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CAUCHY PROBLEM FOR DISPERSIVE EQUATIONS IN α-MODULATION SPACES

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Abstract. In this article, we consider the Cauchy problem for dispersive equations in α-Modulation spaces. For this purpose, we find a method for estimating u^k in α-modulation spaces when k is not an integer, and develop a Strichartz estimate in $M_{p,q}^{s,\alpha}$ which is based on semigroup estimates. In the local case, we that the domain of p is independent of α, which is also the case in the Modulation spaces and in the Besov space.

1. Introduction and statement of main results

In this article we study the Cauchy problem for the following nonlinear Klein-Gordon equation (NLKG):

$$\begin{gathered} u_{tt} + (I-\Delta)u = \pm|u|^k u, \\ u(0) = u_0, \quad u_t(0) = u_1, \end{gathered} \tag{1.1}$$

and the Cauchy problem for the nonlinear Heat equation (NLH)

$$\begin{gathered} u_t + \Delta u = |u|^k u, \\ u(0) = u_0\,, \end{gathered} \tag{1.2}$$

where $k \in (0,+\infty)\backslash\mathbb{Z}$, $\Delta = \partial^2/\partial^2 x_1 + \cdots + \partial^2/\partial^2 x_n$. By Duhamel's formula, (1.1) has the equivalent integral form

$$u(t) = K'(t)u_0 + K(t)u_1 - \int_0^t K(t-\tau)|u|^k u d\tau\,, \tag{1.3}$$

where $\omega = (I-\Delta)$,

$$K(t) = \frac{\sin(t\omega^{1/2})}{\omega^{1/2}}, \quad K'(t) = \cos(t\omega^{1/2})\,.$$

The solution of (1.2) is

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}|u|^k u d\tau\,. \tag{1.4}$$

We study the local and global well posedness of (1.1) in α-modulation spaces. As we know, the frequency-dyadic-decomposition technique plays an important role in

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the theory of function space. Using this technique we can define the Besove space and the Trieble-Lizorkin space [14]. On the other hand, Modulation spaces was fist introduced by Feichtinger in [5] where he used short-time Fouier transform and window function to define it. His initial motivation was using this space to measure smoothness for some function or distribution spaces. Then, Wang and Hudzik [18] used the frequency-uniform-decomposition operators to give another equivalent definition of modulation spaces and considered global solution for the nonlinear Schrödinger equation and the nonlinear Klein-Gordon equation in modulation spaces. Based on this equivalent definition, there are many studies of PDEs in modulation spaces, such as [10, 13, 16, 17, 19]. Then, Gröbner [6] introduce a new decomposition called α-covering, he used this decomposition to define α-modulation space $M^{s,\alpha}_{p,q}$ which is an intermediate function space to connect modulation spaces and Besov space with respect to a parameter $\alpha \in [0,1)$. When $\alpha = 0$, $M^{s,0}_{p,q}$ is equivalent to modulation space $M^s_{p,q}$ which is define in [18]. Besov space can be regarded as the limit case of α-modulation space as $\alpha \to 1$. Later, Han and Wang [8] give another equivalent definition which is more convenient for calculations. In this paper, we will use the Han and Wang's definition.

For equations (1.3) and (1.4), we focus on the case $k \notin \mathbb{Z}$. Because when k is an integer, we can only use the algebra proposition or analysis on $\Box^\alpha_k$ simply. When k is not an integer, we use Besov space as an auxiliary space to estimate $|u|^k u$ in α-modulation space which was first introduced in our previous paper[9]. We also find an interesting phenomenon that the domain of p is independent of α which is same as modulation space and Besov space. The following theorems are the main results in this paper:

Theorem 1.1. *For any $1 \le q < \infty$, $2 \le p < \infty$, $[s] < k$, $\theta \in [0,1]$ we define*

$$\delta = (\frac{1}{2} - \frac{1}{p})(\theta(n+2) + (1-\theta)2n\alpha), \quad \mu = \theta n(\frac{1}{2} - \frac{1}{p}) \tag{1.5}$$

and denate

$$S_1 = \max\big\{1 - \frac{\delta}{2} - n(1-\alpha)(\frac{1}{q} - \frac{1}{p}),\ R(p,q) - \frac{1}{k}(1-\delta) + \frac{\delta}{2}\big\}.$$

where $R(p,q) = \alpha\frac{n}{p} + (1-\alpha)\frac{n}{q'}$. When $q \in [p', p] \cap [\gamma', \gamma]$, $(1 - \frac{2}{p})n < 1 - \delta$ and $S_1 < s < R(p,q) - \frac{n(1+\alpha)}{k}(1 - \frac{2}{p})n + \frac{\delta}{2}$. Then for any $(u_0, u_1) \in M^{s,\alpha}_{2,q} \times M^{s-1,\alpha}_{2,q}$, there exist $T > 0$ such that equation (1.3) has an unique solution in

$$X = L^\infty(0,T; M^s_{2,q}) \cap L^\gamma(0,T; M^{s-\frac{\delta}{2}}_{p,q}) \tag{1.6}$$

where $\gamma = 2/\mu > (k+2)$.

