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NONLINEAR ELLIPTIC PROBLEM OF 2-q-LAPLACIAN TYPE WITH ASYMMETRIC NONLINEARITIES

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Abstract. In this article, we study the nonlinear elliptic problem of 2-q-Laplacian type

$$\begin{gathered} -\Delta u - \mu\Delta_q u = -\lambda|u|^{r-2}u + au + b(u^+)^{\theta-1} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain. For a is between two eigenvalues, we show the existence of three nontrivial solutions.

1. Introduction

In this article, we are interested in finding the multiple nontrivial weak solutions to the nonlinear elliptic problem of 2-q-Laplacian type,

$$\begin{gathered} -\Delta u - \mu\Delta_q u = -\lambda|u|^{r-2}u + au + b(u^+)^{\theta-1} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with samooth boundary $\partial\Omega$, $\lambda, \mu > 0$ are two parameters, $N > 2$, $1 < \min\{q, r\} \le \max\{q, r\} < 2 < \theta \le 2^* = \frac{2N}{N-2}$, $a \in \mathbb{R}$, $b > 0$, and $u^+ = \max\{u, 0\}$. $\Delta_q u = \operatorname{div}(|\nabla u|^{q-2}\nabla u)$ is the q-Laplacian of u.

Paiva and Presoto [12] studied the semilinear elliptic problem with asymmetric nonlinearities,

$$\begin{gathered} -\Delta u = -\lambda|u|^{q-2}u + au + b(u^+)^{p-1} \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

Where $N \ge 3$, $1 < q < 2 < p \le 2^*$, $a \in \mathbb{R}$, $b > 0$ and λ is a positive parameter.

Problem (1.2) is also closely related to the class of superlinear Ambrosetti-Prodi problems [6],

$$-\Delta u = au + (u^+)^p + f(x) \quad \text{in } \Omega. \tag{1.3}$$

Further results for problem (1.3) can be found in [4, 5, 11, 14] and references cited therein.

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Marano and Papageorgiou [10] obtained the existence of three solutions of the (p,q)-Laplacian problem

$$\begin{gathered} -\Delta_p u - \mu\Delta_q u = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.4}$$

by using variational methods and truncation arguments. Nonlinear elliptic problems involving the p-q-Laplacian operator is an active are of research; see [8, 9, 13, 15, 17, 18] and the references therein.

Motivated by the above works, we shall extend the results of problem (1.2) to problem (1.1). By using variational methods, we obtain three solutions to (1.1). We say that g is asymmetric when g satisfies the Ambrosetti-Prodi type condition

$$g_- := \lim_{t\to-\infty}\frac{g(t)}{t} < \lambda_k < g_+ := \lim_{t\to+\infty}\frac{g(t)}{t}.$$

Since problem (1.1) involves $-\Delta$ and $-\Delta_q$, the arguments will be more complicated, and more analysis and estimates are needed.

The eigenvalue problem of the Laplacian, in $\Omega \subset \mathbb{R}^N$, has the form

$$-\Delta u = \lambda u \quad \text{in } \; H_0^1(\Omega). \tag{1.5}$$

By the Ljusternik-Schnirelman principle it is well known that there exists a non-decreasing sequence of nonnegative eigenvalues $0 < \lambda_1 < \lambda_2 \le \dots \le \lambda_j \le \dots$ and a correspondent eigenfunctions φ_j. Also, the first eigenvalue λ_1 is simple and the eigenfunctions associated with λ_1 do not change sign.

Now we are ready to state our main results.

Theorem 1.1. *Let $N \ge 3$, $1 < \min\{q,r\} \le \max\{q,r\} < 2 < \theta < 2^*$ and $\lambda_k < a < \lambda_{k+1}$. Then, for $\lambda > 0$ and $\mu > 0$ small enough, problem* (1.1) *has at least three nontrivial solutions.*

Theorem 1.2. *Let $N \ge 4$, $1 < \min\{q,r\} \le \max\{q,r\} < 2 < \theta = 2^*$ and $\lambda_k < a < \lambda_{k+1}$. Then, for $\lambda > 0$ and $\mu > 0$ small enough, problem* (1.1) *has at least three nontrivial solutions.*

This article is organized as follows. In Section 2, we show some geometric conditions to establish the Mountain-Pass levels and give a technical lemma which is crucial in the proof of our main results. In Section 3, we establish the existence of three nontrivial solutions for the nonlinear elliptic problem (1.1).

2. Preliminaries

In this article, $\|\cdot\|_p$ and $|\cdot|_p$ denote the norms on $W_0^{1,p}(\Omega)$ and $L^p(\Omega)$, respectively;

$$\|u\|_p = \Big(\int_\Omega |\nabla u|^p dx\Big)^{1/p}, \quad |u|_p = \Big(\int_\Omega |u|^p dx\Big)^{1/p}.$$

For convenience, we substitute $\|\cdot\|$ for $\|\cdot\|_2$. The best Sobolev constant S of the embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is denoted by

$$S = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|^2}{|u|_{2^*}^2}.$$

It is known that S is independent of Ω and is never achieved except when $\Omega = \mathbb{R}^N$ (see [16]). Consider the energy functional $I_{\lambda,\mu}$ defined on $H_0^1(\Omega)$ given by

$$I_{\lambda,\mu}(u) = \frac{1}{2}\|u\|^2 + \frac{\mu}{q}\|u\|_q^q + \frac{\lambda}{r}\int_\Omega |u|^r dx - \frac{a}{2}\int_\Omega |u|^2 dx - \frac{b}{\theta}\int_\Omega (u^+)^\theta dx. \tag{2.1}$$

It is easy to know that $I_{\lambda,\mu}$ is of class $\mathcal{C}^2$ and there exists a one to one correspondence between the weak solutions of (1.1) and the critical points of $I_{\lambda,\mu}$ on $H_0^1(\Omega)$. By a weak solution of (1.1) we mean that $u \in H_0^1(\Omega)$ satisfying

$$\begin{aligned}\langle I'_{\lambda,\mu}(u), v\rangle &= \int_\Omega [\nabla u \nabla v + \mu|\nabla u|^{q-2}\nabla u\nabla v]dx + \lambda\int_\Omega |u|^{r-2}uvdx \\ &\quad - a\int_\Omega uvdx - b\int_\Omega (u^+)^{\theta-1}vdx = 0\end{aligned}$$

for all $v \in H_0^1(\Omega)$.

