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EXISTENCE AND NON-EXISTENCE OF GLOBAL SOLUTIONS FOR A SEMILINEAR HEAT EQUATION ON A GENERAL DOMAIN

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Abstract. We consider the parabolic problem $u_t - \Delta u = h(t)f(u)$ in $\Omega\times(0,T)$ with a Dirichlet condition on the boundary and $f, h \in C[0,\infty)$. The initial data is assumed in the space $\{u_0 \in C_0(\Omega); u_0 \geq 0\}$, where Ω is a either bounded or unbounded domain. We find conditions that guarantee the global existence (or the blow up in finite time) of nonnegative solutions.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be either a bounded or unbounded domain with smooth boundary. Meier [11] considered the blow up phenomenon of the solutions of the parabolic problem

$$\begin{gathered} u_t - Lu = h(x,t)f(u) \quad \text{in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 \geq 0 \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where

$$L = \sum_{i,j=1}^{N} a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x,t)\frac{\partial}{\partial x_i}$$

is an uniformly elliptic operator in Ω with bounded coefficients $a_{ij} = a_{ji}$ and h is a continuous function with $h(\cdot,t)$ bounded. The assumptions on the functions f are the following:

$$f \in C^1[0,\infty); \quad f(s) > 0 \text{ for } s > 0; \quad f(0) \geq 0; \quad f' \geq 0; \tag{1.2}$$

$$G(w) = \int_w^\infty \frac{d\sigma}{f(\sigma)} < \infty \quad \text{if } w > 0. \tag{1.3}$$

When $h(x,t) = h(t)$ we have the following result which follows from [11, Theorem 2]. In this article, we denote by $(S(t))_{t\geq 0}$ the heat semigroup with the homogeneous Dirichlet condition on the boundary.

Theorem 1.1 ([11]). *Assume that f satisfies conditions* (1.2) *and* (1.3) *and* $h(x,\cdot) = h(\cdot) \in C[0,\infty)$.

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(i) *Let f be convex with $f(0)=0$. Then the solution u of* (1.1) *blows up in finite time, if there exists $\tau>0$ such that*
$$G(\|S(\tau)u_0\|_\infty) \le \int_0^\tau h(\sigma)d\sigma. \tag{1.4}$$
(ii) *Let $f(0)>0$. If there exists $\tau>0$ such that*
$$G(0) \le \|S(\tau)u_0\|_\infty \int_0^\tau \frac{h(\sigma)}{\|S(t)u_0\|_\infty} d\sigma, \tag{1.5}$$
then the solution of (1.1) *blows up in finite time.*

Meier [10] also considered the semilinear parabolic equation
$$\begin{gathered} u_t - \Delta u = h(t)u^p \quad \text{in } \Omega\times(0,T), \\ u=0 \quad \text{in } \partial\Omega\times(0,T), \\ u(0)=u_0\ge 0 \quad \text{in } \Omega, \end{gathered} \tag{1.6}$$
where $h\in C[0,\infty)$, $p>1$ and $u_0\in L^\infty(\Omega)$. He studied the existence of the Fujita critical exponent p^* of (1.6), that is, a number such that if $1<p\le p^*$, then any nontrivial solution of problem (1.6) blows up in finite time, and if $p>p^*$, then there exists a nontrivial global solution of problem (1.6).

Determining the value of the Fujita critical for problem (1.6) and its extensions has been objective of research of many authors, see for instance [2, 3, 8, 9, 10, 11, 14, 15]. Below we list some values of p^*, which depend of the domain Ω and the function h. For instance,

(i) If $\Omega=\mathbb{R}^N$ and $h=1$, then Fujita's result in [3] means that $p^*=1+2/N$;
(ii) If $\Omega=R^N_k=\{x; x_i>0, i=1,...,k\}$ and $h(t)\sim t^q$ for t large(i.e. there exist constants $c_0,c_1>0$ such that $c_0t^q\le h(t)\le c_1t^q$ for t large) and $q>-1$, then $p^*=1+2(q+1)/(N+k)$, see [11];
(iii) If Ω bounded and $h(t)\sim e^{\beta t}$ for t large, $\beta>0$, then $p^*=1+\beta/\lambda_1$, where λ_1 is the first Dirichlet eigenvalue of the Laplacian in Ω, see [10].

