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ALMOST PERIODIC SOLUTIONS OF ANISOTROPIC ELLIPTIC-PARABOLIC EQUATIONS WITH VARIABLE EXPONENTS OF NONLINEARITY

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Abstract. We prove the well-posedness of Fourier problems for anisotropic elliptic-parabolic equations with variable exponents of nonlinearity without any restrictions at infinity. We obtain estimates of the weak solutions and conditions for the existence of periodic and almost periodic solutions. In addition, some properties of the weak solutions of the Fourier problem are considered.

1. Introduction

We examine the question of well-posedness of the Fourier problem (the problem without initial conditions) for anisotropic second order elliptic-parabolic equations with variable exponents of nonlinearity. These equations are defined on unbounded cylindrical domains which are the Cartesian products of bounded space domains and the whole time axis. Also the existence conditions of periodic and almost periodic solutions are investigated. Moreover, we examine the conditions on input data that guarantee the specific behavior of the solutions at infinity.

The Fourier problem for evolution equations are examined in many papers; see, e.g., [2, 3, 4, 5, 6, 13, 17, 18, 20, 21, 24]. A fairly good survey of results concerning these problems can be found in [2]. It is worth to mention that Fourier problem for linear and a plenty of nonlinear evolution equations are correct only under some restrictions on the growth of solutions and input data as the time variable converges to $-\infty$, in addition to boundary conditions [2, 13, 17, 18, 20, 21, 24]. However, there are the nonlinear parabolic equations for which the Fourier problem are uniquely solvable with no conditions at infinity [3] – [6]. This case for anisotropic elliptic-parabolic equations with variable exponents of nonlinearity is considered here.

It is known that the problem to find time periodic and almost periodic solutions of evolution equations is close to Fourier problem for this equations [5, 8, 10, 12, 14, 18, 25]. Note that the degenerated parabolic equations, in particular, including elliptic-parabolic are examined in [5, 6, 21, 22, 23] and other. Differential equations with variable exponents of nonlinearity are considered in many papers. Solutions of this equations belong to the generalized Lebesgue and Sobolev spaces. More

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information on these spaces and its applications can be received from [1, 4, 9, 11, 15, 16, 19].

This article can be viewed as a natural continuation of the paper [5] for the case of equations with variable exponents of nonlinearity. It consists of three parts: in the first part the formulation of problem and main results are presented, the second part encloses the auxiliary statements while the proofs of main results are in the third part.

2. Setting of the problem and main results

Let Ω be a bounded domain in $\mathbb{R}^n$ with the piecewise smooth boundary $\partial\Omega$. Suppose that $\partial\Omega$ is divided into two subsets Γ_0 and Γ_1, where Γ_0 is closed. The cases $\Gamma_0 = \emptyset$ and $\Gamma_0 = \partial\Omega$ are also possible. We denote by $\nu = (\nu_1, \dots, \nu_n)$ the unit outward pointing normal vector on $\partial\Omega$. Set $Q := \Omega \times \mathbb{R}$, $\Sigma_0 := \Gamma_0 \times \mathbb{R}$, $\Sigma_1 := \Gamma_1 \times \mathbb{R}$, and $Q_{t_1,t_2} := \Omega \times (t_1, t_2)$ for arbitrary real t_1 and t_2. Here and subsequently, we assume that $t_1 < t_2$.

Consider the problem of finding a function $u : \overline{Q} \to \mathbb{R}$ satisfying (in some sense) the equation

$$(b(x)u)_t - \sum_{i=1}^{n} \big(a_i(x,t)|u_{x_i}|^{p_i(x)-2}u_{x_i}\big)_{x_i} + a_0(x,t)|u|^{p_0(x)-2}u = f(x,t), \tag{2.1}$$

for $(x,t) \in Q$, and the boundary conditions

$$u\big|_{\Sigma_0} = 0, \quad \frac{\partial u}{\partial \nu_a}\big|_{\Sigma_1} = 0, \tag{2.2}$$

where $\partial u(x,t)/\partial \nu_a := \sum_{i=1}^{n} a_i(x,t)|u_{x_i}|^{p_i(x)-2}u_{x_i}\,\nu_i(x)$ is the "conormal" derivative on Σ_1, and the functions $b : \Omega \to [0,+\infty)$, $p_j : \Omega \to (1,\infty)$, $a_j : Q \to (0,\infty)$ $(j = 0,\dots,n)$, $f : Q \to \mathbb{R}$ are given.

Next we are going to define a weak solution of the problem (2.1), (2.2) and formulate the main result of our paper. For this, we need some functional spaces and classes of input data of the given problem.

First we introduce some functional spaces. Suppose that either $G = \Omega$ or $G := \Omega \times S$, where S is an interval in $\mathbb{R}$. Consider a function $r \in L_\infty(\Omega)$ such that $r(x) \geq 1$ for almost each $x \in \Omega$. Denote by $L_{r(\cdot)}(G)$ the generalized Lebesgue space consisting of the functions $v \in L_1(G)$ such that $\rho_{G,r}(v) < \infty$, where $\rho_{G,r}(v) := \int_\Omega |v(x)|^{r(x)}\,dx$ for $G = \Omega$, $\rho_{G,r}(v) := \int_G |v(x,t)|^{r(x)}\,dx\,dt$ for $G = \Omega \times S$. The space is equipped with the norm

$$\|v\|_{L_{r(\cdot)}(G)} := \inf\{\lambda > 0 : \rho_{G,r}(v/\lambda) \leq 1\}$$

[11, p. 599]. If $\operatorname{ess\,inf}_{x\in\Omega} r(x) > 1$, then the dual space $[L_{r(\cdot)}(G)]'$ can be identified with $L_{r'(\cdot)}(G)$, where r' is the function defined by the equality $\frac{1}{r(x)} + \frac{1}{r'(x)} = 1$ for almost each $x \in \Omega$.

Let $G = \Omega \times S$, where S is an unbounded interval in $\mathbb{R}$ or $S = \mathbb{R}$. We denote by $L_{r(\cdot),\mathrm{loc}}(\overline{G})$ the space of measurable functions $g : G \to \mathbb{R}$ such that the restriction of g on Q_{t_1,t_2} belongs to $L_{r(\cdot)}(Q_{t_1,t_2})$ for each $t_1, t_2 \in S$. This space is complete locally convex with respect to the family of seminorms $\big\{\|\cdot\|_{L_{r(\cdot)}(Q_{t_1,t_2})} \mid t_1, t_2 \in S\big\}$. A sequence $\{g_m\}$ is said to be convergent strongly (resp., weakly) in $L_{r(\cdot),\mathrm{loc}}(\overline{G})$ provided the sequences of restrictions $\{g_m|_{Q_{t_1,t_2}}\}$ are convergent strongly (resp., weakly) in $L_{r(\cdot)}(Q_{t_1,t_2})$ for all $t_1, t_2 \in S$. Similarly we can define the space $L_{\infty,\mathrm{loc}}(\overline{G})$.