Moreover, if $\gamma = 2/\mu = (k+2)$. Then there exists a small $\nu > 0$ such that for any $\|u_0\|_{M^s_{p,q}} + \|u_1\|_{M^{s-1}_{p,q}} \le \nu$, equation (1.1) has an unique global solution

$$u \in L^\infty(R; M^s_{p,q}) \cap L^\gamma(R; M^{s-\frac{\delta}{2}}_{p,q}) \tag{1.7}$$

Remark 1.2. Theorem 1.1 extends the local well posedness in modulation space and Besov space. The only difference is that $R(p,q)$ replace n/p in Besov space and n/q' in modulation space. This phenomenon coincides with the nature of α-modulation space (see (2.2)).

Remark 1.3. When $p = 2$, the condition $q \in [p', p]$ means $q = 2$. Han and Wang [8] proved that $M_{2,2}^{s,\alpha}$ is equivalent to H^s for any $\alpha \in [0, 1]$, so this result is meaningless. Actually, when $p = 2$, we do not need to choose $q = 2$. The range of q for $p = 2$ is wider which will be described in Remark 2.7.

For (1.4), we have similar result as in Theorem 1.1. The estimate of the Heat semigroup is different to NLKG's, so its domain of p, q, s and work space are also different. Specifically, we have following result for (1.2).

Theorem 1.4. *For any $1 \le q < \infty$, $2 \le p < \infty$, $[s] < k$ which satisfy $q \in [p', p]$, and $n(1 - (2/p)) < 2$. We denote*

$$S_2 = \max\big\{2 - n(1-\alpha)(\frac{1}{q} - \frac{1}{p}), R(p,q) - \frac{2}{k} + \frac{n}{k}(1-\alpha)(1 - \frac{2}{p})\big\}.$$

When

$$S_2 < s < R(p,q) - \frac{n(1+\alpha)}{k}(1 - \frac{2}{p})n.$$

For any $u_0 \in M_{p,q}^{s,\alpha}$, there exist $T > 0$ such that equation (1.3) has an unique solution in $L^\infty(0, T; M_{p,q}^{s,\alpha})$

2. Preliminaries

In this section we give some definitions and properties of function spaces. Also, we will prove the key lemma to estimate $|u|^k u$ in α-Modulation space when k is not an integer.

Definition 2.1 (α-Modulation spaces)**.** Let ρ be a nonnegative smooth radial bump function supported in $B(0, 2)$, satisfying $\rho(\xi) = 1$ for $|\xi| < 1$ and $\rho(\xi) = 0$ for $|\xi| > 2$. For any $k = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$, we set

$$\rho_k^\alpha(\xi) = \rho\Big(\frac{\xi - \langle k\rangle^{\frac{\alpha}{1-\alpha}} k}{\langle k\rangle^{\frac{\alpha}{1-\alpha}}}\Big)$$

and denote

$$\eta_k^\alpha = \rho_k^\alpha(\xi)\Big(\sum_{l\in\mathbb{Z}^n} \rho_k^\alpha\Big)^{-1}$$

Corresponding to the sequence $\{\eta_k^\alpha\}_{k\in\mathbb{Z}^n}$, we define an operator sequence denoted by $\{\Box_k^\alpha\}_{k\in\mathbb{Z}^n}$

$$\Box_k^\alpha = \mathcal{F}^{-1}\eta_k^\alpha \mathcal{F}$$

where $\mathcal{F}$ denote standard Fourier transform. This type of decomposition on frequency extends the dyadic and the uniform decomposition. Moreover, it still has almost orthogonal property which is the same as that in dyadic and the uniform decomposition. That is, for any $k \in \mathbb{Z}^n$ the number of l which satisfy $\Box_l^\alpha \Box_k^\alpha \neq 0$ is uniformly bounded and independent of k. For any α we use Λ_α to denote this number.

