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GLOBAL ATTRACTIVITY FOR NONLINEAR DIFFERENTIAL EQUATIONS WITH A NONLOCAL TERM

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Abstract. In this article we analyze the dynamics of the problem

$$x'(t) = -(\delta + \beta(x(t)))x(t) + \theta \int_0^\tau f(a)x(t-a)\beta(x(t-a))da, \quad t > \tau,$$
$$x(t) = \phi(t), \quad 0 \le t \le \tau,$$

where δ, θ are positive constants, and β, ϕ, f are positives continuous functions. The main results obtained in this paper are the following:

1. Using the Laplace transform, we prove the global asymptotic stability of the trivial steady state.
2. Under some additional hypotheses on the data and by constructing a Lyapunov functional, we show the asymptotic stability of the positive steady state.

We conclude by applying our results to mathematical models of hematopoieses and Nicholson's blowflies.

1. Introduction

The main purpose of this work, is to analyze the of dynamical system

$$\begin{gathered} x'(t) = -(\delta + \beta(x(t)))x(t) + \theta \int_0^\tau f(a)x(t-a)\beta(x(t-a))da, \quad t \ge \tau, \\ x(t) = \phi(t), \quad 0 \le t \le \tau, \end{gathered} \tag{1.1}$$

where δ, θ, are positive constants, and f, ϕ are nonnegative continuous functions. We assume that β is a continuous decreasing function mapping $[0, \infty)$ into $(0, \beta(0)]$ and the function $s\beta(s)$ is bounded in $\mathbb{R}^+$.

System (1.1) is used for describing various phenomena in physics, biology, physiology, see [8, 10, 12, 13, 14] and references therein. In particular, system (1.1) is used in some models arising in hematopoieses and Nicholson's blowflies.

The main goal of our study is to get the asymptotic analysis and the global stability for system (1.1). This problem is widely studied in the literature. Adimy

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et al [2, 4, 5] studied the problem

$$
\begin{gathered}
x'(t) = -(\delta + \beta(x(t)))x(t) + \theta \int_0^\tau f(a)p(t,a)da, \quad t \geq 0,\ x(0) = x_0, \\
\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} = 0, \quad t \geq 0,\ 0 \leq a \leq \tau \\
p(t,0) = x(t)\beta(x(t)), \quad p(0,x) = p_0(x),
\end{gathered}
\tag{1.2}
$$

with $x_0 > 0$ and $p_0(a) \geq 0$.

Note that system (1.2) can be reduced, at least for large t, to a nonlinear delay equation of the form (1.1). In fact, the solution p of the second equation in (1.2) is given by

$$
p(t,a) = \begin{cases} x(t-a)\beta(x(t-a)) & \text{if } t \geq a, \\ p_0(a-t) & \text{if } t \leq a. \end{cases}
$$

Hence, at least for $t \geq \tau$, we have

$$
\begin{gathered}
x'(t) = -(\delta + \beta(x(t)))x(t) + \theta \int_0^\tau f(a)x(t-a)\beta(x(t-a))da, \quad \text{for } t > \tau \\
x(t) = \phi(t), \quad \text{for } 0 \leq t \leq \tau
\end{gathered}
\tag{1.3}
$$

where ϕ satisfies

$$
\begin{gathered}
\phi'(t) = -(\delta + \beta(\phi(t)))\phi(t) + \theta \int_0^t f(a)\phi(t-a)\beta(\phi(t-a))da \\
+ \theta \int_t^\tau f(a)p_0(a-t)da, \quad \text{for } 0 < t \leq \tau, \\
\phi(0) = x_0.
\end{gathered}
$$

Let

$$
\bar{f}(a) = \begin{cases} f(a) & \text{if } a \leq \tau, \\ 0 & \text{if } a > \tau, \end{cases}
$$

and define

$$
\bar{x}(t) = x(t+\tau). \tag{1.4}
$$

Then, going back to the definition of x, we note that $\bar{x}$ solves

$$
\bar{x}'(t) = -(\delta + \beta(\bar{x}(t)))\bar{x}(t) + \theta \int_0^\tau \bar{f}(a)\bar{x}(t-a)\beta(\bar{x}(t-a))da. \tag{1.5}
$$

Henceforth it is sufficient to know the asymptotic behavior of $\bar{x}$.

Taking into consideration the structure of (1.1), we will use the Laplace transform to get the asymptotic stability and to prove some a priori estimates in some cases. Under additional hypotheses, we are able to construct a Lyapunov functional, and then we obtain a global asymptotic stability.

Note that under suitable hypotheses on the data, the authors in [2, 4], proved local stability for the positive steady state.

