Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 156, pp. 1–39.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

NODAL SOLUTIONS FOR SINGULAR SECOND-ORDER BOUNDARY-VALUE PROBLEMS

ABDELHAMID BENMEZAÏ, WASSILA ESSERHANE, JOHNNY HENDERSON

Abstract. We use a global bifurcation theorem to prove the existence of nodal solutions to the singular second-order two-point boundary-value problem

$$\begin{gathered}-(pu')'(t) = f(t, u(t)) \quad t \in (\xi, \eta),\\ au(\xi) - b \lim_{t\to\xi} p(t)u'(t) = 0,\\ cu(\eta) + d \lim_{t\to\eta} p(t)u'(t) = 0,\end{gathered}$$

where ξ, η, a, b, c, d are real numbers with $\xi < \eta$, $a, b, c, d \geq 0$, $p : (\xi, \eta) \to [0, +\infty)$ is a measurable function with $\int_\xi^\eta 1/p(s)\,ds < \infty$ and $f : [\xi, \eta] \times [0, +\infty) \to [0, +\infty)$ is a Carathéodory function.

1. Introduction

Many articles concerning the existence of nodal solutions for second-order differential equations subject to various boundary conditions, have appeared during the previous five decades; see for example [4, 5, 8, 10, 11, 12, 16, 17, 18, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29, 30, 31] and references therein.

Ma and Thompson [19, 20, 21] considered the boundary-value problem (bvp for short),

$$\begin{gathered}-u'' = a(t)f(u), \quad t \in (0, 1),\\ u(0) = u(1) = 0\end{gathered} \tag{1.1}$$

where $a : [0, 1] \to [0, +\infty)$ is continuous and does not vanish identically, and $f : \mathbb{R} \to \mathbb{R}$ is continuous with $f(s)s > 0$ for $s \neq 0$. They proved also, that bvp (1.1) admits $2k$ nodal solutions when the interval whose extremities are $\lim_{u\to0} f(u)/u$ and $\lim_{|u|\to+\infty} f(u)/u$ contains k eigenvalues of the linear bvp associated with (1.1),

$$\begin{gathered}-u'' = \lambda a(t)u, \quad t \in (0, 1),\\ u(0) = u(1) = 0.\end{gathered}$$

2000 *Mathematics Subject Classification.* 34B15, 34B16, 34B18.
Key words and phrases. Singular second-order BVPs; nodal solutions; global bifurcation theorem.

Submitted January 27, 2014. Published July 7, 2014.

Articles [17] and [29] were devoted to the multipoint bvp,

$$\begin{gathered} -u'' = f(u), \quad t \in (0,1), \\ u(0) = 0,\ u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \end{gathered} \tag{1.2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is C^1 with $f(0) = 0$, $m \geq 3$, $\eta_i \in (0,1)$ and $\alpha_i > 0$ for $i = 1, \dots, m-2$ with $\sum_{i=1}^{m-2} \alpha_i < 1$, by which Rynne [29] extended the result and filled some gaps in [17].

Roughly speaking, Rynne proved that bvp (1.2) admits $2k$ nodal solutions when the interval whose extremities are $\lim_{u\to 0} f(u)/u$ and $\lim_{|u|\to+\infty} f(u)/u$ contains k eigenvalues of the linear bvp associated with (1.2),

$$\begin{gathered} -u'' = \lambda u, \quad t \in (0,1), \\ u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i). \end{gathered}$$

This result was extended by Genoud and Rynne in [12] to the case with variable coefficients.

Existence and multiplicity of positive solutions for second order bvps having singular dependence on the independent variable, have been considered in many papers; see, for example, [1, 3, 7, 9, 13, 14, 15, 32, 33, 34] and references therein. In particular, it is proved in [9, 15, 32] that, if the function a in bvp (1.1) is just continuous on $(0,1)$ and satisfies

$$\int_0^1 t(1-t)a(t)dt < \infty, \tag{1.3}$$

then (1.1) admits one or more positive solutions under some additional conditions on the behavior of the ratio $f(u)/u$ at 0 and $+\infty$. A natural question becomes,

> Is it possible to obtain existence results for nodal solutions to bvp (1.1) under Hypothesis (1.3)?

So, the main goal of this paper is to give an answer to this question.

In fact, we will give an answer for a more general bvp having a nonlinearity more general than (1.1), under a hypothesis looking like (1.3). This answer will be based on the knowledge of the spectrum of the linear problem associated with the nonlinear bvp. This was the case also for all the works in [4, 11, 12, 16, 17, 18, 19, 20, 21, 27, 29, 30, 31].

We need also in this work to introduce the concept of half-eigenvalue which generalizes the notion of eigenvalue. The definition of half-eigenvalue here is not the same given by Berysticki (see Remark 3.8), and for the role that will be played by this notion, we refer the reader to [4, 6, 11, 28, 29, 30].

A typical example of a weight function satisfying Hypothesis (1.3) is $a(t) = t^{-3/2}(1-t)^{-3/2}$. Note that such a weight a is not integrable near 0 and 1. We have a similar situation in this work and this causes many difficulties in proving existence of half-eigenvalues as well as in proving the main results of this paper. The existence of half-eigenvalues will be obtained by sequential arguments. We will use in this work, the global bifurcation theorem of Rabinowitz to obtain our main results.

2. Main results

This article concerns the existence of nodal solutions for the bvp,

$$\begin{gathered} -(pu')'(t) = f(t, u(t)), \text{ a.e. } t \in (\xi, \eta) \\ au(\xi) - b \lim_{t\to\xi} p(t)u'(t) = 0, \\ cu(\eta) + d \lim_{t\to\eta} p(t)u'(t) = 0, \end{gathered} \tag{2.1}$$

where $\xi, \eta \in \mathbb{R}$ with $\xi < \eta$, $a, b, c, d \in \mathbb{R}^+ = [0, +\infty)$, $p : (\xi, \eta) \to \mathbb{R}^+$ is a measurable function and $f : (\xi, \eta) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function ($f(\cdot, u)$ is measurable for u fixed and $f(t, \cdot)$ is continuous for $t \in (\xi, \eta)$ a.e.).

Throughout this article, we assume that

$$\int_\xi^\eta \frac{d\tau}{p(\tau)} < \infty, \tag{2.2}$$

$$\Delta = ad + ac \int_\xi^\eta \frac{d\tau}{p(\tau)} + bc > 0. \tag{2.3}$$

Let

$$L_G^1[\xi, \eta] = \big\{q : (\xi, \eta) \to \mathbb{R} \text{ measurable}, \int_\xi^\eta G(t, t)|q(t)|dt < \infty\big\}$$

and let K_G be the cone of all functions $q \in L_G^1[\xi, \eta]$ such that $q(t) \geq 0$ a.e. $t \in [\xi, \eta]$ and $q > 0$ in a subset of a positive measure of $[\xi, \eta]$ where

$$G(t, s) = \frac{1}{\Delta}\begin{cases} \Phi_{ab}(s)\Psi_{cd}(t), & \xi \leq s \leq t \leq \eta, \\ \Phi_{ab}(t)\Psi_{cd}(s), & \xi \leq t \leq s \leq \eta. \end{cases}$$

is the Green's function associated with the bvp

$$\begin{gathered} -(pu')'(t) = 0, \text{ a.e.}\, t \in (\xi, \eta), \\ au(\xi) - b \lim_{t\to\xi} p(t)u'(t) = 0, \\ cu(\eta) + d \lim_{t\to\eta} p(t)u'(t) = 0, \end{gathered}$$

and the functions $\Phi_{ab}(t) = b + a\int_\xi^t 1/p(\tau)\, d\tau$ and $\Psi_{cd}(t) = d + c\int_t^\eta 1/p(\tau)\, d\tau$ are well defined on $[\xi, \eta]$.

Note that the space $L_G^1\ [\xi, \eta]$ depends on the parameters b and d. In fact, we have that $L_G^1[\xi, \eta] = L^1[\xi, \eta]$ if $bd \neq 0$ and $L_G^1[\xi, \eta]\backslash L^1[\xi, \eta]$ is nonempty if $bd = 0$. More precisely, we have that $q \in L_G^1\ [\xi, \eta]$ is not integrable at ξ if and only if $b = 0$ and q is not integrable at η if and only if $d = 0$. For example, if $p = 1$ and $b = d = 0$ the function $q(s) = (s(1-s))^{-3/2} \in L_G^1[\xi, \eta]\backslash L^1[\xi, \eta]$. Moreover, we have that $L_G^1[\xi, \eta] \subset L^1_{\rm loc}(\xi, \eta)$.

The main result of this article (Theorem 2.9) will be obtained under the following additional conditions on the nonlinearity f:

There exist functions $\alpha_\infty, \beta_\infty, \gamma_\infty, \delta_\infty$ and q_0 in K_G such that the set

$$\{t \in (\xi, \eta) : \alpha_\infty(t)\beta_\infty(t) > 0\}$$

is of a positive measure,

$$\begin{aligned} \lim_{u\to 0}\frac{f(t,u)}{u} &= q_0(t) \quad \text{for } t\in[\xi,\eta] \text{ a.e.},\\ \lim_{u\to -\infty}\frac{f(t,u)}{u} &= \beta_\infty(t) \quad \text{for } t\in[\xi,\eta] \text{ a.e.},\\ \lim_{u\to +\infty}\frac{f(t,u)}{u} &= \alpha_\infty(t) \quad \text{for } t\in[\xi,\eta] \text{ a.e.} \end{aligned} \tag{2.4}$$

and

$$\delta_\infty(t)\le \frac{f(t,u)}{u}\le \gamma_\infty(t) \quad \text{for all } u\in\mathbb{R} \text{ and } t\in[\xi,\eta] \text{ a.e.} \tag{2.5}$$

From all the above hypotheses, we understand that a solution to bvp (2.1) is a function $u\in C[\xi,\eta]\cap C^1(\xi,\eta)$ with $(pu')'\in L^1_G[\xi,\eta]$, satisfying all equations in (2.1).

Remark 2.1. Note that Hypothesis (2.5) implies that the nonlinearity f satisfies the following sign condition:

$$f(t,u)u\ge 0 \quad \text{for all } u\in\mathbb{R} \text{ and } t\in[\xi,\eta] \text{ a.e.}$$

Example 2.2. A typical example of a nonlinearity satisfying Hypotheses (2.4))-(2.5), when $p=1$ and $b=d=0$, is

$$\begin{aligned} f(t,u) &= At^{-3/2}(1-t)^{-5/4}u + Bt^{-7/6}(1-t)^{-7/4}\frac{u^3}{1+u^2+e^{-u}}\\ &\quad + Ct^{-11/7}(1-t)^{-13/10}\frac{u^3}{1+u^2+e^{u}}, \end{aligned}$$

where A,B,C are positive constants.

Throughout this article, we denote by E the Banach space of all continuous functions defined on $[\xi,\eta]$, equipped with the sup-norm denoted $\|\cdot\|_\infty$ and by Y the Banach space defined as

$$\begin{aligned} Y = \Big\{ & v\in AC[\xi,\eta] : pv'\in C[\xi,\eta] \text{ and}\\ & av(\xi) - b\lim_{t\to\xi}p(t)v'(t) = cv(\eta) + d\lim_{t\to\eta}p(t)v'(t) = 0\Big\} \end{aligned}$$

equipped with the norm $\|v\|_Y = \|v\|_\infty + \|pv'\|_\infty$ for $v\in Y$. In all this paper, $\pounds$ is the differential operator given by

$$\pounds u(x) = -(pu')'(x)$$

with domain

$$D(\pounds) = \big\{v\in AC[\xi,\eta] : pv'\in C(\xi,\eta) \text{ and } (pv')'\in L^1_G[\xi,\eta]\big\}.$$

Set

$$Y_\# = \big\{v\in D(\pounds) : av(\xi) - b\lim_{t\to\xi}p(t)v'(t) = cv(\eta) + d\lim_{t\to\eta}p(t)v'(t) = 0\big\}.$$

We have that $\pounds : Y_\# \to L^1_G[\xi,\eta]$ is one to one, with

$$\pounds^{-1}v(t) = \int_\xi^\eta G(t,s)v(s)ds \text{ for all } v\in L^1_G[\xi,\eta].$$

For $u\in AC[\xi,\eta]$, $u^{[1]}$ is the quasiderivative of u, for $t\in[\xi,\eta]$; that is, $u^{[1]}(t) = \lim_{\tau\to t}p(\tau)u'(\tau)$ when it exists.

For $k \geq 1$, let S_k^+ denote the set of all functions $v \in AC[\xi, \eta]$ with $pv' \in C(\xi, \eta)$, having exactly $(k-1)$ simple zeros in (ξ, η) (if $v(\tau) = 0$ then $v^{[1]}(\tau) \neq 0$) and v is positive in a right neighborhood of ξ, and denote $S_k^- = -S_k^+$ and $S_k = S_k^+ \cup S_k^-$.

Let

$$\rho_0 = \Big(\int_\xi^\eta \frac{ds}{p(s)} \Big)^{-1} (\eta - \xi),$$

$$C^1_{\#}[\xi, \eta] = \big\{ v \in C^1[\xi, \eta] : av(\xi) - b\rho_0\, v'(\xi) = 0 \text{ and } cv(\eta) + d\, \rho_0 v'(\eta) = 0 \big\}$$

equipped with the C^1-norm and, for all $k \geq 1$ let Θ_k^+ be the set of all functions $v \in C^1_{\#}[\xi, \eta]$ having exactly $(k-1)$ simple zeros in (ξ, η) and v is positive in a right neighborhood of $\xi, \Theta_k^- = -\Theta_k^+$ and $\Theta_k = \Theta_k^+ \cup \Theta_k^-$. It is well known that Θ_k^+, Θ_k^- and Θ_k are open sets in $C^1_{\#}[\xi, \eta]$. Since for all $k \geq 1$ and $\nu = +$ or $-$, $\Phi(S_k^\nu \cap Y) = \Theta_k^\nu$ where $\Phi : Y \to C^1_{\#}[\xi, \eta]$ is the homeomorphism between Banach spaces defined by

$$\Phi(u) = u \circ \varphi^{-1} \quad \text{with } \varphi(t) = \xi + \rho_0 \int_\xi^t \frac{ds}{p(s)},$$

we have that $S_k^\nu \cap Y$ is an open set in Y. Moreover, since if $u \in \partial\Theta_k^\nu$ then there exists $\tau \in [\xi, \eta]$ such that $u(\tau) = u'(\tau) = 0$, we have that for all $v \in \partial(S_k^\nu \cap Y)$ there exists $\tau \in [\xi, \eta]$ such that $u(\tau) = u^{[1]}(\tau) = 0$.

For $\nu = +$ or $-$, let $I^\nu : E \to E$ be defined by $I^\nu u(x) = \max(\nu u(x), 0)$, for $u \in E$. For all $u \in E$, we have

$$u = I^+u - I^-u \quad \text{and} \quad |u| = I^+u + I^-u.$$

This implies that, for all $u, v \in E$,

$$\begin{aligned} |I^+u - I^+v| &\leq \frac{|u-v|}{2} + \frac{||u| - |v||}{2} \leq |u - v|\,, \\ |I^-u - I^-v| &\leq \frac{|u-v|}{2} + \frac{||u| - |v||}{2} \leq |u - v|, \end{aligned} \tag{2.6}$$

and the operators I^+, I^- are continuous.

For sake of simplicity, throughout this paper, we will use u^+ and u^- instead of I^+u and I^-u, respectivley. Now we focus our attention on the eigenvalue bvp

$$\begin{gathered} -(pu')'(t) = \lambda(\alpha(t)u^+(t) - \beta(t)u^-(t)), \text{ a.e. } t \in (\xi, \eta), \\ au(\xi) - b \lim_{t \to \xi} p(t)u'(t) = 0, \\ cu(\eta) + d \lim_{t \to \eta} p(t)u'(t) = 0, \end{gathered} \tag{2.7}$$

where α and β are functions in K_G such that the set $\{t \in (\xi, \eta) : \alpha(t)\beta(t) > 0\}$ is of a positive measure, and λ is a real parameter.

Definition 2.3. We say that λ is a half-eigenvalue of (2.7) if there exists a nontrivial solution (λ, u_λ) of (2.7). In this situation, $\{(\lambda, tu_\lambda),\ t > 0\}$ is a half-line of nontrivial solutions of (2.7) and λ is said to be simple if all solutions (λ, v) of (2.7) with v and u having the same sign on a right neighborhood of ξ are on this half-line. There may exist another half-line of solutions $\{(\lambda, tv_\lambda),\ t > 0\}$, but then we say that λ is simple if u_λ and v_λ have different signs on a right neighborhood of ξ and all solutions (λ, v) of (2.7) lie on these two half lines.

Theorem 2.4. *Assume that* (2.2) *and* (2.3) *hold, and* $\alpha, \beta \in K_G \cap L^1[\xi, \eta]$. *Then the set of half-eigenvalues to bvp* (2.7) *consists of two increasing sequences* $(\lambda_k^+)_{k\geq 1}$ *and* $(\lambda_k^-)_{k\geq 1}$ *such that for all* $k \geq 1$ *and* $\nu = +$ *or* $-$,

1. λ_k^ν *is simple and is the unique half-eigenvalue having a half-line of solutions in* $\{\lambda_k^\nu\} \times S_k^\nu$.
2. λ_k^ν *is a nondecreasing function with the respect of each of the weights* α *and* β *lying in* $L^1[\xi, \eta]$.

Theorem 2.5. *Assume that* (2.2) *and* (2.3) *hold. Then the set of half-eigenvalues to bvp* (2.7) *consists of two nondecreasing sequences* $(\lambda_k^+)_{k\geq 1}$ *and* $(\lambda_k^-)_{k\geq 1}$ *such that for all* $k \geq 1$ *and* $\nu = +$ *or* $-$, λ_k^ν *is the unique half eigenvalue having a half-line of solutions in* $\{\lambda_k^\nu\} \times S_k^\nu$. *Moreover, for all* $k \geq 1$ *and* $\nu = +$ *or* $-$, λ_k^ν *is a nondecreasing function with the respect of each of the weights* α *and* β *lying in* $L_G^1[\xi, \eta]$.

Remark 2.6. It is clear that for all $k \geq 1$ and $\nu = +$ or $-$, λ_k^ν depends on the weights p, α, β and on (ξ, η, a, b, c, d). When there is no confusion, we just denote λ_k^ν, and when we need to be more precise, we write $\lambda_k^\nu(\alpha, \beta)$.

Consider the bvp,

$$\begin{gathered} \pounds u(t) = \mu q(t)u(t), \quad \text{a.e. } t \in (\xi, \eta), \\ au(\xi) - b\lim_{t\to\xi} p(t)u'(t) = 0, \\ cu(\eta) + d\lim_{t\to\eta} p(t)u'(t) = 0, \end{gathered} \tag{2.8}$$

where μ is a real parameter, and $q \in K_G$.

It is clear that if μ is an eigenvalue for (2.8) then μ depends on the weights p, q and on (ξ, η, a, b, c, d). When there is no confusion, we just denote $\mu(q)$, and when we need to be more precise, we write $\mu(q, [\xi, \eta])$.

Theorem 2.7. *Assume that* (2.2) *and* (2.3) *hold. Then bvp* (2.8) *admits a sequence of eigenvalues* $(\mu_k(q))_{k\geq 1}$ *such that:*

1. *For all* $k \geq 3$, $\mu_k(q)$ *is simple and the associated eigenfunction* $\phi_k \in S_k$.
2. *For all* $k \geq 3$, $\mu_k(q) < \mu_{k+1}(q)$.
3. *If* $bd \neq 0$, *or* $bd = 0$ *and* $q \in L^1[\xi, \eta]$, *then* $\mu_1(q, [\xi, \eta]) < \mu_2(q, [\xi, \eta])$ *and* $\mu_1(q), \mu_2(q)$ *are simple having eigenvectors respectively in* S_1 *and* S_2. *If* $bd = 0$ *and* $q \notin L^1[\xi, \eta]$, *then* $\mu_1(q) = \mu_2(q)$ *and* $\mu_1(q) = \mu_2(q)$ *is double having two eigenvectors* $\phi_{1,1} \in S_1$ *and* $\phi_{1,2} \in S_2$.
4. *For all* $k \geq 1$ *and* $\theta > 0$, $\mu_k(\theta q) = \frac{\mu_k(q)}{\theta}$.
5. *Let* $q_1 \in K_G$. *We have* $\mu_k(q_1) \geq \mu_k(q)$ *for all* $k \geq 1$ *whenever* $q_1 \leq q$.
6. *If* $[\xi_1, \eta_2] \subset [\xi, \eta]$ *then* $\mu_k(q, [\xi, \eta]) \leq \mu_k(q, [\xi_1, \eta_1])$.
7. *For all* $k \geq 1$, $\mu_k(\cdot, [\xi, \eta]) : K_G \to \mathbb{R}$ *is continuous.*

Remark 2.8. Since the weight q in Theorem 2.7 is not necessarily in $L^1[\xi, \eta]$, Theorem 2.7 is not covered by [36, Theorems 4.3.1, 4.3.2, 4.3.3, 4.3.4].