When $0 \le \alpha < 1$, $1 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, using this decomposition, we defined the α-modulation spaces as

$$M_{p,q}^{s,\alpha} := \{f \in S' : \|f\|_{M_{p,q}^{s,\alpha}} < \infty\},$$

where

$$\|f\|_{M_{p,q}^{s,\alpha}} := \Big(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{sq}{1-\alpha}} \|\Box_k^\alpha\|_{L^p}^q\Big)^{1/q}$$

where $\langle k\rangle = (1+k^2)^{1/2}$, see [8] for details. Also we need another form of α-modulation space as auxiliary tool to prove Stricharz estimate which is defined as follows:

$$\|f\|_{l^{s,\alpha}_{\square,q}(L^r(0,T;L^p(\mathbb{R}^n)))} = \Big(\sum_{k\in\mathbb{Z}^n} \langle k\rangle^{\frac{sq}{1-\alpha}} \|\square^\alpha_k\|^q_{L^r(0,T;L^p(\mathbb{R}^n))}\Big)^{1/q}$$

If the domain of t is $(-\infty,+\infty)$, we denote $l^{s,\alpha}_{\square,q}(L^rL^p)$ for convenience. The idea of $l^{s,\alpha}_{\square,q}(L^r(0,T;L^p(R^n)))$ was first introduced by Planchon [11, 12] when studing the nonlinear Schrödinger equation and the nonlinear wave equation. It seems only change L^r norm and l^q norm, but very important in α-modulation spaces. Because, we can see that the index q will influence the regularity. In many cases, we should deal with q carefully and choose l^q norm in last step.

Proposition 2.2 (Isomorphism [8]). *Let $0<p,\ q\le\infty,\ s,\ \sigma\in R$. Then $J_\sigma = (I-\Delta)^{\sigma/2}$:*

$$M^{s,\alpha}_{p,q}\to M^{s-\sigma,\alpha}_{p,q} \quad \text{and} \quad l^{s,\alpha}_{\square,q}(L^r(0,T;L^p(R^n)))\to l^{s-\sigma,\alpha}_{\square,q}(L^r(0,T;L^p(R^n)))$$

are isomorphic mappings.

Proposition 2.3 (Embedding [8]). *$M^{s_1,\alpha}_{p_1,q_1}\subset M^{s_2,\alpha}_{p_2,q_2}$ and $l^{s_1,\alpha}_{\square,q_1}(L^r(0,T;L^{p_1}))\subset l^{s_2,\alpha}_{\square,q_2}(L^r(0,T;L^{p_2}))$ under each of the following two conditions:*

$$\text{if } p_1\le p_2,\ q_1\le q_2 \text{ and } s_1\ge s_2+n\alpha(\frac{1}{p_1}-\frac{1}{p_2}), \tag{2.1}$$

$$\text{if } p_1\le p_2,\ q_1>q_2, \text{ and } s_1>s_2+n\alpha(\frac{1}{p_1}-\frac{1}{p_2})+n(1-\alpha)(\frac{1}{q_2}-\frac{1}{q_2}) \tag{2.2}$$

When $\alpha=1$, we can see that (2.2) coincides with that in Besov space, although we can not use $\alpha=1$ to define Besov spaces. We always say that index n/p influences the regularity in Besov spaces, and n/q' influences the regularity in modulation space. Actually, in α-modulation space, that index is $\alpha n/p+(1-\alpha)n/q'$ which coincide with Besov space and modulation space. That is why we define

$$R(p,q)=\alpha\frac{n}{p}+(1-\alpha)\frac{n}{q'}$$

in Theorem 1.1. For convenience and we use $R(p,q)$ through out this article.

Lemma 2.4 (Embedding with Besove spaces [8]). *Assume $B^s_{p,q}$ is the standard Besov spaces, and $1\le p,\ q\le\infty,\ s\in\mathbb{R}$, we have following embeddings:*

$$M^{s+\sigma(p,q),\alpha}_{p,q}\subset B^s_{p,q},\quad \sigma(p,q)=\max(0,n(1-\alpha)(\frac{1}{p\wedge p'}-\frac{1}{q})) \tag{2.3}$$

$$B^{s+\tau(p,q)}_{p,q}\subset M^{s,\alpha}_{p,q},\quad \tau(p,q)=\max(0,n(1-\alpha)(\frac{1}{q}-\frac{1}{p\vee p'})) \tag{2.4}$$

Moreover, we need nonlinear estimate in Besov spaces. There are many forms of such estimate, such as [3][4][15]. All of these forms restricted $q=2$, but in α-modulation space q will influence regularity, so we use a new form which is obtained in [9].

Lemma 2.5 (nonlinear estimate in Besove space). *Assume $2\le p<\infty$, $1\le q\le\infty$, $0\le\delta<s<s_1<\infty$, $[s-\delta]<k$ and satisfy*

$$k(\frac{1}{p}-\frac{s}{n})+\frac{1}{p}-\frac{\delta}{n}=\frac{1}{p'},\quad \frac{1}{p}-\frac{s}{n}>0 \tag{2.5}$$

then we have

$$\||u|^k u\|_{B^{s-\delta}_{p',q}} \preceq \|u\|^{k+1}_{B^{s_1}_{p,q}} \tag{2.6}$$

Then, by Lemmas 2.4 and 2.5 we can estimate the nonlinear part in α-modulation spaces which is the crucial lemma in this paper.