Denote by φ_i a normalized eigenvector relative to eigenvalue λ_i of (1.5). Let $V_k = \langle \varphi_1, \dots, \varphi_k\rangle$ and $W_k = V_k^\perp$. Without loss of generality, we suppose $0 \in \Omega$, and $m \in \mathbb{N}$ large enough so that $B_{2/m} \subset \Omega$, where $B_{2/m}$ denotes the ball of radius $2/m$ with center in 0. Consider the functions introduced in [7],

$$\zeta_m(x) = \begin{cases} 0 & \text{if } x \in B_{1/m}, \\ m|x| - 1 & \text{if } x \in A_m = B_{2/m} \setminus B_{1/m}, \\ 1 & \text{if } x \in \Omega \setminus B_{2/m}. \end{cases}$$

Set $\varphi_i^m = \zeta_m \varphi_i$,

$$V_k^m = \langle \varphi_1^m, \varphi_2^m, \dots, \varphi_k^m\rangle$$

and $W_k^m = (V_k^m)^\perp$. For each $m \in \mathbb{N}$, define a positive cut-off function $\eta \in \mathcal{C}_c^\infty(B_{1/m})$ such that $\eta \equiv 1$ in $B_{1/2m}$, $\eta \le 1$ in $B_{1/m}$ and $\|\nabla\eta\|_\infty \le 4m$; take $\varphi_{k+1}^m = \eta\varphi_{k+1}$. Then

$$\operatorname{supp} u \cap \operatorname{supp} \varphi_{k+1}^m = \emptyset \tag{2.2}$$

whenever $u \in V_k^m$. By [7], it is easy to check the following Lemma.

Lemma 2.1. *As $m \to \infty$ we have*

$$\varphi_i^m \to \varphi_i \quad \textit{in } H_0^1(\Omega) \quad \textit{and} \quad \max_{u \in V_k^m : \int_\Omega |u|^2 = 1} \|u\|^2 \le \lambda_k + c_k m^{2-N}.$$

Corollary 2.2. *For m large enough*

$$V_k^m \oplus W_k = H_0^1. \tag{2.3}$$

As an easy consequence of Lemma 2.1 we have the following decomposition of H_0^1.

Lemma 2.3. *Assume $\lambda_1 < a$, $1 < \min\{q, r\} \le \max\{q, r\} < 2 < \theta \le 2^*$ and $\lambda, \mu > 0$. Then every* (PS) *sequence of $I_{\lambda,\mu}$ is bounded.*

Proof. Suppose $\{u_n\} \subset H_0^1(\Omega)$ is a (PS) sequence of $I_{\lambda,\mu}$; i.e., it satisfies

$$\Big|\frac{1}{2}\|u_n\|^2 + \frac{\mu}{q}\|u_n\|_q^q + \frac{\lambda}{r}\int_\Omega |u_n|^r dx - \frac{a}{2}\int_\Omega |u_n|^2 dx - \frac{b}{\theta}\int_\Omega (u_n^+)^\theta dx\Big| \le C, \tag{2.4}$$

$$\begin{aligned}&\Big|\int_\Omega [\nabla u_n \nabla v + \mu|\nabla u_n|^{q-2}\nabla u_n\nabla v]dx + \lambda\int_\Omega |u_n|^{r-2}u_n vdx \\ &- a\int_\Omega u_n vdx - b\int_\Omega (u_n^+)^{\theta-1}vdx\Big| \le \epsilon_n\|v\|, \quad \forall v \in H_0^1(\Omega),\end{aligned} \tag{2.5}$$

where $\epsilon_n \to 0$ as $n \to \infty$. By (2.4) and (2.5), we obtain

$$
\begin{aligned}
& C + \epsilon_n \|u_n\| \\
& \geq \big| I_\lambda(u_n) - \frac{1}{2}\langle I'_\lambda(u_n), u_n\rangle \big| \\
& = \Big| \big(\frac{\mu}{q} - \frac{\mu}{2}\big)\|u_n\|_q^q + \big(\frac{\lambda}{r} - \frac{\lambda}{2}\big)\int_\Omega |u_n|^r dx + \big(\frac{b}{2} - \frac{b}{\theta}\big)\int_\Omega (u_n^+)^\theta dx \Big| \\
& \geq \big(\frac{b}{2} - \frac{b}{\theta}\big)\int_\Omega (u_n^+)^\theta dx.
\end{aligned} \tag{2.6}
$$

Thus, we have

$$\int_\Omega (u_n^+)^\theta dx \leq C + \epsilon_n \|u_n\|. \tag{2.7}$$

Moreover, by Hölder inequality, we have

$$\int_\Omega (u_n^+)^2 dx \leq |\Omega|^{\frac{\theta-2}{\theta}} \Big(\int_\Omega (u_n^+)^\theta dx \Big)^{2/\theta}. \tag{2.8}$$

On the other hand, by (2.5) we have

$$|\langle I'_{\lambda,\mu}(u_n), u_n^-\rangle| = \big| \|u_n^-\|^2 + \mu \|u_n^-\|_q^q + \lambda |u_n^-|_r^r - a|u_n^-|_2^2 \big| \leq \epsilon_n \|u_n^-\|, \tag{2.9}$$

with $u^- = \max\{-u, 0\}$. It follows from (2.4), (2.7), (2.8) and (2.9) that

$$
\begin{aligned}
\frac{1}{2}\|u_n^+\|^2 & \leq \big(\frac{\mu}{2} - \frac{\mu}{q}\big)\|u_n^-\|_q^q + \big(\frac{\lambda}{2} - \frac{\lambda}{r}\big)\int_\Omega |u_n|^r \\
& \quad + \frac{a}{2}\int_\Omega (u_n^+)^2 dx + \frac{b}{\theta}\int_\Omega (u_n^+)^\theta dx + \frac{1}{2}|\langle I'_{\lambda,\mu}(u_n), u_n^-\rangle| + C \\
& \leq \frac{a}{2}\int_\Omega (u_n^+)^2 dx + \frac{b}{\theta}\int_\Omega (u_n^+)^\theta dx + \epsilon_n \|u_n^-\| + C \\
& \leq \epsilon_n \|u_n\| + \epsilon_n \|u_n^-\| + C.
\end{aligned} \tag{2.10}
$$