For the statement of the main results of this paper we introduce the following notation:

$$\theta_\infty(t) = \max(\alpha_\infty(t), \beta_\infty(t)), \quad \vartheta_\infty(t) = \min(\alpha_\infty(t), \beta_\infty(t)),$$

and let $(\mu_k(\theta_\infty))_{k\geq 1}$, $(\mu_k(\vartheta_\infty))_{k\geq 1}$ and $(\mu_k(q_0))_{k\geq 1}$ be respectively the sequences of eigenvalues given by Theorem 2.7 for $q = \theta_\infty$, $q = \vartheta_\infty$ and $q = q_0$.

Theorem 2.9. *Assume that* (2.2)-(2.5) *hold and that there exist two integers* k, l *with* $2 < k < l$ *such that one of the following situations holds:*

$$\mu_l(q_0) < 1 < \mu_k(\theta_\infty)$$

or

$$\mu_l(\vartheta_\infty) < 1 < \mu_k(q_0)\,.$$

Then for all $i \in \{k, \dots, l\}$ *and* $\nu = +$ *or* $-$, (2.1) *has a solution in* S_i^ν.

Consider the separated variables case

$$\begin{gathered} \pounds u(t) = q_{sv}(t)h(u(t)), \quad \text{a.e. } t \in (\xi, \eta) \\ au(\xi) - b\lim_{t\to\xi} p(t)u'(t) = 0 \\ cu(\eta) + d\lim_{t\to\eta} p(t)u'(t) = 0 \end{gathered} \tag{2.9}$$

where q_{sv} is a nonnegative function in $L^1_G[\xi, \eta]$ which does not vanish identically in $[\xi, \eta]$ and $h : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$\begin{gathered} h(x)x > 0 \quad \text{for all } x \neq 0 \\ lim_{x\to 0}\frac{h(x)}{x} = h_0, \quad \lim_{x\to+\infty}\frac{h(x)}{x} = h_{+\infty}, \\ lim_{x\to-\infty}\frac{h(x)}{x} = h_{-\infty} \quad \text{with } h_0, h_{+\infty}, h_{-\infty} \in (0, +\infty). \end{gathered} \tag{2.10}$$

The following corollary provides an answer to a more general situation than those studied in [20] and [21] and also covers [22, Theorems 2 and 3].

Corollary 2.10. *Assume that* (2.10) *holds and there exist two integers* k, l *with* $2 < k < l$, *such that one of the following situations holds:*

$$h_{+\infty}, h_{-\infty} < \mu_k(q_{sv}) < \mu_l(q_{sv}) < h_0$$

or

$$h_0 < \mu_k(q_{sv}) < \mu_l(q_{sv}) < h_{+\infty}, h_{-\infty}\,.$$

Then for all $i \in \{k, \dots, l\}$ *and* $\nu = +$ *or* $-$, (2.9) *has a solution in* S_i^ν.

Proof. Set $f(t, u) = q_{sv}(t)h(u)$. It is easy to check that f satisfies Hypotheses (2.4)-(2.5), with

$$\begin{gathered} \alpha_\infty(t) = h_{+\infty}q_{sv}(t), \quad \beta_\infty(t) = h_{-\infty}q_{sv}(t), \quad q_0(t) = h_0 q_{sv}(t), \\ \gamma_\infty = h_{\sup}q_{sv}(t), \quad \delta_\infty = h_{\inf}q_{sv}(t), \\ h_{\inf} = \inf\{h(u)/u : u \neq 0\}, \\ h_{\sup} = \sup\{h(u)/u : u \neq 0\}. \end{gathered}$$

Also, we have

$$\theta_\infty(t) = \max(h_{+\infty}, h_{-\infty})q_{sv}(t), \quad \vartheta_\infty(t) = \min(h_{+\infty}, h_{-\infty})q_{sv}(t).$$

Since Property 2 of Theorem 2.7 implies that for all $n \geq 1$,

$$\mu_n(\theta_\infty) = \frac{\mu_n(q_{sv})}{\max(h_{+\infty}, h_{-\infty})}, \quad \mu_n(\vartheta_\infty) = \frac{\mu_n(q_{sv})}{\min(h_{+\infty}, h_{-\infty})}, \quad \mu_n(q_0) = \frac{\mu_n(q_{sv})}{h_0},$$

we obtain that $\mu_l(q_0) < 1 < \mu_k(\theta_\infty)$ if $h_{+\infty}, h_{-\infty} < \mu_k(q_{sv}) < \mu_l(q_{sv}) < h_0$ and $\mu_l(\vartheta_\infty) < 1 < \mu_k(q_0)$ if $h_0 < \mu_k(q_{sv}) < \mu_l(q_{sv}) < h_{+\infty}, h_{-\infty}$. Thus, the conclusion of Corollary 2.10 follows from Theorem 2.9. $\square$

3. Background

3.1. A comparison result.

Theorem 3.1. *Let u and v be two functions in S_k^ν. Then, there exist two intervals $[\xi_1, \eta_1]$ and $[\xi_2, \eta_2]$ such that $uv \geq 0$ in $[\xi_i, \eta_i]$ $i = 1, 2$. Moreover if $pu', pv' \in AC[\xi_i, \eta_i]$ $i = 1, 2$ then*

$$\int_{\xi_1}^{\eta_1} v\pounds u - u\pounds v \geq 0, \quad \int_{\xi_2}^{\eta_2} v\pounds u - u\pounds v \leq 0.$$

The proof of this theorem is based on the following lemma.

Lemma 3.2 ([4]). *Let j and k be two integers such that $j \geq k \geq 2$. Suppose that there exist two families of real numbers*

$$\xi_0 = \xi < \xi_1 < \xi_2 < \dots \xi_{k-1} < \xi_k = \eta$$
$$\eta_0 = \xi < \eta_1 < \eta_2 < \dots \eta_{j-1} < \eta_j = \eta.$$

Then, if $\xi_1 \leq \eta_1$ there exist two integers j_0 and k_0 having the same parity, $1 \leq j_0 \leq j-1$, $1 \leq k_0 \leq k-1$ such that

$$\xi_{k_0} \leq \eta_{j_0} \leq \eta_{j_0+1} \leq \xi_{k_0+1}.$$

Proof of Theorem 3.1. The case $k = 1$ is obvious. Let

$$x_0 = \xi < x_1 < x_2 < \dots < x_{k-1} < x_k = \eta,$$
$$z_0 = \xi < z_1 < z_2 < \dots < z_{k-1} < z_k = \eta$$

be the sequences of zeros of u and v, respectively.

Suppose $x_1 \leq z_1$. Then we deduce from Lemma 3.2 the existence of integers k_0 and j_0 having the same parity such that $x_{k_0} \leq z_{j_0} \leq z_{j_0+1} \leq x_{k_0+1}$.

Therefore, we choose $\xi_1 = x_0$, $\eta_1 = x_1$, $\xi_2 = z_{j0}$ and $\eta_2 = z_{j_0+1}$, and since k_0 and j_0 have the same parity, u and v have the same sign in both the intervals $[\xi_1, \eta_1]$ and $[\xi_2, \eta_2]$.

Now if $u, v \in S_k^\nu \cap Y$, then for $i = 1, 2$, $u^{[1]}(\xi_i)$, $v^{[1]}(\xi_i)$, $u^{[1]}(\eta_i)$ and $v^{[1]}(\eta_i)$ exist and are finite. Moreover if $u \geq 0$ and $v \geq 0$ in $[\xi_i, \eta_i]$ $i = 1, 2$ (the other cases can be checked similarly) then we have since u and v satisfy the boundary condition at ξ_1,

$$v(\xi_1)u^{[1]}(\xi_1) - u(\xi_1)v^{[1]}(\xi_1) = 0, \ u^{[1]}(\eta_1) \leq 0,$$

and

$$v^{[1]}(\xi_2) \geq 0, \quad v^{[1]}(\eta_2) \leq 0,$$
$$-v(\eta_2)u^{[1]}(\eta_2) + u(\eta_2)v^{[1]}(\eta_2) = \begin{cases} 0, & \text{if } \eta_2 = \eta, \\ u(\eta_2)v^{[1]}(\eta_2) \leq 0, & \text{if } \eta_2 < \eta. \end{cases}$$

Thus, we obtain

$$\int_{\xi_1}^{\eta_1} v\pounds u - u\pounds v = -v(\eta_1)u^{[1]}(\eta_1) \geq 0,$$
$$\int_{\xi_2}^{\eta_2} v\pounds u - u\pounds v = -v(\eta_2)u^{[1]}(\eta_2) + u(\eta_2)v^{[1]}(\eta_2) - u(\xi_2)v^{[1]}(\xi_2) \leq 0.$$

This completes the proof. □

3.2. **Spectral radius of a positive operator.** Let Z be a real Banach space and $L(Z)$ the Banach space of linear continuous operators from Z into Z. For $L \in L(Z)$, $r(L) = \lim \|L^n\|^{\frac{1}{n}}$ denotes the spectral radius of L.

A nonempty closed convex subset K of Z is said to be an ordered cone if $(tK) \subset K$ for all $t \geq 0$ and $K \cap (-K) = \{0\}$. Moreover if $Z = \overline{K - K}$, the cone K is said to be total. It well known that an ordered cone K induces a partial order on the Banach space Z ($x \leq y$ if and only if $y - x \in K$ for all $x, y \in Z$).

Let K be an ordered cone of Z and $L \in L(Z)$. L is said to be positive if $L(K) \subset K$ and $\mu \in \mathbb{R}$ is said to be a positive eigenvalue of L if there exists $u \in K \setminus \{0\}$ such that $Lu = \mu u$.

Let $L_1, L_2 \in L(Z)$ be two positive operators. We write $L_1 \leq L_2$ if $L_1 u \leq L_2 u$ for all $u \in K$.

We will use in this work the following result known as the Krein-Rutman Theorem.

Theorem 3.3 ([35, Proposition 7.26])**.** *Assume that the cone K is total and $L \in L(Z)$ is compact and positive with $r(L) > 0$. Then $r(L)$ is a positive eigenvalue of L.*

We will use also the following lemma.

Lemma 3.4 ([35, Corollary 7.28])**.** *Assume that the cone K is total and let L_1, L_2 in $L(Z)$ be two compact and positive operators. If $L_1 \leq L_2$, then $r(L_1) \leq r(L_2)$.*

Next we recall a fundamental result proved by Nussbaum in [23] and used in [3].

Lemma 3.5. *Let (L_n) be a sequence of compact linear operators on a Banach space Z and suppose that $L_n \to L$ in operator norm as $n \to \infty$. Then $r(L_n) \to r(L)$.*

3.3. **The linear eigenvalue bvp in the integrable case.**

Theorem 3.6. *Assume that Hypotheses* (2.2) *and* (2.3) *hold and $q \in K_G \cap L^1[\xi, \eta]$. Then the set of eigenvalues to bvp* (2.8) *consists of an increasing sequence of simple eigenvalues $(\mu_k(q))_{k \geq 1}$ tending to $+\infty$, such that for all $k \geq 1$,*

1. *The eigenfunction ϕ_k associated with $\mu_k(q)$ belongs to S_k.*
2. *If $\theta > 0$ then $\mu_k(\theta q) = \frac{\mu_k(q)}{\theta}$.*
3. *Let q_1 be a nonnegative function in $L^1[\xi, \eta]$ which does not vanish identically in $[\xi, \eta]$. We have $\mu_k(q_1) \geq \mu_k(q)$ for all $k \geq 1$ whenever $q_1 \leq q$. Moreover, if $q_1 < q$ in a subset of a positive measure then $\mu_k(q_1) > \mu_k(q)$.*
4. *If $[\xi_1, \eta_1] \subsetneq [\xi, \eta]$ then $\mu_k(q, [\xi, \eta]) < \mu_k(q, [\xi_1, \eta_1])$.*
5. *μ_k is a continuous function with respect to the variable q lying in $L^1[\xi, \eta]$.*

Proof. From Theorem 4.3.2 in [36], the bvp (2.8) has only real and simple eigenvalues and they are ordered to satisfy

$$-\infty < \mu_1 < \mu_2 < \dots < \lim_{k \to \infty} \mu_k = +\infty.$$

Moreover if ϕ_k is an eigenfunction of μ_k, and n_k denotes the number of zeros of ϕ_k in (ξ, η), then $\phi_k \in AC[\xi, \eta]$, $p\phi_k' \in AC[\xi, \eta]$ and $n_{k+1} = n_k + 1$. Now, we have

$$0 < -\phi_k(\eta)\phi_k^{[1]}(\eta) + \phi_k(\xi)\phi_k^{[1]}(\xi) + \int_\xi^\eta p(\phi_k')^2 = \int_\xi^\eta \phi_k \pounds \phi_k = \mu_k \int_\xi^\eta q\phi_k^2$$

leading to $\mu_k > 0$ for all $k \geq 1$.

Let $L_q : E \to E$ be defined by

$$L_q u(t) = \int_\xi^\eta G(t,s)q(s)u(s)ds = \pounds^{-1}(qu)(t).$$

It is easy to see that L_q is a positive operator with respect to the total cone of nonnegative functions in E and λ is an eigenvalue of L_q if and only if λ^{-1} is an eigenvalue of bvp (2.8). Also, the presence of eigenvalues implies that $r(L_q) > 0$. Thus, we deduce from Theorem 3.3 that $r(L_q)$ is the largest and positive eigenvalue of L_q, and so, we have $\mu_1 = 1/r(L_q)$ and $n_1 = 0$ and for all $k \geq 2$, $n_k = k-1$. That is $\phi_k \in S_k$ and Assertion 1, is proved.

Assertion 2 is obvious and since $\mu_k > 0$ for all $k \geq 1$. Assertion 3 follows directly from [36, Theorem 4.9.1].

To prove Property 4, let $[\xi', \eta'] \subsetneq [\xi, \eta]$, and ϕ and ψ be such that

$$\begin{gathered}
\pounds\phi = \mu_k(q)q\phi, \quad \text{a.e. } t \in (\xi,\eta), \\
au(\xi) - b\lim_{t\to\xi} p(t)u'(t) = 0, \\
cu(\eta) + d\lim_{t\to\eta} p(t)u'(t) = 0,
\end{gathered}$$

and

$$\begin{gathered}
\pounds\psi = \mu_k(q,[\xi_1,\eta_1])q\psi, \quad \text{a.e. } t \in (\xi,\eta), \\
au(\xi_1) - b\lim_{t\to\xi_1} p(t)u'(t) = 0, \\
cu(\eta_1) + d\lim_{t\to\eta_1} p(t)u'(t) = 0.
\end{gathered}$$

Denote by $(x_i)_{1\leq i\leq k}$ and $(y_j)_{1\leq j\leq k}$, respectively, the two sequences of zeros of ϕ and ψ. There exist two integers $1 \leq i_0$, $j_0 \leq k$ such that one of the following two situations holds:

$$\begin{gathered}
\xi \leq x_{i_0-1} \leq y_{j_0-1} < y_{j_0} < x_{i_0} \leq \eta, \\
\xi \leq x_{i_0-1} < y_{j_0-1} < y_{j_0} \leq x_{i_0} \leq \eta.
\end{gathered}$$

Without loss of generality, suppose ϕ and ψ are positive, respectively, in (x_{i_0-1}, x_{i_0}) and (y_{j_0-1}, y_{j_0}) and (3.3) holds. Then we have

$$\psi(y_{j_0}) = 0, \quad \phi(y_{j_0}) > 0, \quad \psi^{[1]}(y_{j_0}) < 0, \quad \psi^{[1]}(y_{j_0-1}) \geq 0, \quad \phi(y_{j_0-1}) \geq 0,$$

and

$$\begin{aligned}
&-\psi^{[1]}(y_{j_0-1})\phi(y_{j_0-1}) + \phi^{[1]}(y_{j_0-1})\psi(y_{j_0-1}) \\
&= \begin{cases} 0 & \text{if } y_{j_0-1} = \xi \\ -\psi^{[1]}(y_{j_0-1})\phi(y_{j_0-1}) \leq 0 & \text{if } y_{j_0-1} > \xi. \end{cases}
\end{aligned}$$

From which we obtain

$$\begin{aligned}
&(\mu_k(q,[\xi,\eta]) - \mu_k(q,[\xi_1,\eta_1])) \int_{y_{j_0-1}}^{y_{j_0}} q\psi\phi \\
&= \int_{y_{j_0-1}}^{y_{j_0}} \psi\pounds\phi - \phi\pounds\psi \\
&= \psi^{[1]}(y_{j_0})\phi(y_{j_0}) - \psi^{[1]}(y_{j_0-1})\phi(y_{j_0-1}) + \phi^{[1]}(y_{j_0-1})\psi(y_{j_0-1}) < 0
\end{aligned}$$

leading to

$$\mu_k(q,[\xi,\eta]) < \mu_k(q,[\xi_1,\eta_1]).$$

Finally, Property 5 is obtained from [36, Theorem 3.5.2]. □

3.4. **Berestycki's half-eigenvalue bvp.** Let m, α and β be three continuous functions on $[\xi,\eta]$ with $m > 0$ in $[\xi,\eta]$ and consider the bvp,

$$\begin{gathered}
\pounds u = \lambda m u + \alpha u^+ - \beta u^- \quad \text{in } (\xi,\eta),\\
a u(\xi) - b \lim_{t\to\xi} p(t)u'(t) = 0\\
c u(\eta) + d \lim_{t\to\eta} p(t)u'(\eta) = 0,
\end{gathered} \tag{3.1}$$

where λ is a real parameter.

Bvp (3.1) is called half-linear since it is linear and positively homogeneous in the cones $u \geq 0$ and $u \leq 0$.

Definition 3.7. We say that λ is a half-eigenvalue of (3.1) if there exists a nontrivial solution (λ, u_λ) of (3.1). In this situation, $\{(\lambda, tu_\lambda),\ t > 0\}$ is a half-line of nontrivial solutions of (3.1) and λ is said to be simple if all solutions (λ, v) of (3.1) with v and u having the same sign on a deleted neighborhood of ξ are on this half-line. There may exist another half-line of solutions $\{(\lambda, tv_\lambda),\ t > 0\}$, but then we say that λ is simple if u_λ and v_λ have different signs on a deleted neighborhood of ξ and all solutions (λ, v) of (3.1) lie on these two half lines.

Remark 3.8. Note that the position of the real parameter in the differential equation in (3.1) is not same as in Problem (2.7). Moreover, we have Problem (3.1) coincides with the linear eigenvalue problem when $\alpha = \beta = 0$, even though Problem (2.7) coincides with the linear eigenvalue problem when $\alpha = \beta$.

Berestycki proved in [4] the following theorem.