Let B be a linear space with a norm or a seminorm $\|\cdot\|_B$. Let us denote by $C(S;B)$ the space of functions $v : S \to B$ such that the restriction of v on any interval $[t_1,t_2] \subset S$ belongs to $C([t_1,t_2];B)$. The space $C(S;B)$ is complete locally convex with respect to the family of seminorms $\{\|v\|_{C([t_1,t_2];B)} := \max_{t\in[t_1,t_2]} \|v(t)\|_B \mid t_1,t_2 \in S\}$. Therefore a sequence $\{g_m\}$ converges in $C(S;B)$ provided the sequences of restrictions $\{g_m|_{[t_1,t_2]}\}$ converge in $C([t_1,t_2];B)$ for each $t_1,t_2 \in S$.

Let $p = (p_0,\dots,p_n) : \Omega \to \mathbb{R}^{1+n}$ be a vector-function satisfying the following condition:

(P) the function $p_j : \Omega \to \mathbb{R}$ are measurable for all $j = 0,1,\dots,n$,

$$p_0^- := \operatorname{ess\,inf}_{x\in\Omega} p_0(x) > 2, \quad p_i^- := \operatorname{ess\,inf}_{x\in\Omega} p_i(x) \ge 2 \quad \text{for } i = 1,\dots,n,$$
$$p_j^+ := \operatorname{ess\,sup}_{x\in\Omega} p_j(x) < +\infty \quad \text{for } j = 0,1,\dots,n.$$

We also denote by $p' := (p_0{}',\dots,p_n{}')$ the vector-function whose components are given by the equalities $1/p_j(x) + 1/p_j{}'(x) = 1$ for almost each $x \in \Omega$.

Let $W^1_{p(\cdot)}(\Omega)$ be the generalized Sobolev space consisting of the functions $v \in L_{p_0(\cdot)}(\Omega)$ such that $v_{x_i} \in L_{p_i(\cdot)}(\Omega)$ for all $i = 1,\dots,n$. The space is equipped with the norm

$$\|v\|_{W^1_{p(\cdot)}(\Omega)} := \|v\|_{L_{p_0(\cdot)}(\Omega)} + \sum_{i=1}^n \|v_{x_i}\|_{L_{p_i(\cdot)}(\Omega)}.$$

We denote by $\widetilde{W}^1_{p(\cdot)}(\Omega)$ the closure of the set $\{v \in C^1(\overline{\Omega}) \mid v|_{\Gamma_0} = 0\}$ in the space $W^1_{p(\cdot)}(\Omega)$.

Next, for arbitrary $t_1,t_2 \in \mathbb{R}$, we denote by $W^{1,0}_{p(\cdot)}(Q_{t_1,t_2})$ the set of functions $w \in L_{p_0(\cdot)}(Q_{t_1,t_2})$ such that $w_{x_i} \in L_{p_i(\cdot)}(Q_{t_1,t_2})$ for all $i = 1,\dots,n$. We define the norm

$$\|w\|_{W^{1,0}_{p(\cdot)}(Q_{t_1,t_2})} := \|w\|_{L_{p_0(\cdot)}(Q_{t_1,t_2})} + \sum_{i=1}^n \|w_{x_i}\|_{L_{p_i(\cdot)}(Q_{t_1,t_2})}.$$

We denote by $\widetilde{W}^{1,0}_{p(\cdot)}(Q_{t_1,t_2})$ the subspace of $W^{1,0}_{p(\cdot)}(Q_{t_1,t_2})$ consisting of functions v such that $v(\cdot,t) \in \widetilde{W}^1_{p(\cdot)}(\Omega)$ for a. e. $t \in [t_1,t_2]$.

Let $G = \Omega \times S$, where S is either an unbounded $\mathbb{R}$ interval or the $\mathbb{R}$ axis. Let us denote by $\widetilde{W}^{1,0}_{p(\cdot),\mathrm{loc}}(\overline{G})$ the linear space of measurable functions such that their restrictions on Q_{t_1,t_2} belong to $\widetilde{W}^{1,0}_{p(\cdot)}(Q_{t_1,t_2})$ for all $t_1,t_2 \in S$. This space is complete locally convex with respect to the family of seminorms $\{\|\cdot\|_{W^{1,0}_{p(\cdot)}(Q_{t_1,t_2})} \mid t_1,t_2 \in \mathbb{R}\}$.

The following assumption on the function b will be needed throughout the paper.

(B) $b : \Omega \to \mathbb{R}$ is measurable and bounded, $b(x) \ge 0$ for a.e. $x \in \Omega$.

For each $x \in \Omega$ we define $\widetilde{b}(x) = b(x)$ if $b(x) > 0$, and $\widetilde{b}(x) = 1$ if $b(x) = 0$. We denote by $\widetilde{H}^b(\Omega)$ the linear space of functions of the form $w = \widetilde{b}^{-1/2}v$, where $v \in L_2(\Omega)$. We introduce a seminorm on $\widetilde{H}^b(\Omega)$ by $|\!|\!|w|\!|\!| := \|b^{1/2}w\|_{L_2(\Omega)}$. It is easy to check that $\widetilde{H}^b(\Omega)$ is the completion of $\widetilde{W}^1_{p(\cdot)}(\Omega)$ with respect to the seminorm $|\!|\!|\cdot|\!|\!|$ (see [21, III.6, p. 141]).

Set

$$\mathbb{V}_p := \widetilde{W}^1_{p(\cdot)}(\Omega), \quad \mathbb{U}^b_{p,\mathrm{loc}} := \widetilde{W}^{1,0}_{p(\cdot),\mathrm{loc}}(\overline{Q}) \cap C(\mathbb{R};\widetilde{H}^b(\Omega)).$$

The space $\mathbb{U}^b_{p,\mathrm{loc}}$ is a complete linear local convex space with respect to the family of seminorms

$$\big\{\|w\|_{W^{1,0}_{p(\cdot)}(Q_{t_1,t_2})} + \max_{t\in[t_1,t_2]} \|w(\cdot,t)\|_{L_2(\Omega)} \,\big|\, t_1,t_2 \in \mathbb{R}\big\}.$$

For an interval I we consider the space $C^1_0(I)$ of $C^1(I)$-functions of compact support.

