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EXISTENCE AND UNIQUENESS OF SOLUTIONS TO PARABOLIC FRACTIONAL DIFFERENTIAL EQUATIONS WITH INTEGRAL CONDITIONS

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Abstract. In this article, we establish sufficient conditions for the existence and uniqueness of a solution, in a functional weighted Sobolev space, for partial fractional differential equations with integral conditions. The results are established by applying the energy inequality method, and the density of the range of the operator generated by the problem.

1. Introduction

Fractional differential equations (FDEs) are generalizations of differential equations of integer order to an arbitrary order. These generalizations play a crucial role in engineering, physics and applied mathematics. Therefore, they have generated a lot of interest from engineers and scientist in recent years. Since FDEs have memory, nonlocal relations in space and time, and complex phenomena can be modeled by using these equations. Indeed, we can find numerous applications in viscoelasticity, electro-chemistry, signal processing, control theory, porous media, fluid flow, rheology, diffusive transport, electrical networks, electromagnetic theory, probability, signal processing, and many other physical processes [18, 19, 20, 23, 30]. For recent developments in fractional differential and in partial differential equations, see the monograph by Kilbas et al [24], and the articles [1, 2, 3, 5, 6, 13, 14, 16, 17, 22, 25, 31, 32].

A large number of problems in modern physics and technology are stated using nonlocal conditions for partial differential equations, which are described using integral conditions. Integral boundary conditions receive a lot of attention because of their applications in population dynamics, blood flow models, chemical engineering and cellular systems; see for example [8, 9, 10, 11, 12, 28, 29].

The existence and uniqueness of solutions to initial and boundary-value problems for fractional differential equations has been extensively studied by many authors; see for example [2, 3, 4, 6, 7, 21, 26]. Some of the existence and uniqueness results have been obtained by using the well-known Lax-Milgram theorem, and by fixed point theorems [26, 15, 33].

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A suitable variational formulation is the starting point of many numerical methods, such as finite element methods and spectral methods. Thus the construction of a variational formulation is essential, and relies strongly on the choice of spaces and their norms. Motivated by this, we extend and generalize the study for PDEs with integral conditions to the study of fractional PDEs with integral conditions. Also we expand the works in classical problems of fractional PDEs to non standard problems. Also we extend the application of the energy inequality method for obtaining existence and uniqueness of solutions in functional weighted Sobolev spaces.

2. Preliminaries

Let $\Gamma(\cdot)$ denote the gamma function. For any positive integer $0<\alpha<1$, the Caputo derivative are the Riemann Liouville derivative are, respectively, defined as follows:

(i) The left Caputo derivatives:

$$ {}_0^C\partial_t^\alpha u(x,t) := \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,\tau)}{\partial\tau}\frac{1}{(t-\tau)^\alpha}\,d\tau\,. \tag{2.1}$$

(ii) The left Riemann-Liouville derivatives:

$$ {}_0^R\partial_t^\alpha u(x,t) := \frac{1}{\Gamma(1-\alpha)}\frac{\partial}{\partial t}\int_0^t \frac{u(x,\tau)}{(t-\tau)^\alpha}\,d\tau. \tag{2.2}$$

Many authors consider the Caputo's version to be natural because it allows the handling of inhomogeneous initial conditions in a easier way. Then the two definitions (2.1) and (2.2) are linked by the following relationship, which can be verified by a direct calculation:

$$ {}_0^R\partial_t^\alpha u(x,t) = {}_0^C\partial_t^\alpha u(x,t) + \frac{u(x,0)}{\Gamma(1-\alpha)t^\alpha}. \tag{2.3}$$

In the rectangular domain $\Omega=(0,1)\times(0,T)$, with $T<\infty$, we consider the equation

$$ \mathcal{L}v = {}_0^C\partial_t^\alpha v(x,t) - \frac{\partial}{\partial x}\big(a(x,t)\frac{\partial v}{\partial x}\big) = F(x,t), \tag{2.4}$$

with the initial data

$$ \ell u = v(x,0) = \phi(x), \quad x\in(0,1), \tag{2.5}$$

Neumann boundary condition

$$ \frac{\partial v}{\partial x}(0,t) = \mu(t), \tag{2.6}$$

and the integral condition

$$ \int_0^1 v(x,t)dx = m(t), \quad t\in(0,T), \tag{2.7}$$

where F, ϕ, μ and m are known functions.