Theorem 3.9. *Assume that $p \in C^1[\xi,\eta]$ and $p > 0$ in $[\xi,\eta]$. Then the set of half eigenvalues of bvp (3.1) consists of two increasing sequences of simple half-eigenvalues for bvp (3.1) $(\lambda_k^+)_{k\geq 1}$ and $(\lambda_k^-)_{k\geq 1}$, such that for all $k \geq 1$ and $\nu = +$ or $-$, the corresponding half-lines of solutions are in $\{\lambda_k^\nu\} \times S_k^\nu$.*

Proposition 3.10. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in C([\xi,\eta])$. We have*

- *If $\alpha_1 \leq \alpha_2$, then $\lambda_k^\nu(\alpha_1) \geq \lambda_k^\nu(\alpha_2)$, for all $k \geq 1$ and $\nu = +$ or $-$.*
- *If $\beta_1 \leq \beta_2$, then $\lambda_k^\nu(\beta_1) \geq \lambda_k^\nu(\beta_2)$, for all $k \geq 1$ and $\nu = +$ or $-$.*

Proof. We present the proof of the first assertion. The second one can be proved in a similar way. Let ϕ_1, ϕ_2 be such that

$$\begin{gathered}
\pounds \phi_1 = \lambda_k^\nu(\alpha_1) m \phi_1 + \alpha_1 \phi_1^+ - \beta \phi_1^- \quad \text{in } (\xi,\eta),\\
a\phi_1(\xi) - b \lim_{t\to\xi} p(t)\phi_1'(t) = 0,\\
c\phi_1(\eta) + d \lim_{t\to\eta} p(t)\phi_1'(\eta) = 0,
\end{gathered}$$

and

$$\begin{gathered}
\pounds \phi_2 = \lambda_k^\nu(\alpha_2) m \phi_2 + \alpha_1 \phi_2^+ - \beta \phi_2^- \quad \text{in } (\xi,\eta),\\
a\phi_2(\xi) - b \lim_{t\to\xi} p(t)\phi_2'(t) = 0,\\
c\phi_2(\eta) + d \lim_{t\to\eta} p(t)\phi_2'(\eta) = 0.
\end{gathered}$$

Note that $\phi_1, \phi_2 \in S_k \cap C^2[\xi, \eta]$ and let $[\xi_1, \eta_1]$ be the interval given by Theorem 3.1 for the functions ϕ_1 and ϕ_2. Since ϕ_1 and ϕ_2 have the same sign in (ξ_1, η_1), after simple computations we obtain

$$(\lambda_k^\nu(\alpha_1) - \lambda_k^\nu(\alpha_2)) \int_{\xi_1}^{\eta_1} m\phi_1\phi_2 = -\int_{\xi_1}^{\eta_1} (\alpha_1 - \alpha_2)\phi_1\phi_2 + \int_{\xi_1}^{\eta_1} \phi_2 \pounds\phi_1 - \phi_1 \pounds\phi_2 \geq 0$$

leading to

$$\lambda_k^\nu(\alpha_1) \geq \lambda_k^\nu(\alpha_2).$$

This completes the proof. □

Remark 3.11. Naturally one can ask, is it possible to extend Berestyki's theorem to the case where the weight m, as well as α and β all belong to $L^1_G[\xi, \eta]$?

This is technically difficult since a half-eigenvalue of (3.1) is decreasing with respect to the weight m only if it is positive.

3.5. **Fučik spectrum.** Consider now the bvp,

$$\begin{gathered} -u''(t) = \alpha u^+(t) - \beta u^-(t), \quad t \in (\xi, \eta), \\ au(\xi) - bu'(\xi) = 0, \\ cu(\eta) + du'(\eta) = 0, \end{gathered} \tag{3.2}$$

where α, β are positive real parameters and $a, b, c, d \in \mathbb{R}^+$ with $ac + ad + bc > 0$.

The statement of the next result requires introducing the functions $\Lambda_{a,b,c,d}, \Lambda_{a,b} : (0, +\infty) \to (0, +\infty)$ defined, for $\sigma > 0$, by

$$\Lambda_{a,b,c,d}(\sigma) = \frac{1}{\sqrt{\sigma}}\Big(\pi - \arcsin\Big(\sqrt{\frac{b^2\sigma}{a^2 + b^2\sigma}}\Big) - \arcsin\Big(\sqrt{\frac{d^2\sigma}{c^2 + d^2\sigma}}\Big)\Big),$$

and

$$\Lambda_{a,b}(\sigma) = \frac{1}{\sqrt{\sigma}}\Big(\pi - \arcsin\Big(\sqrt{\frac{b^2\sigma}{a^2 + b^2\sigma}}\Big)\Big).$$

Note that $\Lambda_{a,b} = \Lambda_{a,b,1,0} = \Lambda_{1,0,a,b}$. The sets S_k^+, S_k^- and S_k are those introduced in Section 2 for $p = 1$. The main goal of this subsection is to describe the set

$$F_s = \{(\alpha, \beta) \in \mathbb{R} \times \mathbb{R} : (3.2) \text{ has a solution } \}$$

known as the Fučik spectrum.

Theorem 3.12. *Let S be the set of solutions to bvp (3.2). Then $S \subset \cup_{k \geq 1} S_k$. Moreover bvp (3.2) admits a solution*

(1) *in S_1^+ if and only if $\Lambda_{a,b,c,d}(\alpha) = \eta - \xi$,*
(2) *in S_1^- if and only if $\Lambda_{a,b,c,d}(\beta) = \eta - \xi$,*
(3) *in S_{2l}^+ with $l \geq 1$ if and only if*
$$\Lambda_{a,b}(\alpha) + \Lambda_{c,d}(\beta) + \pi(l - 1)(\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}}) = \eta - \xi,$$
(4) *in S_{2l}^- with $l \geq 1$ if and only if*
$$\Lambda_{a,b}(\beta) + \Lambda_{c,d}(\alpha) + \pi(l - 1)(\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}}) = \eta - \xi,$$
(5) *in S_{2l+1}^+ with $l \geq 1$ if and only if*
$$\Lambda_{a,b}(\alpha) + \Lambda_{c,d}(\alpha) + \frac{\pi(l - 1)}{\sqrt{\alpha}} + \frac{\pi l}{\sqrt{\beta}} = \eta - \xi,$$

(6) *in* S^-_{2l+1} *with* $l \geq 1$ *if and only if*

$$\Lambda_{a,b}(\beta) + \Lambda_{c,d}(\beta) + \frac{\pi(l-1)}{\sqrt{\beta}} + \frac{\pi l}{\sqrt{\alpha}} = \eta - \xi.$$

Proof. First, note that u is a solution to (3.2) if and only if $v(t) = u((\eta - \xi)t + \xi)$ is a solution to the bvp

$$\begin{gathered} -v''(t) = (\eta-\xi)^2\alpha v^+(t) - (\eta-\xi)^2\beta v^-(t), \ t \in (0,1), \\ av(0) - \frac{b}{(\eta-\xi)}v'(0) = 0, \\ cv(1) + \frac{d}{(\eta-\xi)}v'(1) = 0. \end{gathered}$$

Then, Assertions 1 and 2 of Theorem 3.12 follow from Proposition 3.1 in [2].

Now, for the sake of brevity, we prove only Assertion 3 (the others can be proved similarly). Note that $u \in S^\nu_{2l}$ is a solution to (3.2) if and only if there exists a finite sequence $(x_i)_{i=0}^{i=2l}$ such that

$$\xi = x_0 < x_1 < \dots < x_{2l-1} < x_{2l} = \eta$$

and

$$\begin{gathered} u > 0 \quad \text{in } (x_{2i}, x_{2i+1}) \text{ for } i = 0, \dots, (l-1), \\ u < 0 \quad \text{in } (x_{2i-1}, x_{2i}) \text{ for } i = 1, \dots, l. \end{gathered}$$

Moreover, u satisfies

$$\begin{gathered} -u''(t) = \alpha u(t), \quad t \in (\xi, x_1), \\ au(\xi) - bu'(\xi) = u(x_1) = 0, \end{gathered}$$

and for $i = 1, \dots, (l-1)$:

$$-u''(t) = \alpha u(t), \quad t \in (x_{2i}, x_{2i+1}), u(x_{2i}) = u(x_{2i+1}) = 0,$$

and

$$\begin{gathered} -u''(t) = \beta u(t), \quad t \in (x_{2i-1}, x_{2i}), \\ u(x_{2i-1}) = u(x_{2i}) = 0, \end{gathered}$$

and

$$\begin{gathered} -u''(t) = \alpha u^+(t) - \beta u^-(t), \quad t \in (x_{2l-1}, \eta), \\ u(x_{2l-1}) = cu(\eta) + du'(\eta) = 0. \end{gathered}$$

Hence, from Assertions 1 and 2, we obtain

$$\begin{gathered} \frac{1}{\sqrt{\alpha}}\Big(\pi - \arcsin\Big(\sqrt{\frac{b^2\alpha}{a^2 + b^2\alpha}}\Big)\Big) = x_1 - \xi, \\ \frac{\pi}{\sqrt{\alpha}} = (x_{2i+1} - x_{2i}) \quad \text{for } i = 1, \dots, (l-1), \\ \frac{\pi}{\sqrt{\beta}} = (x_{2i} - x_{2i-1}) \quad \text{for } i = 1, \dots, (l-1), \\ \frac{1}{\sqrt{\beta}}\Big(\pi - \arcsin\Big(\sqrt{\frac{d^2\beta}{c^2 + d^2\beta}}\Big)\Big) = (\eta - x_{2l-1}). \end{gathered}$$

Summing the above equalities, we obtain

$$\Lambda_{a,b}(\alpha) + \Lambda_{c,d}(\beta) + \pi(l-1)\Big(\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}}\Big) = \eta - \xi.$$

Conversely, let $\alpha, \beta > 0$ be such that

$$\Lambda_{a,b}(\alpha) + \Lambda_{c,d}(\beta) + \pi(l-1)\Big(\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}}\Big) = \eta - \xi, \tag{3.3}$$

and let $(x_i)_{i=0}^{i=2l}$ be the sequence defined by

$$\begin{gathered} x_0 = \xi, \quad x_1 = \xi + \frac{1}{\sqrt{\alpha}}\Big(\pi - \arcsin\Big(\sqrt{\frac{b^2\alpha}{a^2 + b^2\alpha}}\Big)\Big), \\ x_{2i} = x_{2i-1} + \frac{\pi}{\sqrt{\beta}} \quad \text{for } i = 1, \dots, (l-1), \\ x_{2i+1} = x_{2i} + \frac{\pi}{\sqrt{\alpha}} \quad \text{for } i = 1, \dots, (l-1),\ x_{2l} = \eta. \end{gathered} \tag{3.4}$$

Observe that from (3.4) and (3.3) we have

$$\Lambda_{a,b,1,0}(\alpha) = x_1 - \xi \quad \text{and } \Lambda_{1,0,c,d,}(\beta) = \eta - x_{2l-1},$$

that is, 1 is the smallest eigenvalue of each of the bvps

$$\begin{gathered} -u'' = \alpha u \text{ in } (\xi, x_1), \\ au(\xi) - bu'(\xi) = u(x_1) = 0, \end{gathered} \tag{3.5}$$

and

$$\begin{gathered} -u'' = \beta u \text{ in } (x_{2l-1}, \eta), \\ u(x_{2l-1}) = cu(\eta) + du'(\eta) = 0. \end{gathered} \tag{3.6}$$

Thus, we consider the function

$$\phi(t) = \begin{cases} \phi_1(t) & \text{for } t \in [\xi, x_1], \\ \phi_{2i}(t) & \text{for } t \in [x_{2i-1}, x_{2i}],\ i = 1, \dots, (l-1), \\ \phi_{2i+1}(t) & \text{for } t \in [x_{2i}, x_{2i+1}],\ i = 1, \dots, (l-1), \\ \phi_{2l}(t) & \text{for } t \in [x_{2l-1}, \eta], \end{cases}$$

where ϕ_1 is the positive eigenfunction associated with the eigenvalue 1 of (3.5) satisfying $\phi_1'(x_1) = -1$,

$$\begin{gathered} \phi_{2i}(t) = -\frac{1}{\sqrt{\beta}} \sin\big(\sqrt{\beta}(t - x_{2i-1})\big) \quad \text{for } i = 1, \dots, (l-1), \\ \phi_{2i+1}(t) = \frac{1}{\sqrt{\alpha}} \sin\big(\sqrt{\alpha}(t - x_{2i})\big) \quad \text{for } i = 1, \dots, (l-1), \end{gathered}$$

and ϕ_{2l} is the negative eigenfunction associated with the eigenvalue 1 of (3.6) satisfying $\phi_{2l}'(x_{2l-1}) = -1$.

Thus, by simple computations we find that

$$\begin{gathered} \phi_{2i-1}'(x_{2i-1}) = \phi_{2i}'(x_{2i-1}) = -1 \quad \text{for } i = 1, \dots, l, \\ \phi_{2i}'(x_{2i}) = \phi_{2i+1}'(x_{2i}) = 1 \quad \text{for } i = 1, \dots, (l-1), \\ \phi_{2i}''(x_{2i-1}) = \phi_{2i}''(x_{2i}) = 0 \quad \text{for } i = 1, \dots, (l-1), \\ \phi_{2i+1}''(x_{2i}) = \phi_{2i+1}''(x_{2i+1}) = 0 \quad \text{for } i = 1, \dots, (l-1), \\ \phi_1''(x_1) = \phi_{2l}''(x_{2l-1}) = 0. \end{gathered}$$

All the above equalities make ϕ a function in $S_{2l}^+ \cap C^2[\xi,\eta]$ satisfying bvp (3.2). This completes the proof. $\square$

4. Proofs of main results

4.1. **Auxiliary results.** Let $q \in L^1_G[\xi,\eta]$. For $\varkappa \in (\xi,\eta)$ we define the operators $L_{q,\varkappa,l} : C[\xi,\varkappa] \to C[\xi,\varkappa]$ and $L_{q,\varkappa,r} : C[\varkappa,\eta] \to C[\varkappa,\eta]$ by

$$L_{q,\varkappa,l}u(t) = \int_\xi^\varkappa G_{\varkappa,l}(t,s)q(s)u(s)ds,$$
$$L_{q,\varkappa,r}u(t) = \int_\varkappa^\eta G_{\varkappa,r}(t,s)q(s)u(s)ds,$$

where

$$G_{\varkappa,l}(t,s) = \Big(b + a\int_\xi^\varkappa \frac{d\tau}{p(\tau)}\Big)^{-1} \begin{cases} \big(b + a\int_\xi^s \frac{d\tau}{p(\tau)}\big)\int_t^\varkappa \frac{d\tau}{p(\tau)}, & \xi \le s \le t \le \varkappa,\\ \big(b + a\int_\xi^t \frac{d\tau}{p(\tau)}\big)\int_s^\varkappa \frac{d\tau}{p(\tau)}, & \xi \le t \le s \le \varkappa,\end{cases}$$

and

$$G_{\varkappa,r}(t,s) = \Big(d + c\int_\xi^\varkappa \frac{d\tau}{p(\tau)}\Big)^{-1} \begin{cases} \int_\varkappa^s \frac{d\tau}{p(\tau)}\big(d + c\int_t^\eta \frac{d\tau}{p(\tau)}\big), & \varkappa \le s \le t \le \eta,\\ \int_\varkappa^t \frac{d\tau}{p(\tau)}\big(d + c\int_s^\eta \frac{d\tau}{p(\tau)}\big), & \varkappa \le t \le s \le \eta.\end{cases}$$

Lemma 4.1. *Assume that* (2.2) *and* (2.3) *hold. Then, for every function $q \in K_G$, $\lim_{\varkappa\to\xi} r(L_{q,\varkappa,l}) = 0$ and $\lim_{\varkappa\to\eta} r(L_{q,\varkappa,r}) = 0$.*

Proof. We will prove that $\lim_{\varkappa\to\xi} r(L_{q,\varkappa,l}) = 0$. The other limit can be obtained similarly. We distinguish two cases:

- $b \neq 0$: In this case $q \in L^1[\xi, \frac{\xi+\eta}{2}]$ and we have

$$\begin{aligned} r(L_{q,\varkappa,l}) &\le \int_\xi^\varkappa G_{\varkappa,l}(s,s)q(s)ds\\ &\le \Big(b + a\int_\xi^\varkappa \frac{d\tau}{p(\tau)}\Big)^{-1}\int_\xi^\varkappa \Big(b + a\int_\xi^\varkappa \frac{d\tau}{p(\tau)}\Big)\Big(\int_s^\varkappa \frac{d\sigma}{p(\sigma)}\Big)q(s)ds\\ &\le \int_\xi^\varkappa \frac{d\sigma}{p(\sigma)}\int_\xi^\varkappa q(s)ds, \end{aligned}$$

from which we obtain that $\lim_{\varkappa\to\xi} r(L_{q,\varkappa,l}) = 0$.

- $b = 0$: In this case $a \neq 0$ and $q(s)\int_\xi^s 1/p(\varepsilon)\,d\varepsilon \in L^1[\xi,(\xi+\eta)/2]$ and we have

$$\begin{aligned} r(L_{q,\varkappa,l}) &\le \int_\xi^\varkappa G_{\varkappa,l}(s,s)q(s)ds\\ &\le \int_\xi^\varkappa \Big(\int_\xi^s \frac{d\sigma}{p(\sigma)}\Big)q(s)ds - \Big(\int_\xi^\varkappa \frac{d\sigma}{p(\sigma)}\Big)^{-1}\int_\xi^\varkappa \Big(\int_\xi^s \frac{d\sigma}{p(\sigma)}\Big)^2 q(s)ds\\ &\le 2\int_\xi^\varkappa (q(s)\int_\xi^s \frac{d\sigma}{p(\sigma)})ds \end{aligned}$$

leading to $\lim_{\varkappa\to\xi} r(L_{q,\varkappa,l}) = 0$. This completes the proof. $\square$

Lemma 4.2. *Assume that* (2.2) *and* (2.3) *hold and let α,β be two functions in K_G. If u is a nontrivial solution of*

$$-(pu')'(t) = \lambda(\alpha(t)u^+(t) - \beta(t)u^-(t)),$$
$$u(t_0) = 0$$

with $t_0 \in \{\xi, \eta\}$, then t_0 is an isolated zero of u (i.e. there exists a neighborhood V_0 of t_0 such that $u(t) \neq 0$ for all $t \in V_0$). Moreover, we have that $\lim_{t\to t_0} p(t)u'(t)$ exists.

