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DIFFUSION OF A SINGLE-PHASE FLUID THROUGH A GENERAL DETERMINISTIC PARTIALLY-FISSURED MEDIUM

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Abstract. The sigma convergence method was introduced by G. Nguetseng for studying deterministic homogenization problems beyond the periodic setting and extended by him to the case of deterministic homogenization in general deterministic perforated domains. Here we show that this concept can also model such problems in more general domains. We illustrate this by considering the quasi-linear version of the distributed-microstructure model for single phase fluid flow in a partially fissured medium. We use the well-known concept of algebras with mean value. An important result of de Rham type is first proven in this setting and then used to get a general compactness result associated to algebras with mean value in the framework of sigma convergence. Finally we use these results to obtain homogenized limits of our micro-model in various deterministic settings, including periodic and almost periodic cases.

1. Introduction

A fissured medium is a structure consisting of a matrix of porous and permeable material through which is intertwined a highly developed system of *fissures* with substantially higher flow rates and lower relative volume. The problem of homogenization or *scaling* is to determine from data or local characteristics the *effective* parameters for a description of this medium on a larger scale. Problems of flow and transport through porous media have been investigated over the last century and have continued to receive increasing attention over the years. To describe the flow of fluid in heterogeneous media, several heuristic models have been developed. The classical and most studied *double diffusion model* for fissured porous rock domain was introduced in 1960 by Barenblatt, Zheltov and Kochina [2] and further developed in that decade [10, 17, 19, 28, 35]. It has been recently rigorously derived by homogenization from an exact micro-model [20, 21, 34]. The special *pseudoparabolic* case of this double diffusion model is particularly important for the applications, and it has been recently upscaled by homogenization [30]. In 1990 Arbogast, Douglas and Hornung [1] developed the more realistic *double porosity model* which has been studied by many researchers and extended to include *secondary flux* [29, 42]. We also refer to [6, 40] for the homogenization of some of the previous models in a random environment.

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In [11] a model for diffusion of a single phase fluid through a periodic *partially-fissured medium* was introduced; it was studied by two-scale convergence in [9], and in [40] the random counterpart of the same model is derived by stochastic homogenization. Our objective here is to fill the gap between these periodic and random cases by considering a general deterministic version of that problem. More precisely, we aim to develop a deterministic approach of homogenization for solving homogenization problems (beyond the classical periodic setting) related to some models consisting of fluid-matrix system interaction in flow, especially of fissured porous media. The problem addressed here is the model from [11] of a partially-fissured medium for which both the fissure system and the porous matrix are connected and contribute to the global flow. Our aim is to study this problem in more general settings beyond periodicity.

To illustrate the process, we describe a *general deterministic partially-fissured medium* that will be used in the following. The reference cell is $Y = (0,1)^N$ with non-empty open disjoint connected subsets Y_1 and Y_2 denoting the local fissure system and porous matrix, respectively, such that $\overline{Y} = \overline{Y}_1 \cup \overline{Y}_2$. Let $S \subset \mathbb{Z}^N$ be an infinite subset of $\mathbb{Z}^N$ to be determined below, and set $G_j = \cup_{k\in S}(k + Y_j)$ for $j = 1, 2$. Assume that $\overline{G_1}$ is connected. In the partially-fissured case, $\overline{G_2}$ can be connected also. (This requires that $N \geq 3$.) Examples can be constructed from the periodic case $S = \mathbb{Z}^N$ by deleting (almost periodic) arrays of cells. The deleted cells represent impermeable regions or *obstacles*, $G_0 = \cup_{k\notin S}(k + Y)$.

Given the open bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ and $\varepsilon > 0$, we define

$$\Omega_j^\varepsilon = \Omega \cap \varepsilon G_j, \quad j = 0, 1, 2.$$

Denote by $\Gamma_{i,j}^\varepsilon = \partial\Omega_i^\varepsilon \cap \partial\Omega_j^\varepsilon \cap \Omega$ the interface of Ω_i^ε with Ω_j^ε lying in Ω. The set Ω_1^ε (resp. Ω_2^ε) is the portion of Ω occupied by the fissures (resp. porous matrix), and the flow region is given by the disjoint union $\Omega^\varepsilon = \Omega_1^\varepsilon \cup \Gamma_{1,2}^\varepsilon \cup \Omega_2^\varepsilon$.

Let ν_j denote the unit outward normal on $\partial\Omega_j^\varepsilon$. Note that $\nu_1 = -\nu_2$ on $\Gamma_{1,2}^\varepsilon$. It is worthwhile to note that, when $S = \mathbb{Z}^N$, we get a structure consisting of fissures and matrix equidistributed (or, as in the classical literature, *periodically distributed*) over the entire domain Ω with period εY. But our domain is not necessarily a periodic array of εY as is usually the case in all deterministic situations encountered so far. We shall see that the *fissured cells* may also be *almost periodically distributed* in Ω.

The partially-fissured micro-model. We set up the micro-model for Darcy flow in the partially-fissured medium. The coefficients of the operator involved in the problem are given as follows. For $2 \leq p < \infty$ and for $j = 1, 2, 3$, let $a_j : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ satisfy the following conditions:

$$\text{For each fixed } \lambda \in \mathbb{R}^N, \text{ the function } a_j(\cdot, \lambda) \text{ is measurable;} \tag{1.1a}$$

$$a_j(y, 0) = 0 \text{ almost every } y \in \mathbb{R}^N; \tag{1.1b}$$

There are two constants positive α_0, α_1 such that a.e. $y \in \mathbb{R}^N$,
(i) $(a_j(y,\lambda) - a_j(y,\mu)) \cdot (\lambda - \mu) \geq \alpha_0|\lambda - \mu|^p$
(ii) $|a_j(y,\lambda) - a_j(y,\mu)| \leq \alpha_1(1 + |\lambda| + |\mu|)^{p-2}|\lambda - \mu|$ (1.1c)
for all $\lambda, \mu \in \mathbb{R}^N$, where the dot denotes the usual Euclidean inner product in $\mathbb{R}^N$ and $|\cdot|$ the associated norm;

$$\text{The density function } c_j : \mathbb{R}^N \to \mathbb{R} \text{ is bounded continuous and satisfies } \Lambda^{-1} \le c_j(y) \le \Lambda \text{ for all } y \in \mathbb{R}^N \text{ where } \Lambda \text{ is positive and independent of } y. \tag{1.1d}$$

Let T be a positive real number. With the above assumptions, the existence of the trace functions $(x,t) \mapsto a_j(x/\varepsilon, Du_\varepsilon(x,t))$ and $x \mapsto c_j(x/\varepsilon)$ here denoted respectively by $a_j^\varepsilon(\cdot, Du_\varepsilon)$ and c_j^ε, has been discussed previously (see e.g., [26, 37]). These functions are well-defined as elements of $L^{p'}(Q)^N$ (where $Q = \Omega \times (0,T)$) and $\mathcal{C}(\Omega)$ respectively, and satisfy properties similar to those in (1.1).

We describe the micro-model for diffusion through the partially-fissured porous medium [11, 9]. The Darcy flow potential in the system of fissures Ω_1^ε is denoted by $u_1^\varepsilon(x,t)$ while that in the porous matrix is a convex combination of two components $u_2^\varepsilon(x,t)$ and $u_3^\varepsilon(x,t)$ which account respectively for the global diffusion through the matrix and the high-frequency variations which lead to local storage in the matrix. The flow potential in Ω_2^ε is given by the combination $\alpha u_2^\varepsilon + \delta u_3^\varepsilon$, where $\alpha + \delta = 1$ with $\alpha \ge 0$ and $\delta > 0$. The flux of the flow component $u_1^\varepsilon(x,t)$ in Ω_1^ε is given by $-a_1(x/\varepsilon, \nabla u_1^\varepsilon(x,t))$ while the flow components $u_2^\varepsilon(x,t)$ and $u_3^\varepsilon(x,t)$ in Ω_2^ε are given by $-a_2(x/\varepsilon, \nabla u_2^\varepsilon(x,t))$ and $-\varepsilon a_3(x/\varepsilon, \varepsilon\nabla u_3^\varepsilon(x,t))$. The flow of fluid at the micro-scale is described by the classical conservation of fluid equations and interface conditions in Ω^ε:

$$\frac{\partial}{\partial t}(c_1^\varepsilon u_1^\varepsilon) - \operatorname{div} a_1^\varepsilon(\cdot, \nabla u_1^\varepsilon) = 0 \quad \text{in } \Omega_1^\varepsilon \times (0,T) \tag{1.2a}$$
$$\frac{\partial}{\partial t}(c_2^\varepsilon u_2^\varepsilon) - \operatorname{div} a_2^\varepsilon(\cdot, \nabla u_2^\varepsilon) = 0 \quad \text{in } \Omega_2^\varepsilon \times (0,T) \tag{1.2b}$$
$$\frac{\partial}{\partial t}(c_3^\varepsilon u_3^\varepsilon) - \varepsilon\operatorname{div} a_3^\varepsilon(\cdot, \varepsilon\nabla u_3^\varepsilon) = 0 \quad \text{in } \Omega_2^\varepsilon \times (0,T) \tag{1.2c}$$
$$u_1^\varepsilon = \alpha u_2^\varepsilon + \delta u_3^\varepsilon \quad \text{on } \Gamma_{1,2}^\varepsilon \times (0,T) \tag{1.2d}$$
$$\alpha a_1^\varepsilon(\cdot, \nabla u_1^\varepsilon) \cdot \nu_1 = a_2^\varepsilon(\cdot, \nabla u_2^\varepsilon) \cdot \nu_1 \quad \text{on } \Gamma_{1,2}^\varepsilon \times (0,T) \tag{1.2e}$$
$$\delta a_1^\varepsilon(\cdot, \nabla u_1^\varepsilon) \cdot \nu_1 = \varepsilon a_3^\varepsilon(\cdot, \varepsilon\nabla u_3^\varepsilon) \cdot \nu_1 \quad \text{on } \Gamma_{1,2}^\varepsilon \times (0,T). \tag{1.2f}$$

We assume the Neumann no-flow conditions on the remaining interfaces

$$a_1^\varepsilon(\cdot, \nabla u_1^\varepsilon) \cdot \nu_1 = 0 \quad \text{on } \Gamma_{1,0}^\varepsilon \times (0,T) \tag{1.2g}$$
$$a_2^\varepsilon(\cdot, \nabla u_2^\varepsilon) \cdot \nu_2 = 0 \text{ on } \Gamma_{2,0}^\varepsilon \times (0,T) \tag{1.2h}$$
$$a_3^\varepsilon(\cdot, \varepsilon\nabla u_3^\varepsilon) \cdot \nu_2 = 0 \text{ on } \Gamma_{2,0}^\varepsilon \times (0,T), \tag{1.2i}$$

and on the global boundary

$$a_1^\varepsilon(\cdot, \nabla u_1^\varepsilon) \cdot \nu_1 = 0 \text{ on } (\partial\Omega_1^\varepsilon \cap \partial\Omega) \times (0,T) \tag{1.2j}$$
$$a_2^\varepsilon(\cdot, \nabla u_2^\varepsilon) \cdot \nu_2 = 0 \text{ on } (\partial\Omega_2^\varepsilon \cap \partial\Omega) \times (0,T) \tag{1.2k}$$
$$a_3^\varepsilon(\cdot, \varepsilon\nabla u_3^\varepsilon) \cdot \nu_2 = 0 \text{ on } (\partial\Omega_2^\varepsilon \cap \partial\Omega) \times (0,T). \tag{1.2l}$$

Finally the initial-boundary-value problem is completed by the initial conditions

$$u_1^\varepsilon(\cdot, 0) = u_1^0,\ u_2^\varepsilon(\cdot, 0) = u_2^0,\ u_3^\varepsilon(\cdot, 0) = u_3^0 \tag{1.2m}$$

where $u_j^0 \in L^2(\Omega)$ are given for $j = 1, 2, 3$.

To solve problem (1.2) we define appropriate spaces. For any fixed $\varepsilon > 0$ let

$$H_\varepsilon = L^2(\Omega_1^\varepsilon) \times L^2(\Omega_2^\varepsilon) \times L^2(\Omega_2^\varepsilon)$$

be equipped with inner product

$$((u_1,u_2,u_3),(v_1,v_2,v_3))_{H_\varepsilon} = \int_{\Omega_1^\varepsilon} c_1^\varepsilon u_1 v_1 dx + \sum_{i=2}^{3}\int_{\Omega_2^\varepsilon} c_i^\varepsilon u_i v_i dx,$$

which makes it a Hilbert space. Next, let $\gamma_j^\varepsilon : W^{1,p}(\Omega_j^\varepsilon) \to L^p(\partial\Omega_j^\varepsilon)$ $(j=1,2)$ denote the usual trace maps. Set $V_\varepsilon = H_\varepsilon \cap W_\varepsilon$ where

$$\begin{aligned} W_\varepsilon = \big\{&(u_1,u_2,u_3) \in W^{1,p}(\Omega_1^\varepsilon) \times W^{1,p}(\Omega_2^\varepsilon) \times W^{1,p}(\Omega_2^\varepsilon) : \\ &\gamma_1^\varepsilon u_1 = \alpha\gamma_2^\varepsilon u_2 + \delta\gamma_2^\varepsilon u_3 \text{ on } \Gamma_{1,2}^\varepsilon\big\}. \end{aligned}$$

V_ε is a Banach space under the norm

$$\begin{aligned} \|(u_1,u_2,u_3)\|_{V_\varepsilon} &= \|\chi_1^\varepsilon u_1\|_{L^2(\Omega)} + \|\chi_2^\varepsilon u_2\|_{L^2(\Omega)} + \|\chi_2^\varepsilon u_3\|_{L^2(\Omega)} \\ &\quad + \|\chi_1^\varepsilon \nabla u_1\|_{L^p(\Omega)} + \|\chi_2^\varepsilon \nabla u_2\|_{L^p(\Omega)} + \|\chi_2^\varepsilon \nabla u_3\|_{L^p(\Omega)}, \end{aligned}$$

where χ_j^ε (for $j=1,2$) denotes the *characteristic function* of the open set Ω_j^ε. Letting $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$, the variational formulation of (1.2) amounts to finding $u^\varepsilon \in L^p(0,T;V_\varepsilon)$ such that

$$\big(\frac{\partial u^\varepsilon}{\partial t}, \varphi\big)_{H_\varepsilon} + \langle \mathcal{A}^\varepsilon u^\varepsilon, \varphi\rangle = 0 \text{ for all } \varphi = (\varphi_1,\varphi_2,\varphi_3) \in V_\varepsilon \tag{1.3}$$

where the operator $\mathcal{A}^\varepsilon : V_\varepsilon \to V_\varepsilon'$ is defined by

$$\langle \mathcal{A}^\varepsilon u, \varphi\rangle = \int_{\Omega_1^\varepsilon} a_1^\varepsilon(\cdot,\nabla u_1)\cdot\nabla\varphi_1 dx + \int_{\Omega_2^\varepsilon} (a_2^\varepsilon(\cdot,\nabla u_2)\cdot\nabla\varphi_2 + a_3^\varepsilon(\cdot,\varepsilon\nabla u_3)\cdot\varepsilon\nabla\varphi_3)dx$$

for $u=(u_1,u_2,u_3)$, $\varphi = (\varphi_1,\varphi_2,\varphi_3) \in V_\varepsilon$. This gives rise to the following abstract Cauchy problem: for each $\varepsilon > 0$ and $u^0 = (u_1^0,u_2^0,u_3^0) \in L^2(\Omega)^3$, find $u^\varepsilon = (u_1^\varepsilon,u_2^\varepsilon,u_3^\varepsilon) \in L^p(0,T;V_\varepsilon)$ such that

$$\frac{d}{dt}u^\varepsilon + \mathcal{A}^\varepsilon u^\varepsilon = 0 \quad \text{in } L^{p'}(0,T;V_\varepsilon'), \tag{1.4a}$$

$$u^\varepsilon(0) = u^0 \quad \text{in } H_\varepsilon. \tag{1.4b}$$

Conversely, a sufficiently smooth solution to (1.4) is also a solution to (1.2). The following result holds.