We shall assume that the function ϕ satisfies a compatibility conditions with (2.6) and (2.7), i.e.,

$$ \frac{d\phi(0)}{dx} = \mu(0), \quad \int_0^1 \phi(x)\,dx = m(0). $$

Since the boundary conditions are inhomogeneous, we construct a function

$$U(x,t) = x\big(1-\frac{3}{2}x\big)\mu(t) + 3x^2 m(t),$$

and introduce a new function $\widetilde{v}(x,t) = v(x,t) - U(x,t)$. Then problem (2.4)–(2.7) can be formulated as

$$\mathcal{L}\widetilde{v} = {}_0^C\partial_t^\alpha \widetilde{v}(x,t) - \frac{\partial}{\partial x}\big(a(x,t)\frac{\partial v}{\partial x}\big) = F(x,t) - \mathcal{L}U = \widetilde{F}(x,t), \tag{2.8}$$

$$\ell\widetilde{v} = \widetilde{v}(x,0) = \phi(x) - \ell U = \varphi(x), \quad x \in (0,1), \tag{2.9}$$

$$\frac{\partial \widetilde{v}}{\partial x}(0,t) = 0, \tag{2.10}$$

$$\int_0^1 \widetilde{v}(x,t)dx = 0, \quad t \in (0,T), \tag{2.11}$$

where φ satisfies a compatibility conditions with (2.10) and (2.11).

Again, introducing a new function $u(x,t) = \widetilde{v}(x,t) - \varphi(x)$ and using (2.3), problem (2.8)-(2.11) can be formulated as

$$\mathcal{L}u = {}_0^R\partial_t^\alpha u(x,t) - \frac{\partial}{\partial x}\big(a(x,t)\frac{\partial u}{\partial x}\big) = \widetilde{F}(x,t) + \frac{\partial}{\partial x}\big(a(x,t)\frac{d\varphi(x)}{dx}\big) = f(x,t), \tag{2.12}$$

$$\ell u = u(x,0) = 0, \quad x \in (0,1), \tag{2.13}$$

$$\frac{\partial u}{\partial x}(0,t) = 0, \tag{2.14}$$

$$\int_0^1 u(x,t)dx = 0, \quad t \in (0,T). \tag{2.15}$$

Next we introduce the function spaces that we need in our investigation. $L_2(0,1)$ and $L_2(0,T,L_2(0,1))$ be the standard function spaces. We denote by $C_0(0,1)$ the vector space of continuous functions with compact support in (0.1). Since such functions are Lebesgue integrable with respect to dx, we can define on $C_0(0,1)$ the bilinear form given by

$$(u,w) = \int_0^1 \Im_x u \cdot \Im_x w \, dx, \tag{2.16}$$

where $\Im_x u = \int_0^x u(\xi,\cdot)d\xi$. The previous bilinear form (2.16) is considered as a scalar product on $C_0(0,1)$ for which $C_0(0,1)$ is not complete.

Definition 2.1 ([8])**.** We denote by $B_2(0,1)$ a completion of $C_0(0,1)$, under the scalar product (2.16) which is denoted $(,\cdot,)_{B_2(0,1)}$. It is called the (Bouziani) space of square integrable primitive functions on $(0,1)$. The norm of function u in $B_2(0,1)$, is the non-negative number

$$\|u\|_{B_2(0,1)} = \sqrt{(u\cdot u)_{B_2(0,1)}} = \|\Im_x u\|_{L_2(0,1)}.$$

For $u \in L_2(0,1)$, we have the inequality

$$\|u\|^2_{B_2(0,1)} \leqslant \frac{1}{2}\|u\|^2_{L_2(0,1)}. \tag{2.17}$$

We denote by $L_2(0,T,B_2(0,1)) := B_2(\Omega)$ the space of functions which are square integrable in the Bochner sense, with the scalar product

$$(u,w)_{L_2(0,T,B_2(0,1))} = \int_0^T ((u,\cdot),(w,\cdot))_{B_2(0,1)}dt. \tag{2.18}$$

Since the space $B_2(0,1)$ is a Hilbert space, it can be shown that $L_2(0,T,B_2(0,1))$ is a Hilbert space as well. Let $C^\infty(0,T)$ denote the space of infinitely differentiable functions on $(0,T)$ and $C_0^\infty(0,T)$ denote the space of infinitely differentiable functions with compact support in $(0,T)$.