Firstly, we show that (u_n^+) is bounded in $H_0^1(\Omega)$. Suppose by contradiction that $\|u_n^+\| \to \infty$, by (2.10), we know that (u_n^-) is also unbounded. Let $w_n = u_n/\|u_n\|$. Since $\{w_n\}$ is bounded in $H_0^1(\Omega)$, there exists $w \in H_0^1(\Omega)$ such that

$$
\begin{gathered}
w_n \rightharpoonup w \quad \text{in } H_0^1(\Omega), \\
w_n \to w \quad \text{in } L^s, \ \forall 1 \leq s < 2^*, \\
w_n \to w \quad \text{a.e. in } \Omega.
\end{gathered}
$$

From (2.10), there exists $\sigma > 0$ satisfying

$$\|u_n^-\| \geq \sigma \|u_n^+\|^2 \tag{2.11}$$

whenever n is large. Notice that

$$w_n^+ = \frac{u_n^+}{\|u_n\|} = \frac{u_n^+}{(\|u_n^+\|^2 + \|u_n^-\|^2)^{1/2}} \leq \frac{u_n^+}{(\|u_n^+\|^2 + \sigma^2\|u_n^+\|^4)^{1/2}},$$

which implies that $w \leq 0$. Furthermore, by

$$w_n^- = \frac{u_n^-}{\|u_n\|} = \frac{u_n^-}{(\|u_n^+\|^2 + \|u_n^-\|^2)^{1/2}} = \frac{u_n^-}{\|u_n^-\|} \cdot \frac{\|u_n^-\|}{(\|u_n^+\|^2 + \|u_n^-\|^2)^{1/2}}$$

and (2.11), we obtain $\|w_n^-\| \to 1$. Hence, by (2.9),

$$-\lambda \frac{|u_n^-|_r^r}{\|u_n^-\|^2} + \mu \frac{\|u_n^-\|_q^q}{\|u_n^-\|^2} + a \frac{|u_n^-|^2}{\|u_n^-\|^2} \to 1. \tag{2.12}$$

Recalling that $q, r < 2$, we obtain

$$\frac{\|u_n^-\|_q^q}{\|u_n^-\|^2} \le |\Omega|^{\frac{2-q}{2}} \|u_n^-\|^{q-2} \to 0, \tag{2.13}$$

$$\frac{|u_n^-|_r^r}{\|u_n^-\|^2} \le |\Omega|^{\frac{2^*-r}{2^*}} S^{-\frac{r}{22^*}} \|u_n^-\|^{\frac{r}{2^*}-2} \to 0. \tag{2.14}$$

Moreover, by (2.11) and $\|w_n^-\| \to 1$, we have

$$\frac{u_n^-}{\|u_n^-\|} - \frac{u_n^-}{\|u_n\|} = \frac{u_n^-}{\|u_n\|}\Big(\frac{\|u_n\|}{\|u_n^-\|} - 1\Big) \to 0 \quad \text{in } H_0^1(\Omega).$$

Thus we may exchange $\|u_n^-\|$ for $\|u_n\|$ in (2.12), and substituting (2.13) and (2.14) into it, we obtain $|w_n^-| \to 1/\sqrt{a}$, then $w \neq 0$. Taking $v = \varphi_1$ in (2.5), one has

$$\begin{aligned} &\int_\Omega [\nabla w_n \nabla \varphi_1 dx + \mu \frac{\|u_n\|_q^{q-1}}{\|u_n\|} \int_\Omega |\nabla w_n|^{q-2} \nabla w_n \nabla \varphi_1 dx \\ &+ \frac{\lambda}{\|u_n\|} \int_\Omega |u_n|^{r-2} u_n \varphi_1 dx - a \int_\Omega w_n \varphi_1 dx - \frac{b}{\|u_n\|} \int_\Omega (u_n^+)^{\theta-1} \varphi_1 dx \to 0; \end{aligned}$$

that is,

$$\begin{aligned} &(\lambda_1 - a) \int_\Omega w_n \varphi_1 dx + \mu \frac{\|u_n\|_q^{q-1}}{\|u_n\|} \int_\Omega |\nabla w_n|^{q-2} \nabla w_n \nabla \varphi_1 dx \\ &+ \frac{\lambda}{\|u_n\|} \int_\Omega |u_n|^{r-2} u_n \varphi_1 dx - \frac{b}{\|u_n\|} \int_\Omega (u_n^+)^{\theta-1} \varphi_1 dx \to 0. \end{aligned} \tag{2.15}$$

Since the second, the third and the fourth term above approach zero, it follows that

$$(\lambda_1 - a) \int_\Omega w \varphi_1 dx = 0,$$

which is a contradiction, as $w \le 0$, $w \neq 0$ and $\lambda_1 < a$, so that (u_n^+) is bounded. Finally, assume that $\|u_n\| \to \infty$ and $\|u_n^+\| \le C$ for all $n \in \mathbb{N}$. Taking $v = w_n$ in (2.5), by (2.13) and

$$\frac{1}{\|u_n\|} \int_\Omega (u_n^+)^\theta dx \to 0,$$

for $\theta \le 2^*$, we obtain $a|w_n|_2^2 \to 1$, so that $w_n \to w$ in $L^2(\Omega)$ with $w \neq 0$. Then by (2.5) we obtain

$$\int_\Omega \nabla w \nabla v dx - a \int_\Omega wv dx = 0 \quad \text{for all } v \in H_0^1(\Omega),$$

with $w \neq 0$ and $w \le 0$, which is a contradiction, as a is not the first eigenvalue. Hence, we conclude that $\{u_n\}$ must be bounded in $H_0^1(\Omega)$. $\square$

In the subcritical case, $1 \le \theta < 2^*$, we can easily know according to the lemma above, $I_{\lambda,\mu}$ satisfies the (PS) condition at every level.