Theorem 1.1. *For any fixed $\varepsilon > 0$, the initial-value problem* (1.4) *possesses a unique solution $u^\varepsilon = (u_1^\varepsilon,u_2^\varepsilon,u_3^\varepsilon) \in L^p(0,T;V_\varepsilon)$. Moreover $u^\varepsilon \in \mathcal{C}([0,T];H_\varepsilon)$ and the following a priori estimate holds:*

$$\begin{aligned} &\frac{1}{2}\|u^\varepsilon(t)\|_{H_\varepsilon}^2 + \alpha_0\int_0^t (\|\chi_1^\varepsilon\nabla u_1^\varepsilon\|_{L^p(\Omega)}^p + \|\chi_2^\varepsilon\nabla u_2^\varepsilon\|_{L^p(\Omega)}^p + \|\varepsilon\chi_2^\varepsilon\nabla u_3^\varepsilon\|_{L^p(\Omega)}^p)ds \\ &\le \frac{1}{2}\|(\chi_1^\varepsilon u_1^0, \chi_2^\varepsilon u_2^0, \chi_2^\varepsilon u_3^0)\|_{H_\varepsilon}^2, \quad 0 \le t \le T. \end{aligned} \tag{1.5}$$

Proof. The existence and uniqueness of u^ε follow from the application of [33, Proposition III.4.1] (see also [9]). Estimate (1.5) is an easy consequence of the variational formulation (1.3) in which we take $\varphi = u^\varepsilon(t)$. □

Theorem 1.1 entails that $(u^\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(0,T;H_\varepsilon)$ and that the sequences $(\chi_1^\varepsilon\nabla u_1^\varepsilon)_{\varepsilon>0}$, $(\chi_2^\varepsilon\nabla u_2^\varepsilon)_{\varepsilon>0}$ and $(\varepsilon\chi_2^\varepsilon\nabla u_3^\varepsilon)_{\varepsilon>0}$ are bounded in $L^p(Q)^N$. Finally, from the properties of the functions a_j, the sequences $(\chi_j^\varepsilon a_j^\varepsilon(\cdot,\nabla u_j^\varepsilon))_{\varepsilon>0}$ (for

$j = 1, 2$) and $(\chi_2^\varepsilon a_3^\varepsilon(\cdot, \varepsilon\nabla u_3^\varepsilon))_{\varepsilon>0}$ are bounded in $L^{p'}(Q)^N$. These boundedness properties shall play an essential role in the sequel where we obtain the homogenized limit of the system (1.2).

2. Algebras with mean value and sigma-convergence

In this section we recall some basic facts about algebras with mean value [43] and the concept of sigma-convergence [22] (see also [25, 31]). Using the semigroup theory we present some essential results for these concepts. We refer the reader to [36] for the details regarding most of the results of this section. In the following, all vector spaces are real vector spaces, and scalar functions take real values.

2.1. **Algebras with Mean Value.** A closed subalgebra A of the $\mathcal{C}$*-algebra of bounded uniformly continuous functions $BUC(\mathbb{R}^N)$ is an *algebra with mean value* on $\mathbb{R}^N$ [18, 8, 31, 43] if it contains the constants, is translation invariant ($u(\cdot+a) \in A$ for any $u \in A$ and each $a \in \mathbb{R}^N$) and each of its elements possesses a *mean value* in the following sense:

- For any $u \in A$, the sequence $(u^\varepsilon)_{\varepsilon>0}$ (defined by $u^\varepsilon(x) = u(x/\varepsilon)$, $x \in \mathbb{R}^N$) weak*-converges in $L^\infty(\mathbb{R}^N)$ to some constant real function $M(u)$ as $\varepsilon \to 0$.

It is known that A (endowed with the sup norm topology) is a commutative $\mathcal{C}$*-algebra with identity. We denote by $\Delta(A)$ the spectrum of A and by $\mathcal{G}$ the Gelfand transformation on A. We recall that $\Delta(A)$ (a subset of the topological dual A' of A) is the set of all nonzero multiplicative linear functionals on A, and $\mathcal{G}$ is the mapping of A into $\mathcal{C}(\Delta(A))$ such that $\mathcal{G}(u)(s) = \langle s, u\rangle$ ($s \in \Delta(A)$), where $\langle , \rangle$ denotes the duality pairing between A' and A. When equipped with the relative weak$*$ topology on A' (the topological dual A' of A), $\Delta(A)$ is a compact topological space, and the Gelfand transformation $\mathcal{G}$ is an isometric $*$-isomorphism identifying A with $\mathcal{C}(\Delta(A))$ as $\mathcal{C}$*-algebras. Moreover the mean value M defined on A is a nonnegative continuous linear functional that can be expressed in terms of a Radon measure β (of total mass 1) in $\Delta(A)$ (called the *M-measure* for A [22]) satisfying the property that $M(u) = \int_{\Delta(A)} \mathcal{G}(u)d\beta$ for $u \in A$.

To any algebra with mean value A we define the subspaces: $A^m \equiv \{\psi \in \mathcal{C}^m(\mathbb{R}^N) : D_y^\alpha\psi \in A \ \forall\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N \text{ with } |\alpha| \le m\}$ (where $D_y^\alpha\psi = \partial^{|\alpha|}\psi/\partial y_1^{\alpha_1} \cdots \partial y_N^{\alpha_N}$). Under the norm $|||u|||_m = \sup_{|\alpha|\le m} \|D_y^\alpha\psi\|_\infty$, A^m is a Banach space. We also define the space $A^\infty = \{\psi \in \mathcal{C}^\infty(\mathbb{R}^N) : D_y^\alpha\psi \in A \ \forall\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N\}$, a Fréchet space when endowed with the locally convex topology defined by the family of norms $|||\cdot|||_m$.

Next, let B_A^p ($1 \le p < \infty$) denote the *Besicovitch space* associated to A, that is the closure of A with respect to the Besicovitch seminorm

$$\|u\|_p = \Big(\limsup_{r\to+\infty} \frac{1}{|B_r|} \int_{B_r} |u(y)|^p dy\Big)^{1/p}.$$

It is known that B_A^p is a complete seminormed vector space verifying $B_A^q \subset B_A^p$ for $1 \le p \le q < \infty$. From this last property one may naturally define the space B_A^∞ as follows:

$$B_A^\infty = \{f \in \cap_{1\le p<\infty} B_A^p : \sup_{1\le p<\infty} \|f\|_p < \infty\}.$$

We endow B_A^∞ with the seminorm $[f]_\infty = \sup_{1\le p<\infty} \|f\|_p$, which makes it a complete seminormed space. We recall that the spaces B_A^p ($1 \le p \le \infty$) are not in general Fréchet spaces since they are not separated in general. The following properties are worth noticing [25, 31]:

(1) The Gelfand transformation $\mathcal{G} : A \to \mathcal{C}(\Delta(A))$ extends by continuity to a unique continuous linear mapping (still denoted by $\mathcal{G}$) of B_A^p into $L^p(\Delta(A))$, which in turn induces an isometric isomorphism $\mathcal{G}_1$ of $B_A^p/\mathcal{N} \equiv \mathcal{B}_A^p$ onto $L^p(\Delta(A))$ (where $\mathcal{N} = \{u \in B_A^p : \mathcal{G}(u) = 0\}$). Moreover if $u \in B_A^p \cap L^\infty(\mathbb{R}^N)$ then $\mathcal{G}(u) \in L^\infty(\Delta(A))$ and $\|\mathcal{G}(u)\|_{L^\infty(\Delta(A))} \leq \|u\|_{L^\infty(\mathbb{R}^N)}$.

(2) The mean value M defined on A, extends by continuity to a positive continuous linear form (still denoted by M) on B_A^p satisfying $M(u) = \int_{\Delta(A)} \mathcal{G}(u)d\beta$ ($u \in B_A^p$). Furthermore, $M(\tau_a u) = M(u)$ for each $u \in B_A^p$ and all $a \in \mathbb{R}^N$, where $\tau_a u(y) = u(y+a)$ for almost all $y \in \mathbb{R}^N$. Moreover for $u \in B_A^p$ we have $\|u\|_p = [M(|u|^p)]^{1/p}$, and for $u + \mathcal{N} \in \mathcal{B}_A^p$ we may still define its mean value once again denoted by M, as $M(u+\mathcal{N}) = M(u)$.

Remark 2.1. Based on property (1) above, we set the following notation that will be used throughout the work: For u either in A or in B_A^p, $\widehat{u}$ stands for the function $\mathcal{G}(u)$, while for u in $\mathcal{B}_A^p$, $\widehat{u}$ denotes the function $\mathcal{G}_1(u)$. This last notation is fully justified since any $u \in \mathcal{B}_A^p$ has the form $u = v + \mathcal{N}$ with $v \in B_A^p$, and using the definition of $\mathcal{G}_1$, $\mathcal{G}_1(v+\mathcal{N}) = \mathcal{G}(v) = \widehat{v}$ as $\mathcal{G}(w) = 0$ for any $w \in \mathcal{N}$.

Let $1 \leq p \leq \infty$. To define the *Sobolev spaces* associated to the algebra A, we consider the N-parameter group of isometries $\{T(y) : y \in \mathbb{R}^N\}$ defined by

$$T(y) : \mathcal{B}_A^p \to \mathcal{B}_A^p,\ T(y)(u+\mathcal{N}) = \tau_y u + \mathcal{N} \text{ for } u \in B_A^p.$$

Since the elements of A are uniformly continuous, $\{T(y) : y \in \mathbb{R}^N\}$ is a strongly continuous group in $\mathcal{L}(\mathcal{B}_A^p, \mathcal{B}_A^p)$ (the Banach space of continuous linear functionals of $\mathcal{B}_A^p$ into $\mathcal{B}_A^p$): $T(y)(u+\mathcal{N}) \to u + \mathcal{N}$ in $\mathcal{B}_A^p$ as $|y| \to 0$. We also associate to $\{T(y) : y \in \mathbb{R}^N\}$ the following N-parameter group $\{\overline{T}(y) : y \in \mathbb{R}^N\}$ defined by

$$\overline{T}(y) : L^p(\Delta(A)) \to L^p(\Delta(A));\ \overline{T}(y)\mathcal{G}_1(u+\mathcal{N}) = \mathcal{G}_1(T(y)(u+\mathcal{N})) \quad \text{for } u \in B_A^p.$$

The group $\{\overline{T}(y) : y \in \mathbb{R}^N\}$ is also strongly continuous. The infinitesimal generator of $T(y)$ (resp. $\overline{T}(y)$) along the ith coordinate direction, denoted by $D_{i,p}$ (resp. $\partial_{i,p}$), is defined as

$$D_{i,p}u = \lim_{t\to 0} \Big(\frac{T(te_i)u - u}{t}\Big) \quad \text{in } \mathcal{B}_A^p$$

$$(\text{resp. } \partial_{i,p}v = \lim_{t\to 0} \Big(\frac{\overline{T}(te_i)v - v}{t}\Big) \quad \text{in } L^p(\Delta(A)))$$

where we have used the same letter u to denote the equivalence class of an element $u \in B_A^p$ in $\mathcal{B}_A^p$ and $e_i = (\delta_{ij})_{1\leq j\leq N}$ (δ_{ij} being the Kronecker δ). The domain of $D_{i,p}$ (resp. $\partial_{i,p}$) in $\mathcal{B}_A^p$ (resp. $L^p(\Delta(A))$) is denoted by $\mathcal{D}_{i,p}$ (resp. $\mathcal{W}_{i,p}$). In the sequel we denote by ϱ the canonical mapping of B_A^p onto $\mathcal{B}_A^p$, that is, $\varrho(u) = u + \mathcal{N}$ for $u \in B_A^p$. The following results were obtained in [36].

Proposition 2.2. *$\mathcal{D}_{i,p}$ (resp. $\mathcal{W}_{i,p}$) is a vector subspace of $\mathcal{B}_A^p$ (resp. $L^p(\Delta(A))$), $D_{i,p} : \mathcal{D}_{i,p} \to \mathcal{B}_A^p$ (resp. $\partial_{i,p} : \mathcal{W}_{i,p} \to L^p(\Delta(A))$) is a linear operator, $\mathcal{D}_{i,p}$ (resp. $\mathcal{W}_{i,p}$) is dense in $\mathcal{B}_A^p$ (resp. $L^p(\Delta(A))$), and the graph of $D_{i,p}$ (resp. $\partial_{i,p}$) is closed in $\mathcal{B}_A^p \times \mathcal{B}_A^p$ (resp. $L^p(\Delta(A)) \times L^p(\Delta(A))$).*

The next result allows us to see $D_{i,p}$ as a generalization of the usual partial derivative.

Lemma 2.3 ([36, Lemma 1]). *Let $1 \leq i \leq N$. If $u \in A^1$ then $\varrho(u) \in \mathcal{D}_{i,p}$ and*

$$D_{i,p}\varrho(u) = \varrho(\frac{\partial u}{\partial y_i}). \tag{2.1}$$

From (2.1) we infer that $D_{i,p} \circ \varrho = \varrho \circ \partial/\partial y_i$, that is, $D_{i,p}$ generalizes the usual partial derivative $\partial/\partial y_i$. One may also define higher order derivatives by setting $D_p^\alpha = D_{1,p}^{\alpha_1} \circ \dots \circ D_{N,p}^{\alpha_N}$ (resp. $\partial_p^\alpha = \partial_{1,p}^{\alpha_1} \circ \dots \circ \partial_{N,p}^{\alpha_N}$) for $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$ with $D_{i,p}^{\alpha_i} = D_{i,p} \circ \dots \circ D_{i,p}$, α_i-times. Now, define the *Besicovitch-Sobolev spaces*

$$\mathcal{B}_A^{1,p} = \cap_{i=1}^N \mathcal{D}_{i,p} = \{u \in \mathcal{B}_A^p : D_{i,p}u \in \mathcal{B}_A^p \ \forall 1 \leq i \leq N\},$$
$$\mathcal{D}_A(\mathbb{R}^N) = \{u \in \mathcal{B}_A^\infty : D_\infty^\alpha u \in \mathcal{B}_A^\infty \ \forall \alpha \in \mathbb{N}^N\}.$$

It can be shown that $\mathcal{D}_A(\mathbb{R}^N)$ is dense in $\mathcal{B}_A^p$, $1 \leq p < \infty$. We also have that $\mathcal{B}_A^{1,p}$ is a Banach space under the norm

$$\|u\|_{\mathcal{B}_A^{1,p}} = \Big(\|u\|_p^p + \sum_{i=1}^N \|D_{i,p}u\|_p^p\Big)^{1/p} \quad (u \in \mathcal{B}_A^{1,p}).$$

The counter-part of the above properties also holds with

$$W^{1,p}(\Delta(A)) = \cap_{i=1}^N \mathcal{W}_{i,p} \text{ in place of } \mathcal{B}_A^{1,p}$$

and

$$\mathcal{D}(\Delta(A)) = \{u \in L^\infty(\Delta(A)) : \partial_\infty^\alpha u \in L^\infty(\Delta(A)) \ \forall \alpha \in \mathbb{N}^N\} \text{ in that of } \mathcal{D}_A(\mathbb{R}^N).$$

The following relation between $D_{i,p}$ and $\partial_{i,p}$ holds.