The results above can be obtained from the following general theorem, using only of the asymptotic behavior of the solution $u(t)=S(t)u_0$, $t\ge 0$, of the linear problem $u_t-\Delta u=0$, in $\Omega\times(0,\infty)$ and the function h.

Theorem 1.2 ([10]). *Let $p>1$, $h\in C[0,\infty)$.*
(i) If there exists $u_0\in L^\infty(\Omega)$, $u_0\ge 0$ such that
$$\int_0^\infty h(\sigma)\|S(\sigma)u_0\|_\infty^{p-1} d\sigma < \infty, \tag{1.7}$$
then there exists a global solution of (1.6) *with* $\lim_{t\to\infty}\|u(t)\|_\infty = 0$.
(ii) If
$$\limsup_{t\to\infty} \|S(t)u_0\|^{p-1} \int_0^t h(\sigma)d\sigma = \infty \tag{1.8}$$
for all $u_0\in L^\infty(\Omega), u_0\ge 0$, then every nontrivial nonnegative solution of (1.6) *blows up in finite time.*

Condition (1.7), was used by Weissler [14], when $h=1$ and $\Omega=\mathbb{R}^N$, to find a non negative global solution of (1.6). This is clear since we can choose a_0 so that

$\overline{u}(t) = a(t)S(t)u_0$, where

$$a(t) = \Big[a_0^{-(p-1)} - (p-1)\int_0^t h(\sigma)\|S(\sigma)u_0\|_\infty^{p-1} d\sigma\Big]^{-1/(p-1)},$$

is a supersolution of (1.6) defined for all $t \geq 0$.

In this work we are interested in the parabolic problem

$$\begin{gathered} u_t - \Delta u = h(t)f(u) \quad \text{in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 \geq 0 \quad \text{in } \Omega, \end{gathered} \tag{1.9}$$

where $h \in C[0,\infty)$, $f \in C[0,\infty)$ is a locally Lipschitz function and $u_0 \in C_0(\Omega)$.

Firstly, we are interested in finding conditions that guarantee the global existence of solutions of problem (1.9). In particular, we would like obtain a similar condition to Theorem 1.1(i). In second place, we are interested in the blow up in finite time of nonnegative solutions of (1.9) assuming only f locally Lipschitz, that is, without condition (1.2).

It is well known that if f is locally Lipschitz, $f(0) = 0$ and $u_0 \in C_0(\Omega)$, $u_0 \geq 0$, problem (1.9) has a unique nonnegative solution $u \in C([0,T_{\max}), C_0(\Omega))$ defined in the maximal interval $[0,T_{\max})$ and verifying the equation

$$u(t) = S(t)u_0 + \int_0^t S(t-\sigma)h(\sigma)f(u(\sigma))d\sigma, \tag{1.10}$$

for all $t \in [0,T_{\max})$. Moreover, we have the blow up alternative: either $T_{\max} = \infty$(global solution) or $T_{\max} < \infty$ and $\lim_{t\to T_{\max}} \|u(t)\|_\infty = \infty$ (blow up solution). Throughout this work a nonnegative function $u \in C([0,T), C_0(\Omega))$ is said to be a solution of (1.9) in a interval $[0,T)$ if satisfies equation (1.10).

Our first result is about the existence of a global solution of problem (1.9).