Let us denote by $\mathbb{A}$ the set of ordered arrays of functions $(a_0, a_1, \dots, a_n)$ satisfying the condition

(A) for each $j \in \{0, 1, \dots, n\}$ the function a_j belongs to the space $L_{\infty,\mathrm{loc}}(\overline{Q})$ and the following holds

$$a_j(x,t) \ge K_1 \quad \text{for almost each} \quad (x,t) \in Q \tag{2.3}$$

with some constant $K_1 > 0$ being dependent on $(a_0, a_1, \dots, a_n)$.

Definition 2.1. Suppose that b, p satisfy conditions (B), (P), respectively, $(a_0, a_1, \dots, a_n) \in \mathbb{A}$, and $f \in L_{p_0'(\cdot),\mathrm{loc}}(\overline{Q})$. A function u is called a weak solution of (2.1), (2.2) provided $u \in \mathbb{U}^b_{p,\mathrm{loc}}$ and the following integral identity holds

$$\iint_Q \Big\{ \sum_{i=1}^n \big(a_i |u_{x_i}|^{p_i(x)-2} u_{x_i} \psi_{x_i} + a_0 |u|^{p_0(x)-2} u \psi \big) \varphi - b u \psi \varphi' \Big\}\, dx\, dt = \iint_Q f \psi \varphi \, dx\, dt \tag{2.4}$$

for all $\psi \in \mathbb{V}_p$, $\varphi \in C^1_0(\mathbb{R})$.

We say that the weak solution of (2.1), (2.2) *continuously depends on input data*, if for each sequence $\{f_k\}_{k=1}^\infty \subset L_{p_0'(\cdot),\mathrm{loc}}(\overline{Q})$ such that $f_k \to f$ as $k \to \infty$ in $L_{p_0'(\cdot),\mathrm{loc}}(\overline{Q})$ we have $u_k \to u$ as $k \to \infty$ in $\mathbb{U}^b_{p,\mathrm{loc}}$. Here u_k and u are weak solutions of (2.1), (2.2) with the right-hand sides f_k and f, respectively.

Theorem 2.2. *Suppose that b and p satisfy conditions* (B) *and* (P)*, respectively, $(a_0, a_1, \dots, a_n) \in \mathbb{A}$, and $f \in L_{p_0'(\cdot),\mathrm{loc}}(\overline{Q})$. Then there exists a unique weak solution of* (2.1)*,* (2.2)*, and it continuously depends on the input data. In addition, the estimate*

$$\begin{aligned} &\max_{t\in[t_0-R_0,t_0]} \int_\Omega b(x)|u(x,t)|^2\, dx + \int_{t_0-R_0}^{t_0} \int_\Omega \Big[\sum_{i=1}^n |u_{x_i}(x,t)|^{p_i(x)} + |u(x,t)|^{p_0(x)} \Big]\, dx\, dt \\ &\le C_1 \Big\{ R^{-2/(p_0^+-2)} + \int_{t_0-R}^{t_0} \int_\Omega |f(x,t)|^{p_0'(x)}\, dx\, dt \Big\} \end{aligned} \tag{2.5}$$

holds for each R, R_0 such that $R \ge 1$, $0 < R_0 < R/2$, and $t_0 \in \mathbb{R}$. Here C_1 is a positive constant which depends on K_1 and $p_j^\pm$ $(j = 0, \dots, n)$ only.

Remark 2.3. Note that Theorem 2.2 has no conditions imposed on the behaviour of the solution and the growth of the functions a_j $(j = 0, \dots, n)$ as well as on the behaviour of f as $t \to -\infty$. However, the theorem is not true for the case when $p_0(x) = p_1(x) = \dots = p_n(x) = 2$ for almost each $x \in \Omega$ (see, for example, [2]). Therefore the condition (P) is essential.

A solution u of (2.1), (2.2) is called *bounded*, if $\sup_{t\in\mathbb{R}} \int_\Omega b(x)|u(x,t)|^2\, dx < \infty$.

Corollary 2.4. *Under the assumptions of Theorem 2.2, if* $f \in L_{p_0{}'(\cdot)}(Q)$ *then the weak solution of* (2.1), (2.2) *is bounded; it belongs to* $\widetilde{W}^{1,0}_{p(\cdot)}(Q)$ *and the estimate*

$$\begin{aligned} &\sup_{t\in\mathbb{R}} \int_\Omega b(x)|u(x,t)|^2\,dx + \iint_Q \Big[\sum_{i=1}^n |u_{x_i}(x,t)|^{p_i(x)} + |u(x,t)|^{p_0(x)}\Big]\,dx\,dt \\ &\le C_1 \iint_Q |f(x,t)|^{p_0{}'(x)}\,dx\,dt \end{aligned} \tag{2.6}$$

holds.

Corollary 2.5. *Under the assumptions of Theorem 2.2, if*

$$\sup_{\tau\in\mathbb{R}} \int_{\tau-1}^{\tau} \int_\Omega |f(x,t)|^{p_0{}'(x)}\,dx\,dt \le C_2$$

for some positive constant C_2, *then the weak solution* u *of* (2.1), (2.2) *is bounded. In addition,*

$$\sup_{\tau\in\mathbb{R}} \int_{\tau-1}^{\tau} \int_\Omega \Big[\sum_{i=1}^n |u_{x_i}(x,t)|^{p_i(x)} + |u(x,t)|^{p_0(x)}\Big]\,dx\,dt \le C_3$$

with some positive constant C_3 *being dependent on* $K_1, p_j^\pm$ $(j=0,\dots,n)$ *and* C_2 *only.*

Corollary 2.6. *Under the assumptions of Theorem 2.2, if moreover*

$$\lim_{\tau\to\pm\infty} \int_{\tau-1}^{\tau} \int_\Omega |f(x,t)|^{p_0{}'(x)}\,dx\,dt = 0,$$

then for the weak solution u *of problem* (2.1), (2.2) *the following relations hold:*

$$\lim_{t\to\pm\infty} \|b(\cdot)u(\cdot,t)\|_{L_2(\Omega)} = 0,$$
$$\lim_{\tau\to\pm\infty} \int_{\tau-1}^{\tau} \int_\Omega \Big[\sum_{i=1}^n |u_{x_i}(x,t)|^{p_i(x)} + |u(x,t)|^{p_0(x)}\Big]\,dx\,dt = 0.$$

Theorem 2.7. *Under the assumptions of Theorem 2.2, if* f, $a_0,\dots,a_n$ *are periodic in time with period* $\sigma > 0$, *then the weak solution of* (2.1), (2.2) *is also* σ*-periodic in time.*

A set $X \subset \mathbb{R}$ is called *relatively dense*, if there exists a positive l such that the interval $[a, a+l]$ contains at least one element of the set X for any $a \in \mathbb{R}$, i.e. $X \cap [a, a+l] \ne \emptyset$.