Lemma 2.6 (Nonlinear estimate in α-modulation spaces). *Let* $1 \le q < \infty$, $2 \le p < \infty$, $[s] < k$ *which satisfy* $q \in [p', p]$, $(1 - \frac{2}{p})n < r$, *and*

$$\max\big\{r - n(1-\alpha)(\frac{1}{q} - \frac{1}{p}), R(p,q) - \alpha\frac{n}{k}(1-\frac{2}{p}) - \frac{r}{k}\big\} < s < R(p,q) - (1+\alpha)\frac{n}{k}(1-\frac{2}{p})$$

then we have

$$\|u^{k+1}\|_{M^{s-r,\alpha}_{p',q}} \preceq \|u\|^{k+1}_{M^{s,\alpha}_{p,q}} \tag{2.7}$$

Proof. By Lemma 2.4 we have

$$\|u^{k+1}\|_{M^{s-r,\alpha}_{p',q}} \preceq \|u^{k+1}\|_{B^{s-r+\tau(p',q)}_{p',q}} \tag{2.8}$$

Since $r - n(1-\alpha)(\frac{1}{q} - \frac{1}{p}) < s$, we have $s - r + \tau(p', q) > 0$, so we can use Lemma 2.6 to obtain

$$\|u^{k+1}\|_{B^{s-r+n(1-\alpha)(\frac{1}{q}-\frac{1}{p})}_{p',q}} \preceq \|u\|^{k+1}_{B^{s-n(1-\alpha)(\frac{1}{p'}-\frac{1}{q})}_{p,q}} \tag{2.9}$$

and choose $s_1 = s + \varepsilon$ in (2.6) then s satisfies

$$k(\frac{1}{p} - \frac{1}{n}(s - n(1-\alpha)(\frac{1}{p'} - \frac{1}{q})) + \frac{1}{p} - \frac{1}{n}(r - n(1-\alpha)(1-\frac{2}{p}) - \varepsilon) = \frac{1}{p'}. \tag{2.10}$$

by (2.13) we have

$$s = n(\frac{\alpha}{p} + \frac{1-\alpha}{q'}) - \alpha\frac{n}{k}(1-\frac{2}{p}) - \frac{r}{k} + \frac{\varepsilon}{k} \tag{2.11}$$

because $\tau(p', q) + \sigma(p, q) = n(1-\alpha)(1-\frac{2}{p}) < r$, it is easy to find $0 < \varepsilon < r - n(1-\alpha)(1-\frac{2}{p})$. Combining this with (2.11) and condition of Lemma 2.6, the domain of s is

$$\begin{aligned}&\max\big\{r - n(1-\alpha)(\frac{1}{q} - \frac{1}{p}), n(\frac{\alpha}{p} + \frac{1-\alpha}{q'}) - \alpha\frac{n}{k}(1-\frac{2}{p}) - \frac{r}{k}\big\}\\ &< s < n(\frac{\alpha}{p} + \frac{1-\alpha}{q'}) - (1+\alpha)\frac{n}{k}(1-\frac{2}{p})\end{aligned}$$

Finally, we use Lemma 2.4 again to obtain

$$\|u^{k+1}\|_{B^{s-r+n(1-\alpha)(\frac{1}{q}-\frac{1}{p})}_{p',q}} \preceq \|u\|^{k+1}_{B^{s-n(1-\alpha)(\frac{1}{p'}-\frac{1}{q})}_{p,q}} \preceq \|u\|^{k+1}_{M^{s,\alpha}_{p,q}}.$$

□

Remark 2.7. The condition $q \in [p', p]$ is not necessary, but only for continence in calculation. So it doesn't means q must equal to 2 when $p = 2$, we can use the same method to find the domain of q when $p = 2$. By the same method as above, we get the following result:
when $q < 2$, $p = 2, n(1-\alpha)(\frac{1}{q} - \frac{1}{2}) < r$,

$$\max\{r - (1-\alpha)(\frac{1}{q} - \frac{1}{2})n, \frac{n}{2} - \frac{1}{k}[r - (1-\alpha)(\frac{1}{q} - \frac{1}{2})n]\} < s < \frac{n}{2} \tag{2.12}$$

when $q > 2$, $p = 2, n(1-\alpha)(\frac{1}{2} - \frac{1}{q}) < r$,

$$\max\{r, R(2,q) - \frac{1}{k}[r - n(1-\alpha)(\frac{1}{2} - \frac{1}{q})]\} < s < R(2,q) \tag{2.13}$$

the same conclusion holds.