Theorem 1.3. *Assume that f is locally Lipschitz and $f(0) = 0$. Suppose that there exists $a > 0$ such that the functions f and $g : (0,\infty) \to [0,\infty)$, defined by $g(s) = f(s)/s$, are nondecreasing in $(0,a]$. If $v_0 \in C_0(\Omega)$, $v_0 \geq 0, v_0 \neq 0, \|v_0\|_\infty \leq a$ verifies*

$$\int_0^\infty h(\sigma)g(\|S(\sigma)v_0\|_\infty)d\sigma < 1, \tag{1.11}$$

then there exists $u_0^* \in C_0(\Omega)$, $0 \leq u_0^* \leq v_0$ such that for any $u_0 \in C_0(\Omega)$ $0 \leq u_0 \leq u_0^*, u_0 \neq 0$ the solution of (1.9) is a global solution. Moreover, there exists a constant $\gamma > 0$ so that $u(t) \leq \gamma \cdot S(t)u_0$ for all $t \geq 0$. In particular, $\lim_{t\to\infty} \|u(t)\|_\infty = 0$.

Remark 1.4. (i) In Theorem 1.3 we assume that g is nondecreasing in some interval $(0,a]$. This condition is verified, for instance, if f is a convex function. An analogous condition on g was used also in [10, Theorem 7], but there it is assumed that $f(0) = f'(0) = 0$ and $\Omega = \mathbb{R}_k^N$.

(ii) If $f(t) = t^p$ for all $t \geq 0$ and $p > 1$, we have that $G(w) = w^{1-p}/(p-1)$ and $g(s) = s^{p-1}$. Thus, condition (1.11) reduces to condition (1.7).

Our second result is the following.

Theorem 1.5. *Let f be a locally Lipschitz function, $f(0) = 0$, $f(s) > 0$ for all $s > 0$ and G given by (1.3). Assume that the following conditions are satisfied:*

(i) *The function f is nondecreasing and verifies the following property*
$$f(S(t)v_0) \le S(t)f(v_0), \tag{1.12}$$
for all $v_0 \in C_0(\Omega), v_0 \ge 0$ and $t > 0$.

(ii) *There exist $\tau > 0$ and $u_0 \in C_0(\Omega)$, $u_0 \ge 0, u_0 \ne 0$ such that*
$$G(\|S(\tau)u_0\|_\infty) \le \int_0^\tau h(\sigma)d\sigma. \tag{1.13}$$

Then the solution of problem (1.9) *blows up in finite time $T_{\max} \le \tau$.*

Remark 1.6. Regarding Theorem 1.5 we have the following comments:

(i) By the positivity of the heat semigroup, we have that $S(t)v_0 \ge 0$ if $v_0 \ge 0$. Hence, the left side of (1.12) is well defined.

(ii) If f is a convex function and $\Omega = \mathbb{R}^N$, then (1.12) holds. It is clear, by Jensen's inequality since $S(t)u_0 = k_t \star u_0$, where k_t is a heat kernel.

(iii) If f is twice differentiable and convex, then (1.12) holds. Indeed, if $w(t) = f(S(t)v_0)$, then $w_t - \Delta w = -f''(S(t)v_0)|\nabla S(t)v_0|^2 \le 0$. We then conclude using the maximum principle.

Theorem 1.3 is proved using a monotone sequence method, see [12, 14]. Our arguments for proving Theorem 1.5 are different to the arguments in Meier. Precisely, Meier uses the subsolutions method for problem (1.1), whereas we use the formulation (1.10) to get an ordinary differential inequality, see inequality (2.3).

We now apply our results to the heat equation with logarithmic nonlinearity
$$\begin{gathered} u_t - \Delta u = h(t)(1+u)[\ln(1+u)]^q \quad \text{in } \mathbb{R}^N \times (0,T), \\ u(0) = u_0 \ge 0 \quad \text{in } \mathbb{R}^N, \end{gathered} \tag{1.14}$$
where $q > 1$ and $h : [0,\infty) \to [0,\infty)$ is a continuous function.

Problem (1.14) was introduced in [5], is a particular case of more general quasilinear models with common properties of convergence to Hamilton-Jacobi equations studied in [4], where the asymptotic of global in time solutions were established. For the mathematical theory of blow-up, see [6] and the references therein. We have the following result.

