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INITIAL-VALUE PROBLEMS FOR HYBRID HADAMARD FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we discuss the existence of solutions for an initial-value problem of nonlinear hybrid differential equations of Hadamard type. The main result is proved by means of a fixed point theorem due to Dhage. An example illustrating the existence result is also presented.

1. Introduction

In this article, we study the existence of solutions for an initial-value problem of hybrid fractional differential equations of Hadamard type given by

$$\begin{gathered} {}_HD^\alpha\Big(\frac{x(t)}{f(t,x(t))}\Big) = g(t,x(t)), \quad 1 \le t \le T,\ 0<\alpha\le 1, \\ {}_HJ^{1-\alpha}x(t)|_{t=1} = \eta, \end{gathered} \tag{1.1}$$

where ${}_HD^\alpha$ is the Hadamard fractional derivative, $f \in C([1,T]\times\mathbb{R},\mathbb{R}\setminus\{0\})$ and $g : C([1,T]\times\mathbb{R},\mathbb{R})$, ${}_HJ^{(\cdot)}$ is the Hadamard fractional integral and $\eta\in\mathbb{R}$.

Fractional calculus has evolved into an important and interesting field of research in view of its numerous applications in technical and applied sciences. The mathematical modeling of many real world phenomena based on fractional-order operators is regarded as better and improved than the one depending on integer-order operators. In particular, fractional calculus has played a significant role in the recent development of special functions and integral transforms, signal processing, control theory, bioengineering and biomedical, viscoelasticity, finance, stochastic processes, wave and diffusion phenomena, plasma physics, social sciences, etc. For further details and applications, see [11, 15].

Fractional differential equations involving Riemann-Liouville and Caputo type fractional derivatives have extensively been studied by several researchers. However, the literature on Hadamard type fractional differential equations is not enriched yet. The fractional derivative due to Hadamard, introduced in 1892 [14], differs from the aforementioned derivatives in the sense that the kernel of the integral in the definition of Hadamard derivative contains logarithmic function of arbitrary

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exponent. A detailed description of Hadamard fractional derivative and integral can be found in [1, 5, 6, 7, 15, 16, 17].

Another interesting class of problems involves hybrid fractional differential equations. For some recent work on the topic , we refer to [2, 3, 12, 18, 19, 21] and the references cited therein.

The article is organized as follows: Section 2 contains some preliminary facts that we need in the sequel. In Section 3, we present the main existence result for the given problem whose proof is based on a fixed point theorem due to Dhage [10].

2. Preliminaries

Definition 2.1 ([15]). The Hadamard fractional integral of order q for a continuous function g is defined as

$$ {}_HJ^q g(t) = \frac{1}{\Gamma(q)} \int_1^t \Big(\log\frac{t}{s}\Big)^{q-1} \frac{g(s)}{s} ds,\ q>0. $$

Definition 2.2 ([15]). The Hadamard derivative of fractional order q for a continuous function $g:[1,\infty)\to\mathbb{R}$ is defined as

$$ {}_HD^q g(t) = \frac{1}{\Gamma(n-q)} \Big(t\frac{d}{dt}\Big)^n \int_1^t \Big(\log\frac{t}{s}\Big)^{n-q-1} \frac{g(s)}{s} ds, \quad n-1<q<n,\ n=[q]+1, $$

where $[q]$ denotes the integer part of the real number q and $\log(\cdot)=\log_e(\cdot)$.

Theorem 2.3 ([15, p. 213]). *Let $\alpha>0$, $n=-[-\alpha]$ and $0\le\gamma<1$. Let G be an open set in $\mathbb{R}$ and let $f:(a,b]\times G\to\mathbb{R}$ be a function such that: $f(x,y)\in C_{\gamma,\log}[a,b]$ for any $y\in G$, then the problem*

$$ {}_HD^\alpha y(t) = f(t,y(t)), \quad \alpha>0, \tag{2.1} $$

$$ {}_HJ^{\alpha-k} y(a+) = b_k,\ b_k\in\mathbb{R}, \quad (k=1,\dots,n,\ n=-[-\alpha]), \tag{2.2} $$

satisfies the Volterra integral equation

$$ y(t) = \sum_{j=1}^n \frac{b_j}{\Gamma(\alpha-j+1)}\Big(\log\frac{t}{a}\Big)^{\alpha-j} + \frac{1}{\Gamma(\alpha)}\int_a^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} f(s,y(s))\frac{ds}{s}, \tag{2.3} $$

for $t>a>0$; i.e., $y(t)\in C_{n-\alpha,\log}[a,b]$ satisfies the relations (2.1)-(2.2) *if and only if it satisfies the Volterra integral equation* (2.3).