Definition 2.2 ([27])**.** For any real $\sigma > 0$, we define the semi-norm

$$|u|^2_{{}^RH_0^\sigma(\Omega)} := \|{}_0^R\partial_t^\sigma u\|^2_{L_2(\Omega)},$$

and the norm

$$\|u\|_{{}^RH_0^\sigma(\Omega)} := \Big(\|u\|^2_{L_2(\Omega)} + |u|^2_{{}^RH_0^\sigma(\Omega)}\Big)^{1/2}. \tag{2.19}$$

Then we define ${}^RH_0^\sigma(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{{}^RH_0^\sigma(\Omega)}$.

Definition 2.3. For any real $\sigma > 0$, we define the semi-norm:

$$|u|^2_{{}^RB_0^\sigma(\Omega)} := \|{}_0^R\partial_t^\sigma(\Im_x u)\|^2_{L_2(\Omega)},$$

and the norm

$$\|u\|_{{}^RB_0^\sigma(\Omega)} := \Big(\|u\|^2_{B_2(\Omega)} + |u|^2_{{}^RB_0^\sigma(\Omega)}\Big)^{1/2}. \tag{2.20}$$

Then we define ${}^RB_0^\sigma(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{{}^RB_0^\sigma(\Omega)}$.

Lemma 2.4 ([27])**.** *If $0 < p < 1$, $0 < q < 1$, $u(x,0) = 0$, $t > 0$, then*

$${}_0^R\partial_t^{p+q}u(x,t) = {}_0^R\partial_t^p u(x,t)\cdot{}_0^R\partial_t^q u(x,t) = {}_0^R\partial_t^q u(x,t)\cdot{}_0^R\partial_t^p u(x,t).$$

Lemma 2.5 ([26])**.** *For any real $\sigma > 0$, the space ${}^RH_0^\sigma(\Omega)$ with respect to the norm* (2.20) *is complete.*

3. Energy estimates and uniqueness of solution

The a priori estimate method, also called the energy-integral method, is one of the most efficient functional analysis methods and an important technique for solving partial differential equations with integral conditions. It has been successfully used in proving the existence, uniqueness, and continuous dependence of the solutions of PDE's. This method is essentially based on the construction of multipliers for each specific problem, which provides a priori estimate from which it is possible to establish the solvability of the problem.

Our proof is based on an energy inequality and the density of the range of the operator generated by the abstract formulation of the problem. First we introduce the needed function spaces, and then prove the existence and the uniqueness for solution of (2.12)-(2.15) as a solution of the operator equation

$$Lu = f. \tag{3.1}$$

Here $L = (\mathcal{L}, \ell)$, with domain E consisting of functions $u \in L_2(0,T,L_2(0,1)) := L^2(\Omega)$ such that ${}_0^R\partial_t^\sigma u$, u_x, $u_{xx} \in L^2(\Omega)$ and u satisfies condition (2.15); the operator L is considered from E to $L^2(\Omega)$, where E is a Banach space (it can be verified using Lemma 2.5) consisting of all functions $u(x,t)$ having a finite norm

$$\|u\|^2_E = \|u\|^2_{{}^RB_0^\sigma(\Omega)},$$

and $L^2(\Omega)$ is the Hilbert space consisting of all elements f for which the norm $L^2(\Omega)$ is finite.

Theorem 3.1. *Let $a(x,t)-\frac{1}{2}\frac{\partial^2 a(x,t)}{\partial x^2}-\frac{\varepsilon}{2}>0$, where $\varepsilon \ll 1$. Then for any function $u\in E$ and we have the inequality*

$$\|u\|_E \le c\|Lu\|_{L^2(\Omega)} \tag{3.2}$$

where c is a positive constant independent of u.