Lemma 2.4. *Let $\lambda_1 < a$ and $\theta = 2^*$. For each $\lambda, \mu > 0$, $I_{\lambda,\mu}$ satisfies the* (PS) *condition at level c with $c < \frac{1}{N} b^{\frac{2-N}{2}} S^{N/2}$.*

Proof. Let $\{u_n\} \subset H_0^1(\Omega)$ be a sequence satisfying

$$I_{\lambda,\mu}(u_n) \to c \quad \text{and} \quad |\langle I'_{\lambda,\mu}(u_n), v\rangle| \le \epsilon_n \|v\|_p, \quad \forall v \in H_0^1(\Omega), \tag{2.16}$$

with $\epsilon_n \to 0$ as $n \to \infty$. By Lemma 2.3 we obtain that $\{u_n\}$ is bounded. Thus, by passing to a subsequence, we have

$$\begin{gathered} u_n \rightharpoonup u \quad \text{in } H_0^1(\Omega), \\ u_n \to u \quad \text{in } L^s,\ \forall 1 \le s < 2^*, \\ u_n \to u \quad \text{a.e. in } \Omega. \end{gathered} \tag{2.17}$$

Since $\{u_n^+\}$ is bounded in $H_0^1(\Omega)$, from the Gagliardo-Nirenberg inequality it follows that $\{u_n^+\}$ is also bounded in L^{2^*}. By passing to a subsequence again, we have $u_n^+ \rightharpoonup u^+$ in L^{2^*}. Hence, we obtain by [11, Lemma 2.3] that

$$\begin{gathered} -\Delta u - \mu\Delta_q u = -\lambda|u|^{r-2}u + au + b(u^+)^{2^*-1}, \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{2.18}$$

Thus, by (2.18) we have

$$I_{\lambda,\mu}(u) = \big(\frac{\mu}{q} - \frac{\mu}{2}\big)\|u\|_q^q + \big(\frac{\lambda}{r} - \frac{\lambda}{2}\big)\int_\Omega |u|^r dx + \big(\frac{b}{2} - \frac{b}{2^*}\big)\int_\Omega (u^+)^{2^*} dx \ge 0. \tag{2.19}$$

Set $w_n = u_n - u$. It is easy to check that

$$|u_n^+ - u^+|_s^s \le |(u_n - u)^+|_s^s = |w_n^+|_s^s, \quad 1 \le s \le 2^*. \tag{2.20}$$

By (2.16) and the Brezis-Lieb Lemma, we have

$$\begin{aligned} &\|w_n\|^2 + \mu\|w_n\|_q^q + \lambda|w_n|_r^r - a|w_n|_2^2 - b|u_n^+ - u^+|_{2^*}^{2^*} \\ &= \|u_n\|^2 - \|u\|^2 + \mu(\|u_n\|_q^q - \|u\|_q^q) + \lambda(|u_n|_r^r - |u|_r^r) \\ &\quad - a(|u_n|_2^2 - |u|_2^2) - b\big(|u_n^+|_{2^*}^{2^*} - |u^+|_{2^*}^{2^*}\big) + o_n(1) \\ &= \langle I'_{\lambda,\mu}(u_n), u_n\rangle - \langle I'_{\lambda,\mu}(u), u\rangle + o(1), \end{aligned}$$

which implies that

$$\lim_{n\to\infty}\big[\|w_n\|^2 + \mu\|w_n\|_q^q + \lambda|w_n|_r^r - a|w_n|_2^2 - b|u_n^+ - u^+|_{2^*}^{2^*}\big] = 0. \tag{2.21}$$

Moreover, by (2.17) we have $w_n \to 0$ in L^r and L^2. Thus, we have from (2.20) and (2.21) that

$$\|w_n\|^2 + \mu\|w_n\|_q^q = b|u_n^+ - u^+|_{2^*}^{2^*} + o(1) \le b|w_n^+|_{2^*}^{2^*} + o(1). \tag{2.22}$$

Without loss of generality, we assume that

$$\|w_n\|^2 = d + o(1), \quad \|w_n\|_q^q = h + o(1). \tag{2.23}$$

By (2.22), (2.23) and Sobolev inequality, we obtain

$$d \ge S\Big(\frac{d + \mu h}{b}\Big)^{2/2^*} \ge Sb^{-2/2^*}d^{2/2^*}. \tag{2.24}$$

If $d = 0$, then we complete the proof. Otherwise, (2.24) implies that

$$d \ge S^{N/2}b^{\frac{2-N}{2}}. \tag{2.25}$$

Then by (2.16), (2.19) and the Brezis-Lieb Lemma, we conclude

$$\begin{aligned} c &\geq c - I_{\lambda,\mu}(u) = I_{\lambda,\mu}(u_n) - I_{\lambda,\mu}(u) + o(1) \\ &= \frac{1}{2}\left(\|u_n\|^2 - \|u\|^2\right) + \frac{\mu}{q}\left(\|u_n\|_q^q - \|u\|_q^q\right) \\ &\quad + \frac{\lambda}{r}\left(|u_n|_r^r - |u|_r^r\right) - \frac{a}{2}\left(|u_n|_2^2 - |u|_2^2\right) - \frac{b}{2^*}\left(|u_n^+|_{2^*}^{2^*} - |u^+|_{2^*}^{2^*}\right) + o(1) \\ &= \frac{1}{2}\|w_n\|^2 + \frac{\mu}{q}\|w_n\|_q^q + \frac{\lambda}{r}|w_n|_r^r - \frac{a}{2}|w_n|_2^2 - \frac{b}{2^*}|u_n^+ - u^+|_{2^*}^{2^*} + o(1). \end{aligned} \tag{2.26}$$