Proof. We present the proof for $t_0 = \xi$ the other case is similar. Let $t_* > \xi$ be such that u does not vanish identically in (ξ, t_*) and suppose that $\alpha+\beta > 0$ a.e. in (ξ, t_*) (the case $\alpha + \beta = 0$ a.e. in (ξ, t_*) is obvious). For the purpose of contradiction, suppose that there is a sequence $(\tau_n) \subset (\xi, t_*)$ such that $u(\tau_n) = 0$ for all $n \in \mathbb{N}$ and $\lim \tau_n = \xi$. In this case, u satisfies for all $n \in \mathbb{N}$,

$$\begin{gathered} -(pu')'(t) = \lambda(\alpha(t)u^+(t) - \beta(t)u^-(t)), \quad \text{a. e. } t \in (\xi, \tau_{n+1}), \\ u(\xi) = u(\tau_{n+1}) = 0. \end{gathered} \tag{4.1}$$

Without loss of generality, assume that u is positive in (τ_n, τ_{n+1}) and let $\mu_{n,1}(\alpha)$ be the first eigenvalue given by Theorem 3.6 associated with a positive eigenvector $\psi_{n,1}$ of

$$\begin{gathered} -(p\psi')'(t) = \mu\alpha(t)\psi(t), \quad t \in (\tau_n, \tau_{n+1}), \\ \psi(\tau_n) = \psi(\tau_{n+1}) = 0. \end{gathered}$$

Multiplying the differential equation in (4.1) by $\psi_{n,1}$, we obtain after two integrations

$$0 \leq u^{[1]}(\tau_n)\psi_{n,1}(\tau_n) = \int_{\tau_n}^{\tau_{n+1}} (\lambda - \mu_{n,1}(\alpha))\alpha\psi_{n,1}u$$

leading to

$$\lambda \geq \mu_{n,1}(\alpha). \tag{4.2}$$

Now, let $\mu^*_{n,1} = 1/r(L_{\alpha,\tau_{n+1},l})$ and let $\psi^*_{n,1}$ be the associated positive eigenvector. $\mu^*_{n,1}$ and $\psi^*_{n,1}$ satisfy

$$\begin{gathered} -(p\psi^{*\prime}_{n,1})'(t) = \mu^*_{n,1}\alpha(t)\psi^*_{n,1}(t), \quad t \in (\xi, \tau_{n+1}) \\ \psi^*_{n,1}(\xi) = \psi^*_{n,1}(\tau_{n+1}) = 0. \end{gathered}$$

Again, multiplying the differential equation in (4.1) by $\psi^*_{n,1}$, we obtain after two integrations

$$0 \geq -\psi^{[1]}_{n,1}(\tau_n)\psi^*_{n,1}(\tau_n) = \int_{\tau_n}^{\tau_{n+1}} (\mu^*_{n,1} - \mu_{n,1}(\alpha))\alpha\psi_{n,1}\psi^*_{n,1}$$

leading to

$$\mu_{n,1}(\alpha) \geq \mu^*_{n,1}. \tag{4.3}$$

Thus, from (4.2), (4.3) and Lemma 4.1 we obtain the contradiction

$$\lambda \geq \lim \mu_{n,1}(\alpha) \geq \lim \mu^*_{n,1} = 1/\lim r(L_{\alpha,\tau_{n+1},l}) = +\infty.$$

Now, suppose that $u > 0$ on $(\xi, t_\#)$ for some $t_\# > \xi$. We have by simple integration over $(t, t_\#) \subset (\xi, t_\#)$

$$p(t)u^{'}(t) - p(t_\#)u^{'}(t_\#) = \lambda \int_t^{t_\#} \alpha(s)u(s)ds$$

leading to

$$\lim_{t\to\xi} p(t)u^{'}(t) = p(t_\#)u'(t_\#) + \lim_{t\to\xi} \int_t^{t_\#} \alpha(s)u(s)ds.$$

This completes the proof. $\square$

Let α, β be two functions in $L^1_{\rm loc}(\xi,\eta)$ such that $\alpha(t) \geq 0$, $\beta(t) \geq 0$ for $t \in [\xi,\eta]$ a.e. and each of α and β is positive in a subset of a positive measure; and consider the initial-value problem

$$\begin{gathered} -(pu')'(t) = \lambda(\alpha(t)u^+(t) - \beta(t)u^-(t)), \\ u(t_0) = \lim_{t\to t_0} p(t)u'(t) = 0. \end{gathered} \tag{4.4}$$

By a solution to (4.4) we mean a function $u \in C(\bar{I}) \cap C^1(I)$ with $(pu')' \in L^1_{\rm loc}(I)$ where $I \subset (\xi,\eta)$ is an open interval such that $t_0 \in \bar{I}$ and u satisfies all equations in (4.4).

Lemma 4.3. *Assume that* (2.2) *holds and let α, β be two functions in K_G. Then, for all $t_0 \in [\xi,\eta]$, $u \equiv 0$ is the unique solution of the initial value problem* (4.4).

Proof. The case $\lambda = 0$ is obvious. Let $\lambda \neq 0$ and u be a solution of (4.4) defined on some interval $[t_0, t_*]$ with $t_* \in (t_0,\eta)$ (the case u defined on $[t_*, t_0]$ with $t_* \in (\xi, t_0)$ can be checked similarly). Since $L^1_G[\xi,\eta] \subset L^1_{\rm loc}(\xi,\eta)$ and u is continuous on $[t_0,t_*]$, $(pu')' \in L^1_{\rm loc}(t_0,t_*)$. We distinguish two cases:

- $t_0 \in (\xi,\eta)$. Let $(z_i)_{i=0}^{i=n}$ be such that

$$\begin{gathered} t_0 = z_0 < z_1 < \dots < z_n = t_*, \\ k_i = |\lambda| \int_{z_i}^{z_{i+1}} \frac{d\tau}{p(\tau)} \int_{z_i}^{z_{i+1}} (\alpha(\tau) + \beta(\tau)) d\tau < 1. \end{gathered}$$

Set for $i \in \{0, 1, \dots, (n-1)\}$, $J_i = [z_i, z_{i+1}]$, $X_i = C(J_i)$ equipped with the sup-norm $\|\cdot\|_{i,\infty}$ and $T_i : X_i \to X_i$ with

$$T_i v(t) = -\int_{z_i}^t \Big(\frac{\lambda}{p(s)} \int_{z_i}^s (\alpha(\tau)v^+(\tau) - \beta(\tau)v^-(\tau)) d\tau\Big) ds.$$

Let $v, w \in X_i$, we have

$$\begin{aligned} |T_i v(t) - T_i w(t)| &\leq \int_{z_i}^t \Big(\frac{|\lambda|}{p(s)} \int_{z_i}^s \alpha(\tau)|v^+(\tau) - w^+(\tau)| d\tau\Big) ds \\ &\quad + \int_{z_i}^t \Big(\frac{|\lambda|}{p(s)} \int_{z_i}^s \beta(\tau)|v^-(\tau) - w^-(\tau)| d\tau\Big) ds \end{aligned}$$

then from (2.6)

$$\begin{aligned} \|T_i v - T_i w\|_{i,\infty} &\leq |\lambda| \int_{z_i}^{z_{i+1}} \frac{ds}{p(s)} \int_{z_i}^s (\alpha(\tau) + \beta(\tau)) d\tau \|v - w\|_{i,\infty} \\ &\leq k_i \|v - w\|_{i,\infty}. \end{aligned}$$

So, T_i is a k_i-contraction.

For $i \in \{0, 1, \dots, (n-1)\}$, let u_i be the restriction of u to the interval J_i. We have that u_0 is a fixed point of T_0. Indeed, since $L^1_G[\xi,\eta] \subset L^1_{\rm loc}(\xi,\eta)$ and u is continuous on $[t_0,t_*]$, $(pu')' \in L^1[t_0,t_*]$. So integrating the differential equation in (4.4) over $[z_0, s] \subset [z_0, z_1]$ we obtain from the initial value condition

$$p(s)u_0'(s) = -\int_{z_0}^s (\alpha(\tau)u_0^+(\tau) - \beta(\tau)u_0^-(\tau)) d\tau$$

from which for all $t \in [z_0, z_1]$ we have

$$u_0(t) = -\int_{z_0}^t \Big(\frac{\lambda}{p(s)} \int_{z_i}^s (\alpha(\tau)u_0^+(\tau) - \beta(\tau)u_0^-(\tau)) d\tau\Big) ds = T_0 u_0(t).$$

Thus, the fact that T_0 is a contraction and the trivial function is a fixed point of T_0 lead to $u_0 \equiv 0$, and in particular, we have

$$u_0(z_1) = u_0^{[1]}(z_1) = u_1(z_1) = u_1^{[1]}(z_1) = 0. \tag{4.5}$$

From this equality, u_1 is a fixed of T_1. Then for the same reasons $u_1 \equiv 0$, and in particular,

$$u_1(z_2) = u_1^{[1]}(z_2) = u_2(z_2) = u_2^{[1]}(z_2) = 0.$$

Repeating the above process, we obtain that $u_i \equiv 0$ for all $i \in \{0, 1, \dots, (n-1)\}$; that is, $u \equiv 0$ on $[t_0, t_*]$.

• $t_0 = \xi$: In this case by Lemma 4.2 we can suppose that $u > 0$ in (ξ, t^*). We distinguish two cases:

(a) $\alpha \in L^1[\xi, t^*]$. Let $t_+ \in (\xi, t^*)$ be such that

$$k_+ = |\lambda| \int_\xi^{t_+} \frac{d\tau}{p(\tau)} \int_\xi^{t_+} \alpha(\tau)d\tau < 1.$$

Set $J_+ = [\xi, t_+]$, $X_+ = C(J_+)$ equipped with the sup-norm $\|\cdot\|_{+,\infty}$ and $T_+ : X_+ \to X_+$ with

$$T_+v(t) = -\int_\xi^t \Big(\frac{\lambda}{p(s)} \int_\xi^s \alpha(\tau)v(\tau)d\tau\Big)ds.$$

It is easy to see that T_+ is a k_+-contraction and u_+ the restriction of u to $[\xi, t_+]$ is a fixed point of T_+, so $u_+ \equiv 0$ in $[\xi, t_+]$, and in particular, $u(t_+) = u^{[1]}(t_+) = 0$. Thus, we conclude from the above step that $u \equiv 0$ on its interval of definition contradicting the beginning of this step.

(b) $\alpha \notin L^1[\xi, t^*]$: In this case $b = 0$ and $\int_\xi^{t^*} (\alpha(t) \int_\xi^t \frac{ds}{p(s)})dt < \infty$. Thus, let $t_\infty \in (\xi, t^*)$ be such that

$$k_\infty = |\lambda| \int_\xi^{t_\infty} \Big(\alpha(t) \int_\xi^t \frac{ds}{p(s)}\Big)dt < 1.$$

Set

$$L^1_\alpha[\xi, t_\infty] = \big\{v : (\xi, t_\infty) \to \mathbb{R} \text{ measuable and } \int_\xi^{t_\infty} \alpha(s)|v(s)|ds < \infty\}$$

equipped with the norm

$$\|v\|_{L^1_\alpha[\xi, t_\infty]} = \int_\xi^{t_\infty} \alpha(s)|v(s)|\,ds.$$

Let u_∞ be the restriction of u to the interval $[\xi, t_\infty]$. We claim that u_∞ belongs to $L^1_\alpha[\xi, t_\infty]$. Indeed, integrating the differential equation in (4.4) over $[\epsilon, t_\infty] \subset (\xi, t_\infty]$, we obtain

$$u_\infty^{[1]}(\epsilon) - u_\infty^{[1]}(t_\infty) = \lambda \int_\epsilon^{t_\infty} \alpha(s)u_\infty(s)ds. \tag{4.6}$$

Letting $\epsilon \to \xi$ in (4.6), we obtain

$$|\lambda| \int_\xi^{t_\infty} \alpha(s)u_\infty(s)ds = |u_\infty^{[1]}(t_\infty)|,$$

and then

$$\int_\xi^{t_\infty} \alpha(s)|u_\infty(s)|ds = \int_\xi^{t_\infty} \alpha(s)u_\infty(s)ds = |\frac{u_\infty^{[1]}(t_\infty)}{\lambda}| < \infty.$$

Now, let $T_\infty : L^1_\alpha[\xi, t_\infty] \to L^1_\alpha[\xi, t_\infty]$ be defined by

$$T_\infty v(t) = \int_\xi^t \Big(-\frac{\lambda}{p(s)} \int_\xi^s \alpha(\tau) v(\tau) d\tau \Big) ds.$$

It is easy to see that T_∞ is a k_∞−contraction and u_∞ is a fixed point of T_∞, so $u_\infty \equiv 0$ in $[\xi, t_\infty]$, and in particular, $u(t_\infty) = u^{[1]}(t_\infty) = 0$. Thus, we conclude from the above step that $u \equiv 0$ on its interval of definition contradicting the beginning of this step. This completes the proof. □

Lemma 4.4. *Assume that* (2.2) *and* (2.3) *hold. If λ is a half-eigenvalue of bvp* (2.7) *associated with an eigenvector u then $u \in S_k$ for some $k \geq 1$.*

Proof. If $u(t_0) = u^{[1]}(t_0) = 0$ for some $t_0 \in [\xi, \eta]$ then we have from Lemma 4.3 that $u \equiv 0$, contradicting (λ, u) is a nontrivial solution of (2.7). This shows that u has only simple zeros.

Now, to the contrary, assume that u has an infinite sequence of consecutive zeros (t_n) converging to some $t^* \in [\xi, \eta]$. We have from the continuity of u, $u(t^*) = 0$ and so from Lemma 4.2 $t^* \in (\xi, \eta)$. Because of the simplicity of zeros of u, we have that $(t_n) = (t_n^1) \cup (t_n^2)$ with $u^{[1]}(t_n^1) > 0$ and $u^{[1]}(t_n^2) < 0$. Since $u \in C^1[t^* - \varepsilon, t^* - \varepsilon]$ for some $\varepsilon > 0$ small enough, we obtain that

$$0 \leq \lim u^{[1]}(t_n^1) = u^{[1]}(t^*) = \lim u^{[1]}(t_n^2) \leq 0.$$

Again by Lemma 4.3, $u \equiv 0$, contradicting (λ, u) is a nontrivial solution of (2.7). □

Lemma 4.5. *Assume that* (2.2) *and* (2.3) *hold and $\alpha, \beta \in K_G \cap L^1[\xi, \eta]$. Then, for each integer $k \geq 1$ and $\nu = +$ or $-$, bvp* (2.7) *admits at most one simple half-eigenvalue having an eigenvector in S_k^ν.*

Proof. To the contrary, suppose that $(\lambda_i, \phi_i) \in \mathbb{R} \times S_k^\nu$ satisfy (2.7) for $i = 1, 2$. Then the integrability of $1/p, \alpha$ and β implies that $\phi_i \in S_k^\nu \cap AC[\xi, \eta]$ and $p\phi_i' \in AC[\xi, \eta]$. Let $[\xi_1, \eta_1]$ and $[\xi_2, \eta_2]$ be the intervals given by Theorem 3.2. Since ϕ_1 and ϕ_2 have the same sign in each of $[\xi_1, \eta_1]$ and $[\xi_2, \eta_2]$, we have

$$0 \leq \int_{\xi_1}^{\eta_1} \phi_2 \pounds \phi_1 - \phi_1 \pounds \phi_2 = (\lambda_1 - \lambda_2) \int_{\xi_1}^{\eta_1} \alpha \phi_1^+ \phi_2^+ + \beta \phi_1^- \phi_2^-,$$
$$0 \geq \int_{\xi_2}^{\eta_2} \phi_2 \pounds \phi_1 - \phi_1 \pounds \phi_2 = (\lambda_1 - \lambda_2) \int_{\xi_2}^{\eta_2} \alpha \phi_1^+ \phi_2^+ + \beta \phi_1^- \phi_2^-,$$

leading to $\lambda_1 = \lambda_2$.

Now, suppose that λ is a half-eigenvalue of (2.7) having two eigenvectors ϕ_1 and ϕ_2 with $\phi_1\phi_2 > 0$ in a right neighborhood of ξ, $\phi_1, \phi_2 \in AC[\xi, \eta]$ and $p\phi_1', p\phi_2' \in AC[\xi, \eta]$. Because of the positive homogeneity of bvp (2.7), there exists two eigenvectors ψ_1 and ψ_2 associated with λ such that $\psi_1\psi_2 > 0$ in a right neighborhood of ξ, $\psi_1, \psi_2 \in AC[\xi, \eta]$, $p\psi_1', p\psi_2' \in AC[\xi, \eta]$ and

$$\psi_1(\xi) = \psi_2(\xi) = b, \quad \psi_1^{[1]}(\xi) = \psi_2^{[1]}(\xi) = a.$$

Indeed; Without loss of generality, suppose that $\phi_1 > 0$ and $\phi_2 > 0$ in a right neighborhood of ξ. Then we distinguish the following three cases.

- $\phi_1(\xi) = 0$. In this case we have $b = 0$ and from (2.3) that $a > 0$ (otherwise if $b \neq 0$ we obtain from the boundary condition at ξ that $\phi_1^{[1]}(\xi) = 0$ and Lemma

4.3 leads to $\phi_1 = 0$). The positivity of ϕ_1 near ξ leads to $\phi_1^{[1]}(\xi) > 0$. Since $a > 0$, $b = 0$ and $\phi_2 > 0$ near ξ, we have $\phi_2(\xi) = 0$ and $\phi_2^{[1]}(\xi) > 0$. Thus,

$$\psi_1 = \frac{a\phi_1}{\phi_1^{[1]}(\xi)}, \quad \psi_2 = \frac{a\phi_2}{\phi_2^{[1]}(\xi)}$$

are eigenvectors associated with λ satisfying

$$\psi_1(\xi) = \psi_2(\xi) = b \quad \text{and} \quad \psi_1^{[1]}(\xi) = \psi_2^{[1]}(\xi) = a.$$

- $\phi_1^{[1]}(\xi) = 0$. In this case we have $a = 0$ and from (2.3) that $b > 0$ (otherwise if $a \neq 0$ we obtain from the boundary condition at ξ that $\phi_1(\xi) = 0$ and Lemma 4.3 leads to $\phi_1 = 0$). The positivity of ϕ_1 near ξ leads to $\phi_1(\xi) > 0$. Since $b > 0$, $a = 0$ and $\phi_2 > 0$ near ξ, we have $\phi_2^{[1]}(\xi) = 0$ and $\phi_2(\xi) > 0$. Thus,

$$\psi_1 = \frac{b\phi_1}{\phi_1(\xi)}, \quad \psi_2 = \frac{b\phi_2}{\phi_2(\xi)}$$

are eigenvectors associated with λ satisfying

$$\psi_1(\xi) = \psi_2(\xi) = b \quad \text{and} \quad \psi_1^{[1]}(\xi) = \psi_2^{[1]}(\xi) = a.$$

- $\phi_1(\xi) > 0$ and $\phi_1^{[1]}(\xi) > 0$. This happens only in the case $a > 0$ and $b > 0$ and we have the boundary condition at $\xi, \phi_1(\xi) > 0$ and $\phi_1^{[1]}(\xi) > 0$. Thus,

$$\psi_1 = \frac{a\phi_1}{\phi_1^{[1]}(\xi)} = \frac{b\phi_1}{\phi_1(\xi)}, \quad \psi_2 = \frac{a\phi_2}{\phi_2^{[1]}(\xi)} = \frac{b\phi_2}{\phi_2(\xi)}$$

are eigenvectors associated with λ satisfying

$$\psi_1(\xi) = \psi_2(\xi) = b \text{ and } \psi_1^{[1]}(\xi) = \psi_2^{[1]}(\xi) = a.$$

At this stage, $\psi = \psi_1 - \psi_2$ satisfies

$$-(p\psi')'(t) = \lambda(\alpha(t)\psi^+(t) - \beta(t)\psi^-(t))$$
$$\psi(\xi) = \psi^{[1]}(\xi) = 0,$$

and we have from Lemma 4.3, $\psi = 0$. That is, $\psi_1 = \psi_2$, and then $\phi_1 = \omega\phi_2$ with $\omega > 0$. This shows that the half-eigenvalue λ is simple and completes the proof of Lemma 4.5. □

For $q \in K_G$ we define the linear compact operator $L_q : E \to E$ by

$$L_q u(t) = \int_\xi^\eta G(t,s)q(s)u(s)ds.$$

Since, we will use the global bifurcation theorem of Rabinowitz to prove the main result of this paper, we need to discuss the geometric and algebraic multiplicities of characteristic values of L_q (which are also eingenvalues of bvp (2.8)). Let μ_0 be a characteristic value of L_q and note that $N(\mu_0 L_q - I) \subset N(\mu_0 L_q - I)^2$. Thus, if μ_0 is not simple then μ_0 is of algebraic multiplicity greater than 1. We know from Theorem 3.6 that if $q \in K_G \cap L^1[\xi,\eta]$ then all characteristic values of L_q have the geometric multiplicity equal to one, so let us see what can happens with the algebraic mutiplicity.

