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OPTION PRICING WITH TRANSACTION COSTS AND STOCHASTIC VOLATILITY

IONUŢ FLORESCU, MARIA C. MARIANI, INDRANIL SENGUPTA

Abstract. In a realistic market with transaction costs, the option pricing problem is known to lead to solving nonlinear partial differential equations even in the simplest model. The nonlinear term in these partial differential equations (PDE) reflects the presence of transaction costs. In this article we consider an underlying general stochastic volatility model. In this case the market is incomplete and the option price is not unique. Under a particular market completion assumption where we use a traded proxy for the volatility, we obtain a non-linear PDE whose solution provides the option price in the presence of transaction costs. This PDE is studied and under suitable regularity conditions, we prove the existence of strong solutions of the problem.

1. Introduction

In this work we consider a market model in which trading the asset requires paying transaction fees which are proportional to the quantity and the value of the asset traded. In this market we study the problem of finding option prices when the underlying asset may be approximated using a stochastic volatility model.

This is a problem with a long history in mathematical finance. In a complete frictionless (i.e., without transaction costs) financial market, the Black-Scholes model (1973) [1] provides a hedging strategy for any European type contingent claim. One needs to trade continuously to re-balance the hedging portfolio and therefore, such an operation tends to be infinitely expensive in a market with transaction costs. For example, [25] shows that the best hedging strategy in this case is to simply buy the asset and hold it for the duration of the call or put option. This is the reason why the requirement of replicating the value of the option continuously and exactly has to be relaxed.

As we feel best was described by Dewynne, Whalley and Wilmott (1994) [5], the approaches taken were local in time and global in time. The former approach (which is also the one taken in this paper) pioneered by Leland (1985) [19] (continued e.g., [2, 13]), considers risk and return over a short interval of time. The later approach pioneered by Hodges and Neuberger (1993) [12] (continued for example

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in [3]) adopts 'optimal strategies', in which risk and return are considered over the lifetime of the option.

In the seminal work [19] Leland introduces the idea of using expected transaction costs for a small interval. The author assumes that the portfolio is rebalanced at deterministic, discrete times, δt units apart, and that the transaction costs are proportional to the value of the underlying. Specifically, the cost incurred is $\kappa|\nu|S$, where ν is the number of shares of the underlying asset bought ($\nu > 0$) or sold ($\nu < 0$) at price S, and κ is a proportionality constant characteristic to the individual investor. Leland proposes a hedging strategy based on replicating an option with an adjusted volatility

$$\hat{\sigma} = \sigma\Big(1 + \sqrt{\frac{2}{\pi}}\frac{\kappa}{\sigma\sqrt{\delta t}}\Big)^{1/2}.$$

Leland claimed in his paper that the hedging error approaches zero using this strategy when the length of revision intervals goes to zero, a claim later disproved by many, first being Kabanov and Safarian [16].

Notwithstanding the claim of the hedging error approaching zero for this modified strategy, the idea of using (conditional) expectations when calculating transaction costs proved valuable. This idea was continued by Boyle and Vorst [2] in discrete time and by Hoggard, Whalley and Wilmott [13] (and further [5, 26]) in continuous time. This later, influential line of work derives a nonlinear PDE whose solution provides the option value. For the reader's convenience we replicate their derivation of the PDE using modern notation in section 1.1. This section is illustrating the idea of modeling transaction costs using conditional expectations in a simple model and we advise the knowledgeable reader to skip this section.

The main contribution of the present paper is to extend the transaction costs model when the asset price is approximated using stochastic volatility models. The asset model used is presented in section 2. When working with stochastic volatility models the market is incomplete and contingent claims do not have unique prices. The classical approach is to "complete the market" by fixing a related tradeable asset and deriving the option price in terms of this asset as well as the underlying equity. In section 3 we take this approach and consider the case when we may be able to find a traded asset serving as a proxy for the volatility (such as the case when a volatility index is traded on the market). In this case we propose a market completion solution. We form a portfolio using this asset as well as the underlying and we derive a PDE which may explicitly give the price of options in the option chain. However, the PDE is nonlinear with a very different nonlinear structure from the classical market completion approach. This type of PDE is analyzed in section 4 and we prove an existence result. The proof constructs an approximating sequence which is demonstrated to converge to the strong solution of the PDE. Finally, section 4.1 concludes the article.

1.1. **Option price valuation in the geometric Brownian motion case with transaction costs.** Suppose Π is the value of the hedging portfolio and $C(S,t)$ is the value of the option. The asset follows a geometric Brownian motion. Using a discrete time approximation, Hoggard, Whalley and Wilmott [13] assume the underlying asset follows the process:

$$\delta S = \mu S\delta t + \sigma S\Phi\sqrt{\delta t}, \tag{1.1}$$

where Φ is a standard normal random variable, μ is a measure of the average rate of growth of the asset price also known as the drift, σ is a measure of the fluctuation (risk) in the asset prices and corresponds to the diffusion coefficient. The quantities involving δ denote the increment of processes over the timestep δt. If the portfolio is given by $\Pi = C - \Delta S$, then the change in portfolio value is given by

$$\delta\Pi = \sigma S\Big(\frac{\partial C}{\partial S} - \Delta\Big)\Phi\sqrt{\delta t} + \Big(\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} + \mu S\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} - \mu\Delta S\Big)\delta t - \kappa S|\nu|$$

To derive the portfolio change in the previous equation it looks like we assumed that the quantity of shares Δ is kept constant. This in fact should not be the case, and in the derivation above Δ is in fact stochastic. Nevertheless, the derivation above is correct. To obtain the expression we use the fact that the constructed portfolio Π is self financing. That is, at all time steps when the portfolio is re-balanced no extra funds are added to the portfolio or consumed from the portfolio value. The basic derivation when the asset follows a geometric Brownian motion and the pricing of vanilla option is desired may be found in [24]. In the appendix at the end of this article we provide a generalization of this result under any dynamics for the stock price. In fact, the same rule applies for a portfolio constructed using any number of assets S_i.

The dynamic above leads to the delta hedging strategy. Specifically, let the quantity of asset held short at time t, $\Delta = \frac{\partial C}{\partial S}(S,t)$. The timestep is assumed to be small, thus the number of assets traded after a time δt is

$$\nu = \frac{\partial C}{\partial S}(S+\delta S, t+\delta t) - \frac{\partial C}{\partial S}(S,t) = \delta S\frac{\partial^2 C}{\partial S^2} + \delta t\frac{\partial^2 C}{\partial t\partial S} + \dots$$

Since $\delta S = \sigma S\Phi\sqrt{\delta t} + \mathcal{O}(\delta t)$, keeping only the leading term yields

$$\nu \simeq \frac{\partial^2 C}{\partial S^2}\sigma S\Phi\sqrt{\delta t}.$$

Thus, the expected transaction cost over a timestep is

$$E[\kappa S|\nu|] = \sqrt{\frac{2}{\pi}}\kappa\sigma S^2|\frac{\partial^2 C}{\partial S^2}|\sqrt{\delta t},$$

where $\sqrt{2/\pi}$ is the expected value of $|\Phi|$. Therefore, the expected change in the value of the portfolio is

$$E(\delta\Pi) = \Big(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} - \kappa\sigma S^2\sqrt{\frac{2}{\pi\delta t}}|\frac{\partial^2 C}{\partial S^2}|\Big)\delta t.$$

The authors then use standard no arbitrage arguments to deduce that the portfolio will earn the riskfree interest rate r,

$$E(\delta\Pi) = r\Big(C - S\frac{\partial C}{\partial S}\Big)\delta t.$$

The authors derive the PDE for option pricing with transaction costs as:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial^2 S} + rS\frac{\partial C}{\partial S} - rC - \kappa\sigma S^2\sqrt{\frac{2}{\pi\delta t}}|\frac{\partial^2 C}{\partial S^2}| = 0, \tag{1.2}$$

on the domain $(S,T) \in (0,\infty)\times(0,T)$ with terminal condition

$$C(S,T) = \max(S-E,0), \quad S\in(0,\infty) \tag{1.3}$$

for European call options with strike price E, and a suitable terminal condition for European puts.