Lemma 2.4 ([36, Lemma 2]). *For any $u \in \mathcal{D}_{i,p}$ we have that $\mathcal{G}_1(u) \in \mathcal{W}_{i,p}$ with $\mathcal{G}_1(D_{i,p}u) = \partial_{i,p}\mathcal{G}_1(u)$.*

Now, let $u \in \mathcal{D}_{i,p}$ ($p \geq 1$, $1 \leq i \leq N$). Then the inequality

$$\|t^{-1}(T(te_i)u - u) - D_{i,p}u\|_1 \leq c\|t^{-1}(T(te_i)u - u) - D_{i,p}u\|_p$$

for a positive constant c independent of u and t, yields $D_{i,1}u = D_{i,p}u$, so that $D_{i,p}$ is the restriction to $\mathcal{B}_A^p$ of $D_{i,1}$. Therefore, for all $u \in \mathcal{D}_{i,\infty}$ we have $u \in \mathcal{D}_{i,p}$ ($p \geq 1$) and $D_{i,\infty}u = D_{i,p}u$ for all $1 \leq i \leq N$. It holds that

$$\mathcal{D}_A(\mathbb{R}^N) = \varrho(A^\infty)$$

and we have the following result.

Proposition 2.5 ([36, Proposition 4]). *The following assertions hold.*

- (i) $\int_{\Delta(A)} \partial_\infty^\alpha \widehat{u} d\beta = 0$ *for all* $u \in \mathcal{D}_A(\mathbb{R}^N)$ *and* $\alpha \in \mathbb{N}^N$*;*
- (ii) $\int_{\Delta(A)} \partial_{i,p} \widehat{u} d\beta = 0$ *for all* $u \in \mathcal{D}_{i,p}$ *and* $1 \leq i \leq N$*;*
- (iii) $D_{i,p}(u\phi) = uD_{i,\infty}\phi + \phi D_{i,p}u$ *for all* $(\phi, u) \in \mathcal{D}_A(\mathbb{R}^N) \times \mathcal{D}_{i,p}$ *and* $1 \leq i \leq N$*.*

The formula (iii) in this proposition leads to the equality

$$\int_{\Delta(A)} \widehat{\phi}\partial_{i,p}\widehat{u} d\beta = -\int_{\Delta(A)} \widehat{u}\partial_{i,\infty}\widehat{\phi} d\beta \quad \forall (u, \phi) \in \mathcal{D}_{i,p} \times \mathcal{D}_A(\mathbb{R}^N).$$

This suggests that we define the concepts of distributions on A and of a weak derivative. Before we can do that, let us endow $\mathcal{D}_A(\mathbb{R}^N) = \varrho(A^\infty)$ with its natural topology defined by the family of norms $N_n(u) = \sup_{|\alpha| \leq n} \sup_{y \in \mathbb{R}^N} |D_\infty^\alpha u(y)|$, $n \in \mathbb{N}$. In this topology, $\mathcal{D}_A(\mathbb{R}^N)$ is a Fréchet space. We denote by $\mathcal{D}'_A(\mathbb{R}^N)$ the topological dual of $\mathcal{D}_A(\mathbb{R}^N)$. We endow it with the strong dual topology. The elements of $\mathcal{D}'_A(\mathbb{R}^N)$ are called *the distributions on* A. One can also define the weak derivative of $f \in \mathcal{D}'_A(\mathbb{R}^N)$ as follows: for any $\alpha \in \mathbb{N}^N$, $D^\alpha f$ stands for the distribution defined by the formula

$$\langle D^\alpha f, \phi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha_\infty \phi \rangle \text{ for all } \phi \in \mathcal{D}_A(\mathbb{R}^N).$$

Since $\mathcal{D}_A(\mathbb{R}^N)$ is dense in $\mathcal{B}^p_A$ $(1 \le p < \infty)$, it is immediate that $\mathcal{B}^p_A \subset \mathcal{D}'_A(\mathbb{R}^N)$ with continuous embedding, so that one may define the weak derivative of any $f \in \mathcal{B}^p_A$, and it verifies the following functional equation:

$$\langle D^\alpha f, \phi \rangle = (-1)^{|\alpha|} \int_{\Delta(A)} \widehat{f} \partial^\alpha_\infty \widehat{\phi} d\beta \quad \forall \phi \in \mathcal{D}_A(\mathbb{R}^N).$$

In particular, for $f \in \mathcal{D}_{i,p}$ we have

$$-\int_{\Delta(A)} \widehat{f} \partial_{i,p} \widehat{\phi} d\beta = \int_{\Delta(A)} \widehat{\phi} \partial_{i,p} \widehat{f} d\beta \quad \forall \phi \in \mathcal{D}_A(\mathbb{R}^N),$$

so that we may identify $D_{i,p} f$ with $D^{\alpha_i} f$, $\alpha_i = (\delta_{ij})_{1 \le j \le N}$. Conversely, if $f \in \mathcal{B}^p_A$ is such that there exists $f_i \in \mathcal{B}^p_A$ with $\langle D^{\alpha_i} f, \phi \rangle = -\int_{\Delta(A)} \widehat{f_i} \widehat{\phi} d\beta$ for all $\phi \in \mathcal{D}_A(\mathbb{R}^N)$, then $f \in \mathcal{D}_{i,p}$ and $D_{i,p} f = f_i$. We are therefore justified in saying that $\mathcal{B}^{1,p}_A$ is a Banach space under the norm $\| \cdot \|_{\mathcal{B}^{1,p}_A}$. The same result holds for $W^{1,p}(\Delta(A))$. Moreover it is a fact that $\mathcal{D}_A(\mathbb{R}^N)$ (resp. $\mathcal{D}(\Delta(A))$) is a dense subspace of $\mathcal{B}^{1,p}_A$ (resp. $W^{1,p}(\Delta(A))$).

We need some further notion. A function $f \in \mathcal{B}^1_A$ is said to be *invariant* if for any $y \in \mathbb{R}^N$, $T(y) f = f$. It is immediate that the above notion of invariance is the well-known one relative to dynamical systems. An algebra with mean value will therefore said to be *ergodic* if every invariant function f is constant in $\mathcal{B}^1_A$. As in [7] one may show that $f \in \mathcal{B}^1_A$ is invariant if and only if $D_{i,1} f = 0$ for all $1 \le i \le N$. We denote by I^p_A the set of $f \in \mathcal{B}^p_A$ that are invariant. The set I^p_A is a closed vector subspace of $\mathcal{B}^p_A$ satisfying the following important property:

$$f \in I^p_A \text{ if and only if } D_{i,p} f = 0 \text{ for all } 1 \le i \le N. \tag{2.2}$$

The *gradient mapping* $D_p = (D_{1,p}, \dots, D_{N,p})$ is an isometric embedding of $\mathcal{B}^{1,p}_A$ onto a closed subspace of $(\mathcal{B}^p_A)^N$, so that $\mathcal{B}^{1,p}_A$ is a reflexive Banach space. By duality we define the divergence operator $\operatorname{div}_{p'} : (\mathcal{B}^{p'}_A)^N \to (\mathcal{B}^{1,p}_A)'$ $(p' = p/(p-1))$ by

$$\langle \operatorname{div}_{p'} u, v \rangle = -\langle u, D_p v \rangle \text{ for } v \in \mathcal{B}^{1,p}_A \text{ and } u = (u_i) \in (\mathcal{B}^{p'}_A)^N, \tag{2.3}$$

where $\langle u, D_p v \rangle = \sum_{i=1}^N \int_{\Delta(A)} \widehat{u}_i \partial_{i,p} \widehat{v} d\beta$.

Now if in (2.3) we take $u = D_{p'} w$ with $w \in \mathcal{B}^{p'}_A$ being such that $D_{p'} w \in (\mathcal{B}^{p'}_A)^N$ then this allows us to define the Laplacian operator on $\mathcal{B}^{p'}_A$, denoted here by $\Delta_{p'}$, as follows:

$$\langle \Delta_{p'} w, v \rangle = \langle \operatorname{div}_{p'} (D_{p'} w), v \rangle = -\langle D_{p'} w, D_p v \rangle \quad \text{for all } v \in \mathcal{B}^{1,p}_A.$$

If in addition $v = \phi$ with $\phi \in \mathcal{D}_A(\mathbb{R}^N)$ then $\langle \Delta_{p'} w, \phi \rangle = -\langle D_{p'} w, D_p \phi \rangle$, so that, for $p = 2$, we get

$$\langle \Delta_2 w, \phi \rangle = \langle w, \Delta_2 \phi \rangle \text{ for all } w \in \mathcal{B}^2_A \text{ and } \phi \in \mathcal{D}_A(\mathbb{R}^N). \tag{2.4}$$

By the equality $\mathcal{D}_A(\mathbb{R}^N) = \varrho(A^\infty)$ we infer at once that $\Delta_p \varrho(u) = \varrho(\Delta_y u)$ for any $u \in A^\infty$, where Δ_y denotes the usual Laplacian operator on $\mathbb{R}^N_y$.

Before we state one of the most important results of this section, we still need to introduce some preliminaries and some notation. To this end let $f \in \mathcal{B}_A^p$. We know that $D^{\alpha_i} f$ exists (in the sense of distributions) and that $D^{\alpha_i} f = D_{i,p} f$ if $f \in \mathcal{D}_{i,p}$. So we can drop the subscript p and therefore denote $D_{i,p}$ (resp. $\partial_{i,p}$) by $\overline{\partial}/\partial y_i$ (resp. ∂_i). Thus, $\overline{D}_y \equiv \overline{\nabla}_y$ will stand for the gradient operator $(\overline{\partial}/\partial y_i)_{1\leq i\leq N}$ and $\overline{\operatorname{div}}_y$ for the divergence operator div_p, with $\mathcal{G}_1 \circ \overline{\operatorname{div}}_y = \widehat{\operatorname{div}}$. We will also denote $\partial \equiv (\partial_1, \dots, \partial_N)$. Finally, we shall denote the Laplacian operator on $\mathcal{B}_A^p$ by $\overline{\Delta}_y$.

With all this in mind, let $u \in A$ and let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$. Since u and φ are uniformly continuous and A is translation invariant, we have that $u * \varphi \in A$ ($*$ stands for the usual convolution). More precisely $u * \varphi \in A^\infty$ since $D_y^\alpha(u * \varphi) = u * D_y^\alpha \varphi$ for any $\alpha \in \mathbb{N}^N$. For $1 \leq p < \infty$ let $u \in B_A^p$ and let $\eta > 0$. Let $v \in A$ be such that $\|u - v\|_p < \eta/(\|\varphi\|_{L^1(\mathbb{R}^N)} + 1)$. Then by Young's inequality we have

$$\|u * \varphi - v * \varphi\|_p \leq \|\varphi\|_{L^1(\mathbb{R}^N)} \|u - v\|_p < \eta,$$

hence $u * \varphi \in B_A^p$ as $v * \varphi \in A$. We may therefore define the convolution between $\mathcal{B}_A^p$ and $\mathcal{C}_0^\infty(\mathbb{R}^N)$ as follows: for $g = u + \mathcal{N} \in \mathcal{B}_A^p$ with $u \in B_A^p$, and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$

$$g \circledast \varphi := u * \varphi + \mathcal{N} \equiv \varrho(u * \varphi).$$

Thus, for $g \in \mathcal{B}_A^p$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ we have $g \circledast \varphi \in \mathcal{B}_A^p$ with

$$\overline{D}_y^\alpha(g \circledast \varphi) = \varrho(u * D_y^\alpha \varphi) \quad \text{for all } \alpha \in \mathbb{N}^N. \tag{2.5}$$

We deduce from (2.5) that $g \circledast \varphi \in \mathcal{D}_A(\mathbb{R}^N)$ since $u * \varphi \in A^\infty$. Moreover we have

$$\|g \circledast \varphi\|_p \leq |\operatorname{supp} \varphi|^{1/p} \|\varphi\|_{L^{p'}(\mathbb{R}^N)} \|g\|_p \tag{2.6}$$

where $\operatorname{supp} \varphi$ stands for the support of φ and $|\operatorname{supp} \varphi|$ its Lebesgue measure. Indeed letting $\varphi = \varrho(u)$ with $u \in B_A^p$,

$$\|g \circledast \varphi\|_p = \|\varrho(u * \varphi)\|_p = \Big(\limsup_{r\to+\infty} |B_r|^{-1} \int_{B_r} |(u * \varphi)(y)|^p dy\Big)^{1/p},$$

and

$$\begin{aligned}\int_{B_r} |(u * \varphi)(y)|^p dy &\leq \Big(\int_{B_r} |\varphi| dy\Big)^p \Big(\int_{B_r} |u(y)|^p dy\Big) \\ &\leq |B_r \cap \operatorname{supp} \varphi| \|\varphi\|_{L^{p'}(B_r)}^p \int_{B_r} |u(y)|^p dy,\end{aligned}$$

hence the claim (2.6).

For $u \in A$ and $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N) = \mathcal{D}(\mathbb{R}^N)$ we can also define the convolution $\widehat{u} \circledast \varphi$ (where $\widehat{u} = \mathcal{G}(u)$ and $\tau_y u = u(\cdot + y)$) as follows

$$(\widehat{u} \circledast \varphi)(s) = \int_{\mathbb{R}^N} \widehat{\tau_y u}(s) \varphi(y) dy \quad (s \in \Delta(A)), \tag{2.7}$$

as an element of $\mathcal{C}(\Delta(A))$ (this is easily seen). We have the crucial equality

$$\widehat{u * \varphi} = \widehat{u} \circledast \varphi \quad \text{for all } u \in A \text{ and } \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N). \tag{2.8}$$

In fact for $x \in \mathbb{R}^N$,

$$\begin{aligned}(\widehat{u} \circledast \varphi)(\delta_x) &= \int_{\mathbb{R}^N} \widehat{\tau_y u}(\delta_x) \varphi(y) dy = \int_{\mathbb{R}^N} \tau_y u(x) \varphi(y) dy \\ &= (u * \varphi)(x) = \widehat{u * \varphi}(\delta_x).\end{aligned}$$

By the continuity of both $\widehat{u} \circledast \varphi$ and $\widehat{u * \varphi}$, and the density of $\{\delta_x : x \in \mathbb{R}^N\}$ in $\Delta(A)$ we end up with (2.8). It is important to note that (2.8) allows us to see that $g \circledast \varphi$ is well-defined for $g \in \mathcal{B}_A^p$. In fact we can deduce from (2.8) that $g \circledast \varphi \in \mathcal{N}$ whenever $g \in \mathcal{N}$ (i.e., $\mathcal{G}_1(g \circledast \varphi) = 0$ whenever $\mathcal{G}_1(g) = 0$).