Let $n \to \infty$ in (2.26), we obtain by (2.22), (2.23), (2.25) and $w_n \to 0$ in L^r and L^2 that

$$\begin{aligned} c &\geq \frac{d}{2} + \frac{\mu h}{q} - \frac{d + \mu h}{2^*} \\ &= (\frac{1}{2} - \frac{1}{2^*})d + (\frac{\mu}{q} - \frac{\mu}{2^*})h \\ &\geq (\frac{1}{2} - \frac{1}{2^*})d \\ &\geq \frac{1}{N} S^{N/2} b^{\frac{2-N}{2}}, \end{aligned}$$

which is a contradiction. $\square$

3. Main result

Firstly, we consider the existence of the nonnegative solution of (1.1) . Define the functional $I_{\lambda,\mu}^+ : H_0^1(\Omega) \to \mathbb{R}$ as follows

$$I_{\lambda,\mu}^+(u) = \frac{1}{2}\|u\|^2 + \frac{\mu}{q}\|u\|_q^q + \frac{\lambda}{r}\int_\Omega (u^+)^r dx - \frac{a}{2}\int_\Omega (u^+)^2 dx - \frac{b}{\theta}\int_\Omega (u^+)^\theta dx. \tag{3.1}$$

It follows that $I_{\lambda,\mu}^+ \in C^1$ and the critical points u_+ of $I_{\lambda,\mu}^+$ satisfy $u_+ \geq 0$ and so are critical points of $I_{\lambda,\mu}$ as well, actually, $(I_{\lambda,\mu}^+)'(u_+)[(u_+)^-] = -\|(u_+)^-\|^2 - \mu\|(u_+)^-\|_q^q = 0$.

Similar to the proofs of Lemma 2.3 and Lemma 2.4, we can show that $I_{\lambda,\mu}^+$ satisfies the (PS) condition.

Lemma 3.1. *Let $2 < \theta \leq 2^*$. If $\lambda, \mu > 0$, then $I_{\lambda,\mu}^+$ satisfies the* (PS) *condition at level c with $c < \frac{1}{N} S^{N/2} b^{\frac{2-N}{2}}$.*

Lemma 3.2. *The trivial solution $u \equiv 0$ is a local minimizer for $I_{\lambda,\mu}^+$, for all $\lambda, \mu > 0$.*

Proof. It suffices to show that 0 is a local minimizer of $I_{\lambda,\mu}^+$ in the topology (see [3]). For $u \in C_0^1(\overline{\Omega})$, we have

$$\begin{aligned} I_{\lambda,\mu}^+(u) &= \frac{1}{2}\|u\|^2 + \frac{\mu}{q}\|u\|_q^q + \frac{\lambda}{r}\int_\Omega (u^+)^r dx - \frac{a}{2}\int_\Omega (u^+)^2 dx - \frac{b}{\theta}\int_\Omega (u^+)^\theta dx \\ &\geq \frac{\lambda}{r}\int_\Omega (u^+)^r dx - \frac{a}{2}\int_\Omega (u^+)^2 dx - \frac{b}{\theta}\int_\Omega (u^+)^\theta dx \\ &\geq \Big(\frac{\lambda}{r} - \frac{a}{2}|u|_{C^0}^{2-r} - \frac{b}{\theta}|u|_{C^0}^{\theta-r}\Big)\int_\Omega (u^+)^r dx \geq 0 \end{aligned}$$

whenever

$$\frac{a}{2}|u|_{C^0}^{2-r} + \frac{b}{\theta}|u|_{C^0}^{\theta-r} \le \frac{\lambda}{r}.$$

□

Lemma 3.3. *There exists $t_0 > 0$ such that $I^+_{\lambda,\mu}(t_0\varphi_1) \le 0$, for all λ, μ in a bounded set.*

Proof. Let φ_1 be the positive eigenfunction associated to λ_1, for $t > 0$, we have

$$\begin{aligned} I^+_{\lambda,\mu}(t\varphi_1) &= \frac{t^2}{2}\|\varphi_1\|^2 + \frac{t^q\mu}{q}\|\varphi_1\|_q^q + \frac{t^r\lambda}{r}\int_\Omega \varphi_1^r dx - \frac{t^2 a}{2}\int_\Omega \varphi_1^2 dx - \frac{t^\theta b}{\theta}\int_\Omega \varphi_1^\theta dx \\ &= \frac{t^2}{2}(\lambda_1 - a)\int_\Omega \varphi_1^2 dx + \frac{t^q\mu}{q}\|\varphi_1\|_q^q + \frac{t^r\lambda}{r}\int_\Omega \varphi_1^r dx - \frac{t^\theta b}{\theta}\int_\Omega \varphi_1^\theta dx \end{aligned}$$

Since $\lambda_1 < a$ and $q, r < 2 < \theta$, there exists a choice of $t_0 > 0$ such that $I^+_{\lambda,\mu}(t_0\varphi_1) \le 0$ for λ, μ in a bounded set. □

Define

$$c^+_{\lambda,\mu} = \inf_{\gamma\in\Gamma^+}\sup_{t\in[0,1]} I^+_{\lambda,\mu}(\gamma(t)),$$

where

$$\Gamma^+ = \{\gamma \in \mathcal{C}([0,1], \gamma(0) = 0, \gamma(1) = t_0\varphi_1\}.$$

On the other hand, by the proof of Lemma 3.3, we obtain

$$I^+_{\lambda,\mu}(t\varphi_1) \le \frac{t^q\mu}{q}\|\varphi_1\|_q^q + \frac{t^r\lambda}{r}\int_\Omega \varphi_1^r dx.$$

Then, if λ and μ are small enough, $c^+_{\lambda,\mu} < \frac{1}{N}S^{N/2}b^{\frac{2-N}{2}}$, consequently, by means of the Mountain Pass Theorem, $c^+_{\lambda,\mu}$ is a critical value of $I^+_{\lambda,\mu}$. Thus, we have the following result.