Theorem 1.7. *Assume that $q > 1$, $h : [0,\infty) \to [0,\infty)$ is a continuous function such that $h(t) \sim t^r$ for t large enough and $r > -1$.*

(i) If $1 < q < 1 + \frac{2}{N}(r+1)$, then every nontrivial solution of (1.14) *blows up in finite time.*

(ii) If $q > 1 + \frac{2}{N}(r+1)$, there exists $u_0 \in C_0(\mathbb{R}^N)$, $u_0 \ne 0, u_0 \ge 0$ so that the solution of (1.14) *is a global solution.*

We also apply our results to the exponential reaction model
$$\begin{gathered} u_t - \Delta u = h(t)[\exp(\alpha u) - 1] \quad \text{in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,T), \\ u(0) = u_0 \ge 0 \quad \text{in } \Omega, \end{gathered} \tag{1.15}$$
with $\alpha > 0$, $h \in C[0,\infty)$ and Ω a bounded domain with smooth boundary. These problems are important in combustion theory [16] under the name of solid-fuel model (Frank-Kamenetsky equation).

Theorem 1.8. *Let $\alpha > 0$ and $h \in C[0,\infty)$.*

(i) *If there exists* $\tau > 0$ *such that* $\int_0^\tau h(\sigma)d\sigma \geq 1/\alpha$, *then there exists* $u_0 \in C_0(\Omega), u_0 \geq 0$ *so that the solution of problem* (1.15) *blows up in finite time.*
(ii) *If* $\int_0^\infty h(\sigma)d\sigma < 1/\alpha$, *then there exists* $u_0 \in C_0(\Omega), u_0 \geq 0$ *such that the solution of problem* (1.15) *is global.*

2. Proof of the main results

Lemma 2.1. *Assume* $h, f : [0,\infty) \to [0,\infty)$ *with* h *continuous,* f *locally Lipschitz and nondecreasing. Let* $u, v \in C([0,T], C_0(\Omega))$ *be solutions of problem* (1.9)*(in the sense of* (1.10)*) with* $u(0) = u_0 \geq 0$ *and* $v(0) = v_0 \geq 0$. *If* $u_0 \leq v_0$, *then* $u(t) \leq v(t)$ *for all* $t \in [0,T]$.

Proof. Let $M = \max\{\|u(t)\|_\infty, \|v(t)\|_\infty; t \in [0,T]\}$. Since $u_0 \leq v_0$ we have

$$u(t) - v(t) \leq \int_0^t S(t-\sigma)h(\sigma)[f(u(\sigma)) - f(v(\sigma))]d\sigma. \tag{2.1}$$

On the other hand, since $u \leq u^+$, f is nondecreasing and locally Lipschitz, we have

$$[f(u) - f(v)] \leq [f(u) - f(v)]^+ \leq L_M(u-v)^+,$$

where L_M is the Lipschitz constant in $[0,M]$. Thus, it follows from inequality (2.1) that

$$\|[u(t) - v(t)]^+\|_\infty \leq L_M \int_0^t h(\sigma)\|[u(\sigma) - v(\sigma)]^+\|_\infty.$$

The conclusion follows from Gronwall's inequality. □

Proof of Theorem 1.5. We adopt the argument used in the proof of[13, Lemma 15.6]. Assume that u is a global solution and let $0 < t \leq s$. It follows from (1.10) and (1.12) that

$$\begin{aligned} S(s-t)u(t) &= S(s)u_0 + \int_0^t S(s-\sigma)h(\sigma)f(u(\sigma))d\sigma \\ &\geq S(s)u_0 + \int_0^t h(\sigma)f(S(s-\sigma)u(\sigma))d\sigma. \end{aligned} \tag{2.2}$$