Lemma 4.6. *Assume that* (2.2) *and* (2.3) *hold and $q \in K_G \cap L^1[\xi,\eta]$. Then, all characteristic values of L_q are of algebraic multiplicity one.*

Proof. Let $(\mu_k(q))$ be the sequence of characteristic values of L_q given by Theorem 3.6. Thus the eigenvector ϕ_k associated with $\mu_k(q)$ satisfies

$$\begin{gathered}-(p\phi_k')'(t)=\mu_k(q)q(t)\phi_k(t),\quad \text{a.e. } t\in(\xi,\eta),\\ a\phi_k(\xi)-b\lim_{t\to\xi}p(t)\phi_k'(t)=0,\\ c\phi_k(\eta)+d\lim_{t\to\eta}p(t)\phi_k'(t)=0.\end{gathered}$$

Multiplying by ϕ_k and integrating over $[\xi,\eta]$ we obtain

$$\int_\xi^\eta p(\phi_k')^2=\mu_k(q)\int_\xi^\eta q\phi_k^2 \tag{4.7}$$

leading to

$$\mu_k(q)>0,\quad \text{and}\quad \int_\xi^\eta q\phi^2>0. \tag{4.8}$$

Now, let $u\in N\big((\mu_k(q)L_q-I)^2\big)$ and set $v=(\mu_k(q)L_{q_0}-I)(u)=\mu_k(q)L_qu-u$. We have $\mu_k(q)L_qv-v=0$ leading to $v=x\phi$ and

$$\mu_k(q)L_qu-u=x\phi_k. \tag{4.9}$$

On the other hand we have that u satisfies the bvp

$$\begin{gathered}-(pu')'(t)=\mu_k(q)q(t)u(t)-x\mu_k(q)q(t)\phi_k(t),\quad \text{a.e. } t\in(\xi,\eta),\\ au(\xi)-b\lim_{t\to\xi}p(t)u'(t)=0,\\ cu(\eta)+d\lim_{t\to\eta}p(t)u'(t)=0.\end{gathered} \tag{4.10}$$

Multiplying the differential equation in (4.10) by ϕ_k and integrating on (ξ,η) we obtain

$$x\mu_k(q)\int_\xi^\eta q\phi_k^2=0.$$

Because of (4.7), the above equality leads to $x=0$. Therefore, we obtain from (4.9) that $u=\omega\phi_k\in N(\mu_k(q)L_q-I)$ with $\omega\in\mathbb{R}$. This completes the proof. □

It remains to discuss the geometric and algebraic multiplicities of characteristic values of L_q when $q\in(K_G\setminus L^1[\xi,\eta])$. We need the following lemma which is a version of L'Hopital's rule.

Lemma 4.7. *Let f and g be two differentiables functions on $(\xi,\xi+\epsilon)$ with $\epsilon>0$ such that $\lim_{t\to\xi}f(t)=\lim_{t\to\xi}g(t)=+\infty$. If $\lim_{t\to\xi}\frac{f'(t)}{g'(t)}=l$ then $\lim_{t\to\xi}\frac{f(t)}{g(t)}=l$.*

Lemma 4.8. *Assume that (2.2) and (2.3) hold and $q\in K_G\setminus L^1[\xi,\eta]$. Let μ be a characteristic value of L_q associated with an eigenvector ϕ. We have*
1. *If ϕ does not change sign then μ is double*
2. *If ϕ has more than one zero in (ξ,η) then μ is simple.*

Proof. Suppose that ψ is another eigenvector associated with the caracteristic value μ and let $W=W(\phi,\psi)$ be the Wronksian of ϕ and ψ. By simple computations follows $(pW)'=0$, from which we obtain

$$\phi\psi'-\phi'\psi=\frac{B}{p},\quad B\in\mathbb{R}. \tag{4.11}$$

Considering (4.11) as a linear first order differential equation where the unknown is ψ, we obtain that ψ takes the form

$$\psi(t) = A\phi(t) + B\psi_\varepsilon(t) \quad \text{with } A, B \in \mathbb{R},\ \varepsilon \in (\xi, \eta),$$
$$\psi_\varepsilon(t) = \phi(t) \int_\varepsilon^t \frac{ds}{p(s)\phi^2(s)}.$$

Thus, we have to examine for $\varepsilon \in (\xi, \eta)$ the ability of the function ψ_ε to be an eigenvector associated with μ or not. Without loss of generality, suppose that q is not integrable at ξ and η (the other cases can be checked similarly). This occurs if $b = d = 0$ and in this case the boundary conditions in bvp (2.1) become the Dirichlet conditions

$$u(\xi) = u(\eta) = 0. \tag{4.12}$$

1. Suppose that ϕ is positive in (ξ, η) and let $\varepsilon \in (\xi, \eta)$ be fixed. We have by simple computations

$$p(t)\psi_\varepsilon'(t) = \frac{1}{\phi(t)} + p(t)\phi'(t) \int_\varepsilon^t \frac{ds}{p(s)\phi^2(s)}, \quad \text{for all } t \in (\xi, \eta) \tag{4.13}$$

then

$$-(p\psi_\varepsilon')'(t) = \lambda q(t)\psi_\varepsilon(t), \quad \text{a.e. } t \in (\xi, \eta). \tag{4.14}$$

Moreover, since q is not integrable at ξ and η, from Lemma 4.2 and [36, Theorem 2.3.1] we have

$$\lim_{t\to\xi} p(t)\phi'(t) = \infty, \quad \lim_{t\to\eta} p(t)\phi'(t) = \infty.$$

Thus, from Lemma 4.7 when $\lim_{t\to\xi} \int_\varepsilon^t \frac{ds}{p(s)\phi^2(s)} = \infty$ (the case $\int_\xi^\varepsilon \frac{ds}{p(s)\phi^2(s)} < \infty$ is obvious), we have

$$\lim_{t\to\xi} \psi_\varepsilon(t) = \lim_{t\to\xi} \frac{\big(\int_\varepsilon^t \frac{ds}{p(s)\phi^2(s)}\big)'}{\big(\frac{1}{\phi(t)}\big)'} = \lim_{t\to\xi} \frac{\frac{1}{p(t)\phi^2(t)}}{-\frac{\phi'(t)}{\phi^2(t)}} = -\frac{1}{\lim_{t\to\xi} p(t)\phi'(t)} = 0$$

and also

$$\lim_{t\to\eta} \psi_\varepsilon(t) = -\frac{1}{\lim_{t\to\eta} p(t)\phi'(t)} = 0.$$

That is, ψ_ε satisfies the boundary conditions (4.12) and all the above shows that ψ_ε is an eigenvector of the characteristic value μ of L_q. Moreover, since the function $\varphi_\varepsilon(t) = A + B\int_\varepsilon^t \frac{ds}{p(s)\phi^2(s)} 0$ vanishes at most once in (ξ, η), the eigenvector ψ lies in $S_1 \cup S_2$ and this shows that μ is double.

2. Note that if $\phi(t_1) = 0$ for some $t_1 \in (\xi, \eta)$, we obtain from (4.13) following $t_1 > \varepsilon$ and $t_1 < \varepsilon$ that at least one of the limits

$$\lim_{t>t_1,\, t\to t_1} p(t)\psi_\varepsilon'(t), \quad \lim_{t<t_1,\, t\to t_1} p(t)\psi_\varepsilon'(t)$$

are infinite and this means that $\psi_\varepsilon \notin Y_\#$. So, the function ψ_ε can not be an eigenvector associated with the characteristic value μ of L_q.

By the contrary suppose that the characteristic value μ is not simple and there exists another eigenvector of μ, $\phi_1 \in Y_\#$. In this case, arguing as in the discussion in the beginning of this proof we obtain

$$\phi(t) = A\phi_1(t) + B\phi_1(t) \int_\varepsilon^t \frac{ds}{p(s)\phi_1^2(s)}$$

and

$$p(t)\phi'(t) = A\phi_1(t) + \frac{B}{\phi_1(t)} + p(t)\phi_1'(t)\int_\varepsilon^t \frac{ds}{p(s)\phi_1^2(s)}, \quad \text{for all } t \in (\xi,\eta)$$

where $A, B \in \mathbb{R}$ and $\varepsilon \in (\xi,\eta)$.

Thus, arguing as in the beginning of part 2 of this proof we obtain that the eigenvector ϕ_1 must be positive. Therefore, from

$$\phi(t) = \phi_1(t)\Big(A + B\int_\varepsilon^t \frac{ds}{p(s)\phi_1^2(s)}\Big)$$

yields that ϕ has at most one zero in (ξ,η). Contradicting ϕ has more than one zero. This completes the proof. □

Lemma 4.9. *Assume that* (2.2) *and* (2.3) *hold and* $q \in K_G \setminus L^1[\xi,\eta]$*. If* μ *is a characteristic value of* L_q *associated with an eigenvector* ϕ *vanishing more than one time in* (ξ,η) *then* μ *is of algebraic multiplicity one.*

Proof. In the same way as in the proof of Lemma 4.6, let us show that if u is a solution to

$$\begin{gathered} -(pu')'(t) = \mu q(t)u(t) - x\mu q(t)\phi(t), \quad \text{a.e. } t\in(\xi,\eta), \\ au(\xi) - b\lim_{t\to\xi} p(t)u'(t) = 0, \\ cu(\eta) + d\lim_{t\to\eta} p(t)u'(t) = 0. \end{gathered} \tag{4.15}$$

then $u = \omega\phi$ with $\omega \in \mathbb{R}$. Let u be a solution to (4.15) and $W = W(\phi,u)$ be the Wronksian of ϕ and u. We have that W satisfies

$$(pW)' = (pu'\phi - pu\phi')' = x\mu q\phi^2$$

leading to

$$u'\phi - u\phi' = \frac{B}{p} + \frac{x\mu}{p}\int_\varepsilon^s q(\tau)(\phi(\tau))^2 d\tau, \quad B \in \mathbb{R}. \tag{4.16}$$

Considering (4.16) as a linear first order differential equation where the unknown is u, we obtain that u takes the form

$$\begin{aligned} u(t) &= A\phi(t) + B\phi(t)\int_\varepsilon^t \frac{ds}{p(s)\phi^2(s)} \\ &\quad + x\mu\phi(t)\int_\varepsilon^t \Big(\frac{1}{p(s)\phi^2(s)}\int_\varepsilon^s q(\tau)(\phi(\tau))^2 d\tau\Big) ds, \end{aligned} \tag{4.17}$$

for $A, B \in \mathbb{R}$ and $\varepsilon \in (\xi,\eta)$.

Arguing as in 2 of the proof of Lemma 4.8, we see that the expression for u given in (4.17) is a solution of (4.10) if and only if $B = x = 0$. This completes the proof. □

Lemma 4.10. *Assume that* (2.2) *and* (2.3) *hold and let* (q_n) *be a sequence in* L_G^1 *converging to* $q \in L_G^1$*. Then* $L_{q_n} \to L_q$ *as* $n \to \infty$ *in operator norm. Moreover, for all* $[\xi_0,\eta_0] \subset (\xi,\eta)$

$$p(L_{q_n}u)' \to p(L_q u)' \quad \text{in } C[\xi_0,\eta_0] \text{ for all } u \in E.$$

Proof. It is easy to check that for all $[\xi_0, \eta_0] \subset (\xi, \eta)$, $q_n \to q$ in $L^1[\xi_0, \eta_0]$, $\Phi_{ab} q_n \to \Phi_{ab} q$ in $L^1[\xi, \xi_0]$, and $\Psi_{cd} q_n \to \Psi_{cd} q$ in $L^1[\eta_0, \eta]$. Thus, we have, for all $[\xi_0, \eta_0] \subset (\xi, \eta)$ and $u \in E$ with $\|u\|_\infty = 1$,

$$\sup_{t\in[\xi,\eta]} |L_{q_n} u(t) - L_q u(t)| \le \int_\xi^\eta G(s,s)|q_n(s) - q(s)|ds = \|q_n - q\|_G,$$

leading to $L_{q_n} \to L_q$ as $n \to \infty$.

Also, for all $[\xi_0, \eta_0] \subset (\xi, \eta)$ and $u \in E$ with $\|u\|_\infty = 1$,, we have

$$\begin{aligned}
&|p(t)(L_{q_n}u)'(t) - p(t)(L_q u)'(t)| \\
&\le \int_{\xi_0}^{\eta_0} \frac{c\Phi_{ab}(s) + a\Psi_{cd}(s)}{\Delta} |q_n(s) - q(s)|ds \\
&\quad + \int_\xi^{\xi_0} \frac{c}{\Delta}\Phi_{ab}(s)|q_n(s) - q(s)|ds + \int_{\eta_0}^{\eta} \frac{a}{\Delta}\Psi_{cd}(s)|q_n(s) - q(s)|ds \\
&= \frac{c}{\Delta}\|\Phi_{ab}(q_n - q)\|_{L^1[\xi,\xi_0].} + \frac{a}{\Delta}\|\Psi_{cd}(q_n - q)\|_{L^1[\eta_0,\eta].} + \|q_n - q\|_{L^1[\xi_0,\eta_0]}
\end{aligned}$$

leading to $p(L_{q_n}u)' \to p(L_q u)'$ in $C[\xi_0, \eta_0]$. The proof is complete. □

Lemma 4.11. *Assume that* (2.2) *and* (2.3) *hold and let* (α_n) *and* (β_n) *be two sequences in* K_G *converging, respectively, to* α *and* β *in* $L^1_G[\xi, \eta]$. *Assume also that for all integers* $n \ge 1$, *there exist* $\lambda_n > 0$ *and* $\phi_n \in E \setminus \{0\}$ *satisfying*

$$\begin{gathered}
-(p\phi_n')'(t) = \lambda_n(\alpha_n(t)\phi_n^+(t) - \beta_n(t)\phi_n^-(t)), \quad t \in (\xi, \eta), \\
a\phi_n(\xi) - b\lim_{t\to\xi} p(t)\phi_n'(t) = 0, \\
c\phi_n(\eta) + d\lim_{t\to\eta} p(t)\phi_n'(t) = 0.
\end{gathered}$$

We have $\phi_n = \lambda_n A_n \phi_n$ *where* $A_n = L_{\alpha_n} I^+ - L_{\beta_n} I^-$.

If (λ_n) *converges to* $\widetilde{\lambda} > 0$, *then there exists* $\psi \in E \setminus \{0\}$ *such that* $\psi = \widetilde{\lambda} A\psi$ *where* $A = L_\alpha I^+ - L_\beta I^-$ *(i.e.* $\widetilde{\lambda}$ *is a half-eigenvalue to bvp* (2.7)*).*

Proof. First, note that Lemma 4.10 guarantee that $L_{\alpha_n} \to L_\alpha$ and $L_{\beta_n} \to L_\beta$ in operator norm. Let ϕ_n be the eigenvector corresponding to λ_n with $\|\phi_n\|_\infty = 1$ and set $\psi_n = \lambda_n A\phi_n$ and $\psi = \lim \psi_n$ (up to a subsequence). We have

$$\begin{aligned}
\|\phi_n - \psi\|_\infty &= \|\lambda_n A_n(\phi_n) - \psi\|_\infty \\
&\le |\lambda_n| \|A_n(\phi_n) - A(\phi_n)\|_\infty + \|\lambda_n A(\phi_n) - \psi\|_\infty \\
&\le |\lambda_n| \|L_{\alpha_n} - L_\alpha\| + |\lambda_n| \|L_{\beta_n} - L_\beta\| + \|\lambda_n A(\phi_n) - \psi\|_\infty,
\end{aligned}$$

leading to $\lim \phi_n = \psi$ and $\|\psi\|_\infty = 1$. Also we have

$$\begin{aligned}
\|\lambda_n A_n(\phi_n) - \widetilde{\lambda} A(\psi)\|_\infty &\le \|\lambda_n A_n(\phi_n) - \lambda_n A_n(\psi)\|_\infty + \|\lambda_n A_n(\psi) - \lambda_n A(\psi)\|_\infty \\
&\quad + \|\lambda_n A(\psi) - \widetilde{\lambda} A(\psi)\|_\infty \\
&\le |\lambda_n| \|A_n\| \|\phi_n - \psi\|_\infty + |\lambda_n| \|A_n - A\| + |\lambda_n - \widetilde{\lambda}| \|A\|,
\end{aligned}$$

leading to $\lim \lambda_n A_n(\phi_n) = \widetilde{\lambda} A(\psi)$. At the end, letting $n \to \infty$ in the equation $\phi_n = \lambda_n A_n(\phi_n)$ we obtain $\psi = \widetilde{\lambda} A\psi$. □

Remark 4.12. Arguing as in the proof of Lemma 4.10, one can prove that $p\phi_n' \to p\psi'$ in $C[\xi_0, \eta_0]$ for all $[\xi_0, \eta_0] \subset (\xi, \eta)$ where ϕ_n and ψ are those of the above proof.

Let $\Lambda_{a,b,c,d}$ and $\Lambda_{a,b}$ be the functions defined in Subsection 3.5. We deduce from Theorem 3.12 a first result for existence of half-eigenvalues in the case where $p \equiv 1$ and the functions α and β are constants.

Corollary 4.13. *Assume that $p \equiv 1$ and α and β are positive constants. Then bvp (2.7) admits two sequences of half eigenvalues (λ_k^+) and (λ_k^-) such that*

- λ_1^+ *is the unique solution of* $\Lambda_{a,b,c,d}(\alpha\sigma) = \eta - \xi$,
- λ_1^- *is the unique solution of* $\Lambda_{a,b,c,d}(\beta\sigma) = \eta - \xi$,
- λ_{2l}^+ *with $l \geq 1$ is the unique solution of*
$$\Lambda_{a,b}(\alpha\sigma) + \Lambda_{c,d}(\beta\sigma) + \pi(l-1)\Big(\frac{1}{\sqrt{\alpha\sigma}} + \frac{1}{\sqrt{\beta\sigma}}\Big) = \eta - \xi,$$
- λ_{2l}^- *with $l \geq 1$ is the unique solution of*
$$\Lambda_{a,b}(\beta\sigma) + \Lambda_{c,d}(\alpha\sigma) + \pi(l-1)\Big(\frac{1}{\sqrt{\alpha\sigma}} + \frac{1}{\sqrt{\beta\sigma}}\Big) = \eta - \xi,$$
- λ_{2l+1}^+ *with $l \geq 1$ is the unique solution of*
$$\Lambda_{a,b}(\alpha\sigma) + \Lambda_{c,d}(\alpha\sigma) + \frac{\pi(l-1)}{\sqrt{\alpha\sigma}} + \frac{\pi l}{\sqrt{\beta\sigma}} = \eta - \xi,$$
- λ_{2l+1}^- *with $l \geq 1$ is the unique solution of*
$$\Lambda_{a,b}(\beta\sigma) + \Lambda_{c,d}(\beta\sigma) + \frac{\pi(l-1)}{\sqrt{\beta\sigma}} + \frac{\pi l}{\sqrt{\alpha\sigma}} = \eta - \xi.$$

Proposition 4.14. *Assume that $p \equiv 1$ and α and β are positive and continuous on $[\xi, \eta]$. Then the set of half-eigenvalues of bvp (2.7) consists of two increasing sequences of simple half-eigenvalues $(\lambda_k^+)_{k\geq 1}$ and $(\lambda_k^-)_{k\geq 1}$, such that for all $k \geq 1$ and $\nu = +$ or $-$, the corresponding half-lines of solutions are in $\{\lambda_k^\nu\}\times S_k^\nu$. Moreover for all $k \geq 1$ and $\nu = +$ or $-$, λ_k^ν is a decreasing function with respect to the weights α and β lying in $C[\xi, \eta]$.*

Proof. Consider the bvp
$$\begin{gathered} -u''(t) = \theta(\alpha(t) + \beta(t))u^+(t) + \lambda\alpha(t)u^-(t) - \lambda\beta(t)u^+(t),\ t \in (\xi, \eta), \\ au(\xi) - b\lim_{t\to\xi} p(t)u'(t) = 0, \\ cu(\eta) + d\lim_{t\to\eta} p(t)u'(t) = 0. \end{gathered} \tag{4.18}$$
From Theorem 3.9 We have that for each integer $k \geq 1$, $\nu = +$ or $-$ and all $\lambda \geq 0$, there exists a unique $\theta_k^\nu(\lambda)$ such that (4.18) has a solution in S_k^ν.

Note that $\theta_k^\nu(0) = \mu_k(\alpha + \beta) > 0$. Now we claim that there exists $\lambda_0 > 0$ such that $\theta_k^\nu(\lambda_0) \leq \lambda_0$. To the contrary, assume that for all $\lambda \geq 0$, $\theta_k^\nu(\lambda) > \lambda$. Thus we have from Proposition 3.10 that
$$\lambda < \theta_k^\nu(\lambda) < \theta_k^\nu(\lambda, \alpha_+, \beta_+) = \theta^*(\lambda)$$
where for $k \geq 1$ and $\nu = +$ or $-$, $\theta_k^\nu(\lambda, \alpha_+, \beta_+)$ is the unique real number for which
$$\begin{gathered} -u''(t) = \theta(\alpha_+ + \beta_+)u^+(t) + \lambda\alpha_+u^-(t) - \lambda\beta_+u^+(t),\ t \in (\xi, \eta), \\ au(\xi) - b\lim_{t\to\xi} p(t)u'(t) = 0, \\ cu(\eta) + d\lim_{t\to\eta} p(t)u'(t) = 0. \end{gathered}$$

has a solution in S_k^ν.