3. Proof of main results

First, we estimate the Klein-Gordon semigroup $G(t) = e^{it\omega^{1/2}}$ in α-Modulation spaces, where $\omega = I - \Delta$. It is well known that $G(t)$ have the following estimate (cf.[1, 2])

$$\|(I-\Delta)^{-\sigma(p)/2}G(t)f\|_p \preceq |t|^{-n(1/2-1/p)}\|f\|_{p'} \tag{3.1}$$

where

$$2 \le p < \infty, \quad \sigma(p) = (n+2)\big(\frac{1}{2} - \frac{1}{p}\big) \tag{3.2}$$

By Bernstein's multiplier estimate

$$\|\Box_k^\alpha (I-\Delta)^{\sigma/2} g\|_p \preceq \langle k\rangle^{\frac{\sigma}{1-\alpha}}\|g\|_p \tag{3.3}$$

So, by (3.1) and (3.3), we have

$$\begin{aligned} \|\Box_k^\alpha G(t)f\|_p &\preceq \langle k\rangle^{\frac{\sigma(p)}{1-\alpha}} \sum_{l\in\Lambda_\alpha} \|(I-\Delta)^{-\sigma(p)/2}\Box_{k+l}^\alpha G(t)f\|_p \\ &\preceq \langle k\rangle^{\frac{\sigma(p)}{1-\alpha}} |t|^{-n(1/2-1/p)} \sum_{l\in\Lambda_\alpha} \|\Box_{k+l}^\alpha f\|_{p'} \end{aligned} \tag{3.4}$$

On the other hand, by Hölder's and Young's inequalities

$$\begin{aligned} \|\Box_k^\alpha G(t)f\|_p &\preceq \|\eta_k^\alpha e^{it(1+|\xi|^2)^{1/2}}\widehat{f}\|_{p'} \\ &\preceq \sum_{l\in\Lambda_\alpha} \|\eta_k^\alpha e^{it(1+|\xi|^2)^{1/2}}\mathcal{F}\Box_{k+l}^\alpha f\|_{p'} \\ &\preceq \sum_{l\in\Lambda_\alpha} \|\eta_k^\alpha e^{it(1+|\xi|^2)^{1/2}}\|_{\frac{2p}{p-2}} \|\mathcal{F}\Box_{k+l}^\alpha f\|_p \\ &\preceq \langle k\rangle^{\frac{\alpha n}{1-\alpha}(1-\frac{2}{p})} \sum_{l\in\Lambda_\alpha} \|\Box_{k+l}^\alpha f\|_{p'}\,. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5) we have

$$\|\Box_k^\alpha G(t)f\|_p \preceq \langle k\rangle^{\frac{\delta(p)}{1-\alpha}} |t|^{-n\theta(1/2-1/p)} \sum_{l\in\Lambda_\alpha} \|\Box_{k+l}^\alpha f\|_{p'} \tag{3.6}$$

where

$$\delta(p) = (\frac{1}{2} - \frac{1}{p})(\theta(n+2) + (1-\theta)2n\alpha), \quad \theta \in [0,1] \tag{3.7}$$

Multiplying by $\langle k\rangle^{\frac{s}{1-\alpha}}$ and taking the l^q norm in both sides of (3.6), we obtain the following estimate.

Theorem 3.1. *Let $s \in \mathbb{R}$, $2 \le p < \infty$, $1/p + 1/p' = 1$, $0 < q < \infty$, $\delta(p)$ be as in (3.7). Then we have*

$$\|G(t)f\|_{M_{p,q}^{s,\alpha}} \preceq |t|^{-\theta n(1/2-1/p)}\|f\|_{M_{p',q}^{s+\delta(p),\alpha}} \tag{3.8}$$

Then we establish the Stricharz estimates in $M^{s,\alpha}_{p,q}$ which is more general than the estimate in [7]. Motivate by Theorem 3.1, we assume $U(t)$ is a dispersive semigroup:

$$U(t) = \mathcal{F}^{-1} e^{itP(\xi)} \mathcal{F} \tag{3.9}$$

which satisfy

$$\|U(t)f\|_{M^{s,\alpha}_{p,q}} \preceq |t|^{-\mu(p)} \|f\|_{M^{s+\delta(p),\alpha}_{p,q}} \tag{3.10}$$

where $2 \le p < \infty$, $1 \le q < \infty, \delta(p) \in \mathbb{R}$, $0 < \mu(p) < 1$, and $P(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is a real valued function.