In particular, if $0<\alpha\le1$, problem (2.1)-(2.2) *is equivalent to the equation*

$$ y(t) = \frac{b}{\Gamma(\alpha)}\Big(\log\frac{t}{a}\Big)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} f(s,y(s))\frac{ds}{s}, \quad s>a>0. \tag{2.4} $$

Further details can be found in [15]. From Theorem 2.3 we have the following result.

Lemma 2.4. *Given $y\in C([1,T],\mathbb{R})$, the integral solution of initial-value problem*

$$ \begin{gathered} {}_HD^\alpha\Big(\frac{x(t)}{f(t,x(t))}\Big) = y(t), \quad 0<t<1, \\ {}_HJ^{1-\alpha}x(t)|_{t=1} = \eta, \end{gathered} \tag{2.5} $$

is given by

$$ x(t) = f(t,x(t))\Big(\frac{\eta}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{y(s)}{s}ds\Big), \quad t\in[1,T]. $$

The following fixed point theorem due to Dhage [10] is fundamental in the proof of our main result.

Lemma 2.5. *Let S be a non-empty, closed convex and bounded subset of the Banach algebra X let $A : X \to X$ and $B : S \to X$ be two operators such that:*

(a) *A is Lipschitzian with a Lipschitz constant k,*
(b) *B is completely continuous,*
(c) *$x = AxBy \Rightarrow x \in S$ for all $y \in S$, and*
(d) *$Mk < 1$, where $M = \|B(S)\| = \sup\{\|B(x)\| : x \in S\}$.*

Then the operator equation $x = AxBx$ has a solution.

3. Existence result

Let $C([1, T], \mathbb{R})$ denote the Banach space of all continuous real-valued functions defined on $[1, T]$ with the norm $\|x\| = \sup\{|x(t)| : t \in [1, T]\}$. For $t \in [1, T]$, we define $x_r(t) = (\log t)^r x(t), r \geq 0$. Let $C_r([1, T], \mathbb{R})$ be the space of all continuous functions x such that $x_r \in C([1, T], \mathbb{R})$ which is indeed a Banach space endowed with the norm $\|x\|_C = \sup\{(\log t)^r |x(t)| : t \in [1, T]\}$.

Let $0 \leq \gamma < 1$ and $C_{\gamma,\log}[1, T]$ denote the weighted space of continuous functions defined by

$$C_{\gamma,\log}[1, T] = \{g(t) : (\log t)^\gamma g(t) \in C[1, T],\ \|y\|_{C_{\gamma,\log}} = \|(\log t)^\gamma g(t)\|_C\}.$$

In the following we denote $\|y\|_{C_{\gamma,\log}}$ by $\|y\|_C$.

Theorem 3.1. *Assume that:*

(H1) *the function $f : [1, T] \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is bounded continuous and there exists a positive bounded function ϕ with bound $\|\phi\|$ such that*
$$|f(t, x(t)) - f(t, y(t))| \leq \phi(t)|x(t) - y(t)|,$$
for $t \in [1, T]$ and for all $x, y \in \mathbb{R}$;

(H2) *there exist a function $p \in C([1, T], \mathbb{R}^+)$ and a continuous nondecreasing function $\Omega : [0, \infty) \to (0, \infty)$ such that*
$$|g(t, x(t))| \leq p(t)\Omega(|x|), \quad (t, x) \in [1, T] \times \mathbb{R};$$

(H3) *there exists a number $r > 0$ such that*
$$r \geq K\Big[\frac{|\eta|}{\Gamma(\alpha)} + \log T \frac{1}{\Gamma(\alpha + 1)} \|p\|\Omega(r)\Big]. \tag{3.1}$$
where $|f(t, x)| \leq K,\ \forall (t, x) \in [1, T] \times \mathbb{R}$ and
$$\|\phi\|\Big[\frac{|\eta|}{\Gamma(\alpha)} + \log T \frac{1}{\Gamma(\alpha + 1)} \|p\|\Omega(r)\Big] < 1.$$

Then the initial-value problem (1.1) has at least one solution on $[1, T]$.