Assume that $k = 2l$ with $l \geq 1$ and $\nu = +$ or $-$ (the other cases can be checked similarly). We have from Corollary 4.13 that

$$\begin{aligned}
&\Lambda_{a,b}((\theta^*(\lambda)-\lambda)\beta_+ + \theta^*(\lambda)\alpha_+) + \Lambda_{c,d}((\theta^*(\lambda)-\lambda)\alpha_+ + \theta^*(\lambda)\beta_+) \\
&+ \frac{\pi(l-1)}{\sqrt{(\theta^*(\lambda)-\lambda)\beta_+ + \theta^*(\lambda)\alpha_+}} + \frac{\pi(l-1)}{\sqrt{(\theta^*(\lambda)-\lambda)\alpha_+ + \theta^*(\lambda)\beta_+}} = \eta - \xi.
\end{aligned} \tag{4.19}$$

Taking into consideration the fact that $\Lambda_{a,b}$ and $\Lambda_{c,d}$ are decreasing functions, we obtain from (4.19) that

$$\begin{aligned}
\eta - \xi &\leq \Lambda_{a,b}(\theta^*(\lambda)\alpha_+) + \Lambda_{c,d}(\theta^*(\lambda)\beta_+) + \frac{\pi(l-1)}{\sqrt{\theta^*(\lambda)\alpha_+}} + \frac{\pi(l-1)}{\sqrt{\theta^*(\lambda)\beta_+}} \\
&\leq \pi l\Big(\frac{1}{\sqrt{\theta^*(\lambda)\alpha_+}} + \frac{1}{\sqrt{\theta^*(\lambda)\beta_+}}\Big)
\end{aligned}$$

leading to

$$\theta^*(\lambda) \leq \pi^2 l^2 (\eta-\xi)^2 \Big(\frac{1}{\sqrt{\alpha_+}} + \frac{1}{\sqrt{\beta_+}}\Big)^2,$$

which contradicts $\lim_{\lambda\to+\infty} \theta^*(\lambda) = +\infty$.

Thus there exists λ_k^ν such that $\theta_k^\nu(\lambda_k^\nu) = \lambda_k^\nu$ and λ_k^ν is a half-eigenvalue of (2.7). Uniqueness and simplicity of λ_k^ν follow from Lemma 4.5. Finally, the monotonicity of λ_k^ν with respect of the weights α and β follows directly from Proposition 3.10. □

Proposition 4.15. *Assume that $p \equiv 1$, α, β are nonnegative and continuous on $[\xi, \eta]$ and the set $\{t \in [\xi, \eta] : \alpha(t)\beta(t) > 0\}$ has positive measure. Then the set of half-eigenvalues of bvp (2.7) consists of two increasing sequences of simple half-eigenvalues $(\lambda_k^+)_{k\geq 1}$ and $(\lambda_k^-)_{k\geq 1}$, such that for all $k \geq 1$ and $\nu = +$ or $-$, the corresponding half-lines of solutions are in $\{\lambda_k^\nu\} \times S_k^\nu$. Moreover for all $k \geq 1$ and $\nu = +$ or $-$, λ_k^ν is a decreasing function with respect to the weights α and β lying in $C[\xi, \eta]$.*

Proof. For $n \geq 1$, $\alpha_n = \alpha + \frac{1}{n}$ and $\beta_n = \beta + \frac{1}{n}$, let $\lambda_{k,n}^\nu = \lambda_k^\nu(\alpha_n, \beta_n)$ be the half-eigenvalue given by Proposition 4.14 associated with the eigenvector $\phi_n \in \Theta_k^\nu$. Because (α_n) and (β_n) are decreasing sequences, we have from Proposition 4.14 that $(\lambda_{k,n}^\nu)_n$ is nondecreasing. Now let $I_0 = [\xi_0, \eta_0] \subset (\xi, \eta)$ be such that $\alpha\beta > 0$ in I_0 and set $\vartheta = \min(\alpha, \beta)$. Hence, we have $\vartheta_n = \min(\alpha_n, \beta_n) = \vartheta + \frac{1}{n} \geq \vartheta$ and we deduce, from the montonicity property in Proposition 4.14 and Properties 5 and 6 in Theorem 2.7, that

$$\lambda_{k,n}^\nu = \lambda_k^\nu(\alpha_n, \beta_n) \leq \lambda_k^\nu(\vartheta_n, \vartheta_n) = \mu_k(\vartheta_n, [\xi, \eta]) \leq \mu_k(\vartheta_n, I_0) \leq \mu_k(\vartheta, I_0),$$

and the sequence $(\lambda_{k,n}^\nu)_n$ converges to some $\lambda_k^\nu > 0$, which is by Lemma 4.11 and Lemma 4.5, a simple half-eigenvalue of bvp (2.7) having an eigenvector $\phi = \lim \phi_n \in \overline{\Theta_k^\nu}$ (up to a subsequence). Because the functions $u \in \partial\Theta_k^\nu$ have a double zero, Lemma 4.4 guarantees that $\phi \in \Theta_k^\nu$.

Let α_1 be a nonnegative continuous function such that the set $\{t \in [\xi, \eta] : \alpha_1(t)\beta(t) > 0\}$ has a positive measure and $\alpha \leq \alpha_1$. We have from Proposition 4.14 that

$$\lambda_k^\nu\Big(\alpha + \frac{1}{n}, \beta + \frac{1}{n}\Big) \leq \lambda_k^\nu\Big(\alpha_1 + \frac{1}{n}, \beta + \frac{1}{n}\Big).$$

Letting $n \to \infty$ we obtain $\lambda_k^\nu(\alpha,\beta) \le \lambda_k^\nu(\alpha_1,\beta)$. Similarly we prove that λ_k^ν is nonincreasing with respect to the weight β. The proof is complete. □

4.2. **Proof of Theorem 2.4.** Let φ and ρ_0 be as in Section 2 and note that λ is a half-eigenvalue with an eigenvector u of (2.7) if and only if λ/ρ_0 is a half-eigenvalue with the eigenvector $v = u \circ \varphi^{-1}$ of the bvp

$$\begin{gathered} -v''(t) = \frac{\lambda}{\rho_0}\Big(\widetilde{\alpha}(t)v^+(t) - \widetilde{\beta}(t)v^-(t)\Big), \quad t \in (\xi,\eta), \\ av(\xi) - b\rho_0 v'(\xi) = 0, \\ cv(\eta) + d\rho_0 v'(\eta) = 0, \end{gathered} \tag{4.20}$$

where

$$\widetilde{\alpha}(t) = p(\varphi^{-1}(t))\alpha(\varphi^{-1}(t)), \quad \widetilde{\beta}(t) = p(\varphi^{-1}(t))\beta(\varphi^{-1}(t))$$

are integrable functions. So, it suffices to prove Theorem 2.4 with $p \equiv 1$. To this aim, let (α_n) and (β_n) be two sequences in $C_c(\xi,\eta)$ such that $\lim \alpha_n = \alpha$ and $\lim \beta_n = \beta$ in $L^1(\xi,\eta)$, and let $\lambda_{k,n}^\nu = \lambda_k^\nu(\alpha_n,\beta_n)$ be the half-eigenvalue given by Proposition 4.15 associated with an eigenvector ϕ_n. Let $\vartheta_n = \inf(\alpha_n,\beta_n)$, $\theta_n = \sup(\alpha_n,\beta_n)$, and $\theta = \sup(\alpha,\beta) \ge \vartheta = \inf(\alpha,\beta) > 0$ on some interval $I_0 = [\xi_0,\eta_0] \subset (\xi,\eta)$. We have that $\lim \vartheta_n = \vartheta$ in $L^1(\xi,\eta)$. Then, we deduce from the monotonicity property in Proposition 4.15 and Property 5 in Theorem 3.6 that

$$\begin{aligned} 0 < \mu_k(\vartheta) - \epsilon &= \mu_k(\vartheta_n) \\ &= \lambda_k^\nu(\theta_n,\theta_n) \le \lambda_{k,n}^\nu \\ &= \lambda_k^\nu(\alpha_n,\beta_n) \le \lambda_k^\nu(\vartheta_n,\vartheta_n) \\ &= \mu_k(\vartheta_n) \le \mu_k(\vartheta) + C \end{aligned}$$

where the constant ϵ and C are respectively small enough and large enough. Let $\lambda_{k,s}^\nu = \limsup \lambda_{k,n}^\nu$ and $\lambda_{k,i}^\nu = \liminf \lambda_{k,n}^\nu$. We have from Lemma 4.11 that $\lambda_{k,s}^\nu$ and $\lambda_{k,i}^\nu$ are half-eigenvalues of (2.7). Then we deduce from Lemma 4.5 that $\lambda_k^\nu = \lambda_{k,s}^\nu = \lambda_{k,i}^\nu$ is the unique and simple half-eigenvalue of (2.7). The same arguments as those used in the proof of Proposition 4.15 show that the eigenvector associated with λ_k^ν belongs to $S_k^\nu \cap Y$.

4.3. **Proof of Theorem 2.5.**

4.3.1. *Proof of uniqueness of* λ_1^ν.

Lemma 4.16. *Assume that Hypotheses* (2.2) *and* (2.3) *hold and* $\alpha,\beta \in K_G$. *Then for* $\nu = +$ *or* $-$, *bvp* (2.7) *admits at most one half-eigenvalue having an eigenvector in* S_1^ν.

Proof. Suppose that λ_1^+ is a half-eigenvalue having an eigenvector $\phi_1 \in S_1^+$ (uniqueness of λ_1^- can be proved in the same way), then $1/\lambda_1^+$ is a positive eigenvalue of the positive operator $L_\alpha : E \to E$ defined by

$$L_\alpha u(t) = \int_\xi^\eta G(t,s)\alpha(s)u(s)ds.$$

So, we have that $r(L_\alpha) > 0$ and since the cone of nonnegative functions is total in E, $r(L_\alpha)$ is a positive eigenvalue of L_α and

$$\lambda_1^+ \ge 1/r(L_\alpha). \tag{4.21}$$

Let (ξ_n) and (η_n) be two sequences in (ξ,η) such that $\lim \xi_n = \xi$, $\lim \eta_n = \eta$, (ξ_n) is decreasing, (η_n) is increasing, and set

$$\alpha_n(t) = \begin{cases} \inf(\alpha(t), \alpha(\xi_n)), & \text{if } t \le \xi_n, \\ \alpha(t), & \text{if } t \in (\xi_n, \eta_n), \\ \inf(\alpha(t), \alpha(\eta_n)), & \text{if } t \ge \eta_n, \end{cases}$$

and let $L_n : E \to E$ be the linear operator defined by

$$L_n u(t) = \int_\xi^\eta G(t,s)\alpha_n(s)u(s)ds. \tag{4.22}$$

We see that for all $n \in \mathbb{N}$, $L_n \le L_\alpha$. Then from Lemma 3.4 we have $r(L_n) \le r(L_\alpha)$.

We have that for all $n \in \mathbb{N}$, $\lambda_1^n = 1/r(L_n) > 0$ is the unique positive eigenvalue associated with a positive eigenvector ϕ_1^n to the linear bvp

$$\begin{gathered} -(pu')'(t) = \lambda\alpha_n(t)u(t), \quad \text{a.e. } t \in (\xi,\eta), \\ au(\xi) - b\lim_{t\to\xi} p(t)u'(t) = 0, \\ cu(\eta) + d\lim_{t\to\eta} p(t)u'(t) = 0. \end{gathered}$$

Moreover, $\alpha_n \to \alpha$ in $L^1_G[\xi,\eta]$. So, we have from Lemmas 4.10 and 3.5 that

$$\lim \lambda_1^n = 1/r(L_\alpha) \le \lambda_1^+. \tag{4.23}$$

Before proving uniqueness, note that, if λ is a positive eigenvalue of bvp (2.8) associated with an eigenvector ϕ, then there exists a subinterval $[\gamma,\delta] \subset (\xi,\eta)$ such that $\alpha(t)\phi(t) > 0$ for almost all $t \in [\gamma,\delta]$. Indeed if this does not occur, we obtain the contradiction

$$\phi(t) = \lambda\int_\xi^\eta G(t,s)\alpha(s)\phi(s)ds = 0 \quad \text{for all } t \in (\xi,\eta).$$

This means also that

$$\phi(t) = \lambda\int_\xi^\eta G(t,s)\alpha(s)\phi(s)ds > 0 \text{ for all } t \in (\xi,\eta).$$

Set

$$\psi_n = L_n\phi_1 \le L_\alpha\phi_1 = (\lambda_1^+)^{-1}\phi_1.$$

Observe that ψ_n satisfies

$$\begin{gathered} -(p'\psi_n)'(t) = \alpha_n(t)\phi_1(t) \ge \lambda_1^+\alpha_n(t)\psi_n(t) \quad \text{a.e. } t \in (\xi,\eta), \\ a\psi_n(\xi) - b\lim_{t\to\xi} p(t)\psi_n'(t) = 0, \\ c\psi_n(\eta) + d\lim_{t\to\eta} p(t)\psi_n'(t) = 0. \end{gathered} \tag{4.24}$$

Multiplying the differential inequality in (4.24) by ϕ_1^n (the eigenvector of λ_1^n) and integrating over $[\xi,\eta]$ we obtain

$$\int_\xi^\eta -(p\psi_n')'\phi_1^n \ge \lambda_1^+\int_\xi^\eta \alpha_n\psi_n\phi_1^n.$$

We find, after two integration by parts of the left hand side,

$$\lambda_1^n\int_\xi^\eta \alpha_n\psi_n\phi_1^n \ge \lambda_1^+\int_\xi^\eta \alpha_n\psi_n\phi_1^n,$$

leading to $\lambda_1^+ \leq \lambda_1^n$ for all $n \geq 1$, from which we have

$$\lambda_1^+ \leq \lim \lambda_1^n. \tag{4.25}$$

At the end, combining (4.25) with (4.23), we obtain $\lambda_1^+ = 1/r(L_\alpha)$, that is $1/r(L_\alpha)$ is the unique half eigenvalue of (2.7) having an eigenvector in S_1^+. □

4.3.2. *Proof of uniqueness of λ_k^ν, $k \geq 2$.*

Lemma 4.17. *Assume that Hypotheses* (2.2) *and* (2.3) *hold and $\alpha, \beta \in K_G$. Then for each integer $k \geq 1$ and $\nu = +$ or $-$, bvp* (2.7) *admits at most one half-eigenvalue having an eigenvector in S_k^ν.*

Proof. To the contrary, assume that λ_1 and λ_2 are two half-eigenvalues having, respectively, the eigenvectors $\phi_1, \phi_2 \in S_k^\nu$ with the sequences of simple zeros $(x_i)_{1\leq i\leq k}$ and $(y_i)_{1\leq i\leq k}$. In the spirit of Theorem 3.1, assume that $x_1 \leq y_1$ and let $i_0, j_0 \in \{1, \dots, k\}$ such that $x_{i_0} \leq z_{j_0} \leq z_{j_0+1} \leq x_{i_0+1}$, and without loss of generality, suppose that $\phi_1 \geq 0$ and $\phi_2 \geq 0$ in each of the intervals $[\xi, x_1]$ and $[z_{j_0}, z_{j_0+1}]$. Let (ξ_n) and (η_n) be the sequences given in the proof of Lemma 4.16 and set

$$\alpha_n^1(t) = \begin{cases} \inf(\alpha(t), \alpha(\xi_n)), & \text{if } t \leq \xi_n, \\ \alpha(t), & \text{if } t \in (\xi_n, z_1). \end{cases}$$

From Lemma 4.16, we have $\lambda_1 = \lim \mu_1(\alpha_n^1, [\xi, x_1])$ and $\lambda_2 = \lim \mu_1(\alpha_n^1, [\xi, z_1])$, and from Property 4 of Theorem 3.6, that for all $n \geq 1$, $\mu_1(\alpha_n^1, [\xi, x_1]) \geq \mu_1(\alpha_n^1, [\xi, z_1])$. Letting $n \to \infty$ we obtain $\lambda_1 \geq \lambda_2$.

Now we will discuss the cases $z_{j_0+1} < \eta$ and $z_{j_0+1} = \eta$. If $z_{j_0+1} < \eta$, then integrating on $[z_{j_0}, z_{j_0+1}]$, we obtain

$$0 \geq \int_{z_{j_0}}^{z_{j_0+1}} -(p\phi_1')'\phi_2 + (p\phi_2')'\phi_1 = (\lambda_1 - \lambda_2) \int_{z_{j_0}}^{z_{j_0+1}} \alpha\phi_1\phi_2,$$

leading to $\lambda_1 = \lambda_2$.

If $z_{j_0+1} = \eta$, then considering

$$\alpha_n^2(t) = \begin{cases} \alpha(t), & \text{if } t \in (x_{i_0}, \eta_n), \\ \inf(\alpha(t), \alpha(\nu_n)), & \text{if } t \geq \eta_n, \end{cases}$$

we have that $\lambda_1 = \lim \mu_1(\alpha_n^2, [x_{i_0}, \eta])$ and $\lambda_2 = \lim \mu_1(\alpha_n^2, [z_{j_0}, \eta])$, and from Property 3 of Theorem 2.5, that for all $n \geq 1$, $\mu_1(\alpha_n^2, [z_{j_0}, \eta]) \geq \mu_1(\alpha_n^2, [x_{i_0}, \eta])$. So, letting $n \to \infty$ we obtain also in this case $\lambda_1 = \lambda_2$. This completes the proof. □

4.3.3. *Proof of existence of $(\lambda_k^\nu)_{k\geq 1}$.* Let (ξ_n) and (η_n) be the sequences introduced in the proof of Lemma 4.16 and consider

$$\alpha_n(t) = \begin{cases} \alpha(t), & \text{if } t \in (\xi_n, \eta_n), \\ 0, & \text{if } t \notin (\xi_n, \eta_n), \end{cases}$$

$$\beta_n(t) = \begin{cases} \beta(t), & \text{if } t \in (\xi_n, \eta_n), \\ 0, & \text{if } t \notin (\xi_n, \eta_n). \end{cases}$$

For all $n, k \geq 1$ and $\nu = +$ or $-$, let $\lambda_{k,n}^\nu$ be the unique half-eigenvalue of

$$\begin{gathered} -(pu')'(t) = \lambda(\alpha_n(t)u^+(t) - \beta_n(t)u^-(t)) \quad \text{a.e. } t \in (\xi, \eta), \\ au(\xi) - b\lim_{t\to\xi} p(t)u'(t) = 0, \end{gathered}$$

$$cu(\eta) + d \lim_{t \to \eta} p(t)u'(t) = 0,$$

having eigenvector $\phi_n \in S_k^\nu$ with $\|\phi_n\| = 1$, $\theta_n = \sup(\alpha_n, \beta_n)$ and $\theta = \sup(\alpha, \beta) > 0$ in some closed interval $I_0 \subset (\xi, \eta)$. Since (α_n) and (β_n) are nondecreasing sequences, we have from Property 2 in Theorem 2.4 that, for all $n \geq 1$,

$$\lambda_{k,n}^\nu \geq \lambda_{k,n+1}^\nu,$$

and

$$\lambda_{k,n}^\nu = \lambda_k^\nu(\alpha_n, \beta_n) \geq \lambda_k^\nu(\theta_n, \theta_n) = \mu_k(\theta_n) > \mu_1(\theta_n). \tag{4.26}$$

Because $\mu_1(\theta_n) = 1/r(L_{\theta_n})$, $\mu_1(\theta) = 1/r(L_\theta)$ and $L_{\theta_n} \to L_\theta$ in operator norm, it follows from Lemma 3.5 that, for $\epsilon > 0$ small enough,

$$\lambda_{k,n}^\nu \geq \lambda_{k,n+1}^\nu \geq \mu_1(\theta) - \epsilon > 0.$$

Thus, from Lemma 4.11 we have that $\lambda_k^\nu = \lim_{n\to\infty} \lambda_{k,n}^\nu$ is a half-eigenvalue of (2.7)) having an eigenvector ψ (as it it shown in proof of Lemma 4.11 $\psi = \lim \phi_n$).