In [17], the authors proved the global attractivity of the positive steady state. In our paper we improve, in some cases, the results obtained in [17]. In particular we do not need that $\delta x + x\beta(x)$ to be strictly increasing in $(0, M)$ (for a particular value M satisfying some additional conditions) which is the main hypothesis in Theorem 2.4 of [17].

To be more precise, when dealing with the example $\beta(x) = \frac{\beta_0 b^n}{b^n + x^n}$, we obtain the global stability of the positive steady state under less restrictive assumptions than [17], see Section 4 and compare the hypotheses in Theorem 3-6 in [17] (in particular the point (i)), with Theorem 4.1 of the current article.

The supplementary condition imposed in [17, Theorem 3.6 (i)], is a direct consequence of the fact that $\delta x + x\beta(x)$ is strictly increasing in $(0, x^*)$ with x^* being the positive steady state of problem (1.3).

This article is organized as follows. In Section 2, we investigate the asymptotic stability of the trivial steady state. We begin by proving a priori estimates for solution of (1.1), then the Laplace transform to prove the stability result.

The case of the positive steady state is treated in Section 3, then under suitable hypotheses on the data, we are able to construct a Lyapunov function, and then, to get the global attractivity of the positive steady state.

In section 4 we will apply our result to analyze the hematopoiesis and Nicholson's blowflies models. We provide some explicit conditions for the global asymptotic stability. Finally, we give some numerical simulations to illustrate the stability results in some practical cases.

2. Convergence to the trivial steady state

In this section we prove that under some hypotheses the trivial solution attract all solutions of (1.1). To show this, we denote

$$K := \theta \int_0^\tau f(\sigma) d\sigma.$$

From [9], we know that system (1.1) has a unique solution for each continuous, initial condition. Moreover it is not difficult to prove the boundedness of the solution of (1.1), see for instance [3].

Let us begin by proving that if $x_0 > 0$ and $p_0 \gneqq 0$, then $x(t) > 0$ for all $t > 0$ where x is the solution of (1.2).

Proposition 2.1. *Let x be the solution of* (1.2) *associated with nonnegative initial data, then $x(t) > 0$ for all $t > 0$.*

Proof. Since $x_0 > 0$, then there exists $\eta > 0$ such that $x(t) > 0$ for all $t \in (0, \eta)$. We argue by contradiction. Assume the existence of $T > 0$ such that $x(t) > 0$ for $t < T$ and $x(T) = 0$. Thus $x'(T) \le 0$. Now if $T \le \tau$ then using (1.2), we obtain that

$$\theta \int_0^T f(a) x(T-a)\beta(x(T-a)) da + \theta \int_T^\tau f(a) p_0(a-T) da \le 0,$$

a contradiction with the fact that $p_0 \gneqq 0$ and that $x(t) > 0$ for $t < T$. If $T > \tau$, we obtain

$$\theta \int_0^\tau f(a) x(T-a)\beta(x(T-a)) da \le 0,$$

which is also a contradiction. Hence we obtain the desired result. □

We are now able to prove the main stability result of this section.

Theorem 2.2. *Assume that $K \le 1$, then the trivial steady state attracts all positive solutions of problem* (1.1).

Proof. To get the desired result we just have to show that $\bar{x}(t) \to 0$ as $t \to \infty$ where $\bar{x}$ is defined in (1.4). It is clear that $\bar{x}$ is bounded and nonnegative. Hence to get the main result, we will use the Laplace transform. Recall that for $u \in L^\infty(\mathbb{R}^+)$,

$$\pounds(u(t))(p) = \int_0^\infty u(t)e^{-pt}dt, \quad p > 0$$

It is clear that $\pounds(\bar{x}(t))(p)$ is well defined for all $p > 0$. Taking the Laplace transform of each term in (1.5), it follows that

$$\begin{aligned} p\pounds(\bar{x}(t))(p) - x(0) = -\delta\pounds(\bar{x}(t))(p) + \pounds(\beta(\bar{x}(t))x(t))(p) \\ + \theta\int_0^\tau e^{-pt}\int_0^\tau f(a)\bar{x}(t-a)\beta(\bar{x}(t-a))\,da\,dt \\ + \theta\int_\tau^\infty e^{-pt}\int_0^t \bar{f}(a)\bar{x}(t-a)\beta(\bar{x}(t-a))\,da\,dt \end{aligned}$$