We also have the obvious equality

$$\partial_i(\widehat{u} \circledast \varphi) = \widehat{u} \circledast \frac{\partial \varphi}{\partial y_i} \quad \text{for all } 1 \leq i \leq N. \tag{2.9}$$

2.2. **The de Rham Theorem.**

Theorem 2.6. *Let $1 < p < \infty$. Let L be a bounded linear functional on $(\mathcal{B}_A^{1,p'})^N$ which vanishes on the kernel of the divergence. Then there exists a function $f \in \mathcal{B}_A^p$ such that $L = \overline{\nabla}_y f$, i.e.,*

$$L(v) = -\int_{\Delta(A)} \widehat{f}\, \widehat{\operatorname{div}}\, \widehat{v} d\beta \quad \textit{for all } v \in (\mathcal{B}_A^{1,p'})^N.$$

Moreover f is unique modulo I_A^p, that is, up to an additive function $g \in \mathcal{B}_A^p$ verifying $\overline{\nabla}_y g = 0$.

Proof. Let $u \in A^\infty$ (hence $\varrho(u) \in \mathcal{D}_A(\mathbb{R}^N)$). Define $L_u : \mathcal{D}(\mathbb{R}^N)^N \to \mathbb{R}$ by

$$L_u(\varphi) = L(\varrho(u * \varphi)) \quad \text{for } \varphi = (\varphi_i) \in \mathcal{D}(\mathbb{R}^N)^N$$

where $u * \varphi = (u * \varphi_i)_i \in (A^\infty)^N$. Then L_u defines a distribution on $\mathcal{D}(\mathbb{R}^N)^N$. Moreover if $\operatorname{div}_y \varphi = 0$ then $\overline{\operatorname{div}}_y(\varrho(u * \varphi)) = \varrho(u * \operatorname{div}_y \varphi) = 0$, hence $L_u(\varphi) = 0$, that is, L_u vanishes on the kernel of the divergence in $\mathcal{D}(\mathbb{R}^N)^N$. By the de Rham theorem, there exists a distribution $S(u) \in \mathcal{D}'(\mathbb{R}^N)$ such that $L_u = \nabla_y S(u)$. This defines an operator

$$S : A^\infty \to \mathcal{D}'(\mathbb{R}^N);\ u \mapsto S(u)$$

satisfying the following properties:

(i) $S(\tau_y u) = \tau_y S(u)$ for all $y \in \mathbb{R}^N$ and all $u \in A^\infty$;
(ii) S maps linearly and continuously A^∞ into $L^{p'}_{\rm loc}(\mathbb{R}^N)$;
(iii) There is a positive constant C_r (that is locally bounded as a function of r) such that

$$\|S(u)\|_{L^{p'}(B_r)} \leq C_r \|L\| |B_r|^{1/p'} \|\varrho(u)\|_{p'}.$$

Property (i) easily comes from the obvious equality

$$L_{\tau_y u}(\varphi) = L_u(\tau_y \varphi) \ \ \forall y \in \mathbb{R}^N.$$

Let us check (ii) and (iii). For that, let $\varphi \in \mathcal{D}(\mathbb{R}^N)^N$ with $\operatorname{supp} \varphi_i \subset B_r$ for all $1 \leq i \leq N$. Then

$$\begin{aligned}
|L_u(\varphi)| &= |L(\varrho(u * \varphi))| \\
&\leq \|L\| \|\varrho(u) \circledast \varphi\|_{(\mathcal{B}_A^{1,p'})^N} \\
&\leq \max_{1 \leq i \leq N} |\operatorname{supp} \varphi_i|^{\frac{1}{p'}} \|L\| \|\varrho(u)\|_{p'} \|\varphi\|_{W^{1,p}(B_r)^N},
\end{aligned}$$

the last inequality being due to (2.6). Hence, as $\operatorname{supp} \varphi_i \subset B_r$ $(1 \leq i \leq N)$,

$$\|L_u\|_{W^{-1,p'}(B_r)^N} \leq \|L\| |B_r|^{1/p'} \|\varrho(u)\|_{p'}. \tag{2.10}$$

Now, let $g \in \mathcal{C}_0^\infty(B_r)$ with $\int_{B_r} g dy = 0$; then by [27, Lemma 3.15] there exists $\varphi \in \mathcal{C}_0^\infty(B_r)^N$ such that $\operatorname{div} \varphi = g$ and $\|\varphi\|_{W^{1,p}(B_r)^N} \leq C(p, B_r)\|g\|_{L^p(B_r)}$. We have

$$\begin{aligned}
|\langle S(u), g\rangle| &= |-\langle \nabla_y S(u), \varphi\rangle| = |\langle L_u, \varphi\rangle| \\
&\leq \|L_u\|_{W^{-1,p'}(B_r)^N}\|\varphi\|_{W^{1,p}(B_r)^N} \\
&\leq C(p, B_r)\|L\||B_r|^{\frac{1}{p'}}\|\varrho(u)\|_{p'}\|g\|_{L^p(B_r)},
\end{aligned}$$

and by a density argument, we get that $S(u) \in (L^p(B_r)/\mathbb{R})' = L^{p'}(B_r)/\mathbb{R}$ for any $r > 0$, where $L^{p'}(B_r)/\mathbb{R} = \{\psi \in L^{p'}(B_r) : \int_{B_r} \psi dy = 0\}$. The properties (ii) and (iii) therefore follow from the above series of inequalities. Taking (ii) as granted it follows that

$$L_u(\varphi) = -\int_{\mathbb{R}^N} S(u) \operatorname{div}_y \varphi dy \text{ for all } \varphi \in \mathcal{D}(\mathbb{R}^N)^N. \tag{2.11}$$

We claim that $S(u) \in \mathcal{C}^\infty(\mathbb{R}^N)$ for all $u \in A^\infty$. Indeed let $e_i = (\delta_{ij})_{1\leq j\leq N}$ (δ_{ij} the Kronecker delta). Then owing to (i) and (iii) above, we have

$$\begin{aligned}
\|t^{-1}(\tau_{te_i}S(u) - S(u)) - S(\frac{\partial u}{\partial y_i})\|_{L^{p'}(B_r)} &= \|S(t^{-1}(\tau_{te_i}u - u) - \frac{\partial u}{\partial y_i})\|_{L^{p'}(B_r)} \\
&\leq c\|t^{-1}(\varrho(\tau_{te_i}u - u)) - \varrho(\frac{\partial u}{\partial y_i})\|_{p'}.
\end{aligned}$$

Hence, passing to the limit as $t \to 0$ above leads us to

$$\frac{\partial}{\partial y_i}S(u) = S(\frac{\partial u}{\partial y_i}) \quad \text{for all } 1 \leq i \leq N.$$

Repeating the same process we end up with

$$D_y^\alpha S(u) = S(D_y^\alpha u) \text{ for all } \alpha \in \mathbb{N}^N.$$

So all the weak derivative of $S(u)$ of any order belong to $L^{p'}_{\mathrm{loc}}(\mathbb{R}^N)$. Our claim is therefore a consequence of [32, Theorem XIX, p. 191].

This being so, we derive from the mean value theorem the existence of $\xi \in B_r$ such that

$$S(u)(\xi) = |B_r|^{-1}\int_{B_r} S(u)dy.$$

On the other hand, the map $u \mapsto S(u)(0)$ is a linear functional on A^∞, and by the above equality we get

$$\begin{aligned}
|S(u)(0)| &\leq \limsup_{r\to 0} |B_r|^{-1}\int_{B_r} |S(u)|dy \\
&\leq \limsup_{r\to 0} |B_r|^{-1/p'}\Big(\int_{B_r} |S(u)|^{p'}dy\Big)^{1/p'} \\
&\leq c\|L\|\|\varrho(u)\|_{p'}.
\end{aligned}$$

Hence, defining $\widetilde{S} : \mathcal{D}_A(\mathbb{R}^N) \to \mathbb{R}$ by $\widetilde{S}(v) = S(u)(0)$ for $v = \varrho(u)$ with $u \in A^\infty$, we get that $\widetilde{S}$ is a linear functional on $\mathcal{D}_A(\mathbb{R}^N)$ satisfying

$$|\widetilde{S}(v)| \leq c\|L\|\|v\|_{p'} \quad \forall v \in \mathcal{D}_A(\mathbb{R}^N). \tag{2.12}$$

We infer from both the density of $\mathcal{D}_A(\mathbb{R}^N)$ in $\mathcal{B}_A^{p'}$ and (2.12) the existence of a function $f \in \mathcal{B}_A^p$ with $\|f\|_p \leq c\|L\|$ such that

$$\widetilde{S}(v) = \int_{\Delta(A)} \widehat{f}\widehat{v}d\beta \quad \text{for all } v \in \mathcal{B}_A^{p'}.$$

In particular

$$S(u)(0) = \int_{\Delta(A)} \widehat{f}\widehat{u}d\beta \quad \forall u \in A^\infty$$

where $\widehat{u} = \mathcal{G}(u) = \mathcal{G}_1(\varrho(u))$. Now, let $u \in A^\infty$ and let $y \in \mathbb{R}^N$. By (i) we have

$$S(u)(y) = S(\tau_y u)(0) = \int_{\Delta(A)} \widehat{\tau_y u}\widehat{f}d\beta.$$

Thus

$$\begin{aligned}
L_u(\varphi) = L(\varrho(u * \varphi)) &= -\int_{\mathbb{R}^N} S(u)(y) \operatorname{div}_y \varphi dy \quad \text{(by (2.11))} \\
&= -\int_{\mathbb{R}^N} \Big(\int_{\Delta(A)} \widehat{\tau_y u}\widehat{f}d\beta\Big) \operatorname{div}_y \varphi dy \\
&= -\int_{\Delta(A)} \Big(\int_{\mathbb{R}^N} \widehat{\tau_y u}(s) \operatorname{div}_y \varphi dy\Big)\widehat{f}d\beta \\
&= -\int_{\Delta(A)} \widehat{f}(\widehat{u} \circledast \operatorname{div}_y \varphi)d\beta \quad \text{(by (2.7))} \\
&= -\int_{\Delta(A)} \widehat{f}\ \mathcal{G}(u * \operatorname{div}_y \varphi)d\beta \quad \text{(by (2.8))} \\
&= -\int_{\Delta(A)} \widehat{f}\ \mathcal{G}(\operatorname{div}_y(u * \varphi))d\beta \\
&= -\int_{\Delta(A)} \widehat{f}\ \mathcal{G}_1(\overline{\operatorname{div}}_y(\varrho(u * \varphi)))d\beta \\
&= \langle \overline{\nabla}_y f, \varrho(u * \varphi)\rangle.
\end{aligned}$$

Finally let $v \in (\mathcal{B}_A^{1,p'})^N$ and let $(\varphi_n)_n \subset \mathcal{D}(\mathbb{R}^N)$ be a mollifier. Then $v \circledast \varphi_n \to v$ in $(\mathcal{B}_A^{1,p'})^N$ as $n \to \infty$, where $v \circledast \varphi_n = (v_i \circledast \varphi_n)_i$. We have $v \circledast \varphi_n \in \mathcal{D}_A(\mathbb{R}^N)^N$ and $L(v \circledast \varphi_n) \to L(v)$ by the continuity of L. On the other hand,

$$\int_{\Delta(A)} \widehat{f}\mathcal{G}_1(\overline{\operatorname{div}}_y(v \circledast \varphi_n))d\beta \to \int_{\Delta(A)} \widehat{f}\widehat{\overline{\operatorname{div}}v}d\beta.$$

We deduce that L and $\overline{\nabla}_y f$ agree on $(\mathcal{B}_A^{1,p'})^N$, i.e., $L = \overline{\nabla}_y f$.

For the uniqueness, let f_1 and f_2 in $\mathcal{B}_A^p$ be such that $L = \overline{\nabla}_y f_1 = \overline{\nabla}_y f_2$, then $\overline{\nabla}_y(f_1 - f_2) = 0$, which means that $f_1 - f_2 \in I_A^p$. $\square$

The preceding result together with its proof are still valid mutatis mutandis when the function spaces are complex-valued. In this case, we only require the algebra A to be closed under complex conjugation ($\overline{u} \in A$ whenever $u \in A$). This result has some important consequences as seen below.

Corollary 2.7. *Let $f \in (\mathcal{B}_A^p)^N$ be such that*

$$\int_{\Delta(A)} \widehat{f} \cdot \widehat{g}d\beta = 0 \ \forall g \in \mathcal{D}_A(\mathbb{R}^N)^N \ \textit{with } \overline{\operatorname{div}}_y g = 0.$$

Then there exists a function $u \in \mathcal{B}_A^{1,p}$, uniquely determined modulo I_A^p, such that $f = \overline{\nabla}_y u$.

Proof. Define $L : (\mathcal{B}_A^{1,p'})^N \to \mathbb{R}$ by $L(v) = \int_{\Delta(A)} \widehat{f} \cdot \widehat{v} d\beta$. Then L lies in $[(\mathcal{B}_A^{1,p'})^N]'$, and it follows from Theorem 2.6 the existence of $u \in \mathcal{B}_A^p$ such that $f = \overline{\nabla}_y u$. This shows at once that $u \in \mathcal{B}_A^{1,p}$. The uniqueness is shown as in Theorem 2.6. □

Before we can state the next consequence, however, we need to give some preliminaries. Let G be a measurable subset of $\mathbb{R}^N$ with the property that $\chi_G \in B_A^r$ for some $r \geq \max(p, p')$. We say that a function $f \in \mathcal{B}_A^1$ *vanishes* on G if

$$\int_{\Delta(A)} \widehat{f}\widehat{\psi} d\beta = 0 \text{ for any } \psi \in \mathcal{D}_A(\mathbb{R}^N) \text{ with } \psi = 0 \text{ on } \mathbb{R}^N \backslash G.$$

We denote by $\mathcal{D}_A(G)$ the set of all $\psi \in \mathcal{D}_A(\mathbb{R}^N)$ satisfying $\psi = 0$ on $\mathbb{R}^N \backslash G$. We set

$$\mathcal{V}_{\overline{\operatorname{div}}_y} = \{\psi \in \mathcal{D}_A(\mathbb{R}^N)^N : \overline{\operatorname{div}}_y \psi = 0\}.$$

With this in mind, we have the following corollary.