Proof. Multiplying (2.12) by $Mu=\int_x^1(\int_0^\xi u(\eta,t)d\eta)d\xi$ and integrating over Ω^τ, where $\Omega^\tau=(0,1)\times(0,\tau)$, we obtain

$$\begin{aligned}
\int_{\Omega^\tau}\mathcal{L}u\cdot Mu\,dx\,dt &= \int_{\Omega^\tau} {}_0^R\partial_t^\alpha u(x,t)\Big(\int_x^1\Big(\int_0^\xi u(\eta,t)d\eta\Big)d\xi\Big)\,dx\,dt\\
&\quad -\int_{\Omega^\tau}\frac{\partial}{\partial x}(a(x,t)\frac{\partial u}{\partial x})\Big(\int_x^1\Big(\int_0^\xi u(\eta,t)d\eta\Big)d\xi\Big)\,dx\,dt\\
&= \int_{\Omega^\tau} f(x,t)\Big(\int_x^1\Big(\int_0^\xi u(\eta,t)d\eta\Big)d\xi\Big)\,dx\,dt
\end{aligned}\tag{3.3}$$

integrating by parts each term of the left-hand side of (3.3), and using conditions (2.13) -(2.15), and Lemma 2.4, we obtain

$$\begin{aligned}
&\int_{\Omega^\tau} {}_0^R\partial_t^\alpha u(x,t)\Big(\int_x^1(\int_0^\xi u(\eta,t)d\eta)d\xi\Big)\,dx\,dt\\
&= \int_{\Omega^\tau} {}_0^R\partial_t^\alpha\Big(\int_0^x u(\xi,t)d\xi\Big)\Big(\int_0^x u(\xi,t)d\xi\Big)\,dx\,dt\\
&= \int_{\Omega^\tau}\Big({}_0^R\partial_t^{\frac{\alpha}{2}}(\int_0^x u(\xi,t)d\xi)\Big)^2\,dx\,dt.
\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}
&-\int_{\Omega^\tau}\frac{\partial}{\partial x}(a(x,t)\frac{\partial u}{\partial x})\Big(\int_x^1\Big(\int_0^\xi u(\eta,t)d\eta\Big)d\xi\Big)\,dx\,dt\\
&= -\int_0^\tau\Big(\int_x^1\Big(\int_0^\xi u(\eta,t)d\eta\Big)d\xi\Big)(a(x,t)\frac{\partial u}{\partial x})\big|_{x=0}^{x=1}dt\\
&\quad -\int_{\Omega^\tau} a(x,t)\frac{\partial u}{\partial x}\Big(\int_0^x u(\xi,t)d\xi\Big)\,dx\,dt\\
&= -\int_0^\tau\Big(\int_0^x u(\xi,t)d\xi\Big)(a(x,t)u(x,t))\big|_{x=0}^{x=1}dt\\
&\quad +\int_{\Omega^\tau}\frac{\partial a}{\partial x}u(x,t)(\int_0^x u(\xi,t)d\xi)\,dx\,dt+\int_{\Omega^\tau} a(x,t)(u(x,t))^2\,dx\,dt\\
&= \frac{1}{2}\int_0^\tau\frac{\partial a}{\partial x}\Big(\int_0^x u(\xi,t)d\xi\Big)^2\big|_{x=0}^{x=1}dt-\frac{1}{2}\int_{\Omega^\tau}\frac{\partial^2 a}{\partial x^2}\Big(\int_0^x u(\xi,t)d\xi\Big)^2\,dx\,dt\\
&\quad +\int_{\Omega^\tau} a(x,t)(u(x,t))^2\,dx\,dt\\
&= \int_{\Omega^\tau} a(x,t)(u(x,t))^2\,dx\,dt-\frac{1}{2}\int_{\Omega^\tau}\frac{\partial^2 a}{\partial x^2}\Big(\int_0^x u(\xi,t)d\xi\Big)^2\,dx\,dt.
\end{aligned}\tag{3.5}$$