Set $\psi(t) = S(s)u_0 + \int_0^t h(\sigma)f(S(s-\sigma)u(\sigma))d\sigma$. Since f is nondecreasing, it follows from (2.2) that

$$\psi'(t) = h(t)f(S(s-t)u(t)) \geq h(t)f(\psi(t)). \tag{2.3}$$

Hence, it follows that if $\Psi(t) = \int_t^\infty \frac{d\sigma}{f(\sigma)}$ for all $t > 0$, then

$$\frac{d}{dt}(\Psi(\psi(t))) = -\frac{\psi'(t)}{f(\psi(t))} \leq -h(t).$$

Thus,

$$\int_0^s h(\sigma)d\sigma \leq \Psi(\psi(0)) - \Psi(\psi(s)) = \int_{\psi(0)}^{\psi(s)} \frac{d\sigma}{f(\sigma)} < \int_{S(s)u_0}^\infty \frac{d\sigma}{f(\sigma)} = G(S(s)u_0)$$

for every $s > 0$. This fact, contradicts inequality (1.13). □

Proof of Theorem 1.3. We use the monotone sequence argument (see [12, 14]). Since $\int_0^\infty h(\sigma)g(\|S(\sigma)v_0\|_\infty)d\sigma < 1$, there exists $\beta > 0$ such that

$$\int_0^\infty h(\sigma)g(\|S(\sigma)v_0\|_\infty) < \frac{\beta}{\beta+1} < 1. \tag{2.4}$$

Set

$$0 < \lambda < \frac{1}{\beta + 1} \min\big\{1, \frac{a}{\|v_0\|_\infty}\big\}. \tag{2.5}$$

From Lemma 2.1, it suffices to show that the corresponding solution u of (1.9) with $u(0) = u_0^* = \lambda v_0$ is global.

We define a sequence $(u^n)_{n\geq 1}$ by $u^0 = S(t)u_0^*$ and

$$u^n(t) = S(t)u_0^* + \int_0^t S(t-\sigma)h(\sigma)f(u^{n-1}(\sigma))d\sigma, \tag{2.6}$$

for $n \in \mathbb{N}$ and all $t \geq 0$.

Now, we claim that

$$u^n(t) \leq (1+\beta)S(t)u_0^*, \tag{2.7}$$

for all $t \geq 0$. We argue by induction on n. It is clear that (2.7) holds for $n = 0$. Assume now that inequality (2.7) holds. It follows from (2.5) and (2.7) that

$$\|u^n(t)\|_\infty \leq \lambda(1+\beta)\|v_0\|_\infty < a. \tag{2.8}$$

So, since $(1+\beta)S(t)u_0^* = \lambda(1+\beta)S(t)v_0 \leq \|S(t)v_0\|_\infty \leq a$ and g is nondecreasing in $(0, a)$ we have

$$\begin{aligned}
u^{n+1}(t) &\leq S(t)u_0^* + \int_0^t S(t-\sigma)h(\sigma)f((1+\beta)S(\sigma)u_0^*)d\sigma \\
&\leq S(t)u_0^* + \int_0^t h(\sigma)S(t-\sigma)\left\{(1+\beta)g[(1+\beta)\lambda S(\sigma)v_0]S(\sigma)u_0^*\right\}d\sigma \\
&\leq S(t)u_0^* + (1+\beta)\int_0^t h(\sigma)S(t-\sigma)[g(\|S(\sigma)v_0\|_\infty)S(\sigma)u_0^*]d\sigma \\
&\leq S(t)u_0^* + (1+\beta)S(t)u_0^*\int_0^t h(\sigma)g(\|S(\sigma)v_0\|_\infty)d\sigma.
\end{aligned}$$

It follows from (2.4) that u^{n+1} verifies inequality (2.7).

