 0, \quad \text{on } \partial(\Omega\backslash\overline{\Omega_\sigma}).
\end{gathered}$$

Define

$$v_2(x) := \begin{cases} \psi(k_1\omega_b(x)), & x \in \Omega_\sigma, \\ \omega_2(x) + \psi(k_1\sigma), & x \in \Omega\backslash\Omega_\sigma. \end{cases} \tag{3.25}$$

For a large enough constant k_1, we will prove that v_2 is a supersolution of (1.1) in three steps.

Step 1. When k_1 is large enough, we will check that v_2 is a supersolution of (1.1) in Ω_σ, namely,

$$\int_\Omega |\nabla v_2|^{p(x)-2}\nabla v_2\nabla\phi\,dx \geq \int_\Omega \lambda bg(v_2)\phi\,dx \geq \int_\Omega \lambda f(x,v_2)\phi\,dx,$$

for any $\phi \in C_0^\infty(\Omega_\sigma)$ with $\phi \geq 0$ and $\operatorname{supp}\phi \subset\subset \Omega_\sigma$. As in the proof of Lemma 3.4, we only need to prove that

$$\begin{aligned}
&\int_\Omega (k_1)^{p(x)}|\nabla\omega_b|^{p(x)}\phi(p(x)-1)[H(v_2)]^{p(x)-1}\frac{2^\varepsilon-1}{p^--1}\frac{H(v_2)}{v_2}dx \\
&\geq \int_\Omega (k_1)^{p(x)-1}|\nabla\omega_b|^{p(x)-1}|\nabla p|[\phi\psi'(k_1\omega_b)^{p(x)-1}]|\ln k_1H(v_2)|\,dx,
\end{aligned} \tag{3.26}$$

$$k_1H(v_2(x)) \geq 1, \forall x \in \Omega_\sigma, \tag{3.27}$$

$$\frac{k_1^{p^--1}}{[v_2(x)]^{p^--1}} \geq 1, \forall x \in \Omega_\sigma. \tag{3.28}$$

We can see that (3.26) is valid, provided

$$|\nabla\omega_b|\frac{2^\varepsilon-1}{p^--1}k_1\frac{H(v_2)}{v_2} \geq c_1\frac{2^\varepsilon-1}{p^--1}k_1\frac{H(v_2)}{v_2} \geq |\nabla p||\ln k_1H(v_2)| \quad \text{in } \Omega_\sigma. \tag{3.29}$$

According to the assumption on g, without loss of generality, we assume that

$$g(s) \geq cs^{-1} \text{ for } s \leq 1, \quad \text{and} \quad g(s) = cs^\theta \text{ for } s \geq 1,$$

where $\theta = p^- - 1 - \varepsilon$. Thus

$$\hat{g}(s) \geq cs^{-p^-} \text{ for } s \leq 1, \quad \text{and} \quad \hat{g}(s) = cs^{\theta+1-p^-} \text{ for } s \geq 1,$$

$$H(s) \geq c_1s^{-\frac{p^-}{p^--1}} \text{ for } s \leq 1, \quad \text{and} \quad H(s) = c_2s^{\frac{\theta+1-p^-}{p^--1}} \text{ for } s \geq 2,$$

$$\eta(s) \leq c_3s^{1+\frac{p^-}{p^--1}} \text{ for } s \leq 1, \quad \text{and} \quad c_4s^{2-\frac{\theta}{p^--1}} \leq \eta(s) \leq c_5s^{2-\frac{\theta}{p^--1}} \text{ for } s \geq 3.$$

Then $\psi(s)$ satisfies

$$c_7 s^{\frac{1}{2-\frac{\theta}{p^--1}}} \le \psi(s) \le c_8 s^{\frac{1}{2-\frac{\theta}{p^--1}}} \quad \text{for } s \ge 3.$$

Let $s_0 \ge 3$ such that $\eta(s_0) \ge 3$. Denote

$$\Omega_\sigma^+ = \{x \in \Omega_\sigma : k_1\omega_b(x) \ge \eta(s_0)\}, \quad \Omega_\sigma^- = \{x \in \Omega_\sigma : k_1\omega_b(x) < \eta(s_0)\}.$$