We set $C(\tau) = \theta\int_0^\tau e^{-pt}\int_0^\tau f(a)\bar{x}(t-a)\beta(\bar{x}(t-a))\,da\,dt$. Then taking in consideration that

$$\begin{aligned} &\int_\tau^\infty e^{-pt}\int_0^t \bar{f}(a)\bar{x}(t-a)\beta(\bar{x}(t-a))\,da\,dt \\ &\le \int_0^\infty e^{-pt}\int_0^t \bar{f}(a)\bar{x}(t-a)\beta(\bar{x}(t-a))\,da\,dt \\ &= \pounds(\bar{f}(t))(p)\pounds(\beta(\bar{x}(t))x(t))(p), \end{aligned}$$

it follows that

$$p\pounds(\bar{x}(t))(p) - x(0) \le (\theta\pounds(f(t))(p) - 1)\pounds(\beta(\bar{x}(t))x(t))(p) - \delta\pounds(\bar{x}(t))(p) + C(\tau).$$

Since $K \le 1$, it follows that $\theta\pounds(f(t))(p) \le 1$, hence

$$p\pounds(\bar{x}(t))(p) + \delta\pounds(\bar{x}(t))(p) \le x(0) + C(\tau).$$

Letting $p \to 0$ and using Fatou's lemma, we obtain

$$\int_0^\infty \bar{x}(t)dt \le x(0) + C(\tau).$$

Hence $\bar{x} \in L^1(\mathbb{R}^+)$, going back to (1.1), using the fact that β is a bounded function, there results that

$$\int_0^\infty |\bar{x}'(t)|dt < \infty.$$

Thus $\bar{x} \in W^{1,1}(\mathbb{R}^+)$, where $W^{1,1}(\mathbb{R}^+)$ is a Sobolev space defined by

$$W^{1,1}(\mathbb{R}^+) = \{\phi \in L^1(\mathbb{R}^+) \text{ such that } \phi' \in L^1(\mathbb{R}^+)\},$$

notice that if $\phi \in W^{1,1}(\mathbb{R}^+)$, then $\phi(t) \to 0$ as $t \to \infty$, see for instance [6] for more details about the properties of the Sobolev spaces.

As a consequence, we obtain that $\bar{x}(t) \to 0$ as $t \to \infty$ and then $x(t) \to 0$ as $t \to \infty$. Hence we conclude. $\square$

We deal now with the complementary case, namely we assume that

$$K > 1. \tag{2.1}$$

Theorem 2.3. *Assume* (2.1) *holds, then the trivial steady state attracts all solutions of problem* (1.1) *provided that*

$$\delta > (K-1)\beta(0). \tag{2.2}$$

The above Theorem is already proved in [4] by constructing a suitable Lyapunov functional. However we provide here a simple proof using the Laplace transform.

Proof of Theorem 2.3. As in the proof of Theorem 2.2, taking the Laplace transform and following the same computation as above, it follows that

$$p\pounds(x(t))(p)-x(0) \le (\theta\pounds(f(t))(p)-1)\pounds(\beta(x(t))x(t))(p)-\delta\pounds(x(t))(p)+C(\tau). \tag{2.3}$$

Using hypothesis (2.1), and the fact that β is nondecreasing, we obtain

$$p\pounds(x(t))(p) + (\delta-(K-1)\beta(0))\pounds(x(t))(p) \le x(0)+C(\tau).$$

Hence, from (2.2), we obtain

$$\int_0^\infty x(t)dt \le x(0)+C(\tau).$$

Thus $x \in L^1(0,\infty)$. Now, going back to (1.1) and using the hypothesis on β, we can show that $x' \in L^1(0,\infty)$. Thus $x \in W^{1,1}(\mathbb{R}^+)$ and then $x(t) \to 0$ as $t \to \infty$. □

The next theorem illustrates the situation, where we have instability of the trivial steady state.

Theorem 2.4. *Assume that*

$$\delta < (K-1)\beta(0). \tag{2.4}$$

Then the trivial steady state is unstable.

Proof. Recall that $K := \theta\int_0^\tau f(\sigma)d\sigma$, then (2.4) is equivalent to

$$\delta < \Big(\theta\int_0^\tau f(\sigma)d\sigma - 1\Big)\beta(0). \tag{2.5}$$

Assume by contradiction that $x(t) \to 0$ as $t \to \infty$. Then by a continuity argument, $\beta(x(t)) \to \beta(0)$ as $t \to \infty$. Thus we obtain the existence of large T such that for all $\varepsilon > 0$ and for all $t > T$,

$$\beta(x(t)) \ge \frac{\beta(0)}{1+\varepsilon}. \tag{2.6}$$

Define $x_1(t) \equiv x(t+\tau)$, then $x_1(0) = x(\tau) > 0$ and

$$x_1'(t) = -(\delta+\beta(x_1(t)))x_1(t) + \theta\int_0^t f(a)x_1(t-a)\beta(x_1(t-a))da\,. \tag{2.7}$$