Theorem 3.2. *If $U(t)$ satisfy* (3.9) *and* (3.10)*, for $\gamma = 2/\theta(p)$ we have*

$$\|U(t)f\|_{l^{s,\alpha}_{\Box,q}}(L^\gamma(\mathbb{R}, L^p)) \preceq \|f\|_{M^{s+\delta(p)/2,\alpha}_{2,q}} \tag{3.11}$$

In addition, if $\gamma \ge q$, then

$$\|U(t)f\|_{L^\gamma(\mathbb{R}, M^{s,\alpha}_{p,q})} \preceq \|f\|_{M^{s+\delta(p)/2,\alpha}_{2,q}} . \tag{3.12}$$

Proof. By the standard dual estimate method, we need to prove only that

$$\int_{\mathbb{R}} (U(t)f, \varphi(t))\, dt \preceq \|f\|_{M^{0,\alpha}_{2,q}} \|\varphi\|_{l^{\delta(p)/2,\alpha}_{\Box,q'}(L^{\gamma'}(\mathbb{R},\mathbb{L}^{'}))} \tag{3.13}$$

holds for all $f \in S(\mathbb{R}^n)$, $\varphi \in C_0^\infty(\mathbb{R}, S(\mathbb{R}^n)$. Because $S(\mathbb{R}^n)$ and $C_0^\infty(\mathbb{R}, S(\mathbb{R}^n)$ are dense in $M^{0,\alpha}_{2,q}$ and $l^{\delta(p)/2,\alpha}_{\Box,q'}(L^{\gamma'}(\mathbb{R}, \mathbb{L}^{'}))$. By the duality of α-Modulation (see[8]),

$$\int_{\mathbb{R}} (U(t)f, \varphi(t)) dt \preceq \|f\|_{M^{0,\alpha}_{2,q}} \| \int_{\mathbb{R}} U(-t)\varphi(t) dt\|_{M^{0,\alpha}_{2,q'}} \tag{3.14}$$

then for any $k \in \mathbb{Z}^n$,

$$\|\Box_k \int_{\mathbb{R}} U(-t)\varphi(t)dt\|_2^2 \preceq \|\Box_k \varphi\|_{L^{\gamma'}(\mathbb{R},L^{p'})} \|\Box_k \int_{\mathbb{R}} U(t-s)\varphi(s)ds\|_{L^\gamma(\mathbb{R},L^p)} \tag{3.15}$$

By the almost orthogonal property of $\{\Box^\alpha_k\}$ and Bernstein's multiplier estimate we have

$$\|\Box^\alpha_k U(t) f\|_p \preceq \sum_{l\in\Lambda_\alpha} \|\Box^\alpha_k \Box^\alpha_{k+l} U(t) f\|_p \preceq |t|^{-\theta(p)} \langle k\rangle^{\delta(p)} \|\Box^\alpha_k f\|_{p'} \tag{3.16}$$

Notice that $0 < \theta(p) < 1$, we can use Hardy-Littlewood-Sobolev's inequality to obtain that

$$\|\Box_k \int_{\mathbb{R}} U(t-s)\varphi(s)ds\|_{L^\gamma(\mathbb{R},L^p)} \preceq \langle k\rangle^{\delta(p)} \|\Box_k \varphi\|_{L^{\gamma'}(\mathbb{R},L^{p'})} \tag{3.17}$$

So, in view of (3.15) and (3.17) we have

$$\|\Box_k \int_{\mathbb{R}} U(-t)\varphi(t)dt\|_2 \preceq \langle k\rangle^{\delta(p)/2} \|\Box_k \varphi\|_{L^{\gamma'}(\mathbb{R},L^{p'})} \tag{3.18}$$

Taking the $l^{q'}$ norm in both side of above inequality, we have

$$\| \int_{\mathbb{R}} U(-t)\varphi(t)dt\|_{M^{0,\alpha}_{2,q'}} \preceq \|\varphi\|_{l^{\delta(p)/2,\alpha}_{\Box,q'}(L^{\gamma'}(\mathbb{R},L^{p'}))} \tag{3.19}$$

Together (3.14) with (3.19), we obtain (3.11).

For (3.12), when $\gamma \ge q$, we can find that the left side of (3.12) is controlled by the left side of (3.11) by Minkowski's inequality. □

Then we estimate the nonlinear part, denote

$$(\mathcal{A}f)(t)=\int_0^t U(t-s)f(s,\cdot)ds$$

Theorem 3.3. *Suppose $U(t)$ satisfies (3.9) and (3.10). For $\gamma=\frac{2}{\theta(p)}$, we have*

$$\|\mathcal{A}f\|_{l^{s,\alpha}_{\Box,q}(L^\infty(\mathbb{R},L^2))}\preceq\|f\|_{l^{s+\delta(p)/2,\alpha}_{\Box,q}(L^{\gamma'}(\mathbb{R},L^{p'}))}\tag{3.20}$$

In addition, if $\gamma'\le q$, then

$$\|U(t)f\|_{L^\infty(\mathbb{R},M^{s,\alpha}_{2,q})}\preceq\|f\|_{L^{\gamma'}(\mathbb{R},M^{s+\delta(p)/2,\alpha}_{p',q})}\tag{3.21}$$

Proof. Using the same method as in Theorem 3.2, the crucial inequality

$$\|\Box^\alpha_k f\|_2^2\preceq\langle k\rangle^{\delta(p)}\|\Box^\alpha_k f\|^2_{L^{\gamma'}(\mathbb{R},L^{p'})}$$

implies (3.20). Then applying Minkowski's inequality, we get (3.21) from (3.20). □