Using the Cauchy inequality with ε and integrating by parts the right hand side, we can estimate

$$
\begin{aligned}
&\int_{\Omega^\tau} f(x,t)\Big(\int_x^1 \Big(\int_0^\xi u(\eta,t)d\eta\Big)d\xi\Big)\,dx\,dt \\
&= -\int_0^\tau \Big(\int_x^1 \Big(\int_0^\xi u(\eta,t)d\eta\Big)d\xi\Big)\Big(\int_0^x f(\xi,t)d\xi\big)\Big|_{x=0}^{x=1}dt \\
&\quad + \int_{\Omega^\tau} \Big(\int_0^x f(\xi,t)d\xi\Big)\Big(\int_0^x u(\xi,t)d\xi\Big)\,dx\,dt \\
&\leqslant \frac{\varepsilon}{2}\int_{\Omega^\tau} \Big(\int_0^x u(\xi,t)d\xi\Big)^2\,dx\,dt + \frac{1}{2\varepsilon}\int_{\Omega^\tau} \Big(\int_0^x f(\xi,t)d\xi\Big)^2\,dx\,dt \\
&\leqslant \frac{\varepsilon}{2}\int_{\Omega^\tau} (u(x,t))^2\,dx\,dt + \frac{1}{2\varepsilon}\int_{\Omega^\tau} (f(x,t))^2\,dx\,dt.
\end{aligned}
\tag{3.6}
$$

Substituting (3.4)-(3.6) into (3.3), we obtain

$$
\begin{aligned}
&\int_{\Omega^\tau} \Big({}_0^R\partial_t^{\frac{\alpha}{2}}(\int_0^x u(\xi,t)d\xi)\Big)^2\,dx\,dt + \int_{\Omega^\tau} a(x,t)(u(x,t))^2\,dx\,dt \\
&\leqslant \frac{\varepsilon}{2}\int_{\Omega^\tau} (u(x,t))^2\,dx\,dt + \frac{1}{2\varepsilon}\int_{\Omega^\tau} (f(x,t))^2\,dx\,dt \\
&\quad + \frac{1}{2}\int_{\Omega^\tau} \frac{\partial^2 a}{\partial x^2}\Big(\int_0^x u(\xi,t)d\xi\Big)^2\,dx\,dt \\
&\leqslant \int_{\Omega^\tau} (\frac{1}{2}\frac{\partial^2 a}{\partial x^2} + \frac{\varepsilon}{2})(u(x,t))^2\,dx\,dt + \frac{1}{2\varepsilon}\int_{\Omega^\tau} (f(x,t))^2\,dx\,dt.
\end{aligned}
$$

Since $a(x,t) - \frac{1}{2}\frac{\partial^2 a(x,t)}{\partial x^2} - \frac{\varepsilon}{2} > 0$, we obtain

$$
\begin{aligned}
&\int_{\Omega^\tau} \Big({}_0^R\partial_t^{\frac{\alpha}{2}}\Big(\int_0^x u(\xi,t)d\xi\Big)\Big)^2\,dx\,dt + \int_{\Omega^\tau} (u(x,t))^2\,dx\,dt \\
&\leqslant \frac{1}{2\varepsilon}\int_{\Omega^\tau} (f(x,t))^2\,dx\,dt,
\end{aligned}
\tag{3.7}
$$

from (2.7), we have

$$
\int_{\Omega^\tau} \Big(\int_0^x u(\xi,t)d\xi\Big)^2\,dx\,dt \leqslant \frac{1}{2}\int_{\Omega^\tau} (u(x,t))^2\,dx\,dt,
$$

then, (3.7) becomes

$$
\begin{aligned}
&\int_{\Omega^\tau} \Big({}_0^R\partial_t^{\frac{\alpha}{2}}\Big(\int_0^x u(\xi,t)d\xi\Big)\Big)^2\,dx\,dt + 2\int_{\Omega^\tau} \Big(\int_0^x u(\xi,t)d\xi\Big)^2\,dx\,dt \\
&\leqslant \int_{\Omega^\tau} \Big({}_0^R\partial_t^{\frac{\alpha}{2}}\Big(\int_0^x u(\xi,t)d\xi\Big)\Big)^2\,dx\,dt + \int_{\Omega^\tau} (u(x,t))^2\,dx\,dt \\
&\leqslant \frac{1}{2\varepsilon}\int_{\Omega^\tau} (f(x,t))^2\,dx\,dt.
\end{aligned}
$$

So, we obtain

$$
\begin{aligned}
&\int_{\Omega^\tau} \Big({}_0^R\partial_t^{\frac{\alpha}{2}}\Big(\int_0^x u(\xi,t)d\xi\Big)\Big)^2\,dx\,dt + \int_{\Omega^\tau} \Big(\int_0^x u(\xi,t)d\xi\Big)^2\,dx\,dt \\
&\leqslant \frac{1}{2\varepsilon}\int_{\Omega^\tau} (f(x,t))^2\,dx\,dt,
\end{aligned}
\tag{3.8}
$$