Lemma 3.4. *Let $N > 2$, $1 < \min\{q,r\} \le \max\{q,r\} < 2 < \theta \le 2^*$ and $\lambda_1 < a$. If λ, μ are small enough, then* (1.1) *has at least a nontrivial positive solution.*

To obtain the negative solution, consider the functional $I^-_{\lambda,\mu} : H^1_0(\Omega) \to \mathbb{R}$ given by

$$I^-_{\lambda,\mu}(u) = \frac{1}{2}\|u\|^2 + \frac{\mu}{q}\|u\|_q^q + \frac{\lambda}{r}\int_\Omega (u^-)^r dx - \frac{a}{2}\int_\Omega (u^-)^2 dx. \tag{3.2}$$

Again, $I^-_{\lambda,\mu} \in C^1$ and the critical points u_- of $I^-_{\lambda,\mu}$ satisfy $u_- \le 0$ and so are critical points of $I_{\lambda,\mu}$ as well. We will apply once again the mountain pass theorem to obtain a critical point of $I^-_{\lambda,\mu}$.

Lemma 3.5. *The trivial solution $u \equiv 0$ is a local minimizer for $I^-_{\lambda,\mu}$, for all $\lambda, \mu > 0$.*

Proof. It suffices to show that 0 is a local minimizer of $I^-_{\lambda,\mu}$ in the topology. For $u \in C^1_0(\overline{\Omega})$, we have

$$\begin{aligned} I^-_{\lambda,\mu}(u) &= \frac{1}{2}\|u\|^2 + \frac{\mu}{q}\|u\|_q^q + \frac{\lambda}{r}\int_\Omega (u^-)^r dx - \frac{a}{2}\int_\Omega (u^-)^2 dx \\ &\ge \frac{\lambda}{r}\int_\Omega (u^-)^r dx - \frac{a}{2}\int_\Omega (u^-)^2 dx \end{aligned}$$

$$\geq \Big(\frac{\lambda}{r} - \frac{a}{2}|u|_{C^0}^{2-r}\Big) \int_\Omega (u^-)^r dx \geq 0$$

whenever $\frac{a}{2}|u|_{C^0}^{2-r} \leq \lambda/r$. □

Lemma 3.6. *There exists $t_0 > 0$ such that $I_{\lambda,\mu}^-(-t_0\varphi_1) \leq 0$, for all λ, μ in a bounded set.*

Proof. For $t > 0$, we have

$$\begin{aligned} I_{\lambda,\mu}^-(-t\varphi_1) &= \frac{t^2}{2}\|\varphi_1\|^2 + \frac{t^q\mu}{q}\|\varphi_1\|_q^q + \frac{t^r\lambda}{r}\int_\Omega \varphi_1^r dx - \frac{t^2 a}{2}\int_\Omega \varphi_1^2 dx \\ &= \frac{t^2}{2}(\lambda_1 - a)\int_\Omega \varphi_1^2 dx + \frac{t^q\mu}{q}\|\varphi_1\|_q^q + \frac{t^r\lambda}{r}\int_\Omega \varphi_1^r dx. \end{aligned}$$

Since $\lambda_1 < a$ and $r, q < 2$, there exists a choice of $t_0 > 0$ which proves the lemma. □

As in the nonnegative solution case, we obtain a critical value

$$c_{\lambda,\mu}^- = \inf_{\gamma\in\Gamma^-} \sup_{t\in[0,1]} I_{\lambda,\mu}^-(\gamma(t)),$$

where

$$\Gamma^- = \{\gamma \in \mathcal{C}([0,1] : \gamma(0) = 0, \gamma(1) = -t_0\varphi_1\}.$$

Similar to the proof of Lemma 3.5, we obtain the estimate

$$c_{\lambda,\mu}^- \leq \max_{s\in[0,1]} I_{\lambda,\mu}^-(-st_0\varphi_1) \leq \frac{t_0^q\mu}{q}\|\varphi_1\|_q^q + \frac{t_0^r\lambda}{r}\int_\Omega \varphi_1^r dx,$$

which implies that if λ, μ are small enough, then we obtain the estimate $c_{\lambda,\mu}^- < \frac{1}{N}S^{N/2}b^{\frac{2-N}{2}}$, consequently, by the Mountain Pass Theorem, $c_{\lambda,\mu}^-$ is a critical value of $I_{\lambda,\mu}^-$. Hence, we obtain another important result.

Lemma 3.7. *Let $N > 2$, $1 < \min\{q, r\} \leq \max\{q, r\} < 2 < \theta \leq 2^*$ and $\lambda_1 < a$. If λ, μ small enough, then* (1.1) *has at least a nontrivial negative solution.*

For W_k and V_k^m are as in Section 2, we now consider the existence of the third solution.

Lemma 3.8. *There exist $\alpha > 0$ and $\rho > 0$ such that*

$$I_{\lambda,\mu}(u) \geq \alpha$$

whenever $u \in W_k$ and $\|u\| = \rho$.

Proof. If $u \in W_k$, then

$$\begin{aligned} I_{\lambda,\mu}(u) &= \frac{1}{2}\|u\|^2 + \frac{\mu}{q}\|u\|_q^q + \frac{\lambda}{r}\int_\Omega |u|^r dx - \frac{a}{2}\int_\Omega |u|^2 dx - \frac{b}{\theta}\int_\Omega (u^+)^\theta dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{a}{2}\int_\Omega |u|^2 dx - \frac{b}{\theta}\int_\Omega (u^+)^\theta dx \\ &\geq \Big(\frac{1}{2} - \frac{a}{2\lambda_{k+1}}\Big)\|u\|^2 - \frac{b}{\theta}|u|_\theta^\theta \\ &\geq \|u\|^2\left(A - B\|u\|^{\theta-2}\right), \end{aligned}$$

with $A, B > 0$. Then it suffices to take $\rho < (A/B)^{\frac{1}{\theta-2}}$. □

Lemma 3.9. *Given $\lambda_0 > 0$ and $\mu_0 > 0$, there exist $m_0 \in \mathbb{N}$ and $R > \rho$ such that*
$$I_{\lambda,\mu}(u) \le \frac{\mu}{q}\|u\|_q^q + \frac{\lambda}{r}\int_\Omega |u|^r dx,$$
whenever $u \in \partial Q_m$, where $Q_m = (B_R \cap V_k^m) \oplus [0, R\varphi_{k+1}^m]$, $m \ge m_0$, $\lambda \le \lambda_0$ and $\mu \le \mu_0$. Henceforth ∂ means the boundary relative to subspace V_k^m.