Here we note that $v_2 = \psi(k_1\omega_b)$ on $\overline{\Omega_\sigma}$. Since ψ is strictly increasing, we have $k_1\omega_b \ge \eta(s_0)$ if and only if $v_2 = \psi(k_1\omega_b) \ge \psi(\eta(s_0)) = s_0 \ge 3$. When $v_2 \ge s_0$, we have

$$\begin{gathered}
c_7(k_1\omega_b)^{\frac{1}{2-\frac{\theta}{p^--1}}} \le v_2 \le c_8(k_1\omega_b)^{\frac{1}{2-\frac{\theta}{p^--1}}} \quad \text{on } \Omega_\sigma^+,\\
c_9(k_1\omega_b)^{\frac{1}{2-\frac{\theta}{p^--1}}\frac{\theta+1-p^-}{p^--1}} \le H(v_2) \le c_{10}(k_1\omega_b)^{\frac{1}{2-\frac{\theta}{p^--1}}\frac{\theta+1-p^-}{p^--1}} \quad \text{on } \Omega_\sigma^+,\\
|\nabla\omega_b|\frac{2^\varepsilon-1}{p^--1}k_1\frac{H(v_2)}{v_2} \ge c_{11}k_1(k_1\omega_b)^{\frac{1}{2-\frac{\theta}{p^--1}}(\frac{\theta+1-p^-}{p^--1}-1)} = \frac{c_{11}}{\omega_b} \ge \frac{c_{11}}{\sigma} \text{ on } \Omega_\sigma^+,
\end{gathered}$$

$$\begin{aligned}
|\nabla p|\|\ln k_1 H(v_2)| &\le |\nabla p|(\ln k_1 + |\ln H(v_2)|)\\
&\le |\nabla p|(\ln k_1 + c_{12}|\ln k_1\omega_b|) \le c_{13}\ln k_1 \quad \text{on } \Omega_\sigma^+.
\end{aligned}$$

Denoting $\sigma = \frac{c_{11}}{c_{13}\ln k_1}$, we obtain

$$|\nabla\omega_b|\frac{2^\varepsilon-1}{p^--1}k_1\frac{H(v_2)}{v_2} \ge c_1\frac{2^\varepsilon-1}{p^--1}k_1\frac{H(v_2)}{v_2} \ge |\nabla p|\|\ln k_1 H(v_2)| \quad \text{on } \Omega_\sigma^+.$$

Since ψ is strictly increasing, we have $k_1\omega_b \le \eta(s_0)$ if and only if $v_2 = \psi(k_1\omega_b) \le s_0$. Note that $H(v_2)$ is decreasing. It follows that $H(v_2) \ge H(s_0)$ on Ω_σ^-. Thus (3.29) is valid when k_1 is large enough. Thus (3.29) is valid, and then (3.26) is valid.

Obviously,

$$\begin{aligned}
k_1 H(v_2(x)) &\ge k_1 H(\psi(k_1\sigma))\\
&\ge k_1 c_9\Big(k_1\frac{c_{11}}{c_{13}\ln k_1}\Big)^{\frac{1}{2-\frac{\theta}{p^--1}}\frac{\theta+1-p^-}{p^--1}}\\
&= c_9(k_1)^{\frac{1}{2-\frac{\theta}{p^--1}}}\Big(\frac{c_{11}}{c_{13}\ln k_1}\Big)^{\frac{1}{2-\frac{\theta}{p^--1}}\frac{\theta+1-p^-}{p^--1}} \to +\infty,
\end{aligned}$$

for all $x \in \Omega_\sigma$ as $k_1 \to +\infty$. Thus (3.27) is valid.

Note that $\frac{\theta}{p^--1} < 1$. Then by the above computation,

$$v_2 \le c_8(k_1\omega_b)^{\frac{1}{2-\frac{\theta}{p^--1}}} \le c_8(k_1\sigma)^{\frac{1}{2-\frac{\theta}{p^--1}}} \le k_1$$

as k_1 is large enough. Thus (3.28) is valid.