The right-hand side of (3.8) is independent of τ, hence replacing the left-hand side by its upper bound with respect to τ from 0 to T, we obtain the desired inequality, where $c = (1/(2\varepsilon))^{1/2}$. $\square$

Proposition 3.2. *The operator L from E to F admits a closure.*

Theorem 3.1 is valid for strong solutions; i.e., we have the inequality

$$\|u\|_B \leq c\|\overline{L}u\|_F, \quad \forall u \in D(\overline{L}). \tag{3.9}$$

Hence we obtain

Corollary 3.3. *A strong solution of* (2.12)-(2.15) *is unique if it exists, and depends continuously on* $\mathrm{F} \in F$.

Corollary 3.4. *The range $R(\overline{L})$ of the operator $\overline{L}$ is closed in F, and $R(\overline{L}) = \overline{R(L)}$.*

4. Existence of solutions

To show the existence of solutions, we prove that $R(L)$ is dense in $L^2(\Omega)$ for all $u \in E$ and for arbitrary $f \in L^2(\Omega)$.

Theorem 4.1. *Let the conditions of Theorem 3.1 be satisfied. if, for $\omega \in L^2(\Omega)$ and for all $u \in E$, we have*

$$\int_\Omega \mathcal{L}u.\omega \, dx\, dt = 0, \tag{4.1}$$

then ω vanishes almost everywhere in Ω, this implies that (2.12)-(2.15) *admits a unique solution* $u = L^{-1}\mathrm{F}$.

Proof. The scalar product in F is defined by

$$(Lu, \omega)_{L^2(\Omega)} = \int_\Omega \mathcal{L}u\omega \, dx\, dt, \tag{4.2}$$

then (4.1) can be written as

$$\int_\Omega {}_0^R\partial_t^\alpha u(x,t) \cdot \omega \, dx\, dt = \int_\Omega \frac{\partial}{\partial x}\Big(a(x,t)\frac{\partial u}{\partial x}\Big) \cdot \omega \, dx\, dt. \tag{4.3}$$

If we put

$$u(x,t) = \Im_t(z(x,\tau)) = \int_0^t z(x,\tau)\, d\tau,$$

where $z, \frac{\partial z}{\partial x}, \frac{\partial}{\partial x}\big(a\frac{\partial \Im_t(z(x,\tau))}{\partial x}\big), {}_0^R\partial_t^\alpha z \in L^2(\Omega)$ and z satisfies the same conditions (2.13)-(2.15). As a result of (4.3), we obtain the equality

$$\int_\Omega {}_0^R\partial_t^\alpha(\Im_t(z(x,\tau))) \cdot \omega \, dx\, dt = \int_\Omega \frac{\partial}{\partial x}(a(x,t)\frac{\partial \Im_t(z(x,\tau))}{\partial x})\omega \, dx\, dt. \tag{4.4}$$

In terms of the given function ω, and from the equality (4.4) we give the function ω in terms of z as

$$\omega = \int_x^1 \Big(\int_0^\xi (\Im_t(z(\eta,\tau)\, d\tau))d\eta\Big)d\xi. \tag{4.5}$$

So, $\omega \in L^2(\Omega)$.

Replacing ω in (4.4) by its representation (4.5) and integrating by parts each term of (4.4) and by taking the condition of z, we obtain

$$\begin{aligned}
&\int_\Omega {}_0^R\partial_t^\alpha(\Im_t(z(x,\tau)))\cdot\omega\,dx\,dt \\
&= \int_\Omega \Big({}_0^R\partial_t^{\alpha/2}\Big(\int_0^x \Im_t(z(\xi,\tau)\,d\tau)d\xi\Big)\Big)^2\,dx\,dt,
\end{aligned}\tag{4.6}$$