The portfolio is considered to be revised every δt where δt is a non-infinitesimal fixed time-step not to be taken $\delta t \to 0$. This approach is now classified into the so-called local in time hedging strategy. The equation (1.2) is claimed as one of the first nonlinear PDE's in finance [13]. It also is one of the most studied in Finance, we refer to [15] for analytical solution and numerical implementation, and to [26] for asymptotic analysis for this model and two other models in the presence of transaction costs. Theorem 1 in [15] proves that under the condition $2\kappa\sqrt{\frac{2}{\sigma^2\pi\delta t}} < 1$, equation (1.2) has a solution for any option V with payoff $V(S,T) \approx \alpha S$ when $S \to \infty$. All option types used in practice have this kind of payoff.

2. Stochastic volatility model with transaction costs

A basic assumption in modeling the equity using a geometric Brownian motion as described above is that the volatility is constant. Much of the literature today shows this is an unrealistic assumption. Any model where the volatility is random is called a stochastic volatility model. A possible alternative approach to stochastic volatility models is to use jump diffusion processes or more general Lévy processes. We do not consider jumps in this work as they will lead to *nonlinear* PDE's with an *integral term*, which are very hard to work with.

In this work we consider the stochastic volatility model

$$dS_t = \mu(S_t)dt + \sigma_t S_t dX_1(t), \tag{2.1}$$

$$d\sigma_t = \alpha(\sigma_t)dt + \beta\sigma_t dX_2(t). \tag{2.2}$$

where the two Brownian motions $X_1(t)$ and $X_2(t)$ are correlated with correlation coefficient ρ:

$$\mathrm{E}(dX_1(t)dX_2(t)) = \rho\, dt \tag{2.3}$$

The stochastic volatility model considered is a modified Hull-White process [14, 27], to contain general drift terms in S and σ. These general drift terms do not influence the PDE derivation. We note that the process above may also be viewed as a generalization of the SABR process [9] which is the stochastic volatility model most used in the financial industry.

The market is arbitrage free and incomplete when using stochastic volatility models. The fundamental theorem of asset pricing [10, 4] guarantees no-arbitrage if an equivalent martingale measure exists, and completeness of the market if the equivalent martingale measure is unique.

In the case of stochastic volatility models (with the exception of the trivial case when the Brownian motions are perfectly correlated $\rho = \pm 1$) there exist an infinite number of equivalent martingale measures [7] and therefore the market is arbitrage free but not complete. This means that the traded asset price does not uniquely determine the derivative prices.

In our previous work we have fixed the price of a particular derivative as a given asset and express all the other derivative prices in terms of the price of the given derivative. For the present work an alternative is discussed in Section 3. This is the case when the volatility is a traded asset, e.g., for S&P500 and its associated volatility index VIX. In this case one may read the volatility information (e.g., VIX) from the market and produce the entire chain of option values.

3. The PDE derivation when the volatility is a traded asset

The results in this section are applicable when there exists a proxy for the stochastic volatility which is actively traded. An example of such a case in today's financial derivative market is the Standard and Poor 500 equity index (in fact the exchange traded fund that replicates it: either SPX or SPY), and the associated volatility index (VIX). The VIX is a traded asset, supposed to represent the implied volatility of an option with strike price exactly at the money (equal with the spot value of SPX) and with maturity exactly one month from the current date. The VIX is calculated using an interpolating formula from the (out-of-money) options available and traded on the market. In our setting we view the VIX as a traded asset, a proxy for the value of the stochastic volatility process in the model we propose here. Using the traded volatility index as a proxy provides a further advantage. The problem of parameter estimation in the stochastic volatility specification (2.2) is much simpler since the volatility process becomes observable. The volatility distribution may be further estimated using a filtering methodology as described for example in [7].

In the future, it is possible that more volatility indices will be traded on the market, and we denote in what follows S as the spot equity price and with σ the matching spot volatility. It is important that this σ be traded (sold and bought). In the present section we are considering the volatility index σ as a perfect proxy for the stochastic volatility. In depth analysis about the appropriateness of this assumption is beyond the scope of the current paper.

We consider a portfolio Π that contains one option, with value $V(S,\sigma,t)$, and quantities Δ and Δ_1 of S and σ respectively. That is,

$$\Pi = V - \Delta S - \Delta_1\sigma. \tag{3.1}$$

We apply Itô's formula to get the dynamics of V. A derivation to find the portfolio dynamics is presented in the Appendix 5. Applying this derivation we obtain the change in value of the portfolio Π as,

$$\begin{aligned} d\Pi &= \Big(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\beta^2\sigma^2\frac{\partial^2 V}{\partial \sigma^2} + \rho\sigma^2\beta S\frac{\partial^2 V}{\partial S\partial\sigma}\Big)dt \\ &\quad + \big(\frac{\partial V}{\partial S} - \Delta\big)dS + \big(\frac{\partial V}{\partial \sigma} - \Delta_1\big)d\sigma - \kappa S|\nu| - \kappa_1\sigma|\nu_1|, \end{aligned}$$

where $\kappa S|\nu|$ and $\kappa_1\sigma|\nu_1|$ represent the transaction cost associated with trading ν of the main asset S and ν_1 of the volatility index σ during the time step δt. It is important to note (see the Appendix) that this equation is an approximation to the exact dynamics of the portfolio. Nevertheless, even though Δ and Δ_1 are treated as constants this derivation is correct and based on the self financing property of the portfolio.

The costs for trading S and σ are different and proportional with quantity transacted. We use k, k_1 to denote cost and ν and ν_1 to denote quantity transacted respectively for S and σ. We choose Δ and Δ_1 which are the quantities of stock respectively volatility to be owned every time portfolio re-balancing is performed as the solutions of:

$$\big(\frac{\partial V}{\partial S} - \Delta\big) = 0,$$

and

$$\big(\frac{\partial V}{\partial \sigma} - \Delta_1\big) = 0.$$

This choice once again eliminates the drift terms and the portfolio dynamics become

$$d\Pi = \Big(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\beta^2\sigma^2 \frac{\partial^2 V}{\partial \sigma^2} + \rho\sigma^2\beta S \frac{\partial^2 V}{\partial S \partial \sigma}\Big) dt - \kappa S|\nu| - \kappa_1\sigma|\nu_1|. \tag{3.2}$$

3.1. **What is the cost of transaction?** We investigate the costs associated with trading both assets present in the market. We perform a detailed analysis of the cost associated with trading S. We state the costs associated with trading σ while noting that the derivation is similar. This section is concerned with finding an approximate value for the quantities traded ν and ν_1.