Step 2. We will check that v_2 is a supersolution of (1.1) in $\Omega\backslash\overline{\Omega_\sigma}$ when k_1 is large enough; i.e.,

$$\int_\Omega |\nabla v_2|^{p(x)-2}\nabla v_2\nabla\phi\,dx \ge \int_\Omega \lambda b g(v_2)\phi\,dx \ge \int_\Omega \lambda f(x,v_2)\phi\,dx,$$

for all $\phi \in C_0^\infty(\Omega\backslash\overline{\Omega_\sigma})$, $\phi \ge 0$. By the definition of ω_2 and Lemmas 3.5 and 3.6, we have

$$\omega_2 \le C_1(k_2)^{\frac{p^--1-\frac{\varepsilon}{2}}{p^--1}}, \quad |\nabla\omega_2| \le C_2(k_2)^{\frac{p^--1-\frac{\varepsilon}{2}}{p-1}}.$$

Since $v_2 = \omega_2 + \psi(k_1\sigma)$ in $\Omega\backslash\overline{\Omega_\sigma}$, we have

$$c_7(k_1\sigma)^{\frac{1}{2-\frac{\theta}{p^--1}}} \leq \psi(k_1\sigma) \leq v_2 \leq C_1(k_2)^{\frac{p^--1-\frac{\varepsilon}{2}}{p^--1}} + \psi(k_1\sigma) \leq (C_1+1)\psi(k_1\sigma).$$

Since $k_1\sigma = \frac{c_{11}k_1}{c_{13}\ln k_1}$ is large enough (as long as k_1 is large enough) and $v_2(\cdot)$ is large enough in $\Omega\backslash\overline{\Omega_\sigma}$, the assumption (ii) of Theorem 1.2 implies that $\frac{g(v_2(x))}{[v_2(x)]^{p^--1-\frac{\varepsilon}{2}}}$ is small enough. Therefore, we see that

$$\frac{1}{\lambda} > \frac{g(v_2(x))}{[v_2(x)]^{p^--1-\frac{\varepsilon}{2}}}.$$

Note that $k_2 = \psi(k_1\sigma)$. We have

$$v_2(x)^{p^--1-\frac{\varepsilon}{2}} \leq [C_1(k_2)^{\frac{p^--1-\frac{\varepsilon}{2}}{p^--1}} + \psi(k_1\sigma)]^{p^--1-\frac{\varepsilon}{2}} \leq C_3k_2^{p^--1-\frac{\varepsilon}{2}}, \quad \forall x \in \Omega\backslash\overline{\Omega_\sigma},$$

which implies

$$(k_2)^{p^--1-\frac{\varepsilon}{2}} \geq C_3\lambda(k_2)^{p^--1-\frac{\varepsilon}{2}}\frac{g(v_2(x))}{[v_2(x)]^{p^--1-\frac{\varepsilon}{2}}} \geq \lambda g(v_2(x)), \quad \forall x \in \Omega\backslash\overline{\Omega_\sigma}.$$

We can see that v_2 is a supersolution of (1.1) in $\Omega\backslash\overline{\Omega_\sigma}$; i.e., for any $\phi \in C_0^\infty(\Omega\backslash\overline{\Omega_\sigma})$ with $\phi \geq 0$, we have

$$\begin{aligned}
\int_\Omega |\nabla v_2|^{p(x)-2}\nabla v_2\nabla\phi\,dx &= \int_\Omega bk_2^{p^--1-\frac{\varepsilon}{2}}\phi\,dx \\
&\geq \int_\Omega \lambda bC_3k_2^{p^--1-\frac{\varepsilon}{2}}\frac{g(v)}{{v_2}^{p^--1-\frac{\varepsilon}{2}}}\phi\,dx \\
&\geq \int_\Omega \lambda f(x,v_2)\phi\,dx.
\end{aligned}$$

Step 3. When k_1 is large enough, we will prove that v_2 is a supersolution of (1.1) in Ω. When $\omega_b(x) = \sigma$, it is easy to check that

$$\begin{aligned}
k_1\psi'(k_1\omega_b) &= k_1H(v_2) \\
&\geq k_1c_9(k_1\omega_b)^{\frac{1}{2-\frac{\theta}{p^--1}}\frac{\theta+1-p^-}{p^--1}} \\
&= c_9(k_1)^{\frac{1}{2-\frac{\theta}{p^--1}}}\sigma^{\frac{1}{2-\frac{\theta}{p^--1}}\frac{\theta+1-p^-}{p^--1}} \\
&= c_9(k_1)^{\frac{1}{2-\frac{\theta}{p^--1}}}\Big(\frac{c_{11}}{c_{13}\ln k_1}\Big)^{\frac{1}{2-\frac{\theta}{p^--1}}\frac{\theta+1-p^-}{p^--1}}.
\end{aligned}$$