Proof. Let m be large enough and $a_k < a$ such that
$$\lambda_k + c_k m^{2-N} \le a_k < a. \tag{3.3}$$
For $u \in V_k^m$, by Lemma 2.1 and (3.3) one can obtain
$$\begin{aligned} I_{\lambda,\mu}(u) &= \frac{1}{2}\|u\|^2 + \frac{\mu}{q}\|u\|_q^q + \frac{\lambda}{r}\int_\Omega |u|^r dx - \frac{a}{2}\int_\Omega |u|^2 dx - \frac{b}{\theta}\int_\Omega (u^+)^\theta dx \\ &\le (\frac{1}{2} - \frac{a}{2a_k})\|u\|^2 + \frac{\mu}{q}\|u\|_q^q + \frac{\lambda}{r}\int_\Omega |u|^r dx - \frac{b}{\theta}\int_\Omega (u^+)^\theta dx \\ &\le \frac{\mu}{q}\|u\|_q^q + \frac{\lambda}{r}\int_\Omega |u|^r dx, \end{aligned} \tag{3.4}$$
and
$$\begin{aligned} &I_{\lambda,\mu}(\xi\varphi_{k+1}^m) \\ &= \frac{\xi^2}{2}\|\varphi_{k+1}^m\|^2 + \frac{\mu\xi^q}{q}\|\varphi_{k+1}^m\|_q^q + \frac{\lambda\xi^r}{r}\int_\Omega |\varphi_{k+1}^m|^r dx \\ &\quad - \frac{a\xi^2}{2}\int_\Omega |\varphi_{k+1}^m|^2 dx - \frac{b\xi^\theta}{\theta}\int_\Omega ((\varphi_{k+1}^m)^+)^\theta dx \\ &\le \frac{\xi^2}{2}\|\varphi_{k+1}^m\|^2 + \frac{\mu_0\xi^q}{q}\|\varphi_{k+1}^m\|_q^q + \frac{\lambda_0\xi^r}{r}\int_\Omega |\varphi_{k+1}^m|^r dx - \frac{b\xi^\theta}{\theta}\int_\Omega ((\varphi_{k+1}^m)^+)^\theta dx. \end{aligned} \tag{3.5}$$
Since $\varphi_{k+1}^m \to \varphi_{k+1}$ in $W_0^{1,2}(\Omega)$ as $m \to \infty$, φ_{k+1} changes of sign, and $\theta > 2, q, r$, there exist $m_0 \in \mathbb{N}$ and $R > 0$ such that
$$I_{\lambda,\mu}(R\varphi_{k+1}^m) \le 0 \quad \forall m \ge m_0. \tag{3.6}$$
Then combining (2.2), (3.4) and (3.6) leads to
$$I_{\lambda,\mu}(u) \le \frac{\mu}{q}\|u\|_q^q + \frac{\lambda}{r}\int_\Omega |u|^r dx, \tag{3.7}$$
whenever $u \in V_k^m \cup (V_k^m \oplus R\varphi_{k+1}^m)$. By (3.5), there exists $\beta > 0$ satisfying
$$I_{\lambda,\mu}(\xi\varphi_{k+1}^m) \le \beta, \tag{3.8}$$
for all $\xi \ge 0$ and $m \ge m_0$. Since $a > \lambda_k$, we may take $R > 0$ such that
$$\begin{aligned} I_{\lambda,\mu}(u) &\le (\frac{1}{2} - \frac{a}{2\lambda_k})\|u\|^2 + \frac{\mu}{q}\|u\|_q^q + \frac{\lambda}{r}\int_\Omega |u|^r dx \\ &\le -\beta + \frac{\mu}{q}\|u\|_q^q + \frac{\lambda}{r}\int_\Omega |u|^r dx. \end{aligned} \tag{3.9}$$
Hence, by (2.2), (3.8) and (3.9) we obtain
$$I_{\lambda,\mu}(u + \xi\varphi_{k+1}^m) = I_{\lambda,\mu}(u) + I_{\lambda,\mu}(\xi\varphi_{k+1}^m) \le \frac{\mu}{q}\|u\|_q^q + \frac{\lambda}{r}\int_\Omega |u|^r dx \tag{3.10}$$
for all $m \ge m_0$ and $u \in \partial(B_R \cap V_k^m)$. Thus, by (3.7) and (3.10), we complete the proof. $\square$

Proof of Theorem 1.1. For the subcritical case, if $\theta < 2^*$, α is given by Lemma 3.8. Take λ and μ small enough in order that

$$\frac{\mu}{q}\|u\|_q^q + \frac{\lambda}{r}\int_\Omega |u|^r dx < \alpha$$

for all $u \in \partial Q_m$. Then by Lemma 3.9 we have

$$I_{\lambda,\mu}(u) < \alpha$$

whenever $u \in \partial Q_m$ and $m \geq m_0$. Applying the Linking Theorem, $I_{\lambda,\mu}$ possesses a critical point u at level $c_{\lambda,\mu}$, where

$$c_{\lambda,\mu} = \inf_{\Gamma}\max_{u\in Q_m} I_{\lambda,\mu}(\eta(u)),$$

$$\Gamma = \{\eta \in \mathcal{C}(\overline{Q_m}, W_0^{1,p}(\Omega)); \eta = Id \quad \text{on } \partial Q_m\},$$

Finally, since $c_{\lambda,\mu} \geq \alpha$, $I_{\lambda,\mu}(u) \geq \alpha > 0$ and $c^{\pm}_{\lambda,\mu} \to 0$ as $\lambda, \mu \to 0$. Hence, if λ, μ are small enough $c^{\pm}_{\lambda,\mu} < \alpha \leq c_{\lambda,\mu}$, and we know that u may be neither of the critical points found above for $I^+_{\lambda,\mu}$ and $I^-_{\lambda,\mu}$; that is, u is the third solution of (1.1). Thus, combining Lemmas 3.4 and 3.7, we conclude that (1.1) has at least three nontrivial solutions. □