Corollary 2.8. *Let $G \subset \mathbb{R}^N$ be as above where $1 < p < \infty$. Let L be a linear functional on $\mathcal{D}_A(G)^N$, bounded in the $(\mathcal{B}_A^{1,p'})^N$-norm. Assume that L vanishes on $\mathcal{D}_A(G)^N \cap \mathcal{V}_{\overline{\operatorname{div}}_y}$. Then there exists a function $f \in \mathcal{B}_A^p$ such that $L = \overline{\nabla}_y f$ on $\mathcal{D}_A(G)^N$.*

Proof. By the Hahn-Banach theorem, L can be extended to a bounded linear functional on $(\mathcal{B}_A^{1,p'})^N$ which moreover vanishes on $\mathcal{V}_{\overline{\operatorname{div}}_y}$. An application of Theorem 2.6 leads at once to the result. □

Remark 2.9. Let $u \in \mathcal{B}_A^{1,p}$ be such that $\overline{\nabla}_y u = 0$; then $u \in I_A^p$. This shows that the mapping

$$u + I_A^p \mapsto \|\overline{\nabla}_y u\|_p \tag{2.13}$$

is a norm on $\mathcal{B}_A^{1,p}/I_A^p$. Since I_A^p is closed, $\mathcal{B}_A^{1,p}/I_A^p$ is a Banach space under the above norm. For the uniqueness argument, we shall always choose the function u in Corollary 2.7 to belong to the space $\mathcal{B}_A^{1,p}/I_A^p$, which we shall henceforth equip with the norm (2.13).

2.3. **Sigma-Convergence.** Let A be an algebra with mean value on $\mathbb{R}^N$. Let Ω be an open subset of $\mathbb{R}^N$ and $T > 0$ a real number. We set $Q = \Omega \times (0, T)$. The concept of sigma-convergence is defined as follows.

Definition 2.10. A sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^p(Q)$ $(1 \leq p < \infty)$ is said to weakly Σ-converge in $L^p(Q)$ to some $u_0 \in L^p(Q; \mathcal{B}_A^p)$ if as $\varepsilon \to 0$, we have

$$\int_Q u_\varepsilon(x,t) f(x,t,\frac{x}{\varepsilon})\, dx\, dt \to \iint_{Q\times\Delta(A)} \widehat{u}_0(x,t,s)\widehat{f}(x,t,s)\, dx\, dt\, d\beta$$

for every $f \in L^{p'}(Q; A)$ $(1/p' = 1 - 1/p)$. We express this by writing $u_\varepsilon \to u_0$ in $L^p(Q)$-weak Σ.

We recall here that $\widehat{u}_0 = \mathcal{G}_1 \circ u_0$ and $\widehat{f} = \mathcal{G} \circ f$, $\mathcal{G}_1$ being the isometric isomorphism sending $\mathcal{B}_A^p$ onto $L^p(\Delta(A))$ and $\mathcal{G}$, the Gelfand transformation on A.

In the sequel the letter E will throughout denote a *fundamental sequence*, that is, any ordinary sequence $E = (\varepsilon_n)_n$ (integers $n \geq 0$) with $0 < \varepsilon_n \leq 1$ and $\varepsilon_n \to 0$ as $n \to \infty$. The following result holds.

Theorem 2.11. *Let $1 < p < \infty$. Any bounded ordinary sequence in $L^p(Q)$ admits a Σ-convergent subsequence.*

The next result is of central interest in the homogenization process.

Theorem 2.12. *Let $1 < p < \infty$. Let $(u_\varepsilon)_{\varepsilon\in E}$ be a bounded sequence of functions in $L^p(0,T;W^{1,p}(\Omega))$. Then there exist a subsequence E' of E, and a couple (u_0,u_1) in $L^p(0,T;W^{1,p}(\Omega;I_A^p)) \times L^p(Q;\mathcal{B}_A^{1,p})$ such that, as $E' \ni \varepsilon \to 0$,*

$$u_\varepsilon \to u_0 \quad \text{in } L^p(Q)\text{-weak } \Sigma;$$

$$\frac{\partial u_\varepsilon}{\partial x_i} \to \frac{\partial u_0}{\partial x_i} + \frac{\overline{\partial} u_1}{\partial y_i} \quad \text{in } L^p(Q)\text{-weak } \Sigma,\ 1 \le i \le N.$$

Proof. Since the sequences $(u_\varepsilon)_{\varepsilon\in E}$ and $(\nabla u_\varepsilon)_{\varepsilon\in E}$ are bounded respectively in $L^p(Q)$ and in $L^p(Q)^N$, there exist a subsequence E' of E and $u_0 \in L^p(Q;\mathcal{B}_A^p)$, $v = (v_j)_j \in L^p(Q;\mathcal{B}_A^p)^N$ such that $u_\varepsilon \to u_0$ in $L^p(Q)$-weak Σ and $\frac{\partial u_\varepsilon}{\partial x_j} \to v_j$ in $L^p(Q)$-weak Σ. For $\Phi \in (\mathcal{C}_0^\infty(Q) \otimes A^\infty)^N$ we have

$$\int_Q \varepsilon \nabla u_\varepsilon \cdot \Phi^\varepsilon \, dx\, dt = -\int_Q (u_\varepsilon (\operatorname{div}_y \Phi)^\varepsilon + \varepsilon u_\varepsilon (\operatorname{div} \Phi)^\varepsilon)\, dx\, dt.$$

Letting $E' \ni \varepsilon \to 0$ we get

$$-\iint_{Q\times\Delta(A)} \widehat{u}_0 \widehat{\operatorname{div}} \widehat{\Phi}\, dx\, dt\, d\beta = 0.$$

This shows that $\overline{\nabla}_y u_0 = 0$, which means that $u_0(x,t,\cdot) \in I_A^p$ (see (2.2)), that is, $u_0 \in L^p(Q;I_A^p) = L^p(0,T;L^p(\Omega;I_A^p))$. Next let $\Phi_\varepsilon(x,t) = \varphi(x,t)\Psi(x/\varepsilon)$ ($(x,t) \in Q$) with $\varphi \in \mathcal{C}_0^\infty(Q)$ and $\Psi = (\psi_j)_{1\le j\le N} \in (A^\infty)^N$ with $\operatorname{div}_y \Psi = 0$. Clearly

$$\sum_{j=1}^N \int_Q \frac{\partial u_\varepsilon}{\partial x_j} \varphi \psi_j^\varepsilon \, dx\, dt = -\sum_{j=1}^N \int_Q u_\varepsilon \psi_j^\varepsilon \frac{\partial \varphi}{\partial x_j}\, dx\, dt$$

where $\psi_j^\varepsilon(x) = \psi_j(x/\varepsilon)$. Passing to the limit in the above equation when $E' \ni \varepsilon \to 0$ we get

$$\sum_{j=1}^N \iint_{Q\times\Delta(A)} \widehat{v}_j \varphi \widehat{\psi}_j \, dx\, dt\, d\beta = -\sum_{j=1}^N \iint_{Q\times\Delta(A)} \widehat{u}_0 \widehat{\psi}_j \frac{\partial \varphi}{\partial x_j}\, dx\, dt\, d\beta. \tag{2.14}$$

First, taking $\Phi = (\varphi\delta_{ij})_{1\le i\le N}$ with $\varphi \in \mathcal{C}_0^\infty(Q)$ (for each fixed $1 \le j \le N$) in (2.14) we obtain

$$\int_Q M(v_j)\varphi \, dx\, dt = -\int_Q M(u_0) \frac{\partial \varphi}{\partial x_j}\, dx\, dt\, d\beta \tag{2.15}$$

and reminding that $M(v_j) \in L^p(Q)$ we have by (2.15) that $\frac{\partial u_0}{\partial x_j} \in L^p(Q;I_A^p) = L^p(0,T;L^p(\Omega;I_A^p))$, where $\frac{\partial u_0}{\partial x_j}$ is the distributional derivative of u_0 with respect to x_j. We deduce that $u_0 \in L^p(0,T;W^{1,p}(\Omega;I_A^p))$. Coming back to (2.14) we get

$$\iint_{Q\times\Delta(A)} (\widehat{\mathbf{v}} - \nabla \widehat{u}_0) \cdot \widehat{\Psi} \varphi \, dx\, dt\, d\beta = 0,$$

and so, as φ is arbitrarily fixed,

$$\int_{\Delta(A)} (\widehat{\mathbf{v}}(x,t,s) - \nabla \widehat{u}_0(x,t,s)) \cdot \widehat{\Psi}(s) d\beta = 0$$

for all Ψ as above and for a.e. (x,t). Therefore we infer from Corollary 2.7 the existence of a function $u_1(x,t,\cdot) \in \mathcal{B}_A^{1,p}$ such that

$$\mathbf{v}(x,t,\cdot) - \nabla u_0(x,t,\cdot) = \overline{\nabla}_y u_1(x,t,\cdot)$$

for a.e. (x,t). From which the existence of a function $u_1 : (x,t) \mapsto u_1(x,t,\cdot)$ with values in $\mathcal{B}_A^{1,p}$ such that $\mathbf{v} = \nabla u_0 + \overline{\nabla}_y u_1$. $\square$

Remark 2.13. If we assume the algebra A to be ergodic, then I_A^p consists of constant functions, so that the function u_0 in Theorem 2.12 does not depend on y, that is, $u_0 \in L^p(0,T;W^{1,p}(\Omega))$. We thus recover the already known result proved in [31] in the case of ergodic algebras. However, in the case that the algebra is not ergodic, the function u_0 may depend on the microscopic variable y.

3. Homogenization results

Throughout this section A will denote an algebra with mean value on $\mathbb{R}^N$.

3.1. Preliminary results.

We describe a set of conditions which suffice for the homogenization of (1.2). Under these conditions we develop preliminary results that will be essential for the sequel. First recall that χ_j (for $j=1,2$) denotes the characteristic function of the set G_j in $\mathbb{R}^N$. This being so, we aim at studying the asymptotics of the sequence of solutions of (1.2) under the assumptions

$$\chi_j \in B_A^r \text{ with } r \geq \max(p,p') \text{ and } M(\chi_j) > 0,\ j=1,2; \tag{3.1a}$$

$$c_j \in A \text{ for } j=1,2,3; \tag{3.1b}$$

$$a_j(\cdot,\lambda) \in (B_A^{p'})^N \text{ for all } \lambda \in \mathbb{R}^N\ (j=1,2,3) \tag{3.1c}$$

where $p' = p/(p-1)$ with $2 \leq p < \infty$. The first result follows exactly as its analogue in [24] (see also [38, Lemma 3.3]).

Lemma 3.1. *Let $j=1,2$. Under assumption* (3.1a) *there exist β-measurable sets $\widehat{G}_j \subset \Delta(A)$ such that $\chi_{\widehat{G}_j} = \widehat{\chi}_j$ where $\widehat{\chi}_j = \mathcal{G}(\chi_j)$ and $\chi_{\widehat{G}_j}$ denotes the characteristic function of $\widehat{G}_j$ in $\Delta(A)$.*

The next result is fundamental.

Lemma 3.2. *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^p(Q)$ ($1<p<\infty$) which weakly Σ-converges in $L^p(Q)$ to $u_0 \in L^p(Q;\mathcal{B}_A^p)$. For $j=1,2$, let G_j be as above. Then, as $\varepsilon \to 0$,*

$$u_\varepsilon \chi_j^\varepsilon \to u_0 \chi_j \quad \textit{in } L^p(Q)\textit{-weak } \Sigma.$$

Now let

$$u^\varepsilon = \chi_1^\varepsilon u_1^\varepsilon + \chi_2^\varepsilon(\alpha u_2^\varepsilon + \delta u_3^\varepsilon). \tag{3.2}$$

By (1.5) in Theorem 1.1 there is a positive constant C such that

$$\sup_{\varepsilon>0} \|u^\varepsilon(t)\|_{L^2(\Omega)} \leq C \quad \text{for all } 0 \leq t \leq T. \tag{3.3}$$

Also, the interface condition (1.2d) together with Green's formula give

$$\nabla u^\varepsilon = \chi_1^\varepsilon \nabla u_1^\varepsilon + \chi_2^\varepsilon(\alpha \nabla u_2^\varepsilon + \delta \nabla u_3^\varepsilon),$$

and still from (1.5) we have

$$\varepsilon \|\nabla u^\varepsilon\|_{L^p(Q)} \leq C \tag{3.4}$$

for some constant $C>0$ independent of ε. This being so we have the

Proposition 3.3. *Let $(u^\varepsilon)_{\varepsilon\in E}$ be as in (3.2). There exist a subsequence E' of E, a pair of functions $u_j \in L^p(0,T;W^{1,p}(\Omega;I_A^p))$ $(j=1,2)$ and two triples of functions $U_j \in L^p(Q;\mathcal{B}_A^{1,p})$ $(j=1,2,3)$ and u_1^*, u_2^*, $U_3^* \in L^2(\Omega;\mathcal{B}_A^2)$ such that, as $E' \ni \varepsilon \to 0$,*

$$u^\varepsilon \to \chi_1 u_1 + \chi_2(\alpha u_2 + \delta U_3) \quad \text{in } L^2(Q)\text{-weak } \Sigma; \tag{3.5}$$

$$\chi_j^\varepsilon \nabla u_j^\varepsilon \to \chi_j(\nabla u_j + \overline{\nabla}_y U_j) \quad \text{in } L^p(Q)^N\text{-weak } \Sigma,\ j=1,2; \tag{3.6}$$

$$\varepsilon\chi_2^\varepsilon \nabla u_3^\varepsilon \to \chi_2 \overline{\nabla}_y U_3 \quad \text{in } L^p(Q)^N\text{-weak } \Sigma; \tag{3.7}$$

$$\chi_j^\varepsilon u_j^\varepsilon(T) \to \chi_j u_j^* \quad \text{in } L^2(\Omega)\text{-weak } \Sigma, j=1,2; \tag{3.8}$$

$$\chi_2^\varepsilon u_3^\varepsilon(T) \to \chi_2 U_3^* \quad \text{in } L^2(\Omega)\text{-weak } \Sigma. \tag{3.9}$$

Proof. Let denote by $\widetilde{\cdot}$ the zero-extension of any of the above sequences on the whole of Ω. For $j=1,2$ the sequences $\widetilde{u_j^\varepsilon}$ and $\widetilde{\nabla u_j^\varepsilon}$ verify $\widetilde{u_j^\varepsilon} = \chi_j^\varepsilon u_j^\varepsilon$ and $\widetilde{\nabla u_j^\varepsilon} = \chi_j^\varepsilon \nabla u_j^\varepsilon$. It follows that $\widetilde{u_j^\varepsilon}$ and $\widetilde{\nabla u_j^\varepsilon}$ are bounded respectively in $L^\infty(0,T;L^2(\Omega))$ and $L^p(Q)^N$. Therefore, given an ordinary sequence E, there exist a subsequence E' of E and some functions v_j and $w_j = (w_j^k)_{1\le k\le N}$ in $L^2(Q;\mathcal{B}_A^2)$ and $L^p(Q;(\mathcal{B}_A^p)^N)$ respectively, such that, as $E' \ni \varepsilon \to 0$, $\widetilde{u_j^\varepsilon} \to v_j$ in $L^2(Q)$-weak Σ and $\widetilde{\nabla u_j^\varepsilon} \to w_j$ in $L^p(Q)^N$-weak Σ. Lemma 3.2 entails

$$\chi_j^\varepsilon u_j^\varepsilon \to \chi_j v_j \quad \text{in } L^2(Q)\text{-weak } \Sigma, \tag{3.10}$$

$$\chi_j^\varepsilon \nabla u_j^\varepsilon \to \chi_j w_j \quad \text{in } L^p(Q)^N\text{-weak } \Sigma. \tag{3.11}$$