Theorem 3.4. *Let $U(t)$ satisfy (3.9) and (3.10). For $\gamma=2/\theta(p)$, we have*

$$\|\mathcal{A}f\|_{l^{s,\alpha}_{\Box,q}(L^\gamma(\mathbb{R},L^p))}\preceq\|f\|_{l^{s+\delta(p)/2,\alpha}_{\Box,q}(L^1(\mathbb{R},L^2))}\tag{3.22}$$

Proof. Let $f,\varphi\in C_0^\infty(\mathbb{R},S(\mathbb{R}^n))$. From Theorem 3.2, we have

$$\begin{aligned}|\int_{\mathbb{R}}\Big(\int_0^t U(t-\tau)f(\tau)d\tau,\varphi(t)\Big)dt|&\preceq\|f\|_{L^1(\mathbb{R},M^{0,\alpha}_{2,q})}\|\int_\cdot^\infty U(\cdot-t)\varphi(t)dt\|_{L^\infty(\mathbb{R},M^{0,\alpha}_{2,q'})}\\&\preceq\|f\|_{L^1(\mathbb{R},M^{0,\alpha}_{2,q})}\|\varphi\|_{l^{\delta(p)/2,\alpha}_{\Box,q'}(L^{\gamma'}(\mathbb{R},L^{p'}))}\end{aligned}$$

By the density and duality we obtain the desired result. □

From (3.20) and (3.22), using Minkowski's inequality, we immediately obtain the next result.

Theorem 3.5. *Let $U(t)$ satisfy (3.9) and (3.10). For $\gamma=2/\theta(p)$, we have*

$$\|\mathcal{A}f\|_{l^{s,\alpha}_{\Box,q}(L^\gamma(\mathbb{R},L^p))}\preceq\|f\|_{l^{s+\delta(p)/2,\alpha}_{\Box,q}(L^{\gamma'}(\mathbb{R},L^{p'}))}\tag{3.23}$$

In addition, if $q\in[\gamma',\gamma]$, we have

$$\|\mathcal{A}f\|_{L^\gamma(\mathbb{R},M^{s-\delta(p)/2,\alpha}_{p,q})}\preceq\|f\|_{L^{\gamma'}(\mathbb{R},M^{s+\delta(p)/2,\alpha}_{p',q})}\tag{3.24}$$

Proof of Theorem 1.1. We consider the integral equation:

$$\Phi(u)=K'(t)u_0+K(t)u_1-\int_0^t K(t-\tau)|u|^k u d\tau$$

in the Banach space

$$X=L^\infty(0,T;M^s_{2,q})\cap L^\gamma(0,T;M^{s-\delta(p)}_{p,q})\tag{3.25}$$

where $\gamma=2/\mu(p)$, $\mu(p)=n\theta(1/2-1/p)$ and $\delta(p)$ is defined by (3.7). For all $2<p\le\infty$, we can choose θ such that $0<\theta(p)<1$. So by (3.10) and (3.12), we have

$$\|\Phi(u)\|_{X_1}\preceq\|u_1\|_{M^{s,\alpha}_{2,q}}+\|u_0\|_{M^{s-1,\alpha}_{p,q}}+\||u|^k u\|_{L^{\gamma'}(0,T;M^{s+\delta(p)/2-1,\alpha}_{p',q})}\tag{3.26}$$

Choosing $r=1-\delta(p)$ $s=s-\frac{\delta(p)}{2}$ in Lemma 2.6 and using Hölder inequality,

$$\||u|^k u\|_{L^{\gamma'}(0,T;M^{s+\delta(p)/2-1,\alpha}_{p',q})}\preceq\|u\|^{k+1}_{L^{(k+1)\gamma'}(0,T;M^{s-\delta(p),\alpha}_{p,q})}\preceq T^{1-\frac{k+2}{\gamma}}\|u\|^{k+1}_X\tag{3.27}$$

Together with (3.11) and (3.12), yield

$$\|u\|_X \preceq \|u_1\|_{M^{s,\alpha}_{2,q}} + \|u_0\|_{M^{s-1,\alpha}_{p,q}} + T^{1-\frac{k+2}{\gamma}}\|u\|^{k+1}_X \tag{3.28}$$

By standard way of contraction mapping, Theorem 1.1 can be proved easily.

For the global case, when $1-\frac{k+2}{\gamma}=0$, we obtain the global solution with small initial value by replacing $(0,T)$ by $\mathbb{R}$ in (3.25).