Let B be a linear space with a norm or a seminorm $\|\cdot\|_B$. A function $v \in C(\mathbb{R}; B)$ is *Borh almost periodic* , if for each $\varepsilon > 0$ the set $\{\sigma \mid \sup_{t\in\mathbb{R}} \|v(t+\sigma) - v(t)\|_B \le \varepsilon\}$ is relatively dense. A function $f \in L_{p_0(\cdot),\mathrm{loc}}(\overline{Q})$ is *Stepanov almost periodic* provided the set $\{\sigma \mid \sup_{\tau\in\mathbb{R}} \int_{\tau-1}^{\tau} \int_\Omega |f(x,t+\sigma) - f(x,t)|^{p_0(x)}\,dx\,dt \le \varepsilon\}$ is relatively dense for each positive ε. We say that $w \in \widetilde{W}^{1,0}_{p(\cdot),\mathrm{loc}}(\overline{Q})$ is *Stepanov almost periodic*, if for each $\varepsilon > 0$ the set $\{\sigma : \sup_{\tau\in\mathbb{R}} \int_{\tau-1}^{\tau} \int_\Omega \big[\sum_{i=1}^n |w_{x_i}(x,t+\sigma) - w_{x_i}(x,t)|^{p_i(x)}$ $+ |w(x,t+\sigma) - w(x,t)|^{p_0(x)}\big]\,dx\,dt \le \varepsilon\}$ is relatively dense. We refer to [8, 12, 18] for the detailed information on the theory of almost periodic functions.

Theorem 2.8. *Let the hypotheses of Theorem 2.2 hold. In addition, suppose that $a_0, \dots, a_n$ are Borh almost periodic functions in $C(\mathbb{R}; L_\infty(\Omega))$. Assume also that f is Stepanov almost periodic in $L_{p_0(\cdot),\mathrm{loc}}(\overline{Q})$. Moreover, the set*

$$F_\varepsilon := \Big\{\sigma : \sup_{\tau\in\mathbb{R}} \int_{\tau-1}^{\tau} \int_\Omega |f(x,t+\sigma) - f(x,t)|^{p_0'(x)}\,dx\,dt \le \varepsilon, \\ \max_{j\in\{0,\dots,n\}} \sup_{t\in\mathbb{R}} \|a_j(\cdot,t+\sigma) - a_j(\cdot,t)\|_{L_\infty(\Omega)} \le \varepsilon\Big\}$$

is relatively dense for each $\varepsilon > 0$. Then the (unique) weak solution of (2.1), (2.2) *is Borh almost periodic in $C(\mathbb{R}; \widetilde{H}^b(\Omega))$ and Stepanov almost periodic in $\widetilde{W}^{1,0}_{p(\cdot),\mathrm{loc}}(\overline{Q})$.*

3. Auxiliary statements

We start with some auxiliary results, which will be used below.

Lemma 3.1. *Suppose that b, p satisfy conditions* (B), (P)*, respectively. Given $t_1, t_2 \in \mathbb{R}$, we assume that a function $w \in \widetilde{W}^{1,0}_{p(\cdot)}(Q_{t_1,t_2})$ satisfies the equality*

$$\int_{t_1}^{t_2} \int_\Omega \Big\{ \Big(\sum_{i=1}^n g_i \psi_{x_i} + g_0 \psi \Big) \varphi - b w \psi \varphi' \Big\}\,dx\,dt = 0, \quad \psi \in \mathbb{V}_p,\ \varphi \in C_0^1(t_1,t_2), \tag{3.1}$$

for some functions $g_j \in L_{p_j{}'(\cdot)}\big(Q_{t_1,t_2}\big)$ $(j = 0,\dots,n)$. Then $w \in C([t_1,t_2]; \widetilde{H}^b(\Omega))$ and the equality

$$\begin{aligned} &\theta(t) \int_\Omega b(x)|w(x,t)|^2\,dx\Big|_{t=\tau_1}^{t=\tau_2} - \int_{\tau_1}^{\tau_2} \int_\Omega b|w|^2 \theta'\,dx\,dt \\ &+ 2\int_{\tau_1}^{\tau_2} \int_\Omega \Big(\sum_{i=1}^n g_i w_{x_i} + g_0 w \Big) \theta\,dx\,dt = 0 \end{aligned} \tag{3.2}$$

holds for all $\tau_1, \tau_2 \in [t_1,t_2]$ $(\tau_1 < \tau_2)$, $\theta \in C^1([t_1,t_2])$.

This statement can be proved similarly to [4, Lemma 1].

Lemma 3.2. *Suppose that b and p satisfy conditions* (B) *and* (P)*, respectively. Given $t_1, t_2 \in \mathbb{R}$ such that $t_2 - t_1 \ge 1$ and $a \in \mathbb{A}$, we suppose that functions u_1 and u_2 from $\widetilde{W}^{1,0}_{p(\cdot)}(Q_{t_1,t_2}) \cap C([t_1,t_2]; \widetilde{H}^b(\Omega))$ satisfy the equality*

$$\begin{aligned} &\int_{t_1}^{t_2} \int_\Omega \Big\{ \Big(\sum_{i=1}^n a_i |u_{l,x_i}|^{p_i(x)-2} u_{l,x_i} \psi_{x_i} + a_0 |u_l|^{p_0(x)-2} u_l \psi \Big) \varphi - b u_l \psi \varphi' \Big\}\,dx\,dt \\ &= \int_{t_1}^{t_2} \int_\Omega \Big(\sum_{i=1}^n f_{i,l} \psi_{x_i} + f_{0,l} \psi \Big) \varphi\,dx\,dt, \quad \psi \in \mathbb{V}_p,\ \varphi \in C_0^1(t_1,t_2) \end{aligned} \tag{3.3}$$