It follows at once that $v_j = \chi_j v_j$ and $w_j = \chi_j w_j$. Now, let us analyze the case $j=1$ (the case $j=2$ will be carried out in a same manner). Let $\Phi \in (\mathcal{C}_0^\infty(Q)\otimes A^\infty)^N$ be such that $\Phi(x,t,y)=0$ for $y\in G_2$. Then, $\Phi^\varepsilon = 0$ in Ω_2^ε, hence $\Phi^\varepsilon \in \mathcal{C}_0^\infty(\Omega_1^\varepsilon \times (0,T))^N$ and

$$\begin{aligned}
\varepsilon\int_Q \chi_1^\varepsilon \nabla u_1^\varepsilon \cdot \Phi^\varepsilon \,dx\,dt &= \varepsilon\int_{\Omega_1^\varepsilon\times(0,T)} \nabla u_1^\varepsilon \cdot \Phi^\varepsilon \,dx\,dt \\
&= -\int_{\Omega_1^\varepsilon\times(0,T)} u_1^\varepsilon[\varepsilon(\operatorname{div}_x\Phi)^\varepsilon + (\operatorname{div}_y\Phi)^\varepsilon]\,dx\,dt \\
&= -\int_Q \chi_1^\varepsilon u_1^\varepsilon[\varepsilon(\operatorname{div}_x\Phi)^\varepsilon + (\operatorname{div}_y\Phi)^\varepsilon]\,dx\,dt.
\end{aligned}$$

Letting $E' \ni \varepsilon \to 0$,

$$-\iint_{Q\times\Delta(A)} \widehat{v}_1 \widehat{\operatorname{div}}\widehat{\Phi}\,dx\,dt\,d\beta = 0$$

for all $\Phi \in (\mathcal{C}_0^\infty(Q)\otimes A^\infty)^N$ satisfying $\Phi(x,t,y)=0$ for $y\in G_2$. This means that $\overline{\nabla}_y v_1 = 0$ in G_1. Also since $v_1 = \chi_1 v_1$, the value of v_1 on G_2 is of no effect and hence may be chosen arbitrarily, so that, in view of the equality $\overline{\nabla}_y v_1 = 0$ in G_1, one may choose $u_1 \in L^2(Q;I_A^p)$ such that $v_1 = \chi_1 u_1$ on $Q\times\mathbb{R}^N$.

Next we seek the relationship between w_1 and u_1. For that, let Φ be as above and further satisfying $\operatorname{div}_y\Phi = 0$. Then

$$\int_Q \chi_1^\varepsilon \nabla u_1^\varepsilon \cdot \Phi^\varepsilon\,dx\,dt = -\int_Q \chi_1^\varepsilon u_1^\varepsilon (\operatorname{div}_x\Phi)^\varepsilon\,dx\,dt.$$

Passing to the limit as $E' \ni \varepsilon \to 0$, it comes from (3.10) and (3.11) (with $v_1 = \chi_1 u_1$) that

$$\iint_{Q\times\Delta(A)} \widehat{\chi}_1 \widehat{w}_1 \cdot \widehat{\Phi}\, dx\, dt\, d\beta = -\iint_{Q\times\Delta(A)} \widehat{\chi}_1 \widehat{u}_1 \mathrm{div}_x \widehat{\Phi}\, dx\, dt\, d\beta. \tag{3.12}$$

Starting from the above equation and proceeding as in the proof of Theorem 2.12 we end up with $u_1 \in L^p(0,T;W^{1,p}(\Omega;I_A^p))$. Coming back to (3.12) we get

$$\iint_{Q\times\widehat{G}_1} (\widehat{w}_1 - \nabla\widehat{u}_1)\cdot\widehat{\Phi}\, dx\, dt\, d\beta = 0$$

for all $\Phi \in (\mathcal{C}_0^\infty(Q)\otimes A^\infty)^N$ satisfying $\Phi(x,t,y) = 0$ for $y \in G_2$, $\mathrm{div}_y\Phi = 0$ and $\Phi(x,t,y)\cdot\nu = 0$ on $\partial\Omega$, where ν denote the unit outward normal to $\partial\Omega$. We deduce from Corollary 2.8 the existence of $U_1 \in L^p(Q;\mathcal{B}_A^{1,p})$ such that $w_1 = \chi_1(\nabla u_1 + \overline{\nabla}_y U_1)$.

We have just derived the existence of u_j and U_j $(j = 1,2)$ such that (3.6) holds true. We need to find U_3 such that (3.5) and (3.7) are satisfied. To this end, since the sequences $(\widetilde{u_3^\varepsilon})_{\varepsilon\in E}$ and $(\varepsilon\widetilde{\nabla u_3^\varepsilon})_{\varepsilon\in E}$ are bounded in $L^2(Q)$ and in $L^p(Q)^N$ respectively, there exist a subsequence of E' not relabeled, and $U_3 \in L^2(Q;\mathcal{B}_A^2)$ and $w_3 \in L^p(Q;\mathcal{B}_A^p)^N$ such that, as $E' \ni \varepsilon \to 0$,

$$\begin{gathered}\widetilde{u_3^\varepsilon} \to U_3 \quad \text{in } L^2(Q)\text{-weak } \Sigma,\\ \varepsilon\widetilde{\nabla u_3^\varepsilon} \to w_3 \quad \text{in } L^p(Q)^N\text{-weak } \Sigma.\end{gathered}$$

Then in view of Lemma 3.2 we have that

$$\chi_2^\varepsilon u_3^\varepsilon = \chi_2^\varepsilon \widetilde{u_3^\varepsilon} \to \chi_2 U_3 \quad \text{in } L^2(Q)\text{-weak } \Sigma, \tag{3.13}$$

$$\varepsilon\chi_2^\varepsilon \nabla u_3^\varepsilon = \varepsilon\chi_2^\varepsilon \widetilde{\nabla u_3^\varepsilon} \to \chi_2 w_3 \quad \text{in } L^p(Q)^N\text{-weak } \Sigma. \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$\chi_2 U_3 = U_3 \quad \text{and} \quad \chi_2 w_3 = w_3, \tag{3.15}$$

i.e., w_3 and U_3 do not depend on y in G_1. So we may take a test function not depending upon $y \in G_1$ in the following sense. Let $\Phi \in (\mathcal{C}_0^\infty(Q)\otimes A^\infty)^N$ with $\Phi(x,t,y) = 0$ for $y \in G_1$. Then as seen previously, $\Phi^\varepsilon \in (\mathcal{C}_0^\infty(\Omega_2^\varepsilon \times (0,T)))^N$ and

$$\begin{aligned}\int_Q \varepsilon\chi_2^\varepsilon \nabla u_3^\varepsilon \cdot \Phi^\varepsilon\, dx\, dt &= \int_{\Omega_2^\varepsilon\times(0,T)} \varepsilon\nabla u_3^\varepsilon \cdot \Phi^\varepsilon\, dx\, dt\\ &= -\int_{\Omega_2^\varepsilon\times(0,T)} u_3^\varepsilon[(\mathrm{div}_y\Phi)^\varepsilon + \varepsilon(\mathrm{div}\Phi)^\varepsilon]\, dx\, dt\\ &= -\int_Q \chi_2^\varepsilon u_3^\varepsilon (\mathrm{div}_y\Phi)^\varepsilon\, dx\, dt - \int_Q \varepsilon\chi_2^\varepsilon u_3^\varepsilon (\mathrm{div}\Phi)^\varepsilon\, dx\, dt.\end{aligned}$$

Passing to the limit as $E' \ni \varepsilon \to 0$ (using (3.13)-(3.14)),

$$\iint_{Q\times\Delta(A)} \widehat{\chi}_2 \widehat{w}_3 \cdot \widehat{\Phi}\, dx\, dt\, d\beta = -\iint_{Q\times\Delta(A)} \widehat{\chi}_2 \widehat{U}_3 \widehat{\mathrm{div}}\widehat{\Phi}\, dx\, dt\, d\beta;$$

that is,

$$\iint_{Q\times\Delta(A)} \widehat{\chi}_2 (\widehat{w}_3 - \widehat{\overline{\nabla}_y U}_3)\cdot\widehat{\Phi}\, dx\, dt\, d\beta = 0$$

for all $\Phi \in (\mathcal{C}_0^\infty(Q)\otimes A^\infty)^N$ with $\Phi(x,t,y) = 0$ for $y \in G_1$. Hence $\chi_2(w_3 - \overline{\nabla}_y U_3) = 0$, or, in view of (3.14), $w_3 = \chi_2\overline{\nabla}_y U_3$. We therefore deduce (3.5) and (3.7).

Finally, (3.8)-(3.9) follow from the boundedness property of the those sequences in $L^2(\Omega)$. $\square$

It follows from (3.3)-(3.4) that the sequences $(u^\varepsilon)_{\varepsilon>0}$ (defined in (3.2) by $u^\varepsilon = \chi_1^\varepsilon u_1^\varepsilon + \chi_2^\varepsilon(\alpha u_2^\varepsilon + \delta u_3^\varepsilon))$ and $(\varepsilon\nabla u^\varepsilon)_{\varepsilon>0}$ are bounded in $L^2(Q)$ and $L^p(Q)^N$ (hence also $L^2(Q)^N$), respectively. The results in Proposition 3.3 show that

$$\begin{gathered} u^\varepsilon \to \chi_1 u_1 + \chi_2(\alpha u_2 + \delta U_3) \quad \text{in } L^2(Q)\text{-weak } \Sigma, \\ \varepsilon\nabla u^\varepsilon \to \delta\chi_2\overline{\nabla}_y U_3 \text{ in } L^p(Q)^N \quad (\text{hence in } L^2(Q)^N)\text{-weak } \Sigma \end{gathered}$$

up to a subsequence of any ordinary sequence E. It follows directly that

$$\delta\chi_2\overline{\nabla}_y U_3 = \overline{\nabla}_y(\chi_1 u_1 + \chi_2(\alpha u_2 + \delta U_3)). \tag{3.16}$$

3.2. **Homogenization results.** Let

$$\begin{gathered} \mathbb{F}^{1,p} = L^p(0,T;W^{1,p}(\Omega;I_A^p))^2 \times L^p(Q;\mathcal{B}_A^{1,p})^3, \\ \mathcal{F}^\infty = [\mathcal{C}_0^\infty(0,T)\otimes\mathcal{C}^\infty(\overline{\Omega};I_A^p)]^2 \times [\mathcal{C}_0^\infty(Q)\otimes\mathcal{D}_A(\mathbb{R}^N)]^3. \end{gathered}$$

Then it can be easily checked that $\mathcal{F}^\infty$ is dense in $\mathbb{F}^{1,p}$. Moreover, we see from Proposition 3.3 that $\boldsymbol{u} = (u_1, u_2, U_1, U_2, U_3) \in \mathbb{F}^{1,p}$ and satisfies (3.16). This suggests us to take as smooth test functions any function $\Phi = (\phi_1, \phi_2, \psi_1, \psi_2, \psi_3) \in \mathcal{F}^\infty$ satisfying

$$\delta\chi_2\nabla_y\psi_3 = \nabla_y(\chi_1\phi_1 + \chi_2(\alpha\phi_2 + \delta\psi_3)). \tag{3.17}$$

For such a Φ set

$$\Phi_{j,\varepsilon} = \phi_j + \varepsilon\psi_j^\varepsilon \ (j=1,2) \quad \text{and} \quad \Phi_{3,\varepsilon} = \psi_3^\varepsilon + \frac{\varepsilon}{\delta}\psi_1^\varepsilon - \frac{\varepsilon\alpha}{\delta}\psi_2^\varepsilon. \tag{3.18}$$

Then, because of (3.17) and Green's theorem, it is an easy exercise to see that $\Phi_\varepsilon = (\psi_{1,\varepsilon}, \psi_{2,\varepsilon}, \psi_{3,\varepsilon}) \in \mathcal{C}_0^\infty((0,T);V_\varepsilon)$; that is,

$$\gamma_1^\varepsilon\psi_{1,\varepsilon} = \alpha\gamma_2^\varepsilon\psi_{2,\varepsilon} + \delta\gamma_2^\varepsilon\psi_{3,\varepsilon} \quad \text{on } \Gamma_{1,2}^\varepsilon.$$

Moreover the following convergence results hold for any $1 < r < \infty$:

$$\begin{gathered} \chi_j^\varepsilon\psi_{j,\varepsilon} \to \chi_j\phi_j \quad \text{in } L^r(Q)\text{-weak } \Sigma,\ j=1,2 \\ \nabla\psi_{j,\varepsilon} \to \nabla\phi_j + \nabla_y\psi_j \quad \text{in } L^r(Q)^N\text{-weak } \Sigma,\ j=1,2 \\ \chi_j^\varepsilon\nabla\psi_{j,\varepsilon} \to \chi_j(\nabla\phi_j + \nabla_y\psi_j) \quad \text{in } L^r(Q)^N\text{-weak } \Sigma, j=1,2 \\ \chi_2^\varepsilon\psi_{3,\varepsilon} \to \chi_2\psi_3 \quad \text{in } L^r(Q)\text{-weak } \Sigma \\ \varepsilon\chi_2^\varepsilon\nabla\psi_{3,\varepsilon} \to \chi_2\nabla_y\psi_3 \quad \text{in } L^r(Q)^N\text{-weak } \Sigma. \end{gathered} \tag{3.19}$$

With this in mind, let $\boldsymbol{u} = (u_1, u_2, U_1, U_2, U_3) \in \mathbb{F}^{1,p}$ be determined by Proposition 3.3. For $j = 1, 2$, we set $\mathbb{D}u_j = \nabla\widehat{u}_j + \partial\widehat{U}_j$ where $\partial\widehat{U}_j = \mathcal{G}_1(\overline{\nabla}_y U_j)$. The following result holds.

Theorem 3.4. *The quintuple $\boldsymbol{u}=(u_1,u_2,U_1,U_2,U_3)\in\mathbb{F}^{1,p}$ solves the variational problem*

$$
\begin{gathered}
-\sum_{j=1}^{2}\Big[\iint_{Q\times\Delta(A)}\widehat{\chi}_j\widehat{c}_j\widehat{u}_j\widehat{\frac{\partial\phi_j}{\partial t}}\,dx\,dt\,d\beta-\iint_{Q\times\Delta(A)}\widehat{\chi}_j\widehat{a}_j(\cdot,\mathbb{D}u_j)\cdot\mathbb{D}\Phi_j\,dx\,dt\,d\beta\Big]\\
-\iint_{Q\times\Delta(A)}\widehat{\chi}_2\widehat{c}_3\widehat{U}_3\widehat{\frac{\partial\psi_3}{\partial t}}\,dx\,dt\,d\beta+\iint_{Q\times\Delta(A)}\widehat{\chi}_2\widehat{a}_3(\cdot,\partial\widehat{U}_3)\cdot\partial\widehat{\psi}_3\,dx\,dt\,d\beta=0\\
\text{for all } \Phi=(\phi_1,\phi_2,\psi_1,\psi_2,\psi_3)\in\mathcal{F}^\infty \text{ satisfying (3.17)}
\end{gathered}
\tag{3.20}
$$

where $\mathbb{D}\Phi_j=\nabla\widehat{\phi}_j+\partial\widehat{\psi}_j$ for $j=1,2$.