It is clear that $x_1(t) \to 0$ as $t \to \infty$. Taking the Laplace transform in (2.7) and using the fact that

$$\begin{aligned}
&\theta\int_0^\tau e^{-pt}\int_0^\tau \bar{f}(a)\bar{x}(t-a)\beta(\bar{x}(t-a))\,da\,dt\\
&\ge \theta\int_0^\tau e^{-pt}\int_0^t \bar{f}(a)\bar{x}(t-a)\beta(\bar{x}(t-a))\,da\,dt,
\end{aligned}$$

which implies

$$\theta\int_0^\infty e^{-pt}\int_0^\tau \bar{f}(a)\bar{x}(t-a)\beta(\bar{x}(t-a))\,da\,dt$$

$$\geq \theta \int_0^\infty e^{-pt} \int_0^t \bar{f}(a)\bar{x}(t-a)\beta(\bar{x}(t-a))\,da\,dt,$$

we conclude that

$$p\pounds(x_1(t))(p) - x_1(0) \geq (\theta\pounds(f(t))(p) - 1)\pounds(\beta(x_1(t))x(t))(p) - \delta\pounds(x_1(t))(p).$$

In view of (2.6), we obtain

$$p\pounds(x_1(t))(p) - x_1(0) \geq \big(\frac{\beta(0)}{1+\varepsilon}(\theta\pounds(f(t))(p) - 1) - \delta\big)\pounds(x_1(t))(p). \tag{2.8}$$

Note that, as $p \to 0$, we have

$$\frac{\beta(0)}{1+\varepsilon}(\theta\pounds(f(t))(p) - 1) - \delta \to \frac{\beta(0)}{1+\varepsilon}(K-1) - \delta.$$

Letting $p \to 0$ in (2.8), using the fact that $\lim_{p\to 0} p\pounds(x_1(t))(p) = \lim_{t\to\infty} x_1(t) = 0$, and by Fatou's lemma, we conclude that

$$0 \geq \Big(\frac{\beta(0)}{1+\varepsilon}(K-1) - \delta\Big)\int_0^\infty x_1(t)dt.$$

Using (2.5) and choosing ε very small, we reach a contradiction with the positivity of x_1. Hence the result follows. □

3. Convergence to positive steady state

In this section, we tudy the local and global asymptotic stability of the positive steady state. To obtain a steady state for (1.1), we just have to solve the equation

$$\big(\delta - (K-1)\beta(x^*)\big)x^* = 0. \tag{3.1}$$

It is clear that positive steady state exists if and only if

$$\delta < (K-1)\beta(0). \tag{3.2}$$

In this case x^* satisfies

$$\beta(x^*) = \frac{\delta}{K-1}. \tag{3.3}$$

Since β is a positive, decreasing function mapping $[0,\infty)$ into $(0,\beta(0)]$, then β^{-1} is well defined and then we reach the existence and the uniqueness of x^*. Thus

$$x^* = \beta^{-1}\big(\frac{\delta}{K-1}\big).$$

We set $p^* = x^*\beta(x^*)$.

Let us begin by proving the local asymptotic stability of the positive steady state. The proof of the following proposition was shown in [4], [2] and [17]. For the sake of completeness, we provide here a simple proof.

Proposition 3.1. *Assume that*

$$-\beta^*(K+1) \leq \delta < (K-1)\beta(0), \tag{3.4}$$

where

$$\beta^* = \beta(x^*) + x^*\beta'(x^*). \tag{3.5}$$

Then the positive steady state of (1.1) *is locally asymptotically stable.*

Proof. Let x^* be the positive solution of (3.1), then to obtain the local asymptotic stability we use a linearisation argument. We set $u(t) = x(t) - x^*$, then dropping all high order terms, we reach

$$u'(t) = -(\delta + \beta^*)u(t) + \theta\beta^* \int_0^\tau f(a)u(t-a)da, \tag{3.6}$$

with β^* defined by (3.5). Note that the characteristic equation of (3.6) is

$$F(\lambda) := \lambda + \delta - \beta^*(\theta \int_0^\tau f(a)e^{-a\lambda}da - 1) = 0. \tag{3.7}$$

To get the desired result we just have to show that if for some $\lambda \in \mathbb{C}$ we have $F(\lambda) = 0$, then $\mathrm{Re}(\lambda) < 0$. We argue by contradiction. Assume the existence of $\lambda \in \mathbb{C}$ such that $\mathrm{Re}(\lambda) \geq 0$ and $F(\lambda) = 0$. We divide the proof into two cases:

Case 1: $-\beta^*(K+1) < \delta$**.** Suppose that $\mathrm{Re}(\lambda) > 0$, then using the fact that β is decreasing, $K > 1$ and (3.1), we can proof that $\beta^* K < \delta + \beta^*$.