with the functions $f_{j,l} \in L_{p_j{}'(\cdot)}(Q_{t_1,t_2})$ $(j = 0,\dots,n; l = 1,2)$, respectively. Then the inequality

$$\begin{aligned} &\max_{t\in[t_0-R_0,t_0]} \int_\Omega b(x)|u_1(x,t) - u_2(x,t)|^2\,dx \\ &\quad + \int_{t_0-R_0}^{t_0} \int_\Omega \Big(\sum_{i=1}^n |u_{1,x_i} - u_{2,x_i}|^{p_i(x)} + |u_1 - u_2|^{p_0(x)} \Big)\,dx\,dt \\ &\le C_4 \Big\{ R^{-2/(p_0^+-2)} + \int_{t_0-R}^{t_0} \int_\Omega \sum_{j=0}^n |f_{j,1}(x,t) - f_{j,2}(x,t)|^{p_j{}'(x)}\,dx\,dt \Big\} \end{aligned} \tag{3.4}$$

holds for each R, R_0 and t_0 such that $R \geq 1$, $0 < R_0 < R/2$, and $t_1 \leq t_0 - R < t_0 \leq t_2$. Here C_4 is a positive constant which depends on K_1 and $p_j^{\pm}$ $(j = 0, \dots, n)$ only.

Proof. Let R, R_0, t_0 be such as in the formulation of the lemma, and $\eta(t) := t - t_0 + R$, $t \in \mathbb{R}$ (see [7]). For given $\psi \in \mathbb{V}_p$, $\varphi \in C_0^1(t_1, t_2)$ we subtract equality (3.3) when $l = 1$, and the same equality when $l = 2$. Then, putting

$$\begin{gathered} u_{12}(x,t) := u_1(x,t) - u_2(x,t), \quad f_{j,12}(x,t) := f_{j,1}(x,t) - f_{j,2}(x,t), \\ a_{0,12}(x,t) := a_0(x,t)\big(|u_1(x,t)|^{p_0(x)-2}u_1(x,t) - |u_2(x,t)|^{p_0(x)-2}u_2(x,t)\big), \\ a_{i,12}(x,t) := a_i(x,t)\big(|u_{1,x_i}(x,t)|^{p_i(x)-2}u_{1,x_i}(x,t) - |u_{2,x_i}(x,t)|^{p_i(x)-2}u_{2,x_i}(x,t)\big) \\ (i = 1, \dots, n;\ j = 0, \dots, n;\ (x,t) \in Q), \end{gathered}$$

we have an equality. From this equality using Lemma 3.1 with $w = u_{12}$, $g_j = a_{j,12} - f_{j,12}$ $(j = 0, \dots, n)$, $\theta = \eta^s$, $s := p_0^-/(p_0^- - 2)$, $\tau_1 = t_0 - R$, $\tau_2 = \tau \in (t_0 - R, t_0]$, we obtain the equality

$$\begin{aligned} &\eta^s(\tau) \int_\Omega b(x)|u_{12}(x,\tau)|^2 dx + 2\int_{t_0-R}^{\tau} \int_\Omega \Big\{ \sum_{i=1}^n a_{i,12}(u_{12})_{x_i} + a_{0,12}u_{12} \Big\} \eta^s \,dx\,dt \\ &= s\int_{t_0-R}^{\tau} \int_\Omega b|u_{12}|^2 \eta^{s-1}\,dx\,dt + 2\int_{t_0-R}^{\tau} \int_\Omega \Big(\sum_{i=1}^n f_{i,12}(u_{12})_{x_i} + f_{0,12}u_{12} \Big) \eta^s \,dx\,dt. \end{aligned} \tag{3.5}$$

We make the corresponding estimates of the integrals of equality (3.5). First we note if $r \in L_\infty(\Omega)$ and $\operatorname{ess\,inf}_{x\in\Omega} r(x) \geq 2$, then on the basis of [3, Lemma 1.2] we have the inequality

$$(|s_1|^{r(x)-2}s_1 - |s_2|^{r(x)-2}s_2)(s_1 - s_2) \geq 2^{2-r^+}|s_1 - s_2|^{r(x)}$$

for each $s_1, s_2 \in \mathbb{R}$ and for almost each $x \in \Omega$ (here $r^+ := \operatorname{ess\,sup}_{x\in\Omega} r(x)$). Using this inequality we obtain

$$\begin{aligned} &\int_{t_0-R}^{\tau} \int_\Omega \Big\{ \sum_{i=1}^n a_{i,12}(u_{12})_{x_i} + a_{0,12}\,u_{12} \Big\} \eta^s dx\,dt \\ &\geq C_5 \int_{t_0-R}^{\tau} \int_\Omega \Big(\sum_{i=1}^n |(u_{12})_{x_i}|^{p_i(x)} + |u_{12}|^{p_0(x)} \Big) \eta^s \,dx\,dt, \end{aligned} \tag{3.6}$$

where $C_5 > 0$ is a constant depending only on K_1 and p_j^+ $(j = 0, \dots, n)$.

Further we need the inequality

$$a\,c \leq \varepsilon|a|^q + \varepsilon^{-1/(q-1)}\,|c|^{q'}, \quad a, c \in \mathbb{R},\ q > 1,\ 1/q + 1/q' = 1,\ \varepsilon > 0, \tag{3.7}$$

which is a corollary from standard Young's inequality: $a\,c \leq |a|^q/q + |c|^{q'}/q'$.

Putting (for almost each $x \in \Omega$) $q = p_0(x)/2$, $q' = p_0(x)/(p_0(x) - 2)$, $a = |u_{12}|^2\eta^{s/q}$, $c = b\eta^{s/q'-1}$, $\varepsilon = \varepsilon_1 > 0$, under (3.7) we obtain

$$\begin{aligned} &\int_{t_0-R}^{\tau} \int_\Omega b|u_{12}|^2\eta^{s-1} dx\,dt \\ &\leq \varepsilon_1 \int_{t_0-R}^{\tau} \int_\Omega |u_{12}|^{p_0(x)}\eta^s\,dx\,dt + \varepsilon_1^{-2/(p_0^- - 2)} \\ &\quad\times \Big(\operatorname{ess\,sup}_{x\in\Omega} |b(x)|^{p_0(x)/(p_0(x)-2)} \Big) \int_{t_0-R}^{\tau} \int_\Omega \eta^{s - p_0(x)/(p_0(x)-2)}\,dx\,dt, \end{aligned} \tag{3.8}$$

where $\varepsilon_1 \in (0,1)$ is an arbitrary number.