Proof. Let $\Phi=(\phi_1,\phi_2,\psi_1,\psi_2,\psi_3)\in\mathcal{F}^\infty$ satisfy (3.17) and define Φ_ε as above. Using Φ_ε as test function in the variational formulation of (1.2) we obtain

$$
\begin{gathered}
-\sum_{j=1}^{2}\Big[\int_Q\chi_j^\varepsilon c_j^\varepsilon u_j^\varepsilon\frac{\partial\psi_{j,\varepsilon}}{\partial t}\,dx\,dt-\int_Q a_j^\varepsilon(\cdot,\nabla u_j^\varepsilon)\cdot\chi_j^\varepsilon\nabla\psi_{j,\varepsilon}\,dx\,dt\Big]\\
-\int_Q\chi_2^\varepsilon c_3^\varepsilon u_3^\varepsilon\frac{\partial\psi_{3,\varepsilon}}{\partial t}\,dx\,dt+\int_Q a_3^\varepsilon(\cdot,\varepsilon\nabla u_3^\varepsilon)\cdot\varepsilon\chi_2^\varepsilon\nabla\psi_{3,\varepsilon}\,dx\,dt=0.
\end{gathered}
\tag{3.21}
$$

Since the sequences $\chi_j^\varepsilon a_j^\varepsilon(\cdot,\nabla u_j^\varepsilon)$ ($j=1,2$) and $\chi_2^\varepsilon a_3^\varepsilon(\cdot,\varepsilon\nabla u_3^\varepsilon)$ are bounded in $L^{p'}(Q)^N$, given an ordinary sequence E, there exist a subsequence E' of E and a triple $g_j\in L^{p'}(Q;\mathcal{B}_A^{p'})^N$ ($j=1,2,3$) such that, as $E'\ni\varepsilon\to0$,

$$
\chi_j^\varepsilon a_j^\varepsilon(\cdot,\nabla u_j^\varepsilon)\to g_j,\quad \chi_2^\varepsilon a_3^\varepsilon(\cdot,\varepsilon\nabla u_3^\varepsilon)\to g_3\quad\text{in } L^{p'}(Q)\text{-weak }\Sigma.
$$

Passing to the limit in (3.21) leads to

$$
\begin{gathered}
-\sum_{j=1}^{2}\Big[\iint_{Q\times\Delta(A)}\widehat{\chi}_j\widehat{c}_j\widehat{u}_j\widehat{\frac{\partial\phi_j}{\partial t}}\,dx\,dt\,d\beta-\iint_{Q\times\Delta(A)}\widehat{\chi}_j\widehat{g}_j\cdot\mathbb{D}\Phi_j\,dx\,dt\,d\beta\Big]\\
-\iint_{Q\times\Delta(A)}\widehat{\chi}_2\widehat{c}_3\widehat{U}_3\widehat{\frac{\partial\psi_3}{\partial t}}\,dx\,dt\,d\beta+\iint_{Q\times\Delta(A)}\widehat{\chi}_2\widehat{g}_3g\cdot\partial\widehat{\psi}_3\,dx\,dt\,d\beta=0
\end{gathered}
$$

We may proceed exactly as in [40, pp. 821-822] to get $g_j=a_j(\cdot,\mathbb{D}u_j)$ ($j=1,2$) and $g_3=a_3(\cdot,\overline{\nabla}_yU_3)$. The result follows. $\square$

Let us decompose (3.20). Before we can do this, let us first set, for $j=1,2$,

$$
\begin{gathered}
\widetilde{a}_j(\cdot,\boldsymbol{v})=\int_{\widehat{G}_j}\widehat{a}_j(\cdot,\widehat{\boldsymbol{v}})d\beta \text{ for } \boldsymbol{v}\in L^p(Q;\mathcal{B}_A^p)^N,\\
\widetilde{a}_3(\cdot,\boldsymbol{v})=\int_{\widehat{G}_2}\widehat{a}_3(\cdot,\widehat{\boldsymbol{v}})d\beta \text{ for } \boldsymbol{v}\in L^p(Q;\mathcal{B}_A^p)^N.
\end{gathered}
$$

This being so, taking in (3.20) $\Phi=(0,0,0,0,V_3)$ then we obtain

$$
-\iint_{Q\times\Delta(A)}\widehat{\chi}_2\widehat{c}_3\widehat{U}_3\widehat{\frac{\partial\psi_3}{\partial t}}\,dx\,dt\,d\beta+\iint_{Q\times\Delta(A)}\widehat{a}_3(\cdot,\partial\widehat{U}_3)\cdot\widehat{\chi}_2\partial\widehat{V}_3\,dx\,dt\,d\beta=0
$$

for all $V_3\in\mathcal{C}_0^\infty(Q)\otimes\mathcal{D}_A(\mathbb{R}^N)$ with $\overline{\nabla}_y(\chi_2V_3)=\chi_2\overline{\nabla}_yV_3$.

Taking in particular $V_3=\varphi\otimes v_3$ with $\varphi\in\mathcal{C}_0^\infty(Q)$ and $v_3\in\mathcal{D}_A(\mathbb{R}^N)$ we obtain

$$
-\int_Q\Big(\int_{\Delta(A)}\widehat{\chi}_2\widehat{c}_3\widehat{U}_3\widehat{v}_3d\beta\Big)\varphi'\,dx\,dt+\int_Q\Big(\int_{\Delta(A)}\widehat{a}_3(\cdot,\partial\widehat{U}_3\Big)\cdot\widehat{\chi}_2\partial\widehat{v}_3d\beta)\varphi\,dx\,dt=0
$$

for all $\varphi \in \mathcal{C}_0^\infty(Q)$ and $v_3 \in \mathcal{D}_A(\mathbb{R}^N)$ with $\overline{\nabla}_y(\chi_2 v_3) = \chi_2 \overline{\nabla}_y v_3$.

$$\langle \chi_2 c_3 \frac{\partial U_3}{\partial t}, v_3\rangle + \int_{\widehat{G}_2} \widehat{a}_3(\cdot, \partial \widehat{U}_3) \cdot \partial \widehat{v}_3 \, dx\, dt\, d\beta = 0$$

for all $v_3 \in \mathcal{B}_A^{1,p}$ with $\overline{\nabla}_y(\chi_2 v_3) = \chi_2 \overline{\nabla}_y v_3$.

Choosing $v_3 = \varrho(\omega_3)$ with $\omega_3 \in A^\infty$ satisfying $\omega_3(y) = 0$ for $y \in G_1$, we are led to the *cell problem*

$$\begin{gathered} \chi_2 c_3 \frac{\partial U_3}{\partial t} - \overline{\operatorname{div}}_y(\chi_2 a_3(\cdot, \overline{\nabla}_y U_3)) = 0 \quad \text{in } G_2 \times (0,T) \\ \delta \chi_2 \overline{\nabla}_y U_3 = \overline{\nabla}_y(\chi_1 u_1 + \chi_2(\alpha u_2 + \delta U_3)) \quad \text{on } \mathbb{R}_y^N. \end{gathered} \tag{3.22}$$

Coming back to (3.20) and taking there $V = (v_1, 0, 0, 0, V_3)$ with $v_1 \in \mathcal{C}_0^\infty(Q) \otimes I_A^p$ and $V_3 \in \mathcal{C}_0^\infty(Q) \otimes \mathcal{D}_A(\mathbb{R}^N)$ satisfying $\delta V_3(x,t,y) = v_1(x,t,y)$ for $(x,t,y) \in Q \times G_2$ and $V_3(x,t,y) = 0$ for $(x,t,y) \in Q \times G_1$. Then $\chi_2 \nabla_y V_3 = 0$ in $Q \times \mathbb{R}^N$ so that

$$\begin{aligned} &-\iint_{Q \times \Delta(A)} \widehat{\chi}_1 \widehat{c}_1 \widehat{u}_1 \widehat{\frac{\partial v_1}{\partial t}} \, dx\, dt\, d\beta + \iint_{Q \times \Delta(A)} \widehat{\chi}_1 \widehat{a}_1(\cdot, \mathbb{D} u_1) \cdot \nabla \widehat{v}_1 \, dx\, dt\, d\beta \\ &-\iint_{Q \times \Delta(A)} \widehat{\chi}_2 \widehat{c}_3 \widehat{U}_3 \widehat{\frac{\partial v_1}{\partial t}} \, dx\, dt\, d\beta = 0, \end{aligned}$$

which leads to

$$\frac{\partial}{\partial t} M(\chi_1 c_1 u_1) - \operatorname{div} \widetilde{a}_1(\cdot, \mathbb{D} u_1) + \frac{1}{\delta} \frac{\partial}{\partial t} \Big(\int_{\widehat{G}_2} \widehat{c}_3 \widehat{U}_3 d\beta \Big) = 0 \quad \text{in } Q. \tag{3.23}$$

Next we take $V = (0, v_2, 0, 0, V_3)$ with $v_2 \in \mathcal{C}_0^\infty(Q) \otimes I_A^p$ and $V_3 = \varphi \otimes w_3 \in \mathcal{C}_0^\infty(Q) \otimes \mathcal{D}_A(\mathbb{R}^N)$ satisfying $\delta V_3 = -\alpha v_2$ in $Q \times G_2$. Then $\chi_2 \nabla_y V_3 = 0$ in $Q \times \mathbb{R}^N$, hence, proceeding as above we obtain

$$\frac{\partial}{\partial t} M(\chi_2 c_2 u_2) - \operatorname{div} \widetilde{a}_2(\cdot, \mathbb{D} u_2) - \frac{\alpha}{\delta} \frac{\partial}{\partial t} \Big(\int_{\widehat{G}_2} \widehat{c}_3 \widehat{U}_3 d\beta \Big) = 0 \quad \text{in } Q. \tag{3.24}$$

Choosing successively $V = (0, 0, V_1, 0, 0)$ and $V = (0, 0, 0, V_2, 0)$ with $V_j = \varphi_j \otimes w_3^j \in \mathcal{C}_0^\infty(Q) \otimes \mathcal{D}_A(\mathbb{R}^N)$ $(j = 1, 2)$ we obtain

$$\int_{\Delta(A)} \widehat{\chi}_j \widehat{a}_j(\cdot, \mathbb{D} u_j) \cdot \partial \widehat{V}_j \, dx\, dt\, d\beta = 0,$$

hence

$$\overline{\operatorname{div}}_y(\chi_j a_j(\cdot, \nabla u_j + \overline{\nabla}_y U_j)) = 0 \quad \text{in } \mathbb{R}^N,\ j = 1, 2,\ \text{for a.e. } (x,t) \in Q. \tag{3.25}$$

Now, substituting (3.22)-(3.25) into (3.20), we deduce from Green's formula that

$$\begin{gathered} \widetilde{a}_j(\cdot, \nabla u_j + \overline{\nabla}_y U_j) \cdot \nu = 0 \quad \text{on } \partial\Omega, \\ u_j(\cdot, 0) = \chi_j u_j^0 \text{ in } \Omega\ (j = 1, 2),\, U_3(\cdot, 0, \cdot) = \chi_2 u_3^0 \quad \text{in } \Omega. \end{gathered}$$

Let us now analyze (3.25). We know that it is equivalent to

$$\int_{\Delta(A)} \widehat{\chi}_j \widehat{a}_j(\cdot, \mathbb{D} u_j) \cdot \partial \widehat{V} d\beta = 0 \quad \text{for all } V \in \mathcal{D}_A(\mathbb{R}^N),$$

and by density, to

$$\int_{\Delta(A)} \widehat{\chi}_j \widehat{a}_j(\cdot, \mathbb{D} u_j) \cdot \partial \widehat{V} d\beta = 0 \quad \text{for all } V \in \mathcal{B}_A^{1,p}. \tag{3.26}$$

Let $\xi \in \mathbb{R}^N$ and consider the following cell problem: find $\pi_j(\xi) \in \mathcal{B}_A^{1,p}$ such that

$$\int_{\Delta(A)} \widehat{\chi}_j \widehat{a}_j(\cdot, \xi + \partial\widehat{\pi}_j(\xi)) \cdot \partial\widehat{V} d\beta = 0 \quad \text{for all } V \in \mathcal{B}_A^{1,p}. \tag{3.27}$$

It is an easy task (using the properties of the function a_j) to see that Eq. (3.27) possesses at least one solution. But if $\pi_j^1(\xi)$ and $\pi_j^2(\xi)$ are two solutions of (3.27) then using them as test functions in (3.27) and subtracting the resulting equalities, we end up with $\overline{\nabla}_y(\pi_j^1(\xi) - \pi_j^2(\xi)) = 0$ on G_j, which shows that the solution is unique up to a function $g_j \in \mathcal{B}_A^{1,p}$ satisfying $\overline{\nabla}_y g_j = 0$ on G_j. Comparing (3.27) in which we take the particular $\xi = \nabla u_j(x,t,y)$ ($(x,t) \in Q \times \mathbb{R}_y^N$) with (3.26), we see by the above uniqueness argument that there exists $g_j \in \mathcal{B}_A^{1,p}$ with $\overline{\nabla}_y g_j = 0$ on G_j, such that

$$U_j(x,t,y) = \pi_j(\nabla u_j(x,t,y)) + g_j.$$

Now, for $j = 1, 2$, set

$$\widetilde{b}_j(\xi) = \int_{\Delta(A)} \widehat{\chi}_j \widehat{a}_j(\cdot, \xi + \partial\widehat{\pi}_j(\xi)) d\beta, \quad \xi \in \mathbb{R}^N,$$

$$b_j(\nabla u_j) = \widetilde{b}_j(\nabla \widehat{u}_j).$$

First, as we can see, the function $\widetilde{b}_j$ does not depend on the choice of the g_j, so it is therefore well-defined. Secondly, b_j satisfies properties similar to a_j. With this in mind, coming back to the equations (3.23) and (3.24) we rewrite them as follows:

$$\frac{\partial}{\partial t} M(\chi_1 c_1 u_1) - \mathrm{div} b_1(\nabla u_1) + \frac{1}{\delta}\frac{\partial}{\partial t}\Big(\int_{\widehat{G}_2} \widehat{c}_3 \widehat{U}_3 d\beta\Big) = 0 \quad \text{in } Q; \tag{3.28a}$$

$$\frac{\partial}{\partial t} M(\chi_2 c_2 u_2) - \mathrm{div} b_2(\nabla u_2) - \frac{\alpha}{\delta}\frac{\partial}{\partial t}\Big(\int_{\widehat{G}_2} \widehat{c}_3 \widehat{U}_3 d\beta\Big) = 0 \quad \text{in } Q. \tag{3.28b}$$

The above equations are complemented with the equation

$$\chi_2 c_3 \frac{\partial U_3}{\partial t} - \overline{\mathrm{div}}_y(\chi_2 a_3(\cdot, \overline{\nabla}_y U_3)) = 0 \quad \text{in } G_2 \times (0,T) \tag{3.28c}$$

and the boundary and initial conditions

$$b_j(\nabla u_j) \cdot \nu = 0 \quad \text{on } \partial\Omega,\ j = 1, 2 \tag{3.28d}$$

and

$$u_j(\cdot, 0) = \chi_j u_j^0 \text{ in } \Omega\ (j = 1, 2), \quad U_3(\cdot, 0, \cdot) = \chi_2 u_3^0 \text{ in } \Omega. \tag{3.28e}$$

Finally u_1, u_2 and U_3 are subjected to the following important condition arising from Green's formula

$$\delta\chi_2 \overline{\nabla}_y U_3 = \overline{\nabla}_y(\chi_1 u_1 + \chi_2(\alpha u_2 + \delta U_3)) \quad \text{in } \mathbb{R}_y^N. \tag{3.28f}$$

Arguing exactly as in [40, Theorem 4.4] (see also [9, Theorem 5.1]) we show that the problem (3.28a)-(3.28f) possesses a unique solution. We can now state the main homogenization result.