Proof. Set $X = C([1, T], \mathbb{R})$ and define a subset S of X as

$$S = \{x \in X : \|x\|_C \leq r\},$$

where r satisfies inequality (3.1).

Clearly S is closed, convex and bounded subset of the Banach space X. By Lemma 2.4, the initial-value problem (1.1) is equivalent to the integral equation

$$x(t) = f(t, x(t))\Big(\frac{\eta}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\log \frac{t}{s}\Big)^{\alpha-1} \frac{g(s, x(s))}{s} ds\Big), \tag{3.2}$$

for $t \in [1,T]$.

Define two operators $\mathcal{A} : X \to X$ by

$$\mathcal{A}x(t) = f(t, x(t)), \quad t \in [1,T], \tag{3.3}$$

and $\mathcal{B} : S \to X$ by

$$\mathcal{B}x(t) = \frac{\eta}{\Gamma(\alpha)}(\log t)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\log \frac{t}{s}\Big)^{\alpha-1} \frac{g(s, x(s))}{s} ds, \quad t \in [1,T]. \tag{3.4}$$

Then $x = \mathcal{A}x\mathcal{B}x$. We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Lemma 2.5. For the sake of clarity, we split the proof into a sequence of steps.

Step 1. We first show that $\mathcal{A}$ is a Lipschitz on X, i.e. (a) of Lemma 2.5 holds.

Let $x, y \in X$. Then by (H1) we have

$$\begin{aligned} |(\log t)^{1-\alpha}\mathcal{A}x(t) - (\log t)^{1-\alpha}\mathcal{A}y(t)| &= (\log t)^{1-\alpha}|f(t, x(t)) - f(t, y(t))| \\ &\le \phi(t)(\log t)^{1-\alpha}|x(t) - y(t)| \\ &\le \|\phi\|\|x - y\|_C \end{aligned}$$

for all $t \in [1,T]$. Taking the supremum over the interval $[1,T]$, we obtain

$$\|\mathcal{A}x - \mathcal{A}y\|_C \le \|\phi\|\|x - y\|_C$$

for all $x, y \in X$. So $\mathcal{A}$ is a Lipschitz on X with Lipschitz constant $\|\phi\|$.

Step 2. The operator $\mathcal{B}$ is completely continuous on S, i.e. (b) of Lemma 2.5 holds.

First we show that $\mathcal{B}$ is continuous on S. Let $\{x_n\}$ be a sequence in S converging to a point $x \in S$. Then by Lebesque dominated convergence theorem,

$$\begin{aligned} &\lim_{n\to\infty}(\log t)^{1-\alpha}\mathcal{B}x_n(t) \\ &= \lim_{n\to\infty}\Big(\frac{\eta}{\Gamma(\alpha)} + (\log t)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{g(s, x_n(s))}{s}ds\Big) \\ &= \Big(\frac{\eta}{\Gamma(\alpha)} + (\log t)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{\lim_{n\to\infty} g(s, x_n(s))}{s}ds\Big) \\ &= \Big(\frac{\eta}{\Gamma(\alpha)} + (\log t)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{g(s, x(s))}{s}ds\Big) \\ &= (\log t)^{1-\alpha}\mathcal{B}x(t), \end{aligned}$$

for all $t \in [1,T]$. This shows that $\mathcal{B}$ is continuous os S. It is sufficient to show that $\mathcal{B}(S)$ is a uniformly bounded and equicontinuous set in X. First we note that

$$\begin{aligned} (\log t)^{1-\alpha}|\mathcal{B}x(t)| &= \Big|\frac{\eta}{\Gamma(\alpha)} + (\log t)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{g(s, x(s))}{s}ds\Big| \\ &\le \Big[\frac{|\eta|}{\Gamma(\alpha)} + \|p\|\Omega(r)(\log T)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{1}{s}ds\Big] \\ &= \frac{|\eta|}{\Gamma(\alpha)} + (\log T)\frac{1}{\Gamma(\alpha+1)}\|p\|\Omega(r), \end{aligned}$$

for all $t \in [1,T]$. Taking supremum over the interval $[1,T]$, the above inequality becomes

$$\|\mathcal{B}x\|_C \le \frac{|\eta|}{\Gamma(\alpha)} + (\log T)\frac{1}{\Gamma(\alpha+1)}\|p\|\Omega(r),$$

for all $x \in S$. This shows that $\mathcal{B}$ is uniformly bounded on S.