Again using inequality (3.7), we obtain

$$\begin{aligned} &\int_{t_0-R}^{\tau}\int_\Omega \Big(\sum_{i=1}^n f_{i,12}(u_{12})_{x_i} + f_{0,12}u_{12}\Big)\eta^s dx\,dt \\ &\le \varepsilon_2 \int_{t_0-R}^{\tau}\int_\Omega \Big(\sum_{i=1}^n |(u_{12})_{x_i}|^{p_i(x)} + |u_{12}|^{p_0(x)}\Big)\eta^s\,dx\,dt \\ &\quad + \int_{t_0-R}^{\tau}\int_\Omega \Big(\sum_{j=0}^n \varepsilon_2^{-1/(p_j^- -1)}|f_{j,12}|^{p_j{}'(x)}\Big)\eta^s dx\,dt, \end{aligned} \tag{3.9}$$

where $\varepsilon_2 \in (0,1)$ is an arbitrary number.

From (3.5) using (3.6), (3.8), (3.9), if ε_1 and ε_2 are sufficiently small, we obtain

$$\begin{aligned} &\eta^s(\tau)\int_{\Omega_R} b(x)|u_{12}(x,\tau)|^2\,dx + \int_{t_0-R}^{\tau}\int_\Omega \Big\{\sum_{i=1}^n |(u_{12})_{x_i}|^{p_i(x)} + |u_{12}|^{p_0(x)}\Big\}\eta^s\,dx\,dt \\ &\le C_6\Big[\int_{t_0-R}^{\tau}\int_\Omega \eta^{s-p_0(x)/(p_0(x)-2)}\,dx\,dt + \int_{t_0-R}^{\tau}\int_\Omega \Big(\sum_{j=0}^n |f_{j,12}|^{p_j{}'(x)}\Big)\eta^s\,dx\,dt\Big], \end{aligned} \tag{3.10}$$

where C_6 is a positive constant depending only on K_1 and $p_j^{\pm}$ $(j = 0,\dots,n)$.

Note that $0 \le \eta(t) \le R$, if $t \in [t_0-R, t_0]$, and $\eta(t) \ge R - R_0$, if $t \in [t_0-R_0, t_0]$, where $R_0 \in (0,R)$ is an arbitrary number. Using this and that $R \ge \max\{1; 2R_0\}$ (then, in particular, we have $R/(R-R_0) = 1 + R_0/(R-R_0) \le 2$), from (3.10) we obtain the required statement. □

4. Proof of the main results

Proof of Theorem 2.2. First we prove that there exists at most one weak solution of problem (2.1), (2.2). Assume the contrary. Let u_1, u_2 be (distinct) weak solutions of this problem. Using Lemma 3.2 we obtain

$$\int_{t_0-R_0}^{t_0}\int_\Omega |u_1-u_2|^{p_0(x)}\,dx\,dt \le C_4R^{-2/(p_0^+-2)}, \tag{4.1}$$

where R, R_0, t_0 are arbitrary numbers such that $R \ge 1$, $0 < R_0 < R/2$, $t_0 \in \mathbb{R}$.

We fix arbitrary numbers $R_0 > 0$, $t_0 \in \mathbb{R}$, and take the limit when $R \to +\infty$ in (4.1). As a result we receive that $u_1 = u_2$ almost everywhere on $Q_{t_0-R_0,t_0}$. Since R_0 and t_0 are arbitrary numbers, we obtain $u_1 = u_2$ almost everywhere on Q. The obtained contradiction proves our statement.

Now we are turn to the proof of the existence of a weak solution of problem (2.1), (2.2). For each $m \in \mathbb{N}$ we consider an initial-boundary value problem for equation (2.1) in the domain $Q_m = \Omega \times (-m,+\infty)$ with a homogeneous initial condition and boundary conditions (2.2), namely: we are searching a function $u_m \in \widetilde{W}^{1,0}_{p(\cdot),loc}(\overline{Q_m}) \cap C([-m,+\infty);\widetilde{H}^b(\Omega))$ which satisfies the initial condition:

$b^{1/2}u_m|_{t=-m}=0$ and the integral equality

$$\begin{aligned}
&\iint_{Q_m}\Big\{\big(\sum_{i=1}^n a_i|u_{m,x_i}|^{p_i(x)-2}u_{m,x_i}\psi_{x_i}+a_0|u_m|^{p_0(x)-2}u_m\psi\big)\varphi-bu_m\psi\varphi'\Big\}\,dx\,dt\\
&=\iint_{Q_m} f_m\psi\varphi\,dx\,dt
\end{aligned}\tag{4.2}$$

for each $\psi\in\mathbb{V}_p$, $\varphi\in C_0^1(-m,+\infty)$, where $f_m(x,t):=f(x,t)$ if $(x,t)\in Q_m$, and $f_m(x,t):=0$ if $(x,t)\in Q\setminus Q_m$. The existence and uniqueness of the function u_m follows from a well-known fact (see, for example, [9]).

We extend u_m on Q by zero and this extension is denoted by u_m again. Further we prove that the sequence $\{u_m\}$ converges in $\mathbb{U}^b_{p,\mathrm{loc}}$ to a weak solution of problem (2.1), (2.2). Indeed, note that for each $m\in\mathbb{N}$ the fuction u_m is a weak solution of the problem which differs from problem (2.1), (2.2) in f_m instead of f. Using Lemma 3.2 for each natural numbers m and k we have

$$\begin{aligned}
&\max_{t\in[t_0,t_0-R_0]}\int_\Omega b(x)|u_m(x,t)-u_k(x,t)|^2dx\\
&+\int_{t_0-R_0}^{t_0}\int_\Omega\Big[\sum_{i=1}^n|u_{m,x_i}-u_{k,x_i}|^{p_i(x)}+|u_m-u_k|^{p_0(x)}\Big]\,dx\,dt\\
&\le C_4\Big\{R^{-2/(p_0^+-2)}+\int_{t_0-R}^{t_0}\int_\Omega|f_m-f_k|^{p_0(x)}\,dx\,dt\Big\},
\end{aligned}\tag{4.3}$$

where R,R_0,t_0 are arbitrary numbers such that $t_0\in R,\ R\ge1,\ 0<R_0<R/2$.