If the number of assets held short at time t is

$$\Delta_t = \frac{\partial V}{\partial S}(S, \sigma, t), \tag{3.3}$$

after a time step δt and re-hedging, the number of assets we hold short is

$$\Delta_{t+\delta t} = \frac{\partial V}{\partial S}(S + \delta S, \sigma + \delta\sigma, t + \delta t).$$

Since the time step δt is assumed small, the changes in asset and the volatility are also small, and applying the Taylor's formula to expand $\Delta_{t+\delta t}$ yields

$$\Delta_{t+\delta t} \simeq \frac{\partial V}{\partial S}(S, \sigma, t) + \delta t \frac{\partial^2 V}{\partial t \partial S}(S, \sigma, t) + \delta S \frac{\partial^2 V}{\partial S^2}(S, \sigma, t) + \delta\sigma \frac{\partial^2 V}{\partial \sigma \partial S}(S, \sigma, t) + \dots$$

Since $\delta S = \sigma S \delta X_1 + \mathcal{O}(\delta t)$ and $\delta\sigma = \beta\sigma\delta X_2 + \mathcal{O}(\delta t)$,

$$\Delta_{t+\delta t} \simeq \frac{\partial V}{\partial S} + \sigma S \delta X_1 \frac{\partial^2 V}{\partial S^2} + \beta\sigma\delta X_2 \frac{\partial^2 V}{\partial \sigma \partial S}. \tag{3.4}$$

Subtracting (3.3) from (3.4), we find the number of assets traded during a time step:

$$\nu = \sigma S \delta X_1 \frac{\partial^2 V}{\partial S^2} + \beta\sigma\delta X_2 \frac{\partial^2 V}{\partial \sigma \partial S}. \tag{3.5}$$

Note that ν is a random variable. We base our estimation of quantity traded on the expectation of this variable and we use it to calculate the expected transaction cost. Since X_1 and X_2 are correlated Brownian motions, we consider Z_1 and Z_2 two independent normal variables with mean 0 and variance 1 and thus we may write the distribution of X_1, X_2 as

$$\begin{gathered} \delta X_1 = Z_1\sqrt{\delta t} \\ \delta X_2 = \rho Z_1 \sqrt{\delta t} + \sqrt{1-\rho^2} Z_2 \sqrt{\delta t}. \end{gathered}$$

Substituting these expressions in ν and denoting

$$\begin{gathered} \alpha_1 = \sigma S \sqrt{\delta t} \frac{\partial^2 V}{\partial S^2} + \beta\sigma\rho\sqrt{\delta t} \frac{\partial^2 V}{\partial \sigma \partial S} \\ \beta_1 = \beta\sigma\sqrt{1-\rho^2}\sqrt{\delta t} \frac{\partial^2 V}{\partial \sigma \partial S}, \end{gathered} \tag{3.6}$$

we write the change in the number of shares over a time step δt as

$$\nu = \alpha_1 Z_1 + \beta_1 Z_2.$$

We calculate the expected value of the transaction costs associated with trading the asset S:

$$\mathbb{E}[\kappa S|\nu| \mid S] = \sqrt{\frac{2}{\pi}} \kappa S \sqrt{\alpha_1^2 + \beta_1^2}.$$

Analyzing the transaction costs associated with trading the volatility index σ proceeds in an entirely similar way and produces a similar formula to (3.5):

$$\nu_1 = \sigma S \delta X_1 \frac{\partial^2 V}{\partial S \partial \sigma} + \beta \sigma \delta X_2 \frac{\partial^2 V}{\partial \sigma^2}. \tag{3.7}$$

Therefore (3.2) leads to the nonlinear PDE

$$\begin{aligned}
&\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\beta^2\sigma^2 \frac{\partial^2 V}{\partial \sigma^2} + \rho\sigma^2\beta S \frac{\partial^2 V}{\partial S \partial \sigma} + rS\frac{\partial V}{\partial S} + r\sigma \frac{\partial V}{\partial \sigma} - rV \\
&- \kappa S \sqrt{\frac{2}{\pi \delta t}} \sqrt{\sigma^2 S^2 \big(\frac{\partial^2 V}{\partial S^2}\big)^2 + 2\rho\beta\sigma^2 S \frac{\partial^2 V}{\partial S^2}\frac{\partial^2 V}{\partial S \partial \sigma} + \beta^2\sigma^2 \big(\frac{\partial^2 V}{\partial S \partial \sigma}\big)^2} \\
&- \kappa_1 \sigma \sqrt{\frac{2}{\pi \delta t}} \sqrt{\sigma^2 S^2 \big(\frac{\partial^2 V}{\partial S \partial \sigma}\big)^2 + 2\rho\beta\sigma^2 S \frac{\partial^2 V}{\partial S \partial \sigma}\frac{\partial^2 V}{\partial \sigma^2} + \beta^2\sigma^2 \big(\frac{\partial^2 V}{\partial \sigma^2}\big)^2} = 0.
\end{aligned} \tag{3.8}$$

The two final radical terms in the resulting PDE above are coming from transaction costs. As noted in section 1, in this equation δt is a non-infinitesimal fixed time-step not to be taken $\delta t \to 0$. It is the time period for re-balancing and again if it is too small this term will explode the solution of the equation.

As is the case of all PDE's in finance this is a terminal value problem. The specific boundary condition depends on the particular type of option priced but in all cases is expressed at $t = T$ the maturity of the option. For example, for an European Call the condition is $V(S, \sigma, T) = \max(S - K, 0)$ for all σ, where K is the particular option's strike. The general treatment presented in the next section is applicable to any option with boundary condition at T a function of S_T and T only. In fact the theorems stated apply to options whose payoff value is a function of σ_T as well. This is very valuable for certain types of non-vanilla options such as variance swaps.

The next section is devoted to the study of this type of nonlinear equations. In the next section we transform the final value boundary problem (FVBP) to an initial value boundary problem (IVBP) by changing the time variable from t to $\tau = T - t$. Note that this change will only modify the time derivative $\partial V/\partial t$ which becomes negative in the IVBP. Theorem 4.8 is the key result which is then extended in Theorem 4.9 to the full domain characterizing the PDE (3.8).

4. The analysis of the resulting nonlinear PDE

We need the full PDE treatment to prove the existence of a solution for (3.8). The proof we give in the following lemmas and the final theorems 4.8 and 4.9 is constructive and hidden within the proof is the approximating solution of the PDE. Please note that the equation (3.8) has two very similar nonlinear terms, which will be treated in a similar way.

We use the following change of variables

$$S = e^x, \quad \sigma = e^y, \quad t = T - \tau, \quad V(S, \sigma, t) = v(x, y, \tau).$$

Since $S, \sigma \in [0, \infty)$ this transformation gives $x, y \in (-\infty, \infty)$. The PDE (3.8) is transformed into the forward PDE

$$\begin{aligned}
&- \frac{\partial v}{\partial \tau} + \frac{1}{2} e^{2y}\Big(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\Big) + \frac{1}{2}\beta^2\Big(\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y}\Big) + \rho e^y \beta \frac{\partial^2 v}{\partial x \partial y} + r\frac{\partial v}{\partial x} + r\frac{\partial v}{\partial y} - rv \\
&- \kappa \sqrt{\frac{2}{\pi \delta t}} \sqrt{e^{2y}\Big(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\Big)^2 + 2\rho\beta e^y \Big(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\Big)\frac{\partial^2 v}{\partial x \partial y} + \beta^2\Big(\frac{\partial^2 v}{\partial x \partial y}\Big)^2}
\end{aligned}$$

$$- \kappa_1 \sqrt{\frac{2}{\pi \delta t}} \sqrt{\beta^2 \Big(\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y} \Big)^2 + 2\rho\beta e^y \Big(\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y} \Big) \frac{\partial^2 v}{\partial x \partial y} + e^{2y} \Big(\frac{\partial^2 v}{\partial x \partial y} \Big)^2} = 0.$$

or,

$$\begin{aligned} &- \frac{\partial v}{\partial \tau} + \frac{1}{2} e^{2y} \frac{\partial^2 v}{\partial x^2} + \frac{1}{2} \beta^2 \frac{\partial^2 v}{\partial y^2} + \rho e^y \beta \frac{\partial^2 v}{\partial x \partial y} + (r - \frac{1}{2} e^{2y}) \frac{\partial v}{\partial x} + (r - \frac{1}{2} \beta^2) \frac{\partial v}{\partial y} - rv \\ &= \mathfrak{F}_1 \Big(y, \frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 v}{\partial x \partial y} \Big) + \mathfrak{F}_2 \Big(y, \frac{\partial v}{\partial y}, \frac{\partial^2 v}{\partial y^2}, \frac{\partial^2 v}{\partial x \partial y} \Big), \end{aligned} \tag{4.1}$$