(1) If $\beta^* \geq 0$, then $|\lambda + \delta + \beta^*| \leq \beta^* K < \delta + \beta^*$.
(2) When $\beta^* < 0$, if $\mathrm{Re}(\lambda) > 0$, using (3.7), we obtain

$$|\lambda + \delta + \beta^*| < -\theta\beta^* \int_0^\tau f(a)da := -\beta^* K.$$

Owing to (3.4), we reach that $-\beta^* K \leq \delta + \beta^*$.

Consequently, in both cases it follows that $\mathrm{Re}(\lambda) < 0$, a contradiction with the main hypothesis in this case.

It is not difficult to show that the same conclusion is obtained if $\delta > -\beta^*(K+1)$, and $\mathrm{Re}(\lambda) = 0$.

Case 2: $\delta = -\beta^*(K+1)$**.** Using the same arguments as in the first case we can prove that the case $\mathrm{Re}(\lambda) > 0$ can not occur. Hence we just have to analyze the case where $\mathrm{Re}(\lambda) = 0$.

Suppose that $\lambda = iw$, with $w \in \mathbb{R}$ and $F(\lambda) = 0$. Taking the real part in (3.7), we obtain

$$\int_0^\tau f(a)cos(wa)da = \frac{\delta + \beta^*}{\theta\beta^*}.$$

Moreover, since $\delta = -\beta^*(K+1)$, there results that

$$\int_0^\tau f(a)da = -\frac{\delta + \beta^*}{\theta\beta^*}. \tag{3.8}$$

Using the fact that $f \gneqq 0$ and by (3.8), we obtain the existence of $(\tau_1, \tau_2) \subset (0, \tau)$ such that $1 + cos(wa) = 0$ for all $a \in (\tau_1, \tau_2)$ which is impossible. Hence the conclusion follows. □

To prove that the positive steady state is globally attractive, we need some additional hypothesis on β. More precisely we suppose that one of the following hypotheses holds:

(B1) $s\beta(s)$ is a nondecreasing function, or
(B2) there exists a positive constant r_0 such that

$$\max_{\mathbb{R}^+}(s\beta(s)) = r_0\beta(r_0) \quad \text{and} \quad \beta(r_0) < \frac{\delta}{K-1} \leq \beta(0)$$

Let us begin by proving the following technical lemma.

Lemma 3.2. *Assume that* (2.1), *(B2) hold, then $x^* < r_0$ and there exists a positive constant T such that*
$$x(t) < r_0 \quad \text{for all } t \geq T, \tag{3.9}$$
where $x(t)$ and x^* are the solutions of problems (1.1), (3.1), *respectively.*

Proof. Let us begin by proving that $x^* < r_0$. Recall that β is a decreasing function. Hence using (3.3) and (B2) we easily get that $x^* < r_0$.

We prove now (3.9). We argue by contradiction, hence we have to analyze two cases:

Case I. Assume there exists $T > 0$ such that $x(t) \geq r_0$ for all $t \geq T$. Without loss of generality we can assume that $T > \tau$. We set $x_1(t) = x(t+T)$, then x_1 solves
$$x_1'(t) = -(\delta + \beta(x_1(t)))x_1(t) + \theta \int_0^\tau f(a)x_1(t-a)\beta(x_1(t-a))da. \tag{3.10}$$
Note that x_1 is a bounded function, then using the properties of β, taking the Laplace transform in (3.10) and following closely the same computations as in the proof of Theorem 2.2, we obtain
$$p\pounds(x_1(t))(p) + \big(\delta - \beta(r_0)(K-1)\big)\pounds(x_1(t))(p) \leq x_1(0) + C(T),$$
in view of (B2), by letting $p \to 0$, we reach that $x_1 \in L^1(\mathbb{R}^+)$. Going back to (3.10), we obtain $x_1' \in L^1(\mathbb{R}^+)$. Hence $x_1 \in W^{1,1}(\mathbb{R}^+)$ and then $x_1(t) \to 0$ as $t \to \infty$, a contradiction with the fact that $x(t) \geq r_0$.