We show that for fixed t_0 and R_0 the left side of inequality (4.3) converges to zero when $m,k\to+\infty$. Actually, let $\varepsilon>0$ be an arbitrary small number. We choose $R\ge\max\{1,2R_0\}$ to be big enough such that the following inequality holds

$$C_4R^{-2/(p_0^+-2)}<\varepsilon.\tag{4.4}$$

This is possible as $p_0^+-2>0$. Under (4.4) for arbitrary $m,k\in\mathbb{N}$ such that $\max\{-m,-k\}\le t_0-R$ (then $f_m=f_k$ almost everywhere on $\Omega\times(t_0-R,t_0)$) the right side of inequality (4.3) is less than ε. From this it follows that the restriction of the terms of the sequence $\{u_m\}$ on $Q_{t_0-R_0,t_0}$ is a Cauchy sequence in $\widetilde{W}^{1,0}_{p(\cdot)}(Q_{t_0-R_0,t_0})\cap C([t_0-R_0,t_0];\widetilde{H}^b(\Omega))$. Therefore, since t_0 and R_0 are arbitrary, it follows that there exists a function $u\in\mathbb{U}^b_{p,\mathrm{loc}}$ such that $u_m\to u$ in $\mathbb{U}^b_{p,\mathrm{loc}}$. Assuming that in (4.2) the integration on Q_m can be replaced by integration on Q, we take the limit of this equality for $m\to\infty$. As a result we obtain (2.4) for all $\psi\in\mathbb{V}_p$ and $\varphi\in C_0^1(\mathbb{R})$. It means that the function u is a weak solution of problem (2.1), (2.2). Estimate (2.5) directly follows from Lemma 3.2 putting $u_1=u$, $u_2=0$, $f_{0,1}=f$, $f_{i,1}=0$ $(i=1,\dots,n)$, $f_{j,2}=0$ $(j=0,\dots,n)$. Continuous dependence of a weak solution of problem (2.1), (2.2) on input data is easily proved using Lemma 3.2 with u_k and f_k instead of u_1 and $f_{0,1}$ respectively, and also u and f instead of u_2 and $f_{0,2}$ respectively, putting $f_{i,1}=f_{i,2}=0\,(i=1,\dots,n)$. □

The Proofs of Corollaries 2.4–2.6 follow from estimate (2.5).

Proof of Theorem 2.7. Let u denote a weak solution of problem (2.1), (2.2). Put $u^{(\mu)}(x,t):=u(x,t+\mu)$, $f^{(\mu)}(x,t):=f(x,t+\mu)$, $a_j^{(\mu)}(x,t):=a_j(x,t+\mu)$, $(x,t)\in Q$, where $\mu \in \mathbb{R}$. Replace variable t by $t+\mu$ ($\mu \in \mathbb{R}$ is arbitrary at present) in (2.4). As a result we obtain an identity which we will write in the form

$$\begin{aligned} &\iint_Q \Big\{ \big(\sum_{i=1}^n a_i^{(0)} |u_{x_i}^{(\mu)}|^{p_i(x)-2} u_{x_i}^{(\mu)} \psi_{x_i} + a_0^{(0)} |u^{(\mu)}|^{p_0(x)-2} u^{(\mu)} \psi \big) \varphi \\ &\quad - b u^{(\mu)} \psi \varphi' \Big\} \,dx\,dt \\ &= \iint_Q \Big(\sum_{i=1}^n (a_i^{(0)} - a_i^{(\mu)}) |u_{x_i}^{(\mu)}|^{p_i(x)-2} u_{x_i}^{(\mu)} \psi_{x_i} \\ &\quad + (a_0^{(0)} - a_0^{(\mu)}) |u^{(\mu)}|^{p_0(x)-2} u^{(\mu)} \psi \Big) \varphi \,dx\,dt + \iint_Q f^{(\mu)} \psi \varphi \,dx\,dt \end{aligned} \tag{4.5}$$

for all $\psi \in \mathbb{V}_p$, $\varphi \in C_0^1(\mathbb{R})$. From this, putting $\mu = \sigma$ and using periodicity of the functions a_j $(j = 0, \dots, n)$ and f, we obtain that the function $u^{(\sigma)}$ is a weak solution of problem (2.1), (2.2). Taking this into consideration and the fact of uniqueness of a weak solution of the problem (2.1), (2.2), we get $u^{(0)} = u^{(\sigma)}$ almost everywhere on Q. Therefore the statement of Theorem 2.7 is proved. $\square$

Proof of Theorem 2.8. Similarly as in the proof of Theorem 2.7 we arrive to equality (4.5). Let $\delta_* := \min\{1; K_1/2\}$ and $\sigma \in F_{\delta_*}$, where F_ε is defined in the formulation of given theorem. We consider the identity (4.5) at first for $\mu = 0$ and afterwards for $\mu = \sigma$. Then using Lemma 3.2 with $u_1 = u^{(0)}$, $u_2 = u^{(\sigma)}$, $a_j = a_j^{(0)}$ $(j = 0, \dots, n)$, $f_{0,1} = f^{(0)}$, $f_{0,2} = (a_0^{(0)} - a_0^{(\sigma)}) |u^{(\sigma)}|^{p_0(x)-2} u^{(\sigma)} + f^{(\sigma)}$, $f_{i,1} = 0$, $f_{i,2} = (a_i^{(0)} - a_i^{(\sigma)}) |u_{x_i}^{(\sigma)}|^{p_i(x)-2} u_{x_i}^{(\sigma)}$, $(i = 1, \dots, n)$, $t_0 = \tau \in \mathbb{R}$, $R_0 = 1$, $R = l \in \mathbb{N}$ $(l \ge 2)$, we obtain

$$\begin{aligned} &\max_{t \in [\tau-1,\tau]} \int_\Omega b(x) |u^{(\sigma)}(x,t) - u^{(0)}(x,t)|^2 dx \\ &+ \int_{\tau-1}^\tau \int_\Omega \Big[\sum_{i=1}^n |u_{x_i}^{(\sigma)} - u_{x_i}^{(0)}|^{p_i(x)} + |u^{(\sigma)} - u^{(0)}|^{p_0(x)} \Big] \,dx\,dt \\ &\le C_4 \Big(l^{-2/(p_0^+ - 2)} + \int_{\tau-l}^\tau \int_\Omega \Big\{ \big(|f^{(\sigma)} - f^{(0)}| + |a_0^{(\sigma)} - a_0^{(0)}| |u^{(\sigma)}|^{p_0(x)-1} \big)^{{p_0}'(x)} \\ &\quad + \sum_{i=1}^n |a_i^{(\sigma)} - a_i^{(0)}|^{{p_i}'(x)} \cdot |u_{x_i}^{(\sigma)}|^{p_i(x)} \Big\} \,dx\,dt \Big). \end{aligned} \tag{4.6}$$