Next we show that $\mathcal{B}$ is an equicontinuous set in X. Let $\tau_1, \tau_2 \in [1,T]$ with $\tau_1 < \tau_2$ and $x \in S$. Then we have

$$\begin{aligned}
&\big|(\log\tau_2)^{1-\alpha}(\mathcal{B}x)(\tau_2) - (\log\tau_1)^{1-\alpha}(\mathcal{B}x)(\tau_1)\big| \\
&\le \frac{\|p\|\Omega(r)}{\Gamma(\alpha)}\Big|\int_1^{\tau_2}(\log\tau_2)^{1-\alpha}\Big(\log\frac{\tau_2}{s}\Big)^{\alpha-1}\frac{1}{s}ds - \int_1^{\tau_1}(\log\tau_1)^{1-\alpha}\Big(\log\frac{\tau_1}{s}\Big)^{\alpha-1}\frac{1}{s}ds\Big| \\
&\le \frac{\|p\|\Omega(r)}{\Gamma(\alpha)}\Big|\int_1^{\tau_1}\Big[(\log\tau_2)^{1-\alpha}\Big(\log\frac{\tau_2}{s}\Big)^{\alpha-1} - (\log\tau_1)^{1-\alpha}\Big(\log\frac{\tau_1}{s}\Big)^{\alpha-1}\Big]\frac{1}{s}ds\Big| \\
&\quad + \frac{\|p\|\Omega(r)}{\Gamma(\alpha)}\Big|\int_{\tau_1}^{\tau_2}(\log\tau_2)^{1-\alpha}\Big(\log\frac{\tau_2}{s}\Big)^{\alpha-1}\frac{1}{s}ds\Big|.
\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in S$ as $t_2 - t_1 \to 0$. Therefore, it follows from the Arzelá-Ascoli theorem that $\mathcal{B}$ is a completely continuous operator on S.

Step 3. Next we show that hypothesis (c) of Lemma 2.5 is satisfied. Let $x \in X$ and $y \in S$ be arbitrary elements such that $x = \mathcal{A}x\mathcal{B}y$. Then we have

$$\begin{aligned}
(\log t)^{1-\alpha}|x(t)| &= (\log t)^{1-\alpha}|\mathcal{A}x(t)||\mathcal{B}y(t)| \\
&= |f(t,x(t))|\Big|\Big(\frac{\eta}{\Gamma(\alpha)} + (\log t)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{g(s,y(s))}{s}ds\Big)\Big| \\
&\le K\Big|\Big(\frac{\eta}{\Gamma(\alpha)} + (\log t)^{1-\alpha}\frac{1}{\Gamma(\alpha)}\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{g(s,y(s))}{s}ds\Big)\Big| \\
&\le K\Big[\frac{|\eta|}{\Gamma(\alpha)} + (\log T)^{1-\alpha}\|p\|\Omega(r)\frac{1}{\Gamma(\alpha)}\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{1}{s}ds\Big] \\
&\le K\Big[\frac{|\eta|}{\Gamma(\alpha)} + (\log T)\frac{1}{\Gamma(\alpha+1)}\|p\|\Omega(r)\Big].
\end{aligned}$$

Taking supremum for $t \in [1,T]$, we obtain

$$\|x\|_C \le K\Big[\frac{|\eta|}{\Gamma(\alpha)} + (\log T)\frac{1}{\Gamma(\alpha+1)}\|p\|\Omega(r)\Big] \le r,$$

that is, $x \in S$.

Step 4. Now we show that $Mk < 1$, that is, (d) of Lemma 2.5 holds.

This is obvious by (H_4), since we have $M = \|B(S)\| = \sup\{\|\mathcal{B}x\| : x \in S\} \le \frac{|\eta|}{\Gamma(\alpha)} + (\log T)\frac{1}{\Gamma(\alpha+1)}\|p\|\Omega(r)$ and $k = \|\phi\|$.