Case II. Let us consider now the oscillatory case; namely, we assume the existence of a sequence $\{t_n\}_n$ such that $t_n \to \infty$, $x(t_n) = r_0$ and $x'(t_n) \geq 0$. Fix t_{n_0} such that $t_{n_0} > \tau$, using (1.1), it follows that
$$0 \leq -(\delta + \beta(r_0))r_0 + \theta \int_0^\tau f(a)x(t_{n_0} - a)\beta(x(t_{n_0} - a))da,$$
by definition of r_0, we have
$$\delta \leq \beta(r_0)(K-1),$$
which is a contradiction with hypothesis (B2). $\square$

We now can state the main result on the global attractivity of the positive steady state. Let $p(t,a) = x(t-a)\beta(x(t-a))$ (for $t \geq \tau$) and $p^* = x^*\beta(x^*)$. It is clear that (for t so large) problem (1.1) is equivalent to the system
$$\begin{gathered} x'(t) = -(\delta + \beta(x(t)))x(t) + \theta \int_0^\tau f(a)p(t,a)da, \\ \frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} = 0, \quad 0 \leq a \leq \tau \\ p(t,0) = x(t)\beta(x(t)). \end{gathered} \tag{3.11}$$
Therefore, to treat the global attractivity for (1.1), it is sufficient to deal with the same question for system (3.11).

Theorem 3.3. *Assume that either the condition* (B1) *or* (B2) *holds. Then the positive steady state attracts all positive solutions of* (3.11).

Proof. We set $\eta = \liminf_{t\to\infty} x(t)$, then following closely the same arguments as in the proof of [17, Lemma 2.2], we obtain that $\eta > 0$.

Define the set $C_+ = \{h \in L^1(0,\tau), \int_0^\tau f(a)h(a)da > 0\}$. If $(x_0, p_0) \in [0,\infty)\times C_+$, then for (x,p) the corresponding solution to the system (3.11), we can define the Lyapunov functional V by

$$V(x(t), p(t,.)) = \int_0^\tau \phi(a) H(\frac{p(t,a)}{p^*})da + g(\frac{x(t)}{x^*}),$$

where

$$H(s) = s - ln(s) - 1, \tag{3.12}$$

$$g(s) = \frac{\phi(0)}{K\beta(x^*)}\big(s - \beta(x^*)\int_1^s \frac{d\sigma}{\sigma\beta(\sigma x^*)}\big), \tag{3.13}$$

$$\phi(a) = \frac{\phi(0)}{c}\int_a^\tau f(a)da, \tag{3.14}$$

with $c = \int_0^\tau f(a)da$. Then we set

$$I := \frac{d}{dt}\int_0^\tau \phi(a)H(\frac{p(t,a)}{p^*})da. \tag{3.15}$$

By straightforward computations,

$$I = \big(\frac{x\beta(x)}{x^*\beta(x^*)} - \ln(\frac{x\beta(x)}{x^*\beta(x^*)})\big)\phi(0) + \int_0^\tau \phi'(a)\big(\frac{p(t,a)}{p^*} - \ln(\frac{p(t,a)}{p^*})\big)da.$$

Also from (3.13), we obtain

$$\begin{aligned} J &:= \frac{d}{dt} g(\frac{x(t)}{x^*}) \\ &= \Big(\frac{\theta}{x^*}\int_0^\tau f(a)p(t,a)da - (\delta + \beta(x(t)))\frac{x(t)}{x^*}\Big)\frac{\phi(0)}{K\beta(x^*)}(1 - \frac{x^*\beta(x^*)}{x(t)\beta(x(t))}). \end{aligned} \tag{3.16}$$

Now, adding and subtracting the term

$$\frac{c\theta\phi(0)}{Kx^*\beta(x^*)}(1 - \frac{x^*\beta(x^*)}{x(t)\beta(x(t))})p(t,0),$$

summing the equations (3.15)-(3.16) and using the fact that $\delta = (K-1)\beta(x^*)$, there results that

$$\begin{aligned} I + J &= \Big(\frac{x\beta(x)}{x^*\beta(x^*)} - \ln(\frac{x\beta(x)}{x^*\beta(x^*)})\Big)\phi(0) + \int_0^\tau \phi'(a)\big(\frac{p(t,a)}{p^*} - \ln(\frac{p(t,a)}{p^*})\big)da \\ &\quad + \frac{\theta\phi(0)}{Kx^*\beta(x^*)}(1 - \frac{x^*\beta(x^*)}{x(t)\beta(x(t))})\int_0^\tau f(a)(p(t,a) - p(t,0))da \\ &\quad + \frac{x(t)}{x^*}\frac{\phi(0)}{K\beta(x^*)}(1 - \frac{x^*\beta(x^*)}{x(t)\beta(x(t))})\big((K-1)(\beta(x(t)) - \beta(x^*)). \end{aligned}$$