where we use the notation

$$\begin{aligned} &\mathfrak{F}_1 \Big(y, \frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 v}{\partial x \partial y} \Big) \\ &= \kappa \sqrt{\frac{2}{\pi \delta t}} \sqrt{e^{2y} \Big(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x} \Big)^2 + 2\rho\beta e^y \Big(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x} \Big) \frac{\partial^2 v}{\partial x \partial y} + \beta^2 \Big(\frac{\partial^2 v}{\partial x \partial y} \Big)^2}, \end{aligned}$$

and

$$\begin{aligned} &\mathfrak{F}_2 \Big(y, \frac{\partial v}{\partial y}, \frac{\partial^2 v}{\partial y^2}, \frac{\partial^2 v}{\partial x \partial y} \Big) \\ &= \kappa_1 \sqrt{\frac{2}{\pi \delta t}} \sqrt{\beta^2 \Big(\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y} \Big)^2 + 2\rho\beta e^y \Big(\frac{\partial^2 v}{\partial y^2} - \frac{\partial v}{\partial y} \Big) \frac{\partial^2 v}{\partial x \partial y} + e^{2y} \Big(\frac{\partial^2 v}{\partial x \partial y} \Big)^2}. \end{aligned}$$

for the two nonlinear terms.

Lemma 4.1. *There exists a constant $C^* > 0$, independent of variables in $\mathfrak{F}_1$ and $\mathfrak{F}_2$ such that*

$$\begin{aligned} &\big| \mathfrak{F}_1 \Big(y, \frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 v}{\partial x \partial y} \Big) + \mathfrak{F}_2 \Big(y, \frac{\partial v}{\partial y}, \frac{\partial^2 v}{\partial y^2}, \frac{\partial^2 v}{\partial x \partial y} \Big) \big| \\ &\leq C^* e^{|y|} \Big(|\frac{\partial v}{\partial x}| + |\frac{\partial v}{\partial y}| + |\frac{\partial^2 v}{\partial x^2}| + |\frac{\partial^2 v}{\partial y^2}| + 2 |\frac{\partial^2 v}{\partial x \partial y}| \Big). \end{aligned}$$

Proof. We analyze the two terms in a similar way. For the first term we have

$$\begin{aligned} &\big| \mathfrak{F}_1 \Big(y, \frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 v}{\partial x \partial y} \Big) \big| \\ &= \big| \kappa \sqrt{\frac{2}{\pi \delta t}} \sqrt{e^{2y} \Big(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x} \Big)^2 + 2 e^y \rho\beta \big(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x} \Big) \frac{\partial^2 v}{\partial x \partial y} + \beta^2 \Big(\frac{\partial^2 v}{\partial x \partial y} \Big)^2} \big| \\ &\leq \big| \kappa \sqrt{\frac{2}{\pi \delta t}} \big| \sqrt{\Big(e^y |\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}| + |\rho\beta \frac{\partial^2 v}{\partial x \partial y}| + |\beta \sqrt{1-\rho^2} \frac{\partial^2 v}{\partial x \partial y}| \Big)^2} \\ &\leq \big| \kappa \sqrt{\frac{2}{\pi \delta t}} \big| \Big(e^y |\frac{\partial^2 v}{\partial x^2}| + e^y |\frac{\partial v}{\partial x}| + (|\rho\beta| + |\beta \sqrt{1-\rho^2}|) |\frac{\partial^2 v}{\partial x \partial y}| \Big) \\ &\leq \big| \kappa \sqrt{\frac{2}{\pi \delta t}} \big| \Big(e^{|y|} |\frac{\partial^2 v}{\partial x^2}| + e^{|y|} |\frac{\partial v}{\partial x}| + e^{|y|} (|\rho\beta| + |\beta \sqrt{1-\rho^2}|) |\frac{\partial^2 v}{\partial x \partial y}| \Big). \end{aligned}$$

Therefore, there exists $C_1 > 0$ such that

$$\big| \mathfrak{F}_1 \Big(y, \frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 v}{\partial x \partial y} \Big) \big| \leq C_1 e^{|y|} \Big(|\frac{\partial v}{\partial x}| + |\frac{\partial^2 v}{\partial x^2}| + |\frac{\partial^2 v}{\partial x \partial y}| \Big).$$

The second term $\mathfrak{F}_2$ produces a similar expression with some different constant C_2. Taking $C^* = \max\{C_1, C_2\}$ we will obtain the stated result. $\square$

Lemma 4.2. *Suppose $|\rho| < 1$. Then the equation* (4.1) *is of parabolic type.*

Proof. For $(v_i, v_j) \in \mathbb{R}^2$ and $\theta > 0$, we have

$$\begin{aligned} &(\sigma^2 - \theta)v_i v_i + (\beta^2 - \theta)v_j v_j + 2\rho\sigma\beta v_i v_j \\ &= \Big[\Big(\sqrt{\sigma^2 - \theta}v_i + \frac{\rho\sigma\beta}{\sqrt{\sigma^2 - \theta}}v_j\Big)^2 + v_j^2\Big(\beta^2(1 - \frac{\rho^2\sigma^2}{\sigma^2 - \theta}) - \theta\Big)\Big] \end{aligned}$$

Therefore,

$$\lim_{\theta \to 0}\big(\beta^2(1 - \frac{\rho^2\sigma^2}{\sigma^2 - \theta}) - \theta\big) = \beta^2(1 - \rho^2).$$

Since $|\rho| < 1$ and $\beta \neq 0$ we have

$$\lim_{\theta \to 0}\big(\beta^2(1 - \frac{\rho^2\sigma^2}{\sigma^2 - \theta}) - \theta\big) > 0.$$

Thus there exists $\theta_1 > 0$ in the neighborhood of 0 such that

$$\big(\beta^2(1 - \frac{\rho^2\sigma^2}{\sigma^2 - \theta_1}) - \theta_1\big) > 0.$$

Therefore with this θ_1, for all $(v_i, v_j) \in \mathbb{R}^2$,

$$\big(\sigma^2 v_i v_i + \beta^2 v_j v_j + 2\rho\sigma\beta v_i v_j\big) > \theta_1(|v_i|^2 + |v_j|^2).$$

This proves that equation (4.1) is parabolic. $\square$

4.1. **Solution of** (4.1)**.** To analyze the main PDE (4.1) we need the following definitions related to spaces with classical derivatives, known as Hölder spaces. We define $C^k_{\rm loc}(\Omega)$ to be the set of all real-valued functions $u = u(x)$ with continuous classical derivatives $D^\alpha u$ in Ω, where $0 \le |\alpha| \le k$. Next, we set

$$\begin{gathered} |u|_{0;\Omega} = [u]_{0;\Omega} = \sup_\Omega |u|, \\ [u]_{k;\Omega} = \max_{|\alpha|=k} |D^\alpha u|_{0;\Omega}. \end{gathered}$$

Definition 4.3. The space $C^k(\Omega)$ is the set of all functions $u \in C^k_{\rm loc}(\Omega)$ such that the norm

$$|u|_{k;\Omega} = \sum_{j=0}^k [u]_{j;\Omega}$$

is finite. With this norm, it can be shown that $C^k(\Omega)$ is a Banach space.

If the seminorm

$$[u]_{\delta;\Omega} = \sup_{x,y\in\Omega,\, x\neq y} \frac{|u(x) - u(y)|}{|x - y|^\delta}$$

is finite, then we say the real-valued function u is Hölder continuous in Ω with exponent δ. For a k-times differentiable function, we will set

$$[u]_{k+\delta;\Omega} = \max_{|\alpha|=k} \left[D^\alpha u\right]_{\delta;\Omega}.$$

Definition 4.4. The Hölder space $C^{k+\delta}(\overline{\Omega})$ is the set of all functions $u \in C^k(\Omega)$ such that the norm
$$|u|_{k+\delta;\Omega} = |u|_{k;\Omega} + [u]_{k+\delta;\Omega}$$
is finite. With this norm, it can be shown that $C^{k+\delta}(\overline{\Omega})$ is a Banach space.