From the inequality $(a+c)^q \le 2^{q-1}(a^q + c^q)$, $a \ge 0$, $c \ge 0$, $q \ge 1$, we have

$$\begin{aligned} &\int_{\tau-l}^\tau \int_\Omega \big(|f^{(\sigma)} - f^{(0)}| + |a_0^{(\sigma)} - a_0^{(0)}| |u^{(\sigma)}|^{p_0(x)-1} \big)^{{p_0}'(x)} \,dx\,dt \\ &\le 2^{1/(p_0^- - 1)} \int_{\tau-l}^\tau \int_\Omega \big(|f^{(\sigma)} - f^{(0)}|^{{p_0}'(x)} + |a_0^{(\sigma)} - a_0^{(0)}|^{{p_0}'(x)} |u^{(\sigma)}|^{p_0(x)} \big) \,dx\,dt \\ &\le 2^{1/(p_0^- - 1)} \int_{\tau-l}^\tau \int_\Omega |f^{(\sigma)} - f^{(0)}|^{{p_0}'(x)} \,dx\,dt \\ &\quad + \Big(\sup_{t \in \mathbb{R}} \|a_0^{(\sigma)}(\cdot,t) - a_0^{(0)}(\cdot,t)\|_{L_\infty(\Omega)} \Big)^{(p_0^+)'} \int_{\tau-l}^\tau \int_\Omega |u^{(\sigma)}|^{p_0(x)} \,dx\,dt, \end{aligned} \tag{4.7}$$

$$\begin{aligned}&\int_{\tau-l}^{\tau}\int_\Omega\Big(\sum_{i=1}^n|a_i^{(\sigma)}-a_i^{(0)}|^{p_i{'}(x)}\cdot|u_{x_i}^{(\sigma)}|^{p_i(x)}\Big)\,dx\,dt\\&\le\max_{i\in\{1,\dots,n\}}\Big(\sup_{t\in\mathbb{R}}\|a_i^{(\sigma)}(\cdot,t)-a_i^{(0)}(\cdot,t)\|_{L_\infty(\Omega)}\Big)^{(p_i^+)'}\int_{\tau-l}^{\tau}\int_\Omega\sum_{i=n}^{n}|u_{x_i}^{(\sigma)}|^{p_i(x)}\,dx\,dt,\end{aligned}\tag{4.8}$$

where $(p_j^+)':=p_j^+/(p_j^+-1)$ $(j=0,\dots,n)$.

Since $\sigma\in F_{\delta_*}$ and f is Stepanov almost periodic, it follows that $a^{(\sigma)}(x,t)\ge K_1/2$ $(j=0,\dots,n)$ for a. e. $(x,t)\in Q$ and $\sup_{s\in\mathbb{R}}\int_{s-1}^{s}\int_\Omega|f^{(\sigma)}(x,t)|^{p_0(x)}\,dx\,dt\le C_6$, where $C_6>0$ is a constant independent on σ. From this under Corollary 2.5 we have

$$\sup_{s\in\mathbb{R}}\int_{s-1}^{s}\int_\Omega\Big[|u^{(\sigma)}|^{p_0(x)}+\sum_{i=1}^n|u_{x_i}^{(\sigma)}|^{p_i(x)}\Big]\,dx\,dt\le C_7,\tag{4.9}$$

where $C_7>0$ is a constant independent of σ. Thus, from (4.6) using (4.7) and (4.8), we obtain

$$\begin{aligned}&\int_\Omega b(x)|u^{(\sigma)}(x,\tau)-u^{(0)}(x,\tau)|^2dx\\&+\int_{\tau-1}^{\tau}\int_\Omega\Big[\sum_{i=1}^n|u_{x_i}^{(\sigma)}-u_{x_i}^{(0)}|^{p_i(x)}+|u^{(\sigma)}-u^{(0)}|^{p_0(x)}\Big]\,dx\,dt\\&\le C_8\Big\{l^{-2/(p_0^+-2)}+\sum_{k=1}^{l}\int_{\tau-k}^{\tau-k+1}\int_\Omega|f^{(\sigma)}-f^{(0)}|^{p_0{'}(x)}\,dx\,dt\\&\quad+\max_{j\in\{0,\dots,n\}}\Big(\sup_{t\in\mathbb{R}}||a_j^{(\sigma)}(\cdot,t)-a_j^{(0)}(\cdot,t)||_{L_\infty(\Omega)}\Big)^{(p_j^+)'}\sum_{k=1}^{l}\int_{\tau-k}^{\tau-k+1}\int_\Omega\Big[|u^{(\sigma)}|^{p_0(x)}\\&\quad+\sum_{i=1}^n|u_{x_i}^{(\sigma)}|^{p_i(x)}\Big]\,dx\,dt\Big\},\end{aligned}\tag{4.10}$$

where C_8 is a constant independent of τ,σ and l.

Let $\varepsilon>0$ be an arbitrary small fixed number. We show that the set

$$\begin{aligned}U_\varepsilon:=\Big\{\sigma\in\mathbb{R}:\sup_{t\in\mathbb{R}}\int_\Omega b(x)|u(x,t+\sigma)-u(x,t)|^2\,dx\le\varepsilon,\\ \sup_{\tau\in\mathbb{R}}\int_{\tau-1}^{\tau}\int_\Omega\Big[\sum_{i=1}^n|u_{x_i}(x,t+\sigma)-u_{x_i}(x,t)|^{p_i(x)}\\+|u(x,t+\sigma)-u(x,t)|^{p_0(x)}|\Big]\,dx\,dt\le\varepsilon\Big\}\end{aligned}$$

contains a set F_δ for some $\delta\in(0,\delta_*]$ implying the relative density of the set U_ε. Indeed, choose big enough $l\in\mathbb{N}\,(l\ge2)$ satisfying the inequality

$$C_8l^{-2/(p_0^+-2)}\le\varepsilon/2,\tag{4.11}$$

and fix this value l. Then take $\delta\in(0,\delta_*]$ such that the following inequality remains true

$$C_8\Big(\delta+\max_{j\in\{0,\dots,n\}}\delta^{(p_j^+)'}C_7\Big)l\le\varepsilon/2.\tag{4.12}$$

Therefore, if $\delta \in F_\delta$, then the right side of the inequality (4.10) is less than or equal to ε. This implies that $F_\delta \subset U_\varepsilon$, that is the fact we had to prove. $\square$

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