Theorem 3.5. *For each* $\varepsilon > 0$ *let* $(u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon) \in L^p(0,T; V_\varepsilon)$ *be the unique solution to* (1.2). *Suppose that* (3.1a)-(3.1c) *hold. Then, as* $\varepsilon \to 0$,

$$\chi_j^\varepsilon u_j^\varepsilon \to \chi_j u_j \quad \text{in } L^2(Q)\text{-weak } \Sigma\ (j = 1, 2), \tag{3.29}$$

$$\chi_2^\varepsilon u_3^\varepsilon \to \chi_2 U_3 \quad \text{in } L^2(Q)\text{-weak } \Sigma, \tag{3.30}$$

where (u_1, u_2, U_3) *is the unique solution of the homogenized system* (3.28).

Proof. Given any ordinary sequence E, the existence of a triple (u_1, u_2, U_3) (up to a subsequence of E) derives from Proposition 3.3, and the fact that it solves (3.28a)-(3.28f) comes from the preceding analysis and Theorem 3.4. Since (3.28a)-(3.28f) possesses a unique solution, the convergence results (3.29) and (3.30) hold true for the whole sequence. This completes the proof. □

Remark 3.6. If we assume the algebra A to be ergodic then the functions u_j ($j = 1, 2$) do not depend on y, that is, $u_j \in L^p(0, T; W^{1,p}(\Omega))$. In this case $M(\chi_j c_j u_j) = M(\chi_j c_j)u_j$. Setting $\theta_j = M(\chi_j c_j) > 0$ (see assumption (3.1a)), equations (3.28a) and (3.28b) become

$$\begin{gathered}
\theta_1 \frac{\partial u_1}{\partial t} - \operatorname{div} b_1(\nabla u_1) + \frac{1}{\delta} \frac{\partial}{\partial t}\Big(\int_{\widehat{G}_2} \widehat{c}_3 \widehat{U}_3 d\beta \Big) = 0 \quad \text{in } Q, \\
\theta_2 \frac{\partial u_2}{\partial t} - \operatorname{div} b_2(\nabla u_2) - \frac{\alpha}{\delta} \frac{\partial}{\partial t}\Big(\int_{\widehat{G}_2} \widehat{c}_3 \widehat{U}_3 d\beta \Big) = 0 \quad \text{in } Q.
\end{gathered}$$

respectively.

4. Examples

In this section we present some concrete situations which may occur in the physical framework. We begin with some preliminary results.

4.1. **Preliminaries.** As the cells $(k+Y)_{k\in S}$ are pairwise disjoint, the characteristic function χ_Θ of the set $\Theta = \cup_{k\in S}(k + Y_1)$ in $\mathbb{R}^N$ verifies $\chi_\Theta = \sum_{k\in S} \chi_{k+Y_1}$ or more precisely,

$$\chi_\Theta = \sum_{k\in\mathbb{Z}^N} \theta(k)\chi_{k+Y_1},$$

where θ is the characteristic function of S in $\mathbb{Z}^N$. We refer to θ as the *distribution function of the fissured cells* [24].

Proposition 4.1 ([24, Sec. 3.1] or [38, Prop. 4.1])**.** *Let A be an algebra with mean value on $\mathbb{R}^N$. Suppose that the distribution function of the fissured cells lies in the space of essential functions on $\mathbb{Z}^N$, $ES(\mathbb{Z}^N)$ (see* [23]*). Moreover assume that for every φ in $\mathcal{K}(Y)$ (the space of all continuous complex functions on $\mathbb{R}^N_z$ with compact support contained in $Y = (0, 1)^N$), the function $\sum_{k\in\mathbb{Z}^N} \theta(k)\tau_k\varphi$ (where $\tau_k\varphi(y) = \varphi(y + k)$, $y \in \mathbb{R}^N$) lies in A. Then $\chi_\Theta \in B^p_A(\mathbb{R}^N)$ $(1 \le p < \infty)$ and*

$$M(\chi_\Theta) = \mathfrak{M}(\theta)\lambda(Y_1),$$

λ being the Lebesgue measure on $\mathbb{R}^N$ while $\mathfrak{M}(\theta)$ is the essential mean of θ [23].

Corollary 4.2 ([24, Corollary 3.2])**.** *With the hypotheses of Proposition 4.1,* (3.1a) *is satisfied.*

This leads to some specific examples.

4.2. **Equidistribution of the fissured cells.** We assume here that the distribution of fissured cells is given by $\theta(k) = 1$ for any $k \in \mathbb{Z}^N$. Then $S = \mathbb{Z}^N$, and proceeding as in [24, Sect. 3.2] we obtain

$$\chi_j \in B^r_{\mathcal{C}_{\rm per}(Y)}(\mathbb{R}^N) \quad (1 \le r < \infty) \text{ and } M(\chi_j) > 0 \text{ for } j = 1, 2, \tag{4.1}$$

that is (3.1a), where $\mathcal{C}_{\rm per}(Y)$ denotes the space of Y-periodic continuous functions on $\mathbb{R}^N$. This being so, we can consider the homogenization problem for (1.2) under the following assumptions:

(H1) (*Periodic homogenization*) We assume that the functions c_j and $a_j(\cdot,\lambda)$ are Y-periodic for every $\lambda \in \mathbb{R}^N$ and all $j = 1,2,3$. This leads to the assumptions (3.1b) -(3.1c) with $A = \mathcal{C}_{\rm per}(Y)$. We recover in this special case the results of [9]. Precisely Theorem 3.5 reads as

Theorem 4.3. *For each $\varepsilon > 0$ let $(u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon) \in L^p(0,T;V_\varepsilon)$ be the unique solution to* (1.2). *Under hypothesis* (H1) *and* (4.1) *we have, as $\varepsilon \to 0$,*

$$\chi_j^\varepsilon u_j^\varepsilon \to \chi_j u_j \quad \text{in } L^2(Q)\text{-weak } \Sigma \ (j=1,2),$$
$$\chi_2^\varepsilon u_3^\varepsilon \to \chi_2 U_3 \quad \text{in } L^2(Q)\text{-weak } \Sigma$$

where (u_1, u_2, U_3) is the unique solution of the homogenized system

$$\begin{gathered}
\theta_1 \frac{\partial u_1}{\partial t} - \operatorname{div} b_1(\nabla u_1) + \frac{1}{\delta}\frac{\partial}{\partial t}\Big(\int_{Y_2} c_3 U_3 dy\Big) = 0 \quad \text{in } Q;\\
\theta_2 \frac{\partial u_2}{\partial t} - \operatorname{div} b_2(\nabla u_2) - \frac{\alpha}{\delta}\frac{\partial}{\partial t}\Big(\int_{Y_2} c_3 U_3 dy\Big) = 0 \quad \text{in } Q;\\
\chi_{Y_2} c_3 \frac{\partial U_3}{\partial t} - \operatorname{div}_y(\chi_{Y_2} a_3(\cdot, \nabla_y U_3)) = 0 \quad \text{in } Y_2 \times (0,T);\\
b_j(\nabla u_j)\cdot \nu = 0 \quad \text{on } \partial\Omega,\ j=1,2;\\
\delta U_3 + \alpha u_2 = u_1 \quad \text{on } \Gamma_{1,2} = \partial Y_1 \cap \partial Y_2;\\
u_j(\cdot,0) = \chi_j u_j^0 \quad \text{in } \Omega,\ (j=1,2);\\
U_3(\cdot,0,\cdot) = \chi_2 u_3^0 \quad \text{in } \Omega\,.
\end{gathered} \tag{4.2}$$

where $\theta_j = \int_{Y_j} c_j(y)dy$ for $j = 1,2$.

Proof. Everything has been checked in the preceding section except the interface condition (4.2) which is a consequence of (3.28f) and the Green's formula as in [9]. □

(H2) We also assume that c_j and $a_j(\cdot,\lambda)$ are respectively Bohr and Besicovitch almost periodic functions on $\mathbb{R}^N$ [3, 4]. Then as $\mathcal{C}_{\rm per}(Y) \subset AP(\mathbb{R}^N)$ (the space of Bohr almost periodic continuous functions on $\mathbb{R}^N$) we have $B^r_{\mathcal{C}_{\rm per}(Y)}(\mathbb{R}^N) \subset B^r_{AP(\mathbb{R}^N)}(\mathbb{R}^N)$, and (3.1a)-(3.1c) hold with $A = AP(\mathbb{R}^N)$.

(H3) Denoting by $\mathcal{B}_\infty(\mathbb{R}^N)$ the space of all continuous functions on $\mathbb{R}^N$ that have finite limit at infinity (which is an algebra with mean value on $\mathbb{R}^N$), we may also assume that

$$c_j \in \mathcal{B}_\infty(\mathbb{R}^N), \quad a_j(\cdot,\lambda) \in \mathcal{C}_{\rm per}(Y) \quad \text{for all } \lambda \in \mathbb{R}^N,\ j=1,2,3.$$

This leads to (3.1a)–(3.1c) with $A = \mathcal{B}_\infty(\mathbb{R}^N) + \mathcal{C}_{\rm per}(Y)$ (this is easily verified).

4.3. **Periodic distribution of the fissured cells.** Assume the function θ is periodic; that is, there is a network $\mathcal{R}$ in $\mathbb{R}^N$ with $\mathcal{R} \subset \mathbb{Z}^N$ such that

$$\theta(k+r) = \theta(k) \quad \text{for all } k \in \mathbb{Z}^N \text{ and all } r \in \mathcal{R}.$$

Denoting by $P_{\mathcal{R}}(\mathbb{R}^N)$ the algebra of periodic functions on $\mathbb{R}^N$ represented by the group of periods $\mathcal{R}$, i.e. the algebra of functions $u \in \mathcal{C}(\mathbb{R}^N)$ that verify $u(y + k) = u(y)$ for all $y \in \mathbb{R}^N$ and all $k \in \mathcal{R}$, we argue as in Subsection 5.1 to get $\chi_j \in B^r_{P_{\mathcal{R}}(\mathbb{R}^N)}(\mathbb{R}^N)$ $(1 \le r < \infty)$ and $M(\chi_j) > 0$. We can therefore repeat the arguments of the preceding subsection to solve the homogenization problems for (1.2) under assumptions (H1)-(H3) without slightest change.

4.4. **Almost periodic distribution of the fissured cells.** Assume the function θ is almost periodic; that is, the translates $\tau_h\theta$ ($h \in \mathbb{Z}^N$) form a relatively compact set in $\ell^\infty(\mathbb{Z}^N)$. Then we have

$$\chi_j \in B^r_{AP(\mathbb{R}^N)}(\mathbb{R}^N) \ (1 \le r < \infty) \text{ with } M(\chi_j) > 0, \ j = 1, 2;$$

that is (3.1a) with $A = AP(\mathbb{R}^N)$. Bearing this in mind, we may assume the functions c_j and $a_j(\cdot, \lambda)$ satisfy the following hypotheses.

(H4) (*Almost periodic homogenization*) c_j belongs to $AP(\mathbb{R}^N)$ and $a_j(\cdot, \lambda)$ belongs to $B^r_{AP(\mathbb{R}^N)}(\mathbb{R}^N)$ for any $\lambda \in \mathbb{R}^N$ and $j = 1, 2, 3$, so that (3.1b)–(3.1c) hold with $A = AP(\mathbb{R}^N)$.

(H5) $c_j \in AP(\mathbb{R}^N)$ and $a_j(\cdot, \lambda) \in L^{p'}_{\infty,AP}(\mathbb{R}^N)$ for all $\lambda \in \mathbb{R}^N$ and $j = 1, 2, 3$, where $L^{p'}_{\infty,AP}(\mathbb{R}^N)$ denotes the closure with respect to the Besicovitch seminorm $\|\cdot\|_{p'}$ (defined in Section 2) of the space of finite sums

$$\sum_{\text{finite}} \varphi_i u_i \quad \text{with } \varphi_i \in \mathcal{B}_\infty(\mathbb{R}^N), \, u_i \in AP(\mathbb{R}^N).$$

Then we are led to (3.1b)-(3.1c) with $A = \mathcal{B}_\infty(\mathbb{R}^N) + AP(\mathbb{R}^N)$, an algebra with mean value on $\mathbb{R}^N$ [22, 31].

(H6) (*Homogenization in non ergodic algebra*) Let A_1 be the algebra generated by the function $f(z) = \cos \sqrt[3]{z}$ ($z \in \mathbb{R}$) and all its translates $f(\cdot + a)$, $a \in \mathbb{R}$. It is known that A is an algebra with mean value which is not ergodic; see [18] for details. Now let A be defined as follows: $A_2 = A_1 \odot \ldots \odot A_1$, N times, (the product of N copies of A_1; see [22, 31] for the definition of a product of algebras with mean value) which gives a non ergodic algebra on $\mathbb{R}^N$.

We assume that $c_j \in A_2$ and $a_j(\cdot, \lambda) \in B^{p'}_{A_2}(\mathbb{R}^N)$ ($\lambda \in \mathbb{R}^N$, $j = 1, 2, 3$). Then we are led to (3.1a)-(3.1c) with A being the algebra with mean value generated by $AP(\mathbb{R}^N) \cup A_2$.

(H7) (*Weak almost periodic homogenization*) We assume that $c_j \in WAP(\mathbb{R}^N)$ and $a_j(\cdot, \lambda) \in B^{p'}_{WAP(\mathbb{R}^N)}(\mathbb{R}^N)$ ($\lambda \in \mathbb{R}^N$, $j = 1, 2, 3$) where $WAP(\mathbb{R}^N)$ is the algebra of continuous weakly almost periodic functions on $\mathbb{R}^N$ [13], which is an algebra with mean value on $\mathbb{R}^N$ [25, 31]. Since $AP(\mathbb{R}^N) \subset WAP(\mathbb{R}^N)$, (3.1a)-(3.1c) are satisfied with $A = WAP(\mathbb{R}^N)$.

One may also consider some other hypotheses.

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