For any two points $P_1 = (x_1, t_1)$, $P_2 = (x_2, y_2) \in Q_T$, we define the parabolic distance between them as
$$d(P_1, P_2) = \left(|x_1 - x_2|^2 + |t_1 - t_2|\right)^{1/2}.$$
For a real-valued function $u = u(x,t)$ on Q_T, let us define the semi-norm
$$[u]_{\delta,\delta/2;Q_T} = \sup_{P_1,P_2 \in Q_T,\, P_1 \neq P_2} \frac{|u(x_1,t_1) - u(x_2,t_2)|}{d^\delta(P_1,P_2)}.$$
If this semi-norm is finite for some u, then we say u is Hölder continuous with exponent δ. The maximum norm of u is given by
$$|u|_{0;Q_T} = \sup_{(x,t)\in Q_T} |u(x,t)|.$$

Definition 4.5. The space $C^{\delta,\delta/2}(\overline{Q}_T)$ is the set of all functions $u \in Q_T$ such that the norm
$$|u|_{\delta,\delta/2;Q_T} = |u|_{0;Q_T} + [u]_{\delta,\delta/2;Q_T}$$
is finite. Furthermore, we define
$$C^{2k+\delta,k+\delta/2}(\overline{Q}_T) = \{u : D^\alpha \partial_t^\rho u \in C^{\delta,\delta/2}(\overline{Q}_T),\, 0 \le |\alpha| + 2\rho \le 2k\}.$$
We define a semi-norm on $C^{2k+\delta,k+\delta/2}(\overline{Q}_T)$ by
$$[u]_{2k+\delta,k+\delta/2;Q_T} = \sum_{|\alpha|+2\rho=2k} [D^\alpha \partial_t^\rho u]_{\delta,\delta/2;Q_T},$$
and a norm by
$$|u|_{2k+\delta,k+\delta/2;Q_T} = \sum_{0\le|\alpha|+2\rho\le 2k} |D^\alpha \partial_t^\rho u|_{\delta,\delta/2;Q_T}.$$
Using this norm, it can be shown that $C^{2k+\delta,k+\delta/2}(\overline{Q}_T)$ is a Banach space.

With these tools in place, we prove the existence of a classical solution for (4.1). Let us denote
$$Lu = \frac{1}{2}e^{2y}\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\beta^2\frac{\partial^2 u}{\partial y^2} + \rho e^y \beta \frac{\partial^2 u}{\partial x \partial y} + (r - \frac{1}{2}e^{2y})\frac{\partial u}{\partial x} + (r - \frac{1}{2}\beta^2)\frac{\partial u}{\partial y} - ru.$$
We first consider the following initial-boundary value problem in a bounded parabolic domain $Q_T = \Omega \times (0,T)$, $T > 0$, where Ω is a bounded domain in $\mathbb{R}^2$.
$$\begin{gathered} -u_\tau + Lu = \mathfrak{F}_1\Big(y, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}\Big) + \mathfrak{F}_2\Big(y, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}\Big) \quad \text{in } Q_T, \\ u(x,y,0) = u_0(x,y) \quad \text{on } \Omega, \\ u(x,y,\tau) = g(x,y,\tau) \quad \text{on } \partial\Omega \times (0,T). \end{gathered} \tag{4.2}$$
Then, we extend our results to the corresponding initial-value problem in the unbounded domain $\mathbb{R}^{2+1}_T = \mathbb{R}^2 \times (0,T)$:
$$\begin{gathered} -u_\tau + Lu = \mathfrak{F}_1\Big(y, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}\Big) + \mathfrak{F}_2\Big(y, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}\Big) \quad \text{in } \mathbb{R}^{2+1}_T, \\ u(x,y,0) = u_0(x,y) \quad \text{on } \mathbb{R}^2. \end{gathered} \tag{4.3}$$

Throughout this section, we impose the following assumptions. These assumptions are reasonable for smooth terminal conditions. However, if the terminal conditions are not smooth then they may be approximated by smooth functions for which the following assumptions are true. They are same assumptions as observed in a different problem in [21].

- (A1) The coefficients of L belong to the Hölder space $C^{\delta,\delta/2}(\overline{Q}_T)$;
- (A2) The value of $|\rho| < 1$;
- (A3) $u_0(x,y)$ and $g(x,y,t)$ belong to the Hölder spaces $C^{2+\delta}(\mathbb{R}^2)$ and $C^{2+\delta,1+\delta/2}(\overline{Q}_T)$ respectively;
- (A4) The two consistency conditions
$$g(x,y,0) = u_0(x,y),$$
$$g_\tau(x,y,0) - Lu_0(x,y) = 0$$
are satisfied for all $x \in \partial\Omega$.

We shall prove the existence of a solution to (4.1) using an iterative argument. We will do this by providing estimates based on a Green's function. Afterwards, we will use a standard argument to show that our solution can be extended to a solution to the initial-value problem in $\mathbb{R}^{2+1}_T$.

Let us define the function space $C^{1+1,0+1}(\overline{Q}_T)$ to be the set of all $u \in C^{1,0}(\overline{Q}_T) \cap W^{2,1}_\infty(\overline{Q}_T)$. We will say $u \in C^{1+1,0+1}(\overline{Q}_T)$ is a strong solution to the parabolic initial-boundary value problem (4.1) provided that u satisfies the parabolic equation almost everywhere in Q_T and the initial-boundary conditions in the classical sense. The following lemma follows immediately from [17, Theorem 10.4.1].

Lemma 4.6. *There exists a unique solution $\varphi \in C^{2+\delta,1+\delta/2}(\overline{Q}_T)$ to the problem*
$$\begin{gathered} -u_\tau + Lu = 0 \quad \text{in } Q_T, \\ u(x,y,0) = u_0(x,y) \quad \text{on } \Omega, \\ u(x,y,\tau) = g(x,y,\tau) \quad \text{on } \partial\Omega \times (0,T). \end{gathered} \tag{4.4}$$

For completeness we include (below) [17, Theorem 10.4.1].

Theorem 4.7. *Let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{R}^d_+$ and take a $k \in \{1,2,\dots\}$. Let $p \in [1,\infty)$, $m \in \{0,\dots,k\}$ and $q \in (0,\infty)$ be constants such that*
$$k - \frac{d}{p} = m - \frac{d}{q}. \tag{4.5}$$
Then $q \geq p$ and for any $u \in W^k_p(\Omega)$ we have
$$[u]_{W^m_q(\Omega)} \leq N[u]_{W^k_p(\Omega)}, \tag{4.6}$$
with N independent of u. In particular, if,
$$1 - \frac{d}{p} = -\frac{d}{q},$$
That is, (4.5) *is satisfied with $k=1$ and $m=0$, then*
$$\|u\|_{\mathcal{L}_q(\Omega)} \leq N\|u_x\|_{\mathcal{L}_q(\Omega)}. \tag{4.7}$$

We next state and prove our main theorem.