Then

$$|\nabla\omega_2| \leq C(k_2)^{\frac{p^--1-\frac{\varepsilon}{2}}{p^--1}} \leq C\Big(\frac{c_{11}k_1}{c_{13}\ln k_1}\Big)^{\frac{1}{2-\frac{\theta}{p^--1}}\frac{p^--1-\frac{\varepsilon}{2}}{p^--1}} < |\nabla\psi(k_1\omega_b)|$$

as $k_1 \to +\infty$. Thus we know that

$$(k_1\psi'(k_1\omega_b(x))|\nabla\omega_b(x)|)^{p(x)-1} - |\nabla\omega_2(x)|^{p(x)-1} > 0, \quad \text{when } \omega_b(x) = \sigma.$$

Therefore, when $\sigma = c_{11}/(c_{13}\ln k_1)$ and k_1 is large enough, similar argument as to the step 3 of the proof of Lemma 3.4 implies v_2 is a supersolution of (1.1).

It is easy to see that $\mu\phi$ defined in the proof of Theorem 1.1 is a subsolution of (1.1) and $\mu\phi \le v_2$ when μ is small enough. By Lemma 2.7, we can get the existence of a solution to (1.1). The proof is complete. $\square$

Proof of Theorem 1.3. At first, similar to the proof of Theorem 1.2, we will prove that v_2 defined by (3.25) is also a supersolution of (1.1) for a large enough constant k_1.

Similar to the proof of Theorem 1.2, we consider the solution ω_2 of the problem

$$\begin{gathered} -\Delta_{p(x)}\omega_2 = b(x)k_2^{p^- - 1 - \frac{\varepsilon}{2}}, \quad \text{in } \Omega\backslash\overline{\Omega_\sigma}, \\ \omega_2(x) > 0, \quad \text{in } \Omega\backslash\overline{\Omega_\sigma}, \\ \omega_2(x) = 0, \quad \text{on } \partial(\Omega\backslash\overline{\Omega_\sigma}), \end{gathered} \tag{3.30}$$

where $k_2 = \psi(k_1\sigma)$. We have

$$\omega_2 \le C_1(k_2)^{\frac{p^- - 1 - \frac{\varepsilon}{2}}{p^- - 1}}.$$

Next we only need to prove that

$$|\nabla\omega_2| \le C_2(k_2)^{\frac{p^- - 1 - \frac{\varepsilon}{2}}{p^- - 1}}.$$

Now we consider $(\gamma k_2)^{\frac{p^- - 1 - \frac{\varepsilon}{2}}{p^- - 1}}\omega_b$, where $\gamma \ge 1$ is a constant. Here we note that $\nabla\omega_b \cdot \nu = |\nabla\omega_b|$ on $\partial(\Omega\backslash\overline{\Omega_\sigma})$ and $\nabla p \cdot \nu < 0$ on $\partial(\Omega\backslash\overline{\Omega_\sigma})$, where ν is the inward unit normal vector on $\partial(\Omega\backslash\overline{\Omega_\sigma})$. There exists a small enough positive constant $\delta > 0$ such that $\nabla\omega_b\nabla p < 0$ in $(\Omega\backslash\overline{\Omega_\sigma})^{\#}_{\delta} := \{x \in \Omega\backslash\overline{\Omega_\sigma} : d(x, \partial(\Omega\backslash\overline{\Omega_\sigma})) < \delta\}$. By computations it follows that