and

$$\begin{aligned}
&\int_\Omega \frac{\partial}{\partial x}(a\frac{\partial\Im_t(z(x,\tau))}{\partial x})\omega\,dx\,dt \\
&= \int_\Omega \frac{\partial}{\partial x}\Big(a\frac{\partial\Im_t(z(x,\tau))}{\partial x}\Big)\Big(\int_x^1(\int_0^\xi(\Im_t(z(\eta,\tau)\,d\tau))d\eta)d\xi\Big)\,dx\,dt \\
&= \int_0^T\Big(\int_x^1\Big(\int_0^\xi(\Im_t(z(\eta,\tau)\,d\tau))d\eta\Big)d\xi\Big)\big(a\frac{\partial\Im_t(z)}{\partial x})\big|_{x=0}^{x=1}dt \\
&\quad + \int_\Omega\big(a\frac{\partial\Im_t(z)}{\partial x}\big)\Big(\Big(\int_0^x(\Im_t(z(\xi,\tau)\,d\tau))d\xi\Big)\Big)\,dx\,dt \\
&= \int_0^T\Big(a\int_0^x(\Im_t(z(\xi,\tau)\,d\tau))d\xi\Big)\Big(\Im_t(z)\Big)\big|_{x=0}^{x=1}dt \\
&\quad - \int_\Omega \Im_t(z)\Big(\int_0^x\Im_t(z)\Big)\,dx\,dt - \int_\Omega a(\Im_t(z))^2\,dx\,dt \\
&= -\frac12\int_0^T\frac{\partial a}{\partial x}\Big(\int_0^x\Im_t(z)\Big)^2\big|_{x=0}^{x=1}dt + \frac12\int_\Omega\frac{\partial^2 a}{\partial x^2}\Big(\int_0^x\Im_t(z)\Big)^2\,dx\,dt \\
&\quad - \int_\Omega a(\Im_t(z))^2\,dx\,dt \\
&= \frac12\int_\Omega\frac{\partial^2 a}{\partial x^2}\Big(\int_0^x\Im_t(z(\xi,\tau)\,d\tau)d\xi\Big)^2\,dx\,dt - \int_\Omega a(x,t)(\Im_t(z))^2\,dx\,dt.
\end{aligned}$$

By combining the above expression and (4.6), we obtain

$$\begin{aligned}
&\int_\Omega\Big({}_0^R\partial_t^{\frac{\alpha}{2}}\Big(\int_0^x\Im_t(z(\xi,\tau)\,d\tau)d\xi\Big)\Big)^2\,dx\,dt \\
&= \frac12\int_\Omega\frac{\partial^2 a}{\partial x^2}\Big(\int_0^x\Im_t(z(\xi,\tau)\,d\tau)d\xi\Big)^2\,dx\,dt - \int_\Omega a(x,t)(\Im_t(z))^2\,dx\,dt,
\end{aligned}\tag{4.7}$$

estimated the right-hand side of (4.7), we obtain

$$\begin{aligned}
&\int_\Omega\Big({}_0^R\partial_t^{\alpha/2}\Big(\int_0^x\Im_t(z(\xi,\tau)\,d\tau)d\xi\Big)\Big)^2\,dx\,dt \\
&= \frac12\int_\Omega\frac{\partial^2 a}{\partial x^2}\Big(\int_0^x\Im_t(z(\xi,\tau)\,d\tau)d\xi\Big)^2\,dx\,dt - \int_\Omega a(x,t)\big(\Im_t(z)\big)^2\,dx\,dt \\
&\leqslant \frac12\int_\Omega\big[\frac12\frac{\partial^2 a}{\partial x^2} - a(x,t)\big](\Im_t z(\xi,\tau)\,d\tau)^2\,dx\,dt,
\end{aligned}$$

since $a(x,t)-\frac12\frac{\partial^2 a(x,t)}{\partial x^2}-\frac{\varepsilon}{2}>0$, we obtain

$$\int_\Omega\Big({}_0^R\partial_t^{\frac{\alpha}{2}}\Big(\int_0^x\Im_t(z(\xi,\tau)\,d\tau)d\xi\Big)\Big)^2\,dx\,dt$$

$$\leqslant \frac{1}{2}\int_{\Omega}\big[\frac{1}{2}\frac{\partial^2 a}{\partial x^2}-a(x,t)\big](\Im_t z(\xi,\tau)\,d\tau)^2\,dx\,dt\leqslant 0.$$

And thus $z=0$ in Ω, then $\omega=0$ in Ω. This completes the proof. □

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