On the other hand, since u^n verifies inequality (2.8) and f is nondecreasing on $(0, a]$, we can prove using induction that $u^n \leq u^{n+1}$ for all $n \in \mathbb{N}$. Therefore, if $u(t) = \lim u^n(t)$ for all $t \geq 0$, from monotone convergence theorem and (2.6), we conclude that u is a global solution of (1.9). □

Proof of Theorem 1.7. Let $f : [0,\infty) \to [0,\infty)$ defined by

$$f(s) = (1+s)[\ln(1+s)]^q, \tag{2.9}$$

for all $s \geq 0$. Then $f''(s) > 0$ for all $s > 0$. By Remark 1.6(iii), condition (1.12) is verified. Set $G(w) = \int_w^\infty \frac{ds}{(s+1)[\ln(1+s)]^q} = \frac{[\ln(1+w)]^{1-q}}{q-1}$. From here,

$$[G(\|S(t)u_0\|_\infty)]^{-1}\int_0^t h(\sigma)d\sigma = (q-1)[\ln(1+\|S(t)u_0\|_\infty)]^{q-1}\int_0^t h(\sigma)d\sigma. \tag{2.10}$$

To verify condition (1.13), we use the following result, which follows directly from L'Hôpital's rule:

$$\lim_{t\to\infty}\frac{\ln(1+c_0t^{-\beta})}{t^{-\alpha}} = \begin{cases} (c_0\beta)/\alpha & \text{if } \alpha = \beta, \\ 0 & \text{if } \beta > \alpha, \\ \infty & \text{if } \beta < \alpha, \end{cases} \tag{2.11}$$

for $\alpha, \beta, c_0 > 0$. From [7](Lemma 2.12), we know that $\|S(t)u_0\|_\infty \geq c_0 t^{-N/2}$ for t large and $u_0 \in C_0(\mathbb{R}^N), u_0 \geq 0, u_0 \neq 0$. Therefore, it follows from (2.10) and (2.11) that if $h(t) \geq c_1 t^r, r > -1$, for t large enough then there exists a constant $c > 0$ so that

$$\begin{aligned}
[G(\|S(t)u_0\|_\infty)]^{-1} \int_0^t h(\sigma)d\sigma &\geq c[\ln(1 + c_0 t^{-\frac{N}{2}})]^{q-1} t^{r+1} \\
&\geq c(c_0 t^{-\frac{N}{2}})^{q-1} t^{r+1} > 1,
\end{aligned}$$

if $q < 1 + \frac{2}{N}(r+1)$. Hence, condition (1.13) is verified and the conclusion follows of Theorem 1.5.

We now analyze global existence using Theorem 1.3. It is clear that f and $g(s) = f(s)/s$, where f is given by (2.9) are nondecreasing functions. Let $\psi \in C_0(\mathbb{R}^N)$ with $\|\psi\|_\infty = 1$. From [7](Lemma 2.12) there exists $c_1, t_0 > 0$ such that

$$\|S(t)\psi\|_\infty \leq c_1 t^{-N/2}, \tag{2.12}$$

for all $t \geq t_0$. Let $\epsilon > 0$ so that $1 + r - \frac{N}{2}(q-1) + \epsilon q < 0$. From (2.11) there exists $t_1 > 0$ such

$$\ln(1 + c_1 t^{-N/2}) \leq t^{N/2-\epsilon}, \tag{2.13}$$

for all $t \geq t_1$. Let $t_2 > 0$ such that

$$h(t) \leq c_2 t^r, \tag{2.14}$$

for all $t \geq t_2$ and fix $t_3 > \max\{1, t_0, t_1, t_2\}$ satisfying

$$c_4 t_3^{1+r-\frac{N}{2}(q-1)+\epsilon q} < \frac{1}{2}, \tag{2.15}$$

where $c_4 = c_3 c_2/[N(q-1)/2 - r - 1 - \epsilon q] > 0$ and $c_3 = (1 + 1/c_1)$.

Consider $v_0 = \mu\psi$ with $0 < \mu \leq 1$ and

$$c_5(t_3) g(\mu) < \frac{1}{2}, \tag{2.16}$$

where $c_5(t_3) = \int_0^{t_3} h(\sigma)d\sigma$. This fact is possible because $\lim_{\mu \to 0^+} g(\mu) = 0$.