Finally, we find the domain of p in the local problem. When we use Lemma 2.5, p should satisfy

$$n(1-\alpha)(1-\frac{2}{p}) < 1-\theta(p);$$

that is,

$$n(1-\alpha)(1-\frac{2}{p}) < 1-(n+2)\theta(\frac{1}{2}-\frac{1}{p}) - n\alpha(1-\frac{2}{p})$$

by simply calculations, p should satisfy $(\frac{1}{2}-\frac{1}{p}) < \frac{1}{2n+(n+2)\theta}$. Then we choose $\theta=0$, the domain of p is $n(1-\frac{2}{p})<1$ which is dependence of α and also same as that in Modulation space and Besove space.

We briefly interpret this interesting phenomenon. In the Modulation space case, when choose $\alpha=0$ and $\theta=0$ in (3.7), we can see that there is no any regularity lost in semigroup estimate. As the result, there is also no regularity lost in Stricharz estimate. But in the estimate of nonlinear term, we lost $n(1-\frac{2}{p})$ regularity when embedding between Modulation space and Besov space. In the Besov case, we lost $n(1-\frac{2}{p})$ regularity in semigroup estimate and Stricharz estimate. And there is no any regularity lost when estimate $|u|^k u$. For α-Modulation case, in semigroup estimate and Stricharz estimate we lost $n\alpha(1-\frac{2}{p})$ and in nonlinear estimate the number is $n(1-\alpha)(1-\frac{2}{p})$. So, the total lost is $n(1-\frac{2}{p})$ for any α. Also, see the integral form of the (1.1), there is a $(I-\Delta)^{-\frac{1}{2}}$ in its nonlinear part. So the domain is $n(1-\frac{2}{p})<1$ for any α. □

Proof of Theorem 1.4. we first prove that

$$\|e^{t\Delta}f\|_{M^{s_1,\alpha}_{p,q}} \preceq (1+t^{-\frac{s_1-s_2}{2}})\|f\|_{M^{s_2,\alpha}_{p,q}} \tag{3.29}$$

for any $s_1 \ge s_2$. For the low frequency part; that is, $|k|\le 100\sqrt{n}$, we have

$$\sum_{|k|\le 100\sqrt{n}} \langle k\rangle^{\frac{s_1 q}{1-\alpha}} \|\Box^\alpha_k e^{t\Delta}f\|^q_{L^p} \preceq \sum_{|k|\le 100\sqrt{n}} \langle k\rangle^{\frac{s_2 q}{1-\alpha}} \|\Box^\alpha_k f\|^q_{L^p} \preceq \|f\|^q_{M^{s_2,\alpha}_{p,q}} \tag{3.30}$$

For the high frequency part, we use Bernstein's inequality

$$\|\Box^\alpha_k e^{t\Delta}f\| \preceq e^{-ct|k|^{\frac{2}{1-\alpha}}}\|\Box^\alpha_k f\|_{L^p}$$

when $|k|\ge 100\sqrt{n}$, we have

$$\begin{aligned}\langle k\rangle^{\frac{s_1}{1-\alpha}}\|\Box^\alpha_k e^{t\Delta}f\|_{L^p} &\preceq \langle k\rangle^{\frac{s_1-s_2}{1-\alpha}} e^{-ct|k|^{\frac{2}{1-\alpha}}}\langle k\rangle^{\frac{s_2}{1-\alpha}}\|\Box^\alpha_k f\|_{L^p} \\ &\preceq t^{\frac{s_1-s_2}{2}}\langle k\rangle^{\frac{s_2}{1-\alpha}}\|\Box^\alpha_k\|_{L^p}\end{aligned} \tag{3.31}$$

Taking l^q norm in both side of (3.31) and combing with (3.30), we obtain (3.29).

Then, we prove the existence of a local solution to (1.4) in the Banach space

$$X_2 = \{u : \|u\|_{L^\infty(0,T;M^{s,\alpha}_{p,q})} \le C_0\}, \quad d(u,v) = \|u-v\|_{L^\infty(0,T;M^{s,\alpha}_{p,q})}$$

Choosing s_1 such that $s - s_1 \to 2^-$, By (3.29) and Lemma 2.6, we have

$$\begin{aligned}
\|\Phi(u)\|_{X_2} &\preceq \|u_0\|_{M^{s,\alpha}_{p,q}} + \|\int_0^t e^{(t-\tau)\Delta}|u|^k u d\tau\|_{X_2} \\
&\preceq \|u_0\|_{M^{s,\alpha}_{p,q}} + \|\int_0^t (t-\tau)^{-\frac{s-s_1}{2}} d\tau\|_{L^\infty(0,T)}\|u\|^{k+1}_{X_2} \\
&\preceq \|u_0\|_{M^{s,\alpha}_{p,q}} + T^{1-\frac{s-s_1}{2}}\|u\|^{k+1}_{X_2}
\end{aligned}$$

By choosing suitable C_0 and T, we can obtain the conclusion of Theorem 1.4. □

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