Theorem 4.8. *There exists a strong solution $u \in C^{1+1,0+1}(\overline{Q}_T)$ to the problem*

$$\begin{gathered}
-u_\tau + Lu = \mathfrak{F}_1\Big(y, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}\Big) + \mathfrak{F}_2\Big(y, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}\Big) \quad \text{in } Q_T, \\
u(x,y,0) = u_0(x,y) \quad \text{on } \Omega, \\
u(x,y,\tau) = g(x,y,\tau) \quad \text{on } \partial\Omega \times (0,T).
\end{gathered} \tag{4.8}$$

Proof. Let φ be defined as in Lemma 4.6. We choose $g = \varphi$ and introduce a change of variables to transform our problem into a problem with zero boundary condition. If we let

$$\begin{gathered}
v(x,y,\tau) = u(x,y,\tau) - \varphi(x,y,\tau), \\
v_0(x,y) = u_0(x,y) - \varphi(x,y,0) = 0,
\end{gathered}$$

then v will satisfy the initial-boundary value problem

$$\begin{gathered}
-v_\tau + Lv = \mathfrak{F}_1\Big(y, \frac{\partial(v+\varphi)}{\partial x}, \frac{\partial^2(v+\varphi)}{\partial x^2}, \frac{\partial^2(v+\varphi)}{\partial x \partial y}\Big) \\
+ \mathfrak{F}_2\Big(y, \frac{\partial(v+\varphi)}{\partial x}, \frac{\partial^2(v+\varphi)}{\partial x^2}, \frac{\partial^2(v+\varphi)}{\partial x \partial y}\Big) \quad \text{in } Q_T, \\
v(x,y,0) = 0 \quad \text{on } \Omega, \\
v(x,y,\tau) = 0 \quad \text{on } \partial\Omega \times (0,T)
\end{gathered} \tag{4.9}$$

If the problem (4.9) has a strong solution, then (4.8) will have a strong solution as well since $u = v + \varphi$. We use an iteration procedure to construct the solution to (4.9). Consider the problem

$$\begin{gathered}
-\beta_\tau + L\beta = \mathfrak{F}_1\Big(y, \frac{\partial(\alpha+\varphi)}{\partial x}, \frac{\partial^2(\alpha+\varphi)}{\partial x^2}, \frac{\partial^2(\alpha+\varphi)}{\partial x \partial y}\Big) \\
+ \mathfrak{F}_2\Big(y, \frac{\partial(\alpha+\varphi)}{\partial x}, \frac{\partial^2(\alpha+\varphi)}{\partial x^2}, \frac{\partial^2(\alpha+\varphi)}{\partial x \partial y}\Big) \quad \text{in } Q_T, \\
\beta(x,y,0) = 0 \quad \text{on } \Omega, \\
\beta(x,y,\tau) = 0 \quad \text{on } \partial\Omega \times (0,T),
\end{gathered} \tag{4.10}$$

where $\alpha \in C^{2+\delta,1+\delta/2}(\overline{Q}_T)$ is arbitrary. We can show that (with arguments in [17]):

$$\begin{aligned}
&\mathfrak{F}_1\Big(y, \frac{\partial(\alpha+\varphi)}{\partial x}, \frac{\partial^2(\alpha+\varphi)}{\partial x^2}, \frac{\partial^2(\alpha+\varphi)}{\partial x \partial y}\Big) \\
&+ \mathfrak{F}_2\Big(y, \frac{\partial(\alpha+\varphi)}{\partial x}, \frac{\partial^2(\alpha+\varphi)}{\partial x^2}, \frac{\partial^2(\alpha+\varphi)}{\partial x \partial y}\Big) \in C^{\delta,\delta/2}(\overline{Q}_T).
\end{aligned}$$

Thus, by [17, Theorem 10.4.1], there exists a unique solution $\beta \in C^{2+\delta,1+\delta/2}(\overline{Q}_T)$ to problem (4.10).

Using this result, we can now define $v^n \in C^{2+\delta,1+\delta/2}(\overline{Q}_T)$, $n \geq 1$, the unique solution to the linearized problem (suppressing the arguments y, $\frac{\partial(v^{n-1}+\varphi)}{\partial x}$, $\frac{\partial^2(v^{n-1}+\varphi)}{\partial x^2}$, $\frac{\partial^2(v^{n-1}+\varphi)}{\partial x \partial y}$ for $\mathfrak{F}_1$ and $\mathfrak{F}_2$)

$$\begin{gathered}
-\partial_\tau v^n + Lv^n = \mathfrak{F}_1 + \mathfrak{F}_2 \quad \text{in } Q_T, \\
v^n(x,0) = 0 \quad \text{on } \Omega, \\
v^n(x,\tau) = 0 \quad \text{on } \partial\Omega \times (0,T),
\end{gathered} \tag{4.11}$$

where $v^0 = v_0(x) = 0 \in C^{2+\delta,1+\delta/2}(\overline{Q}_T)$. To prove the existence of a solution to problem (4.9), we will show that this sequence converges.

From [18, Chapter IV.16], there exists a Green's function $G(x,y,\tau,\tau')$ for problem (4.11). For $n \geq 1$, the solution v^n can be written as

$$\begin{aligned} v^n(x,y,\tau) &= \int_0^\tau \int_\Omega G(x,y,z,w,\tau,\tau')(\mathfrak{F}_1 + \mathfrak{F}_2)\,dz\,dw\,d\tau' \\ &\quad + \int_\Omega G(x,y,z,w,\tau,0) v_0(z,w)\,dz\,dw \\ &= \int_0^\tau \int_\Omega G(x,y,z,w,\tau,\tau')(\mathfrak{F}_1 + \mathfrak{F}_2)\,dz\,dw\,d\tau', \end{aligned}$$

because $v_0(z,w) = 0$. Also, due to [18, Theorem 16.3] we have several estimates of the Green's function (see [18, page 413 and 414]). For convenience, we will write

$$\begin{aligned} &\mathcal{F}^{n-1}(z,w,\tau') \\ &= \mathfrak{F}_1\Big(w, \frac{\partial(v^{n-1}+\varphi)}{\partial z}(z,w,\tau'), \frac{\partial^2(v^{n-1}+\varphi)}{\partial z^2}(z,w,\tau'), \frac{\partial^2(v^{n-1}+\varphi)}{\partial z \partial w}(z,w,\tau')\Big) \\ &\quad + \mathfrak{F}_2\Big(w, \frac{\partial(v^{n-1}+\varphi)}{\partial z}(z,w,\tau'), \frac{\partial^2(v^{n-1}+\varphi)}{\partial z^2}(z,w,\tau'), \frac{\partial^2(v^{n-1}+\varphi)}{\partial z \partial w}(z,w,\tau')\Big). \end{aligned}$$

Now we take the first and second derivatives of $v^n(x,y,\tau)$ with respect to x and y. For convenience, we use subscripts $x_1 = x$ and $x_2 = y$ to write derivatives.

$$\begin{aligned} v^n_{x_i}(x,y,\tau) &= \int_0^\tau \int_\Omega G_{x_i}(x,y,z,w,\tau,\tau')\,\mathcal{F}^{n-1}(z,w,\tau')\,dz\,dw\,d\tau', \\ v^n_{x_i x_j}(x,y,\tau) &= \int_0^\tau \int_\Omega G_{x_i x_j}(x,y,z,w,\tau,\tau')\,\mathcal{F}^{n-1}(z,w,\tau')\,dz\,dw\,d\tau'. \end{aligned}$$

with $i,j \in \{1,2\}$.

Using the same procedure as obtained in [22], we have

$$\|v^n(\cdot,\cdot,\tau)\|_{W^2_\infty(\Omega)} \tag{4.12}$$

$$\leq C(T,\gamma) + C\int_0^\tau \Big(A + B(\tau-\tau')^{-\frac{1}{2}} + D(\tau-\tau')^{-\gamma}\Big)\,\|v^{n-1}(\cdot,\cdot,\tau')\|_{W^2_\infty(\Omega)}\,d\tau'. \tag{4.13}$$

Observe that there exist an upper bound (ϵ) of the integral

$$\int_0^\tau \big(A + B(\tau-\tau')^{-\frac{1}{2}} + D(\tau-\tau')^{-\gamma}\big)\,d\tau',$$

for $\tau \in [0,T_1]$, with $T_1 \leq T$, so that $|\epsilon C| < 1$. This is possible as C *does not* depend on T. We choose this T_1 to be the initial time for the next time step. We will follow exactly same computation as below to obtain a solution in the interval and we move on to the next interval (until we reach T). When we solve the problem in the interval $[0,T_1]$ by the method described below we will find a solution given by v. $v(T_1)$ will denote the initial value of the same problem in the next interval. If $v(T_1) \neq 0$, in order to get (4.9) we need to use $v - v(T_1)$ as the new variable. This will lead to a constant term in the right hand side of the first equation in (4.9). But that will not change any other subsequent derivations.