$$\begin{aligned} &-\Delta_{p(x)}(\gamma k_2)^{\frac{p^- - 1 - \frac{\varepsilon}{2}}{p^- - 1}}\omega_b \\ &= (\gamma k_2)^{\frac{p^- - 1 - \frac{\varepsilon}{2}}{p^- - 1}(p(x)-1)}(-\Delta_{p(x)}\omega_b - \frac{p^- - 1 - \frac{\varepsilon}{2}}{p^- - 1}|\nabla\omega_b|^{p(x)-2}\nabla\omega_b\nabla p\ln\gamma k_2) \\ &\ge (\gamma k_2)^{\frac{p^- - 1 - \frac{\varepsilon}{2}}{p^- - 1}(p(x)-1)}(-\Delta_{p(x)}\omega_b) \\ &\ge b(\gamma k_2)^{p^- - 1 - \frac{\varepsilon}{2}} \text{ in } (\Omega\backslash\overline{\Omega_\sigma})^{\#}_{\delta}. \end{aligned}$$

Since ω_b is positive and continuous, there exists a large enough positive γ such that $\gamma^{\frac{p^- - 1 - \frac{\varepsilon}{2}}{p^- - 1}}\omega_b > 2C_1$ for $d(x, \partial(\Omega\backslash\overline{\Omega_\sigma})) = \delta$. Therefore $(\gamma k_2)^{\frac{p^- - 1 - \frac{\varepsilon}{2}}{p^- - 1}}\omega_b$ is a supersolution of (3.30) in $(\Omega\backslash\overline{\Omega_\sigma})^{\#}_{\delta}$. By the comparison principle, we have $(\gamma k_2)^{\frac{p^- - 1 - \frac{\varepsilon}{2}}{p^- - 1}}\omega_b \ge \omega_2$ in $(\Omega\backslash\overline{\Omega_\sigma})^{\#}_{\delta}$, and then

$$|\nabla\omega_2| \le |(\gamma k_2)^{\frac{p^- - 1 - \frac{\varepsilon}{2}}{p^- - 1}}\nabla\omega_b| \le C_2(k_2)^{\frac{p^- - 1 - \frac{\varepsilon}{2}}{p^- - 1}} \quad \text{on } \partial(\Omega\backslash\overline{\Omega_\sigma}).$$

Note that

$$\max_{x \in \overline{\Omega\backslash\Omega_\sigma}} \omega_2(x) \le C_3(k_2)^{\frac{p^- - 1 - \frac{\varepsilon}{2}}{p^- - 1}}$$

and $k_2 = \psi(k_1\sigma)$. Since $v_2 = \omega_2 + \psi(k_1\sigma)$ in $\Omega\backslash\overline{\Omega_\sigma}$, we have

$$\begin{aligned} c_7(k_1\sigma)^{\frac{1}{2 - \frac{\theta}{p^- - 1}}} &\le \psi(k_1\sigma) \le v_2 \\ &\le \max_{x \in \overline{\Omega\backslash\Omega_\sigma}} \omega_2(x) + \psi(k_1\sigma) \end{aligned}$$

$$\le C_3(k_2)^{\frac{p^- -1-\frac{\varepsilon}{2}}{p^- -1}} + \psi(k_1\sigma) \le C_4\psi(k_1\sigma).$$

As in the proof of Theorem 1.2, we can see that the v_2 defined in the proof of Theorem 1.2 is a supersolution of (1.1).

It is easy to see that $\mu\phi$ defined in the proof of Theorem 1.1 is a subsolution of (1.1), and $\mu\phi \le v_2$ when μ is small enough. By Lemma 2.7, we obtain the existence of solution of (1.1). The proof is complete. □

Proof of Theorem 1.4. The proof is similar to that of Theorem 1.2. We will prove that v_2 defined in (3.25) is a supersolution of (1.1) for a large enough constant k_1.

Since (1.1) is radial, we may assume the both solutions $\omega_b(\cdot)$ and $\omega_2(\cdot)$ are radial. We only need to prove that v_2 defined in (3.25) is also a supersolution of (1.1) for a large enough constant k_1. Since ω_2 is radial, it is easy to see that

$$\omega_2 \le C_1(k_2)^{\frac{p^- -1-\frac{\varepsilon}{2}}{p^- -1}}, |\nabla\omega_2| \le C_2(k_2)^{\frac{p^- -1-\frac{\varepsilon}{2}}{p^- -1}}.$$

Similar to the proof of Theorem 1.2, we can see that the v_2 defined in (3.25) is a supersolution of (1.1). The proof is complete. □

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