Proof of Theorem 1.2. For the critical case, $\theta = 2^*$. Consider the family of functions taken from [1]:

$$u_\epsilon = \frac{C_N \epsilon^{(N-2)/2}}{(\epsilon^2 + |x|^2)^{(N-2)/2}}, \quad \epsilon > 0,$$

where

$$C_N = (N(N-2))^{(N-2)/4}.$$

Let $u^m_\epsilon = \eta u_\epsilon$, where η is given as section 2, and $Q^\epsilon_m = (B_R \cap V^m_k) \oplus [0, Ru^m_\epsilon]$. Replacing u^m_ϵ by φ^m_{k+1} in Lemma 3.7, we obtain

$$I_{\lambda,\mu}(u) \leq \frac{\mu}{q}\|u\|_q^q + \frac{\lambda}{r}\int_\Omega |u|^r dx, \quad \forall u \in \partial Q^\epsilon_m$$

whenever m is large. Hence, to conclude the proof of Theorem 1.2, it remains to show that

$$\sup_{u\in Q^\epsilon_m} I_{\lambda,\mu}(u) < \frac{1}{N} S^{N/2} b^{\frac{2-N}{2}} \tag{3.11}$$

for all ϵ, λ and μ small enough. Let

$$J(u) = \frac{1}{2}\|u\|^2 - \frac{a}{2}\int_\Omega |u|^2 dx - \frac{b}{2^*}\int_\Omega (u^+)^{2^*} dx.$$

Then, we have

$$I_{\lambda,\mu}(u) = J(u) + \frac{\lambda}{r}\int_\Omega |u|^r dx + \frac{\mu}{q}\|u\|_q^q.$$

It is sufficient to prove that there exist $m_0 > 0$ and $\epsilon_0 > 0$ such that

$$\sup_{u\in Q^\epsilon_m} J(u) < \frac{1}{N} S^{N/2} b^{\frac{2-N}{2}}$$

for all $m \geq m_0$ and $\epsilon < \epsilon_0$. It is not difficult to obtain the following expressions [2]:

$$\int_\Omega |\nabla u^m_\epsilon|^2 dx = S^{N/2} + O(\epsilon^{N-2}), \tag{3.12}$$

$$\int_\Omega |u_\epsilon^m|^{2^*} dx = S^{N/2} + O(\epsilon^N). \tag{3.13}$$

Moreover, we obtain

$$\begin{aligned}
\int_\Omega |u_\epsilon^m|^2 dx &= \int_{B(0,1/m)} |u_\epsilon|^2 dx + O(\epsilon^{N-2}) \\
&\geq \int_{B(0,\epsilon)} \frac{C_N^2 \epsilon^{N-2}}{[2\epsilon^2]^{N-2}} + \int_{\epsilon<|x|<1/m} \frac{C_N^2 \epsilon^{N-2}}{[2|x|^2]^{N-2}} + O(\epsilon^{N-2}) \\
&= \begin{cases} d\epsilon^2 |\ln \epsilon| + O(\epsilon^2), & \text{if } N = 4, \\ d\epsilon^2 + O(\epsilon^{N-2}), & \text{if } N \geq 5, \end{cases}
\end{aligned} \tag{3.14}$$

where d is a positive constant. If $N = 4$, according (3.12), (3.13) and (3.14), one has

$$\begin{aligned}
\frac{\|u_\epsilon^m\|^2 - a|u_\epsilon^m|^2}{|u_\epsilon^m|_{2^*}^2} &\leq \frac{S^2 - ad\epsilon^2|\ln \epsilon| + O(\epsilon^2)}{(S^2 + O(\epsilon^4))^{1/2}} \\
&= S - ad\epsilon^2 |\ln \epsilon| S^{-1} + O(\epsilon^2) < S,
\end{aligned}$$

for $\epsilon > 0$ sufficiently small. And similarly, if $N \geq 5$, we obtain

$$\begin{aligned}
\frac{\|u_\epsilon^m\|^2 - a|u_\epsilon^m|^2}{|u_\epsilon^m|_{2^*}^2} &\leq \frac{S^{N/2} - ad\epsilon^2 + O(\epsilon^{N-2})}{(S^{N/2} + O(\epsilon^N))^{2/2^*}} \\
&= S - ad\epsilon^2 S^{(2-N)/2} + O(\epsilon^{N-2}) < S,
\end{aligned}$$

for $\epsilon > 0$ sufficiently small. Let $u = v + tu_\epsilon^m \in Q_m^\epsilon$. By simple computation, we obtain

$$\max_{t\geq 0} J(tu_\epsilon^m) = \frac{b^{\frac{2-N}{2}}}{N} \Big(\frac{\|u_\epsilon^m\|^2 - a|u_\epsilon^m|^2}{|u_\epsilon^m|_{2^*}^2} \Big)^{N/2} < \frac{1}{N} S^{N/2} b^{\frac{2-N}{2}}. \tag{3.15}$$

Fix $m_0 > 0$ such that $\lambda_k + c_k m_0^{2-N} \leq \sigma < a$. Then, for $m \geq m_0$, we obtain

$$\begin{aligned}
J(v) &= \frac{1}{2}\|v\|^2 - \frac{a}{2}\int_\Omega |v|^2 dx - \frac{b}{2^*}\int_\Omega (v^+)^{2^*} dx \\
&\leq \frac{1}{2}\|v\|^2 - \frac{a}{2}|v|^2 \leq \frac{\sigma}{2}|v|^2 - \frac{a}{2}|v|^2 \leq 0.
\end{aligned} \tag{3.16}$$

From (3.15) and (3.16), we obtain

$$J(u) = J(v + tu_\epsilon^m) = J(v) + J(tu_\epsilon^m) \leq J(tu_\epsilon^m) < \frac{1}{N} S^{N/2} b^{\frac{2-N}{2}}.$$

So, (3.11) holds. □

Letting $\mu \to 0$ in Theorem 1.1 and Theorem 1.2, we easily show that Theorems 1.1 and 1.2 extend the main results in Paiva and Presoto [12].

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