It follows of (2.12) that $\|S(t)v_0\|_\infty \leq c_1 \mu t^{-N/2} \leq c_1 t^{-N/2}$ for all $t \geq t_0$. Thus, $g(\|S(t)v_0\|_\infty) \leq g(c_1 t^{-N/2})$ for all $t \geq t_0$. Hence, by (2.13) - (2.16) we have

$$\begin{aligned}
&\int_0^\infty h(\sigma) g(\|S(\sigma)v_0\|_\infty) d\sigma \\
&\leq g(\|v_0\|_\infty) \int_0^{t_3} h(\sigma)d\sigma + \int_{t_3}^\infty h(\sigma) g(c_1 \sigma^{-N/2}) d\sigma \\
&\leq g(\mu) \int_0^{t_3} h(\sigma)d\sigma + \int_{t_3}^\infty h(\sigma)(1 + \frac{1}{c_1 \sigma^{-N/2}})[\ln(1 + c_1 \sigma^{-N/2})]^q d\sigma \\
&< \frac{1}{2} + c_3 \int_{t_3}^\infty h(\sigma) \sigma^{N/2} [\ln(1 + c_1 \sigma^{-N/2})]^q d\sigma \\
&\leq \frac{1}{2} + c_3 c_2 \int_{t_3}^\infty \sigma^r \sigma^{N/2} \sigma^{-(N/2-\epsilon)q} d\sigma \\
&\leq \frac{1}{2} + c_4 {t_3}^{1+r-\frac{N}{2}(q-1)+\epsilon q} < 1.
\end{aligned}$$

Therefore, estimate (1.11) is satisfied. $\square$

Remark 2.2. We can see from (2.10) (fixing t), that if $u_0 = \lambda\psi$ with $\psi \in C_0(\mathbb{R}^N), \psi \geq 0, \psi \neq 0$, then condition (1.12) is satisfied when $\lambda > 0$ is large. In other words, if initial data is large enough, then the corresponding solution of problem (1.14) blows up in finite time.

Proof of Theorem 1.8. (i) Note that

$$G(w) = \int_w^\infty \frac{d\sigma}{\exp(\alpha\sigma) - 1} = -\frac{1}{\alpha}\ln[1 - \exp(-\alpha w)].$$

Let $w_0 > 0$ such that $\ln(1 - \exp(-\alpha w_0)) = -1$. Set $u_0 = \lambda\varphi_1$, where $\lambda \geq w_0 e^{\lambda_1 \tau}$ and φ_1 is the first eigenfunction associated to first eigenvalue λ_1 of the Laplacian with Dirichlet condition on the boundary $\partial\Omega$. We suppose that $\|\varphi_1\|_\infty = 1$. Hence, $\|S(\tau)u_0\|_\infty = \lambda e^{-\lambda_1\tau} \geq w_0$. Thus, $G(\|S(\tau)u_0\|_\infty) \leq G(w_0) \leq \int_0^\tau h(\sigma)d\sigma$. From Theorem 1.5, the result follows.

(ii) We use Theorem 1.3. Let $g(s) = \frac{e^{\alpha s}-1}{s}$ for all $s > 0$ and let $\epsilon > 0$ so that $\int_0^\infty h(\sigma)d\sigma < 1/(\alpha + \epsilon)$. Since $\lim_{s\to 0^+} g(s) = \alpha$, there exist $s_0 > 0$ such that $g(s) < \alpha + \epsilon$ for all $0 < s < s_0$. Moreover, g is nondecreasing in $(0, \infty)$.

It follows that if $v_0 \in C_0(\Omega), v_0 \geq 0, v_0 \neq 0$ with $\|v_0\|_\infty < s_0$, then

$$\int_0^\infty h(\sigma)g(\|S(\sigma)v_0\|_\infty)d\sigma \leq (\alpha + \epsilon)\int_0^\infty h(\sigma)d\sigma < 1.$$

So, estimate (1.11) is verified. $\square$

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