In view of Lemma 4.4, it remains to show that $\psi \in S_k^\nu$. To the contrary, assume that $\psi \in S_l^+$ with $l \neq k$ and let $(z_j)_{j=1}^{j=l-1}$ be the sequence of interior zeros of ψ and $[\xi_1, \eta_1] \subset (\xi, \eta)$ such that

$$\xi_1 < z_1 < z_2 < \cdots < z_{l-1} < \eta_1.$$

Choose $\delta > 0$ small enough and set $I_j = (z_j - \delta, z_j + \delta)$ for $j \in \{1, \ldots, l-1\}$. There exists $n_* \in \mathbb{N}$ such that for all $n \geq n_*$, $\phi_n\psi > 0$ in all the intervals $[\xi_1, z_1 - \delta]$, $[z_{k-1} + \delta, \eta_1]$, $[z_j + \delta, z_{j+1} - \delta]$, $j \in \{1, \ldots, l-2\}$.

Fix $j \in \{1, \ldots, l-1\}$. There exists $n_j \geq n_*$ such that the function ϕ_n has exactly one zero in I_j. Otherwise if there is a subsequence (ϕ_{n_i}) such that for all $i \geq 1$, ϕ_{n_i} has at least two zeros, then we can choose $x_{n_i}^1$ and $x_{n_i}^2$ in I_j such that

$$\phi_{n_i}^{[1]}(x_{n_i}^1) \leq 0 \leq \phi_{n_i}^{[1]}(x_{n_i}^2).$$

Let

$$\begin{gathered} x_{\inf}^1 = \liminf x_{n_i}^1, \quad x_{\sup}^1 = \limsup x_{n_i}^1 \\ x_{\inf}^2 = \liminf x_{n_i}^1, \quad x_{\sup}^2 = \liminf x_{n_i}^1. \end{gathered}$$

Hence, since $\psi = \lim \phi_n/\|\phi_n\|$ we have

$$\psi(x_{\inf}^1) = \psi(x_{\inf}^2) = \psi(x_{\sup}^1) = \psi(x_{\sup}^2) = 0$$

leading to

$$\lim x_{n_i}^1 = \lim x_{n_i}^2 = z_j.$$

Moreover, from Remark 4.12 it follows that

$$\psi^{[1]}(z_j) = \lim \phi_{n_l}^{[1]}(x_{n_i}^1) = \lim \phi_{n_l}^{[1]}(x_{n_i}^2) = 0,$$

contradicting the simplicity of z_j.

Now, we claim that there exists $n^* \in \mathbb{N}$ such that for all $n \geq n^*$, ϕ_n does not vanish in the intervals (ξ, ξ_1) and (η_1, η). Again, to the contrary, assume that there is a subsequence (ϕ_{n_i}) such that for all $i \geq 1$, ϕ_{n_i} has at least one zero. Let $x_{n_i} \in (\xi, \xi_1)$ be the first zero of ϕ_{n_i}. In this case, we have that

$$\lim x_{n_i} = \xi, \quad \psi(\xi) = \phi_{n_i}(\xi) = 0.$$

Moreover, for all $i \geq 1$, ϕ_{n_i} satisfies

$$\begin{gathered} -(p\phi_{n_i}')'(t) = \mu_{n_i}\alpha_{n_i}(t)\phi_{n_i}(t) \quad \text{a.e. } t \in (\xi, x_{n_i}), \\ \phi_{n_i}(\xi) = \phi_{n_i}(x_{n_i}) = 0 \\ \phi_{n_i} > 0 \ in \ (\xi, x_{n_i}). \end{gathered} \tag{4.27}$$

Clearly, Equation (4.27) implies that $\mu_{n_i} = \mu_1(\alpha_{n_i}, [\xi, x_{n_i}])$. Taking into consideration $\lim \alpha_n = \alpha$ in $L^1_G[\xi, \eta]$ and $\mu_{n_i} = \mu_1(\alpha_{n_i}, [\xi, x_{n_i}]) = 1/r(L^{n_i}_\alpha)$ where $L^{n_i}_\alpha : C[\xi, x_{n_i}] \to C[\xi, x_{n_i}]$ is defined by

$$L^{n_i}_\alpha u(t) = \int_\xi^{x_{n_i}} G_{n_i}(t,s)\alpha(s)u(s)ds,$$

from Lemma 3.5 we obtain

$$\mu_{n_i} = \mu_1(\alpha_{n_i}, [\xi, x_{n_i}]) = \frac{1}{r(L^{n_i}_{\alpha_{n_i}})} \geq \frac{1}{r(L^{n_i}_\alpha)} = \mu_1(\alpha, [\xi, x_{n_i}]). \tag{4.28}$$

Thus, combining Lemma 4.1 with (4.28)), we obtain the contradiction

$$\widetilde{\lambda} = \lim \mu_{n_i} \geq \lim \mu_1(\alpha, [\xi, x_{n_i}]) = \lim \frac{1}{r(L^{n_i}_\alpha)} = +\infty.$$

Hence, we conclude that for all $n \geq n_\infty = \max\{n_*, n^*, n_1, \dots, n_{k-1}\}$, ϕ_n has exactly $(l-1)$ simple zeros in (ξ, η) contradicting $\phi_n \in S^+_k$.

Finally, letting $n \to \infty$ in $\lambda^\nu_{k,n} < \lambda^\nu_{k+1,n}$ we obtain $\lambda^\nu_k \leq \lambda^\nu_{k+1}$.

4.4. **Proof of Theorem 2.7.** The existence of $(\mu_k(q))_{k\geq 1}$ as a nondecreasing sequence follows from Theorem 2.5 when taking $\alpha = \beta = q$ in bvp (2.7) and for all $k \geq 1$, $\mu_k(q)$ has an eigenvector $\phi_k \in S_k$. We have from Lemma 4.5 and assertion 2 in Lemma 4.8 that $\mu_k(q)$ is simple for all $k \geq 3$ and Assertion 1 is proved. Assertion 2 follows from the monotonicity of the sequence $(\mu_k(q))_{k\geq 1}$ and the simplicity of $\mu_k(q)$ for $k \geq 3$. Assertion 3 follows from Lemma 4.5, and assertion 1 in Lemma 4.8 and Lemma 4.16. Assertion 4 is obvious.

Assertion 5 follows from the monotonicity property of half-eigenvalues in Theorem 2.5 and Assertion 6 is obtained when letting $n \to \infty$ in the relation

$$\mu_k(q_n, [\xi, \eta]) \leq \mu_k(q_n, [\xi_1, \eta_1]),$$

where

$$q_n(t) = \begin{cases} q(t), & \text{if } t \in (\xi_n, \eta_n), \\ 0, & \text{if } t \notin (\xi_n, \eta_n), \end{cases}$$

and $(\xi_n)_{n\geq 1}, (\eta_n)_{n\geq 1}$ are those in the proof of Lemma 4.16.

It remains to prove Assertion 7. Let $(q_n) \subset K_G$ be a sequence converging to $q \in K_G$ in $L^1_G[\xi, \eta]$, and $[\xi_0, \eta_0] \subset (\xi, \eta)$ such that $q > 0$ in $[\xi_0, \eta_0]$. We have then from Property 6 and Property 5 in Theorem 3.6

$$\mu_k(q_n, [\xi, \eta]) \leq \mu_k(q_n, [\xi_0, \eta_0]) \leq \mu_k(q, [\xi_0, \eta_0]) + C.$$

Set $\mu^1 = \liminf \mu_k(q_n, [\xi, \eta])$ and $\mu^2 = \limsup \mu_k(q_n, [\xi, \eta])$. There exist two subsequences $(\mu_k(q_{n_i}, [\xi, \eta]))$ and $(\mu_k(q_{n_j}, [\xi, \eta]))$ of $(\mu_k(q_n, [\xi, \eta]))$ converging respectively to μ^1 and μ^2. Applying Lemma 4.11, we obtain that μ^1 and μ^2 are eigenvalues of (2.8). Furthermore, arguing as in Subsection 4.3.3, we see that the eigenvectors associated with μ^1 and μ^2 belongs to S_k. Thus, we deduce from Lemmas 4.16 and 4.17 that

$$\mu^1 = \mu^2 = \lim \mu_k(q_n, [\xi, \eta]) = \mu_k(q, [\xi, \eta]).$$

This completes the proof

4.5. **Proof of Theorem 2.9.** Consider the bifurcation bvp associated with bvp (2.1),

$$\begin{gathered} -(pu')'(t) = \mu q_0(t)u(t) + \mu g(t, u(t)), \quad \text{a.e. } t \in (\xi, \eta), \\ au(\xi) - b \lim_{t\to\xi} p(t)u'(t) = 0, \\ cu(\eta) + d \lim_{t\to\eta} p(t)u'(t) = 0, \end{gathered} \tag{4.29}$$

where μ is a real parameter and $g(t, u) = f(t, u) - q_0(t)u$ and in all that follows, we denote by $(\mu_k(q_0))_{k\geq 1}$ the sequence of eigenvalues obtained from Theorem 2.7 for the bvp

$$\begin{gathered} \pounds u(t) = \mu q_0(t)u(t), \quad \text{a.e. } t \in (\xi, \eta), \\ au(\xi) - b \lim_{t\to\xi} p(t)u'(t) = 0, \\ cu(\eta) + d \lim_{t\to\eta} p(t)u'(t) = 0, \end{gathered}$$

and by $(\chi_k^\nu)_{k\geq 1}$, with $\nu = +$ or $-$, the two sequences of half-eigenvalues of the bvp

$$\begin{gathered} -(pu')'(t) = \chi(\alpha_\infty(t)u^+(t) - \beta_\infty(t)u^-(t)), \quad \text{a.e. } t \in (\xi, \eta), \\ au(\xi) - b \lim_{t\to\xi} p(t)u'(t) = 0, \\ cu(\eta) + d \lim_{t\to\eta} p(t)u'(t) = 0, \end{gathered}$$

given by Theorem 2.5.

Applying $\pounds^{-1}$, we obtain that bvp (4.29) is equivalent to the equation

$$u = \mu L_{q_0} u + \mu H(u)$$

where $H : E \to E$ is defined by

$$Hu(s) = \int_\xi^\eta G(t, s)g(s, u(s))ds$$

and is completely continuous.

Lemma 4.18. *Assume that* (2.2)-(2.4) *hold. Then from each $\mu_k(q_0)$, with $k \geq 3$, bifurcate two unbounded components (in $\mathbb{R} \times E$), Γ_k^+ and Γ_k^- such that for all $k \geq 3$ and $\nu = +$ or $-$, $\Gamma_k^\nu \subset \mathbb{R} \times S_k^\nu$.*

Proof. First, note that Hypothesis (2.4) implies that $H(u) = \circ(\|u\|_\infty)$ near 0. Indeed, we have for $(u_n) \subset E$ with $\lim \|u_n\|_\infty = 0$

$$\frac{|Hu_n(t)|}{\|u_n\|_\infty} \leq \int_\xi^\eta R_n(s)ds \quad \text{where } R_n(s) = G(s, s)\frac{|f(s, u_n(s)) - q_0(s)u_n(s)|}{\|u_n\|_\infty}.$$

Then, it follows from (2.5) that

$$\begin{aligned} R_n(s) &\leq G(s, s)(\gamma_\infty(s) + \delta_\infty(s) + q_0(s))\frac{|u_n(s)|}{\|u_n\|_\infty} \\ &\leq G(s, s)(\gamma_\infty(s) + \delta_\infty(s) + q_0(s)) \in L^1[\xi, \eta] \end{aligned}$$

and from (2.4) that

$$R_n(s) \leq G(s, s)\Big|\frac{f(s, u_n(s))}{u_n(s)} - q_0(s)\Big|\frac{|u_n(s)|}{\|u_n\|_\infty}$$

$$\leq G(s,s)|\frac{f(s,u_n(s))}{u_n(s)} - q_0(s)| \to 0$$

as $n \to +\infty$, $s \in [\xi, \eta]$ a.e..

So, by the Lebesgue dominated convergence theorem, we have

$$\lim \frac{H(u_n)}{\|u_n\|_\infty} = 0;$$

that is, $H(u) = o(\|u\|_\infty)$ at 0.

Since for all $k \geq 3$, $\mu_k(q_0)$ is of algebraic multiplicity one, from [25, Theorem 1.40] we conclude that from each $(\mu_k(q_0), 0)$ with $k \geq 3$, bifurcate two components Γ_k^1 and Γ_k^2 of nontrivial solutions of bvp (4.29) such that for $i = 1, 2$, Γ_k^i is either unbounded in $\mathbb{R} \times E$ or meets $(\mu_l(q_0), 0)$ where $l \neq k$.

Now, note that if $(\lambda, u) \in \Gamma_k^i$ $i = 1, 2$ then all zeros of u are simple. This is due to the fact that (λ, u) satisfies also the bvp

$$\begin{gathered}
-(pv')'(t) = \lambda q_u(t)v(t), \ t \in (\xi, \eta), \\
av(\xi) - b\lim_{t\to\xi} p(t)v'(t) = 0, \\
cv(\eta) + d\lim_{t\to\eta} p(t)v'(t) = 0,
\end{gathered}$$

with

$$q_u(t) = \frac{f(t,u)}{u}.$$

Since Hypothesis (2.5) guarantees that $q_u \in L_G^1[\xi, \eta]$, we deduce from Theorem 2.7 that there exists an integer $j \geq 1$ such that $u \in S_j$.

Also, we claim that for all $k \geq 3$ and $i = 1, 2$, there exists a neighborhood V_k^i of $(\mu_k(q_0), 0)$ such that $\Gamma_k^i \cap V_k^1 \subset \mathbb{R} \times S_k$. Let $(\mu_n, u_n)_{n\geq 1} \subset \Gamma_k^i$ be a sequence converging to $(\mu_k, 0)$. Thus, $v_n = u_n/\|u_n\|_\infty$ satisfies

$$v_n = \mu_n L_{q_0}(v_n) + \mu_n \frac{H(u_n)}{\|u_n\|_\infty} \quad \text{and} \quad \|v_n\|_\infty = 1.$$

Since L_{q_0} is compact and $H(u) = o(\|u\|_\infty)$ near 0, there exists a subsequence (v_{n_j}) of (v_n) converging to v in E satisfying

$$v = \mu_k(q_0)L_{q_0}v \quad \text{and} \quad \|v\|_\infty = 1.$$

So, from Theorem 2.7 we have $v \in S_k$.

Let $(z_j)_{j=1}^{j=k-1}$ be the sequence of interior zeros of v and $[\xi_1, \eta_1] \subset (\xi, \eta)$ such that

$$\xi_1 < z_1 < z_2 < \dots < z_{k-1} < \eta_1.$$

Choose $\delta > 0$ small enough and set $I_j = (z_j - \delta, z_j + \delta)$ for $j \in \{1, \dots, k-1\}$. There exists $n_* \in \mathbb{N}$ such that for all $n_j \geq n_*$, $v_{n_j}v > 0$ in all the intervals $[z_j + \delta, z_{j+1} - \delta]$ $j \in \{1, \dots, k-2\}$, $[\xi_1, z_1 - \delta]$, $[z_{k-1} + \delta, \eta_1]$.

Fix $j \in \{1, \dots, k-1\}$. There exists $n_j \geq n_*$ such that the function v_n has exactly one zero in I_j. Otherwise if there is a subsequence (v_{n_l}) such that for all $l \geq 1$, v_{n_l} has at least two zeros, then we can choose $x_{n_l}^1$ and $x_{n_l}^2$ in I_j such that

$$v_{n_l}^{[1]}(x_{n_l}^1) \leq 0 \leq v_{n_l}^{[1]}(x_{n_l}^2).$$

Hence, we obtain

$$\lim x_{n_l}^1 = \lim x_{n_l}^2 = z_j,$$

and from Lemma 4.10 that

$$v^{[1]}(z_j) = \lim v_{n_l}^{[1]}(x_{n_l}^1) = \lim v_{n_l}^{[1]}(x_{n_l}^2) = 0,$$

contradicting the simplicity of z_j.

Now, we claim that there exists $n^* \in \mathbb{N}$ such that for all $n \geq n^*$, v_n does not vanish in the intervals (ξ, ξ_1) and (η_1, η). To the contrary, assume that there is a subsequence (v_{n_l}) such that, for all $l \geq 1$, v_{n_l} has at least one zero. Let $x_{n_l} \in (\xi, \xi_1)$ be the first zero of v_{n_l}. In this case, we have that

$$\lim x_{n_l} = \xi, \quad v(\xi) = v_n(\xi) = 0.$$

Moreover, for all $l \geq 1$, u_{n_l} satisfies

$$\begin{gathered} -(pu_{n_l}')'(t) = \mu_{n_l} q_{n_l}(t) u_{n_l}(t), \quad t \in (\xi, x_{n_l}), \\ u_{n_l}(\xi) = u_{n_l}(x_{n_l}) = 0, \\ u_{n_l} > 0 \quad \text{in } (\xi, x_{n_l}), \end{gathered} \tag{4.30}$$

where $q_{n_l}(t) = (f(t, u_{n_l}(t))/u_{n_l}(t))$. Clearly, Equation (4.30) implies that $\mu_{n_l} = \mu_1(q_{n_l}, [\xi, x_{n_l}])$. Taking into consideration Hypothesis (2.5), from Property 3 in Theorem 2.7 we obtain that

$$\mu_{n_l} = \mu_1(q_{n_l}, [\xi, x_{n_l}]) \geq \mu_1(\gamma_\infty, [\xi, x_{n_l}]).$$

So, we obtain as in Subsection 4.3.3 the contradiction

$$\mu_k(q_0) = \lim \mu_{n_l} \geq \lim \mu_1(\gamma_\infty, [\xi, x_{n_l}]) = +\infty.$$

At this stage we conclude that, for all $n \geq n_\infty = \max\{n_*, n^*, n_1, \dots, n_{k-1}\}$, v_n has exactly $(k-1)$ simple zeros in (ξ, η) and so the existence of the neighborhood V_k^ν.

Using the same arguments as those used above, we see that for all $(\mu_0, u_0) \in \Gamma_k^i$ with $u_0 \in S_{k_1}$, there exists a neighborhood W_0 of (μ_0, u_0) such that $W_0 \cap \Gamma_k^\nu \subset \mathbb{R} \times S_{k_1}^\nu$. This shows that the number of zeros of functions u lying in the projection of Γ_k^ν onto the space E is locally constant, so it is constant and it is equal to $(k-1)$. Thus, $\Gamma_k^i \subset \mathbb{R} \times S_k$. Set

$$\Gamma_k^+ = (\Gamma_k^1 \cap \mathbb{R} \times S_k^+) \cup (\Gamma_k^2 \cap \mathbb{R} \times S_k^+) \text{and } \Gamma_k^- = (\Gamma_k^1 \cap \mathbb{R} \times S_k^-) \cup (\Gamma_k^2 \cap \mathbb{R} \times S_k^-)$$

and let $\varsigma > 0$ and $\varkappa \in (0,1)$. We have from Theorem 1.25 in [25] that, for $i = 1, 2$, there exists a sequence $(\mu_n^i, u_n^i)_{n \geq 1} \subset \Gamma_k^i$ such that $|\mu_n^i - \mu_k(q_0)| < \varsigma$, $u_n^i = t_n^i v_k + w_n^i, \lim t_n^i = 0$ and $t_n^1 > \varkappa \|u_n^1\|$, $t_n^2 < -\varkappa \|u_n^2\|$. Moreover, from [25, Lemma 1.24] we have that $w_n^i = o(|t_n^i|)$. Arguing as above, we see that $\lim(u_n^i / \|u_n^i\|_\infty) = v^i$ (up to a subsequence) where v^1 and v^2 are eigenvectors associated with $\mu_k(q_0)$ with $v^1 > 0$ near ξ and $v^2 < 0$ near ξ. So $v^1 \in S_k^+$ and $v^2 \in S_k^-$. Since the limits are in $E = C[\xi, \eta]$, arguing as in the proof of existence of neighborhood V_k^i, in the beginning of the proof, we obtain that $u_n^1 \in S_k^+$ and $u_n^2 \in S_k^-$ for n large enough. This shows that for all $k \geq 3$ and $\nu = +$ or $-$, $\Gamma_k^\nu \neq \emptyset$, and again because of the topology of E (if $v_n \to v$ in E and $v > 0$ near ξ then $v_n > 0$ near ξ for n large) and functions u lying in the projection of Γ_k^ν onto the space E have only simple zeros and all have the same number of zeros, Γ_k^ν does not leave $\mathbb{R} \times S_k^\nu$.