Define $s(t) = x(t)/x^*$, then using (3.12), (3.14) and the fact that

$$H(x) - H(y) = H'(y)(x-y) + \frac{1}{2}H''(z)(x-y)^2$$

with $\min(x,y) < z < \max(x,y)$, we obtain

$$I + J = L + \frac{\phi(0)(K-1)}{Kx^*\beta(x^*)\beta(sx^*)}(sx^*\beta(sx^*) - x^*\beta(x^*))\big(\beta(sx^*) - \beta(x^*)\big), \tag{3.17}$$

where

$$L = -\frac{\phi(0)}{2c} \int_0^\tau (\frac{p^*}{z(t,a)})^2 (\frac{p(t,a)}{p^*} - \frac{p(t,0)}{p^*})^2 f(a) da,$$

with

$$\min(\frac{p(t,0)}{p^*}, \frac{p(t,a)}{p^*}) \le z(t,a) \le \max(\frac{p(t,0)}{p^*}, \frac{p(t,a)}{p^*}) \quad \text{for all } (t,a).$$

Having in mind Lemma 3.2 and the fact that the function β is decreasing, we obtain

$$(sx^*\beta(sx^*) - x^*\beta(x^*))\big(\beta(sx^*) - \beta(x^*)\big) \le 0. \tag{3.18}$$

Indeed, inequality (3.18) follows easily in the case when $s\beta(s)$ is increasing.

Let us assume that (B2) holds, if $s \le 1$, then $sx^* := x(t) \le x^* \le r_0$. Using the fact that $s\beta(s)$ is nondecreasing for $s \le r_0$, (r_0 is defined in (B2)), we conclude that $sx^*\beta(sx^*) - x^*\beta(x^*) \le 0$. The same result occurs for $s > 1$.

Note that if $(\frac{d}{dt} V(x(t), p(t,.)) = 0$, then $x(t) = x^*$ and $p(t,a) = p(t,0)$. Going back to the equation of p, we obtain $\frac{\partial p}{\partial t} = 0$, hence $p(t,a) = A$ with $A \in \mathbb{R}$.

Now, by identification with the equation of the positive steady state, we reach that $p(t,.) = p^*$. Therefore, using the LaSalle invariance principle, (see e.g., [13]), it follows that (x^*, p^*) is globally attractive and the result follows. □

4. Application to population dynamics

The blood production process is one of the major biological phenomena occurring in human body. According to [4, 11, 10, 14] stem cells are classified as proliferating phase (population p) and resting phase (population r). To describe the dynamics of the population of proliferating and resting stem cells, the authors in [2, 4], proposed the following age structured model,

$$\begin{gathered}
\frac{\partial r}{\partial t} + \frac{\partial r}{\partial a} = -(\delta + \beta(x(t)))r(t,a), \quad t \ge 0,\ a \ge 0 \\
\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} = -(\gamma + g(a))p, \quad t \ge 0,\ 0 \le a \le \tau \\
r(t,0) = 2\int_0^\tau g(a)p(t,a)da, \quad x(t) = \int_0^\infty r(t,a)da, \\
p(t,0) = x(t)\beta(x(t)), \quad p(0,x) = p_0(x),
\end{gathered} \tag{4.1}$$

where $\beta(x) \equiv \frac{\beta_0 b^n}{b^n + x^n}$ with $\beta_0 > 0$, $b \ge 0$ and $n \ge 0$, see [11, 12, 13].

Notice that β_0 is the maximum production rate and b is the resting population density for which the rate of reentry β attains its maximum rate of change with respect to the resting phase population. The constant n describes the sensitivity of β with the changes.

Notice that the study of asymptotic behavior of system (4.1) can be reduced to analyze problems of the form (1.1), (1.2). To see that, define $x(t) = \int_0^\infty r(t,a)da$, by integrating the first equation in (4.1), we obtain

$$\begin{gathered}
x'(t) = -(\delta + \beta(x(t)))x(t) + 2\int_0^\tau g(a)p(t,a)da, \quad t \ge 0,\ a \ge 0 \\
\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} = -(\gamma + g(a))p, \quad t \ge 0,\ 0 \le a \le \tau \\
p(t,0) = x(t)\beta(x(t)), \quad p(0,x) = p_0(x).
\end{gathered} \tag{4.2}$$

Now, using the characteristic equation we obtain

$$p(t,a)=\begin{cases} p_0(a-t)e^{-\gamma t-\int_0^t g(\sigma+a-t)d\sigma} & t<a\\ x(t-a)\beta(x(t-a))e^{-\gamma a-\int_0^a g(\sigma)d\sigma} & t>a. \end{cases}$$