Thus all the conditions of Lemma 2.5 are satisfied and hence the operator equation $x = \mathcal{A}x\mathcal{B}x$ has a solution in S. In consequence, the problem (1.1) has a solution on $[1,T]$. This completes the proof. $\square$

Example 3.2. Consider the initial-value problem

$$\begin{gathered}
{}_HD^{1/2}\Big(\frac{x(t)}{f(t,x)}\Big) = g(t,x), \quad 1 < t < e, \\
{}_HJ^{1/2}x(t)|_{t=1} = 1,
\end{gathered} \tag{3.5}$$

where

$$f(t,x) = \frac{1}{5\sqrt{\pi}}(\sin t\tan^{-1}x + \pi/2),$$

$$g(t,x)=\frac{1}{10}\Big(\frac{1}{6}|x|+\frac{1}{8}\cos x+\frac{|x|}{4(1+|x|)}+\frac{1}{16}\Big).$$

Obviously $|f(t,x)|\le \frac{\sqrt{\pi}}{5}=K$, $\|\phi\|=\frac{1}{5\sqrt{\pi}}$, $|g(t,x)|\le \frac{1}{10}\big(\frac{1}{6}|x|+\frac{7}{16}\big)$. We choose $\|p\|=\frac{1}{10}$, $\Omega(r)=\frac{1}{6}r+\frac{7}{16}$. By the condition (H3), it is found that $\frac{261}{1192}\le r<\frac{3}{8}(400\pi-87)$. Clearly all the conditions of Theorem 3.1 are satisfied. Hence, by the conclusion of Theorem 3.1, it follows that problem (3.5) has a solution.

4. Discussion

Operator equations such as $x=\mathcal{A}x\mathcal{B}x$ associated with problem (1.1), are known as quadratic integral equations. Some recent works on these kinds of equations can be found in [4, 8, 9, 13, 20] and the references cited therein. It is interesting to note that the involvement of the term $\frac{\eta}{\Gamma(\alpha)}(\log t)^{\alpha-1}$ in the integral solution (3.2) of the problem (1.1) makes it unbounded. In consequence, we have to consider an appropriate space to establish the existence of a solution to the given problem. In this scenario, Banach's fixed point theorem cannot be used in the weighted normed space. However, if we take $\eta=0$, then we can obtain an integral equation $x=\mathcal{F}x$, where the operator $\mathcal{F}:C([1,T],\mathbb{R})\to C([1,T],\mathbb{R})$ is

$$(\mathcal{F}x)(t)=\frac{1}{\Gamma(\alpha)}f(t,x(t))\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{g(s,x(s))}{s}ds,$$

where $C([1,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[1,T]\to\mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by $\|x\|=\sup\{|x(t)|:t\in[1,T]\}$. In this situation, we can apply Banach's contraction mapping principle. For that, assuming that $|f(t,x(t))|\le M_1$, $|g(t,x(t))|\le M_2$, $|f(t,x(t))-f(t,y(t))|\le L_1|x-y|$, $|g(t,x(t))-g(t,y(t))|\le L_2|x-y|$, for all $x,y\in\mathbb{R}$, we obtain

$$\begin{aligned}
&|(\mathcal{F}x)(t)-(\mathcal{F}y)(t)|\\
&=\frac{1}{\Gamma(\alpha)}\Big|f(t,x(t))\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{g(s,x(s))}{s}ds\\
&\quad -f(t,y(t))\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{g(s,y(s))}{s}ds\Big|\\
&=\frac{1}{\Gamma(\alpha)}\Big|[f(t,x(t))-f(t,y(t))]\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{g(s,x(s))}{s}ds\\
&\quad +f(t,y(t))\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{[g(s,x(s))-g(s,y(s))]}{s}ds\Big|\\
&\le\max_{t\in[1,T]}\Big[\frac{1}{\Gamma(\alpha)}\Big\{(L_1M_2+M_1L_2)\int_1^t\Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{1}{s}ds\Big\}\Big]\|x-y\|\\
&\le\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}(L_1M_2+M_1L_2)\|x-y\|.
\end{aligned}$$

Letting $\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}(L_1M_2+M_1L_2)<1$, the operator $\mathcal{F}$ is a contraction. Thus, by Banach's contraction principle, the problem (1.1) with $\eta=0$ has a unique solution.

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