Thus by dividing the interval $[0,T]$ properly we can obtain the required solution. Next, we present the proof of obtaining a solution for the interval $\tau \in [0,T_1]$.

We observe from (4.13) that

$$\|v^n(\cdot,\cdot,\tau)\|_{W^2_\infty(\Omega)} \le C(T,\gamma)\left(1 + C\epsilon + \dots + C^{n-1}\epsilon^{n-1}\right).$$

Since $|\epsilon C| < 1$, we obtain $\|v^n(\cdot,\cdot,\tau)\|_{W^2_\infty(\Omega)} \le \frac{C(T,\gamma)}{1-\epsilon C}$, where $n = 0,1,2,\dots$. Consequently $\|v^n(\cdot,\cdot,\tau)\|_{W^2_\infty(\Omega)}$ is uniformly bounded on the closed interval $[0,T_1]$. Using this result along with (4.9), we can easily show that $\|v^n_\tau(\cdot,\cdot,\tau)\|_{L^\infty(\Omega)}$ is also uniformly bounded on $[0,T]$.

Since $\|v^n(\cdot,\cdot,\tau)\|_{W^2_\infty(\Omega)}$ and $\|v^n_\tau(\cdot,\cdot,\tau)\|_{L^\infty(\Omega)}$ are continuous functions of τ on the closed interval $[0,T_1]$, it follows that $|v^n|$, $|v^n_{x_i}|$, $|v^n_{x_ix_j}|$ and $|v^n_t|$ are uniformly bounded on $\overline{Q}_{[0,T_1]}$. Thus $v^n(\cdot,\cdot,\tau)$ is equicontinuous in $C(\overline{Q}_{[0,T_1]})$. By the Arzelà-Ascoli theorem, there exists a subsequence $\{v^{n_k}\}_{k=0}^\infty$ such that as $k\to\infty$,

$$\begin{gathered} v^{n_k} \to v \in C(\overline{Q}_{[0,T_1]}),\\ v^{n_k}_{x_i} \to v_{x_i} \in C(\overline{Q}_{[0,T_1]}), \end{gathered}$$

where the convergence is uniform. Furthermore, by [6, Theorem 3 in Appendix D],

$$\begin{gathered} v^{n_k}_{x_ix_j} \to v_{x_ix_j} \in L^\infty(\overline{Q}_{[0,T_1]}),\\ v^{n_k}_{\tau} \to v_{\tau} \in L^\infty(\overline{Q}_{[0,T_1]}), \end{gathered}$$

as $k\to\infty$. Here, the convergence is in the weak sense. Therefore, v^{n_k} converges uniformly on the compact set $\overline{Q}_{[0,T_1]}$ to a function $v \in C^{1+1,0+1}(\overline{Q}_{[0,T_1]})$. As mentioned earlier we can extend the solution to $v \in C^{1+1,0+1}(\overline{Q}_T)$ by taking into account all the solutions that we are getting for different (finitely many) *sufficiently small* intervals in $[0,T]$. By a standard argument (see [8, Section 7.4, on page 201]), we have that v satisfies the parabolic equation in (4.9) almost everywhere and the initial-boundary conditions in the classical sense. Hence, v is a strong solution to problem (4.9). Consequently, u is a strong solution to (4.8). □

Now, we show that we can extend this solution to give us a classical solution on the unbounded domain $\mathbb{R}^{2+1}_T = \mathbb{R}^2 \times (0,T)$.

Theorem 4.9. *There exists a classical solution $u \in C^{2,1}(\mathbb{R}^{2+1}_T)$ to the problem*

$$\begin{gathered} -u_\tau + Lu = \mathfrak{F}_1\Big(y, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}\Big) + \mathfrak{F}_2\Big(y, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}\Big) \quad \text{in } \mathbb{R}^{2+1}_T\\ u(x,y,0) = u_0(x,y) \quad \text{on } \mathbb{R}^2 \end{gathered} \tag{4.14}$$

such that the solution $u(x,y,t) \to g(x,y,t)$ as $\sqrt{x^2+y^2}\to\infty$.

Proof. We approximate the domain $\mathbb{R}^2$ by a non-decreasing sequence $\{\Omega_N\}_{N=1}^\infty$ of bounded smooth sub-domains of Ω. For simplicity, we will let $\Omega_N = B(0,N)$ be the open ball in $\mathbb{R}^2$ centered at the origin with radius N. Also, we let $V_N = \Omega_N \times (0,T)$.

Using the previous theorem, we let $u_M \in C^{2,1}(\overline{V}_M)$ be a solution to the problem

$$\begin{gathered} -u_\tau + Lu = \mathfrak{F}_1\Big(y, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}\Big) + \mathfrak{F}_2\Big(y, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}\Big) \quad \text{in } V_M\\ u(x,y,0) = u_0(x,y) \quad \text{on } \Omega_M\\ u(x,y,t) = g(x,y,t) \quad \text{on } \partial\Omega_M \times (0,T). \end{gathered} \tag{4.15}$$

Since $M \ge 1$ is arbitrary, we can use a standard diagonal argument (for details see [23, Theorem 2.1]) to extract a subsequence that converges to a solution u to the problem on the whole unbounded space $\mathbb{R}_T^{2+1}$. Clearly, $u(x,y,0) = u_0(x,y)$ and $u(x,y,t) \to g(x,y,t)$ as $\sqrt{x^2+y^2} \to \infty$. $\square$

We would like to remark that the existence proof provided is constructive. The sequence used in the proof to show convergence to a solution may also be used to approximate the solution numerically. Specifically, the sequence (4.11) started from the boundary condition will converge to the solution of the nonlinear PDE (4.1).

Conclusions

In this article we analyze a market model where the assets are driven by stochastic volatility models and trading assets involves paying proportional transaction costs. We show that the price of an option written on this type of equity may be obtained as a solution to a partial differential equation. We obtain the option pricing PDE for the scenario when the volatility (or a proxy for volatility) is a traded asset. In this case all option prices may be found as solutions to the resulting nonlinear PDE. Furthermore, hidden within this scenario is the case when the option depends on two separate assets and the assets are correlated in the same form as S and σ are in the current paper. The treatment of the option in this case is entirely equivalent with the case discussed in this article.

5. Appendix: Derivation of the option value PDE's in arbitrage free and complete markets

In this appendix we present the correct derivation of portfolio dynamics used when deriving the PDE's (3.2) and (3.8). Suppose that we want to price a claim V which at time t is dependent on S, σ and t. We note that the same approach works if the contingent claim V is contingent on any set of n traded assets $S_1(t), \dots, S_n(t)$, but for clarity we use the specific case presented in this paper.

The market contains two traded assets $S(t)$ and $\sigma(t)$ which have some specific dynamics irrelevant to this derivation, as well as a risk free account that earns the risk free interest rate r. This risk-free account is available from the moment $t = 0$ when the portfolio is constructed. Specifically, one share of this money market account solves:

$$dM(t) = rM(t)dt \quad \text{or} \quad M(t) = e^{rt}$$

Suppose we form a portfolio (any portfolio) containing shares in these assets and in the money account. Divide the interval $[0,t]$ into intervals with endpoints $0 = t_0 < t_1 < \dots < t_N = T$ and for simplicity assume that the times are equally spaced at intervals δt wide. Suppose that at one of these times t_k our portfolio has value:

$$X(k) = \Delta(k)S(k) + \Delta_1(k)\sigma(k) + \Gamma(k)M(k),$$

where $\Delta(k)$ and $\Delta_1(k)$ are the number of shares of respective assets, while $\Gamma(k)$ is the number of shares of the riskless asset we own.