Finally, taking into consideration the claim in the beginning of the proof, and the fact that Γ_k^ν does not leave $\mathbb{R} \times S_k^\nu$ we understand that for all $k \geq 3$ and $\nu = +$ or $-$, Γ_k^ν is unbounded in $\mathbb{R} \times E$. ◻

Lemma 4.19. *Assume that (2.2)-(2.5) hold. Then for all $k \geq 3$ and $\nu = +$ or $-$, the component Γ_k^ν rejoins the point (χ_k^ν, ∞).*

Proof. Because $u \equiv 0$ is the unique solution of bvp (4.29) for $\mu = 0$, we have from Lemma 4.18 that for all $k \geq 1$ and $\nu = +$ or $-$, $(\{0\} \times E) \cap \Gamma_k^\nu = \emptyset$. Therefore, if $(\mu, u) \in \Gamma_k^\nu$ then $\mu > 0$.

Moreover, if $(\mu, u) \in \Gamma_k^\nu$, then $\mu = \mu_k(\frac{f(t,u)}{u}, [\xi, \eta])$, and this together with Hypothesis (2.5) and Property 3 of Theorem 2.7, leads to

$$\mu = \mu_k\Big(\frac{f(t,u)}{u}, [\xi, \eta]\Big) \leq \mu_k(\delta_\infty, [\xi, \eta]).$$

This shows that for all $k \geq 1$ and $\nu = +$ or $-$, the projection of Γ_k^ν onto the real axis is bounded.

Now, let (μ_n, u_n) be a sequence in Γ_k^ν such that $\lim_{n\to+\infty} \|u_n\| = +\infty$. For contradiction purposes, suppose that $\lim_{n\to+\infty} \mu_n \neq \chi_k^\nu$. Then there exist $\varepsilon > 0$ and a subsequence of (μ_n), which will be denoted for convenience by $(\mu_n)_{n\geq 1}$, such that

$$|\mu_n - \chi_k^\nu| \geq \varepsilon.$$

Denote by (v_n) the sequence defined by $v_n = \frac{u_n}{\|u_n\|}$. Note that $\|v_n\|_\infty = 1$ and (μ_n, v_n) satisfies

$$v_n = \mu_n A_\infty v_n + \frac{\Omega u_n}{\|u_n\|_\infty}$$

where $A, K : E \to E$ are defined by

$$A_\infty u(t) = \int_\xi^\eta G(t,s)(\alpha_\infty(s)u^+(s) - \beta_\infty(s)u^-(s))ds,$$
$$\Omega u(t) = \int_\xi^\eta G(t,s) g_\infty(s, u(s))ds,$$

with $g_\infty(t,x) = f(t,x) - \alpha_\infty(t)x^+ + \beta_\infty(t)x^-$. Note that Hypotheses (2.4) and (2.5) imply that $\Omega u_n = o(\|u_n\|_\infty)$ at ∞. Indeed, we have

$$\begin{aligned}
\frac{|\Omega u_n(t)|}{\|u_n\|_\infty} &= \Big|\int_\xi^\eta G(t,s)\Big(\frac{g_\infty(s, u_n(s))}{\|u_n\|_\infty}\Big)ds\Big| \\
&\leq \int_\xi^\eta G(s,s)\Big|\frac{f(s, u_n(s)) - \alpha_\infty(s)u_n^+(s) + \beta_\infty(s)u_n^-(s)}{\|u_n\|_\infty}\Big|ds.
\end{aligned}$$

From (2.4) we have

$$\begin{aligned}
&G(s,s)\Big|\frac{f(s, u_n(s)) - \alpha_\infty(s)u_n^+(s) + \beta_\infty(s)u_n^-(s)}{\|u_n\|_\infty}\Big| \\
&\leq G(s,s)(\gamma_\infty(s) + \delta_\infty(s) + \alpha_\infty(s) + \beta_\infty(s))\frac{|u_n(s)|}{\|u_n\|_\infty} \\
&\leq G(s,s)(\gamma_\infty(s) + \delta_\infty(s) + \alpha_\infty(s) + \beta_\infty(s)) \in L^1[\xi, \eta].
\end{aligned}$$

Now, set

$$P_n(s) = G(s,s)\Big|\frac{f(s, u_n(s)) - \alpha_\infty(s)u_n^+(s) + \beta_\infty(s)u_n^-(s)}{\|u_n\|_\infty}\Big|$$

and let us prove that $\lim P_n(s) = 0$ for $s \in [\xi, \eta]$ a.e..

Let $s \in [\xi, \eta]$ (such s exists a.e.), such that

$$\lim_{x \to +\infty} \frac{f(s,x)}{x} = \alpha_\infty(s) \quad \text{and} \quad \lim_{x \to -\infty} \frac{f(s,x)}{x} = \beta_\infty(s).$$

For such an s we distinguish the following cases:

- $\lim u_n(s) = +\infty$: in this case we have

$$\begin{aligned} P_n(s) &= G(s,s)\Big|\frac{f(s,u_n(s))}{u_n(s)} - \alpha_\infty(s)\Big|\frac{u_n(s)}{\|u_n\|_\infty} \\ &\le G(s,s)\Big|\frac{f(s,u_n(s))}{u_n(s)} - \alpha_\infty(s)\Big| \to 0 \quad \text{as } n \to +\infty. \end{aligned}$$

- $\lim u_n(s) = -\infty$: in this case we have

$$\begin{aligned} P_n(s) &= G(s,s)\Big|\frac{f(s,u_n(s))}{u_n(s)} - \beta_\infty(s)\Big|\frac{|u_n(s)|}{\|u_n\|_\infty} \\ &\le G(s,s)\Big|\frac{f(s,u_n(s))}{u_n(s)} - \beta_\infty(s)\Big| \to 0 \quad \text{as } n \to +\infty. \end{aligned}$$

- $\lim u_n(s) \neq \pm\infty$: in this case there may exist subsequences $(u_{n_k^1}(s))$ and $(u_{n_k^2}(s))$ such that $(u_{n_k^1}(s))$ is bounded and $\lim u_{n_k^2}(s) = \pm\infty$. Arguing as in the above two cases we obtain $\lim P_{n_k^2}(s) = 0$ and we have

$$P_{n_k^1}(s) \le G(s,s)(\gamma_\infty(s) + \delta_\infty(s) + \alpha_\infty(s) + \beta_\infty(s))\frac{|u_{n_k^1}(s)|}{\|u_{n_k^1}\|_\infty} \to 0 \text{ as } k \to +\infty.$$

Thus, we have $\lim P_n(s) = 0$ for $s \in [\xi, \eta]$ a.e. By the Lebesgue dominated convergence theorem, we conclude that $\Omega u_n = o(\|u_n\|_\infty)$ at ∞.

Now, because of the compactness of A_∞ and the boundedness of (v_n), there exists a subsequence (v_{n_j}) converging to $v \in S_k^\nu$ (use the same arguments as in the proof of Lemma 4.18) with $\|v\|_\infty = 1$ satisfying $v = \chi_\infty A v$, where χ_∞ is the limit of some subsequence of (μ_{n_l}) of (μ_n). Thus, we have $\chi_\infty = \chi_k^\nu$ and the contradiction

$$0 = \lim |\mu_{n_l} - \chi_k^\nu| \ge \varepsilon > 0.$$

□

Now, we are able to prove Theorem 2.9. Note that $u \in S_i^\nu$ is a solution to (2.1) if and only if $(1,u) \in \Gamma_i^\nu$, and this occurs if $\lambda_i^\nu < 1 < \mu_i(q_0)$ or $\mu_i(q_0) < 1 < \lambda_i^\nu$.

Assume that $\mu_l(q_0) < 1 < \mu_k(\theta_\infty)$ with $2 < k < l$, and let $i \in \{k, \dots, l\}$. We have $\mu_i(q_0) \le \mu_l(q_0) < 1$, and from the nondecreasing property of λ_i^ν with respect to the functions α and β,

$$\lambda_i^\nu = \lambda_i^\nu(\alpha_\infty, \beta_\infty) \ge \lambda_i^\nu(\theta_\infty, \theta_\infty) = \mu_i(\theta_\infty) \ge \mu_k(\theta_\infty) > 1.$$

Now, assume that $\mu_l(\vartheta_\infty) < 1 < \mu_k(q_0)$ with $2 < k < l$, and let $i \in \{k, \dots, l\}$. We have $\mu_i(q_0) \ge \mu_k(q_0) > 1$, and from the nondecreasing property of λ_i^ν with respect to the functions α and β,

$$\lambda_i^\nu = \lambda_i^\nu(\alpha_\infty, \beta_\infty) \le \lambda_i^\nu(\vartheta_\infty, \vartheta_\infty) = \mu_i(\vartheta_\infty) \ge \mu_k(\vartheta_\infty) > 1.$$

Thus, Theorem 2.9 is proved.

Remark 4.20. Note that if $q_0 \in K_G \cap L^1[\xi, \eta]$, from Lemma 4.6 we have that for all $n \ge 1$, $\mu_n(q_0)$ is of algebraic multiplicity one. Thus, Theorem 2.9 and Corollary 2.10 can be extended to the case $1 \le k < l$.

Remark 4.21. Theorem 2.9 holds if we replace Hypothesis (2.4) by the following assumptions:

$$|f(t,u)-q_0(t)u| \leq \hat{g}_0(t,|u|) \quad \text{for all } |u| \leq \varsigma_0, ,\ t \in [\xi,\eta] \text{ a.e., for some } \varsigma_0 > 0,$$
$$|f(t,u)-\alpha_\infty(t)u^+ + \beta_\infty(t)u^-| \leq \hat{g}_\infty(t,|u|) \quad \text{for all } u \in \mathbb{R},\ t \in [\xi,\eta] \text{ a.e.},$$
$$\lim_{u\to\varrho} \frac{\hat{g}_\varrho(t,u)}{u} = 0 \quad \text{in } L^1_G[\xi,\eta] \text{ for } \varrho = 0,+\infty$$

where $\hat{g}_0, \hat{g}_{+\infty} : [\xi,\eta]\times\mathbb{R}^+ \to \mathbb{R}^+$ are such that for $\varrho = 0,+\infty$, $\hat{g}_\varrho,(\cdot,u) \in L^1_G[\xi,\eta]$ for u fixed and $\hat{g}_\varrho(t,\cdot)$ is nondecreasing for $t \in [\xi,\eta]$ a.e. Indeed, we have for $(u_n) \subset E$ with $\lim \|u_n\|_\infty = 0$

$$\begin{aligned}\frac{|Hu_n(t)|}{\|u_n\|_\infty} &\leq \int_\xi^\eta G(s,s)\frac{|f(s,u_n(s)) - q_0(s)u_n(s)|}{\|u_n\|_\infty}ds \\ &\leq \int_\xi^\eta G(s,s)\frac{\hat{g}_0(s,\|u_n\|_\infty)}{\|u_n\|_\infty}ds \to 0 \quad \text{as } n \to +\infty\end{aligned}$$

leading to $Hu = o(\|u\|_\infty)$ at 0.

Also, for $(u_n) \subset E$ with $\lim \|u_n\|_\infty = +\infty$ we have

$$\begin{aligned}\frac{|\Omega u_n(t)|}{\|u_n\|_\infty} &\leq \int_\xi^\eta G(s,s)\frac{|f(s,u_n(s)) - \alpha_\infty(s)u_n^+(s) + \beta_\infty(s)u_n^-(s)|}{\|u_n\|_\infty}ds \\ &\leq \int_\xi^\eta G(s,s)\frac{\hat{g}_{+\infty}(s,\|u_n\|_\infty)}{\|u_n\|_\infty}ds \to 0 \quad \text{as } n \to +\infty\end{aligned}$$

leading to $\Omega u = o(\|u\|_\infty)$ at ∞.

The function f given in Example 2.2 satisfies the above condition with

$$\begin{gathered} q_0(t) = At^{-3/2}(1-t)^{-5/4}, \\ \alpha_\infty(t) = At^{-3/2}(1-t)^{-5/4} + Bt^{-7/6}(1-t)^{-7/4}, \\ \beta_\infty(t) = At^{-3/2}(1-t)^{-5/4} + Ct^{-11/7}(1-t)^{-13/10}, \end{gathered}$$

and

$$\hat{g}_0(t,u) = (Bt^{-7/6}(1-t)^{-7/4} + Ct^{-11/7}(1-t)^{-13/10})u^3,$$
$$\hat{g}_\infty(t,u) = \begin{cases} MCt^{-11/7}(1-t)^{-13/10} + Bt^{-\frac{7}{6}}(1-t)^{-7/4}, & \text{for } u > 0, \\ Ct^{-11/7}(1-t)^{-13/10} + MBt^{-7/6}(1-t)^{-7/4}, & \text{for } u < 0, \end{cases}$$

where

$$M = \sup\{\frac{x^3}{1+x^2+e^x} : x \geq 0\}.$$

Acknowledgements. The authors want to thank the anonymous referee for the comments and suggestions about this article.

A. Benmezaï and W. Esserhane, would like to thank their laboratory, Fixed Point Theory and Applications, for supporting this work.

References

[1] N. Benkaci-Ali, A. Benmezaï, S. K. Ntouyas; *Eigenvalue criteria for existence of a positive solution to singular three point BVP,* J. Abstr. Differ. Equ. Appl. **2** (2012), No. 2, 48–55.

[2] A. Benmezai; *Positive solutions for a second order two point boundary value problem*, Commun. Appl. Anal. **14** (2010), No. 2, 177–190.

[3] A. Benmezaï, J. Graef, L. Kong; *Positive solutions to a two point singular boundary value problem*, Differ. Equ. Appl. **3** (2011), 347-373.
[4] H. Berestycki; *On some non-linear Sturm-Liouville boundary value problems*, J. Differential Equations **26** (1977), 375-390.
[5] H. Berestycki; *Le nombre de solution de certains problèmes semi-linéaires élliptiques*, J. Functional Anal. **40** (1981), 1-29.
[6] P. A. Binding, B. P. Rynne; *Half-eigenvalues of periodic Sturm-Liouville problems*, J. Differential Equations **206** (2004), 280-305.
[7] M. Chhetri, S. Robinson; *Multiple positive solutions for singular boundary value problems*, Comm. Appl. Nonlinear Anal. **14** (2007) No. 1, 15-29.
[8] Y. Cui, J. Sun, Y. Zou; *Global bifurcation and multiple results for Sturm-Liouville boundary value problems*, J. Comput. Appl. Math. **235** (2011), 2185-2192.
[9] M. Feng, X. Zhang, W. Ge; *New existence theorems of positive solutions for a singular boundary value problem*, Electron. J. Qual. Theory Differ. Equ. **2006**, No. 13, 9 pp.
[10] F. Genoud; *Bifurcation from infinity for an asymptotically linear boundary value problem on the half line*, Nonlinear Anal. **74** (2011), 4533-4543.
[11] F. Genoud, B. P. Rynne; *Half eigenvalues and the Fucik spectrum of multipoint boundary value problems*, J. Differential Equations **252**, No. 9 (2012), 5076-5095.
[12] F. Genoud, B. P. Rynne; *Second order multi-point problems with variable coefficients*, Nonlinear Anal. **74**, No. 18 (2011), 7269-7284.
[13] K. S. Ha, V. H. Lee; *Existence of multiple positive solutions of singular boundary value problems*, Nonlinear Anal. **28**, No. 8 (1997), 1429-1438.
[14] E. R. Kaufmann, N. Kosmatov; *A second order singular boundary value problem*, Comput. Math. Appl. **47** (2004), 1317-1326.
[15] Y. H. Lee; *An existence result of positive solutions for singular superlinear boundary value problems and its applications*, J. Korean Math. Soc. **34** (1997), No. 1, 247-255.
[16] R. Ma, X. Han; *Existence of nodal solutions of nonlinear eigenvalue problems with indefinite weight function*, Nonlinear Anal. **71** (2009), 2119-2125.
[17] R. Ma, D. O'Regan; *Nodal solutions for second order m-point boundary value problems with nonlinearities across several eigenvalues,* Nonlinear Anal. **64** (2006), 1562-1577.
[18] R. Ma, B. Thompson; *A note on bifurcation from an interval*, Nonlinear Anal. **62** (2005), 743-749.
[19] R. Ma, B. Thompson; *Multiplicity results for second-order two point boundary value problems with superlinear or sublinear nonlinearities*, J. Math. Anal. Appl. **303** (2005), 726-735.
[20] R. Ma, B. Thompson; *Multiplicity results for second-order two point boundary value problems with nonlinearities across several eigenvalues*, Appl. Math. Lett. **18** (2005), 587-595.
[21] R. Ma, B. Thompson; *Nodal solutions for a nonlinear eigenvalue problems*, Nonlinear Anal. **59** (2004), 707-718.
[22] Y. Naïto, S. Tanaka; *On the existence of multiple solutions of the boundary value problem for nonlinear second order differential equations*, Nonlinear Anal. **56** (2004), 919-935.
[23] R. D. Nussbaum; *Periodic solutions of some nonlinear integral equations*, Dynamical Systems, (Proc. Internat. Sympos., Univ. Florida, Gainesville, Fla. 1976), Academic Press, New York, (1977), 221-249.
[24] P. H. Rabinowitz; *Nonlinear Sturm-Liouville boundary value problems for second order ordinary differential equations,* Comm. Pure Appl. Math. **23** (1970), 939-962.
[25] P.H. Rabinowitz; *Some global results for nonlinear eigenvalue problems,* J. Functional Anal. **7** (1971), 487-513.
[26] P. H. Rabinowitz; *On bifurcation from infinity*, J. Differential Equations **14** (1973), 462-475.
[27] B. P. Rynne; *Linear second order problems with Sturm-Liouville-type multipoint boundary conditions*, Electron. J. Differential Equations **2012** (2012), No. 146, 21 pp.
[28] B. P. Rynne; *Second order, three point, boundary value problem with jumping nonlinearities*, Nonlinear Anal. **68** (2008), 3294-3306.
[29] B. P. Rynne; *Spectral properties and nodal solutions for second-order m-point boundary value problems*, Nonlinear Anal. **67**, No. 12 (2007), 3318-3327.
[30] B. P. Rynne; *Half-eigenvalues of self adjoint, 2mth order differential operators and semilinear problems with jumping nonlinearities*, Differential Integral Equations **14** (2001), No. 9, 1129-1152.

[31] B. P. Rynne; *The Fucik spectrum of general Sturm-Liouville problems*, J. Differential Equations **161** (2000), 87-109.

[32] Y. Sun, L. Liu, Y. J. Cho; *Positive solutions of singular nonlinear Sturm-Liouville boundary value problems*, ANZIAM J. **45** (2004), 557-571.

[33] S. D. Taliaferro; *A nonlinear singular boundary value problem*, Nonlinear Anal. **3**, No. 6 (1979), 897-904.

[34] L. Xiyu; *Some existence and nonexistence principles for a class of singular boundary value bvps*, Nonlinear Anal. **27**, No. 10 (1996), 1147-1164.

[35] E. Zeidler; *Nonlinear Functional Analysis and its Applications, Vol. I, Fixed Point Theorems*, Springer-Verlag, New York 1986.

[36] A. Zettl; *Sturm-Liouville Theory*, American Mathematical Society, Mathematical Surveys and Monographs, Vol. 121, 2005.

Abdelhamid Benmezaï
Faculty of Mathematics, USTHB, Algiers, Algeria
E-mail address: aehbenmezai@gmail.com

Wassila Esserhane
Graduate School of Statistics and Applied Economics, P.O. Box 11, Doudou Mokhtar, Ben-Aknoun Algiers, Algeria
E-mail address: ewassila@gmail.com

Johnny Henderson
Department of Mathematics, Baylor University, Waco, Texas 76798-7328, USA
E-mail address: Johnny_Henderson@baylor.edu