Hence, at least for $t\geq\tau$, it follows that

$$x'(t)=-(\delta+\beta(x(t)))x(t)+2\int_0^\tau f(a)x(t-a)\beta(x(t-a))da, \tag{4.3}$$

with $f(a)=g(a)e^{-\int_0^a(\gamma+g(\sigma))d\sigma}$. The steady states of problem (4.3) are given by

$$(\delta-(K-1)\beta(x^*))x^*=0.$$

where

$$K=2\int_0^\tau f(a)da.$$

If $\delta\leq(K-1)\beta(0)$, then either $x^*=0$, or

$$\beta(x^*)=\frac{\delta}{K-1}:=\alpha. \tag{4.4}$$

Therefore, $x^*=\beta^{-1}(\alpha)$ is the unique non trivial solution.

Now we give some explicit conditions to get the global stability of the positive steady state. This result improve in some cases [17, Theorem 3.6]. Notice that the proof of [17, Theorem 3.6] is based on the perturbation theory, see [15] and [16], which is different from the one used here.

As a direct application of Lemma 3.2 and Theorem 3.3, we obtain the next result.

Theorem 4.1. *Let $\beta(x)=\frac{\beta_0 b^n}{b^n+x^n}$ where $n>0$. Assume that either $n\leq 1$, or $n>1$ and*

$$\frac{\delta}{K-1}<\beta_0<\frac{n\delta}{(n-1)(K-1)}. \tag{4.5}$$

Then the positive steady state of problem (4.3) *is globally asymptotically stable.*

Proof. If $n\leq 1$, then we can directly prove that $x\beta(x)$ is a nondecreasing function.

Assume that $n>1$, then $\max_{\mathbb{R}^+}(s\beta(s))=r_0\beta(r_0)$ where $r_0=\frac{\delta}{(n-1)^{\frac{1}{n}}}$ and inequality (3.4) is always satisfied. The condition (B2) is satisfied if and only if (4.5) holds. Therefore, the result follows using Lemma 3.2 and Theorem 3.3. □

In the case where $\beta(x)=\beta_0 e^{-nx}$, equation (1.1) is the Nicholson's blowflies model, we refer to [7] for more details in this direction.

Therefore, using Lemma 3.2 and Theorem 3.3, we obtain the next result that gives a sufficient condition related to the parameters K, δ, β_0, in order to get the global asymptotic stability.

Theorem 4.2. *Let $\beta(x)=\beta_0 e^{-nx}$, where $n>0$. Assume that*

$$\frac{\delta}{K-1}<\beta_0<\frac{\delta e}{K-1}.$$

Then the positive steady state of problem (4.3) *is globally asymptotically stable.*

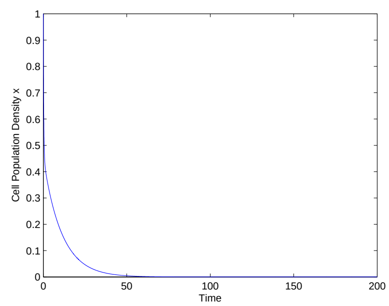

FIGURE 1. Global stability of the trivial solution where $\delta = 1.3$

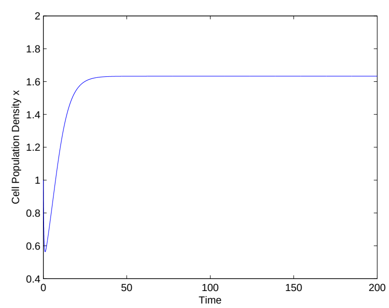

FIGURE 2. Blobal stability of the positive steady state where $\delta = 0.6$

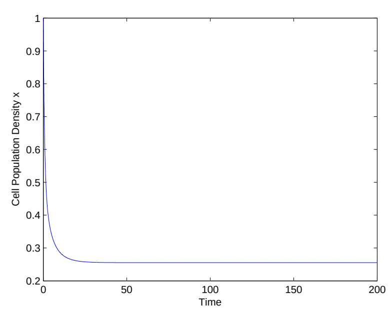

FIGURE 3. Global stability of the positive steady state related to a blowflies model, with $\beta(x) = e^{-2x}$ and $\delta = 0.6$

5. Numerical simulation

In this section, we illustrate our theoretical results with a number of numerical simulations. In Figures 1–3, numerical simulations relating to a model of blood production process are presented. The functions β and the division rates f are respectively given by $\beta(x) = \frac{\beta_0 b^n}{b^n + x^n}$ with $\beta_0 = 1$, $n = 2$ and $f(a) = e^{-a}$.

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