Suppose that at the next time t_{k+1} we need to re-balance this portfolio to contain exactly some other weights $\Delta(k+1)$ and $\Delta_1(k+1)$. To this purpose, we need to trade the assets and thus we pay transaction costs depending on the differences of the type $\Delta(k+1) - \Delta(k)$ as well as on the price of the specific asset traded.

Here we make the assumption that the portfolio is self financing. This means that any extra or missing monetary value resulting from re-balancing the portfolio will be put or borrowed from the money account. Mathematically, we need to have the following two quantities equal:

$$\begin{aligned} X_{k+1} &= \Delta(k)S(k+1) + \Delta_1(k)\sigma(k+1) + \Gamma(k)M(k+1) \\ X(k+1) &= \Delta(k+1)S(k+1) + \Delta_1(k+1)\sigma(k+1) + \Gamma(k+1)M(k+1) \\ &\quad - \nu(k+1). \end{aligned}$$

In this expression all transaction costs incurred at time t_{k+1} are lumped into the term $\nu(k+1)$. Setting the two quantities equal and rearranging the terms gives

$$\begin{aligned} &(\Delta(k+1) - \Delta(k))S(k+1) + (\Delta_1(k+1) - \Delta_1(k))\sigma(k+1) \\ &+ (\Gamma(k+1) - \Gamma(k))M(k+1) - \nu(k+1) = 0 \end{aligned}$$

In this equation we add and subtract $S(k)(\Delta(k+1) - \Delta(k))$ and $\sigma(k)(\Delta_1(k+1) - \Delta_1(k))$, which gives the following self-financing condition:

$$\begin{aligned} &S(k)(\Delta(k+1) - \Delta(k)) + (\Delta(k+1) - \Delta(k))(S(k+1) - S(k)) \\ &+ \sigma(k)(\Delta_1(k+1) - \Delta_1(k)) + (\Delta_1(k+1) - \Delta_1(k))(\sigma(k+1) - \sigma(k)) \\ &+ (\Gamma(k+1) - \Gamma(k))M(k) + (\Gamma(k+1) - \Gamma(k))(M(k+1) - M(k)) - \nu(k+1) = 0 \end{aligned} \tag{5.1}$$

The next step requires some explanation. We plan to sum these expressions over k and to take the mesh of partition $\max_k |t_{k+1} - t_k|$ to converge to zero. However, we need to deal with the transaction costs term. If the re-balancing length of the interval goes to 0 then the transaction costs become infinite. This is why it is important to realize that the actual re-balancing needs to be done at fixed points in time length δt apart. Because of this, the limit is an approximation of the PDE dynamic.

Taking the limit while at the same time bounding the transaction costs we obtain stochastic integrals for all these expressions. Expressing the integrals in differential form for compactness sake we obtain the continuous time self-financing condition

$$S(t)d\Delta(t) + d\langle \Delta, S\rangle_t + \sigma(t)d\Delta_1(t) + d\langle \Delta_1, \sigma\rangle_t + M(t)d\Gamma(t) + d\langle \Gamma, M\rangle_t - \nu(\delta t) \approx 0 \tag{5.2}$$

For no transaction costs (last term zero), this condition is exact for any portfolio which is self financing and for any stochastic dynamics of the weights and assets. In the presence of transaction costs we need to bound the total transaction costs over the interval δt when the subintervals length aproach 0. For this reason the equation (5.2) is only approximately satisfied when dealing with transaction costs. We used the notation $\nu(\delta t)$ to bound the transaction costs over the interval δt. The expected value of this term will be calculated in the paper.

The condition 5.2 is valid for any self-financing portfolios. Next we form a specific portfolio, one that will replicate the payoff of the contingent claim V at time T. Such a portfolio involves stochastic weights and has the form

$$\Pi(t) = V(t) - \Delta(t)S(t) - \Delta_1(t)\sigma(t),$$

since we replicate using only the underlying assets S and σ. The weights are stochastic and they are suitably chosen to replicate the contingent claim V. The dynamics of the portfolio $\Pi(t)$ may be derived using the Itô's lemma properly as

$$\begin{aligned} d\Pi(t) &= dV(t) - \Delta(t)dS(t) - S(t)d\Delta(t) - d\langle \Delta, S\rangle_t \\ &\quad - \Delta_1(t)d\sigma(t) - \sigma(t)d\Delta_1(t) - d\langle \Delta_1, \sigma\rangle_t \end{aligned} \tag{5.3}$$

Now, the idea is that the later two terms will disappear when we use the self financing condition and we will obtain the equations presented in the paper. More specifically, since the suitable choices of Δ terms allow us to replicate the option value $V(t)$, the amount in the money account at time t is exactly $\Pi(t)$. Therefore, the number of shares held in the money account at any time is $\Gamma(t) = \frac{\Pi(t)}{M(t)}$. The last two troublesome terms in (5.3) are substituted using the self financing condition (5.2). When we do so, note that we need to calculate the terms $M(t)d\Gamma(t)+d\langle\Gamma, M\rangle_t$ for this specific $\Gamma(t) = \frac{\Pi(t)}{M(t)}$. Here it pays to know that $M(t)$ as well as $1/M(t) = e^{-rt}$ are deterministic and therefore the terms $d\langle\Pi/M, M\rangle_t$ and $d\langle\Pi, 1/M\rangle_t$ vanish in the resulting expression. Furthermore:

$$M(t)d\Gamma(t) = M(t)d\frac{\Pi(t)}{M(t)} = d\Pi(t) + \Pi(t)(-r)dt + M(t)d\langle\Pi, \frac{1}{M}\rangle_t,$$

and as mentioned the last term is zero. After we perform the calculations and we cancel the terms which are the same we end up with the expression

$$dV(t) - \sum_{i=1}^{n} \Delta_i(t)dS_i(t) - r\Pi(t)dt - \nu(\delta t) = 0. \tag{5.4}$$

This is the expression we use in this paper.

The next step, as presented in section 3, is to use suitable replicating weights to make the stochastic integrals disappear by equating all the terms multiplying dt which gives the PDE's used in the article.

The derivation presented in the Appendix is clearly valid if we use discrete time. However, the self-financing condition needs to hold at all times when we re-balance the portfolio. At the same time we cannot re-balance at every continuous time thus the resulting continuous time equation is questionable. In this appendix we used the continuous time condition (5.2) instead of the discrete one (5.1) simply due to the convenience of working with Itô's lemma and thus vanishing quadratic variations in (5.3), instead of higher dt terms in the Taylor expansion. However, if we go the long route and replace (5.2) with its discrete counterpart from the proof of Itô's lemma the same terms as in the continuous version will also disappear in the discrete time expression. However, since we do not re-balance inside intervals of width δt there are no further transaction costs while taking sub partitions of these original intervals. Therefore when the mesh of the sub partitions is converging to 0 we will obtain the final equation (5.4).

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Ionuţ Florescu
Financial Engineering Division and Hanlon Financial Systems Lab, Stevens Institute of Technology, Castle Point on Hudson, Hoboken, NJ 07030, USA
E-mail address: `ifloresc@stevens.edu`

Maria C. Mariani
Department of Mathematical Sciences, University of Texas at El Paso Bell Hall 124, El Paso, TX 79968-0514, USA
E-mail address: `mcmariani@utep.edu`

Indranil SenGupta
Department of Mathematics, North Dakota State University, NDSU Dept # 2750, Minard Hall 408E12, Fargo, ND 58108-6050, USA
E-mail address: indranil.sengupta@ndsu.edu