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FUNCTIONAL DIFFERENTIAL EQUATIONS WITH UNBOUNDED DELAY IN EXTRAPOLATION SPACES

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Abstract. We study the existence, regularity and stability of solutions for nonlinear partial neutral functional differential equations with unbounded delay and a Hille-Yosida operator on a Banach space X. We consider two nonlinear perturbations: the first one is a function taking its values in X and the second one is a function belonging to a space larger than X, an extrapolated space. We use the extrapolation techniques to prove the existence and regularity of solutions and we establish a linearization principle for the stability of the equilibria of our equation.

1. Introduction

In this work, we study the existence, regularity and stability of solutions for the neutral functional differential equations with infinite delay

$$\begin{gathered} \frac{d}{dt}[x(t) - F(x_t)] = A[x(t) - F(x_t)] + G(x_t) \quad \text{for } t \geq 0, \\ x(t) = \varphi(t), \quad \text{for } t \leq 0,\ \varphi \in \mathcal{B}, \end{gathered} \tag{1.1}$$

where $A : D(A) \to X$ is a linear operator on a Banach space X. We assume that A is not necessarily densely defined and satisfies the Hille-Yosida condition. This means that A satisfies the usual assumptions of the Hille-Yosida theorem characterizing the generator of a C_0-semigroup except the density of the domain $D(A)$ in X: there exist $N_0 \geq 1$ and $\omega_0 \in \mathbb{R}$ such that $(\omega_0, +\infty) \subset \rho(A)$ and

$$\sup\{(\lambda - \omega_0)^n |(\lambda I - A)^{-n}| : n \in \mathbb{N},\ \lambda > \omega_0\} \leq N_0,$$

where $\rho(A)$ is the resolvent set of the operator A. The phase space $\mathcal{B}$ is a linear space of functions from $(-\infty, 0]$ into X satisfying some assumptions which they will be described in the sequel. For every $t \geq 0$, the function $x_t \in \mathcal{B}$ is defined by

$$x_t(\theta) = x(t + \theta) \quad \text{for } \theta \in (-\infty, 0].$$

F is a Lipschitz continuous function from $\mathcal{B}$ to X, and G is a continuous function from $\mathcal{B}$ with values in the space $\mathbb{F}_1$, where $\mathbb{F}_1$, larger than X, is the extrapolation space associated to the C_0-semigroup generated by the part of the operator A in $X_0 = \overline{D(A)}$ (see Section 2).

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Wu and Xia [26, 27] studied a system of partial neutral functional differential equations defined on the unit circle S, which is a model for a continuous circular array of identical resistively coupled transmission lines with mixed initial boundary conditions. This system is

$$\begin{gathered}\frac{\partial}{\partial t}[x(.,t)-qx(.,t-r)]=k\frac{\partial^2}{\partial \xi^2}[x(.,t)-qx(.,t-r)]+f(x_t) \quad \text{for } t\geq 0,\\ x_0=\varphi\in C([-r,0];H^1(S)),\end{gathered}\tag{1.2}$$

where $x_t(\xi,\theta)=x(\xi,t+\theta)$, $-r\leq\theta\leq 0$, $t\geq 0$, $\xi\in S$, k is a positive constant, and $0\leq q<1$. The space of initial data was chosen to be $C([-r,0];H^1(S))$.

Motivated by this work, Hale [13, 14] presented the basic theory of existence, uniqueness and properties of the solution operator of equation (1.2), as well as Hopf bifurcation and conditions for the stability and instability of periodic orbits.

Adimy, Ezzinbi and their collaborators [2, 3, 4, 5, 6] considered (1.1) with finite delay and the function G taking its values in the Banach space X. They established the basic theory of existence, uniqueness, stability and some properties of the solution operator. In the literature devoted to partial functional differential equations with finite delay $r>0$, the state space is always the space of continuous functions on $[-r,0]$, and the variation of constants formula is the main tool for studying the properties of the solution operator. For more details, we refer to Travis and Webb [24], and Wu [25].

When the delay is unbounded the situation is more complicated since the properties of the solutions depend on the phase space $\mathcal{B}$. The choice of this space plays an important role in both quantitative and qualitative studies. A usual choice of $\mathcal{B}$ is a Banach space satisfying some assumptions which make the system well-posed. For the basic theory of functional differential equations with infinite delay in finite dimensional spaces, we refer to Hale and Kato [15], and Hino, Murakami and Naito [18]. This theory was extended to partial functional differential equations with infinite delay by Henriquez [16] in 1994. Since then, many other authors investigated partial functional differential equations with infinite delay by considering different phase spaces $\mathcal{B}$.

In [1, 7, 12, 17], the authors studied some classes of partial neutral functional differential equations with unbounded delay. In [1] Adimy, Bouzahir and Ezzinbi used the theory of integrated semigroups to study the existence and uniqueness of mild solutions for a class of partial neutral functional differential equations with unbounded delay. Chang [7] considered a generator of an analytic compact C_0-semigroup and assumed that the nonlinear part is continuous with respect to fractional powers of this generator. He studied the existence and uniqueness of solutions of partial neutral functional differential equations with unbounded delay. Ezzinbi, Ghnimi and Taoudi [12] introduced a new concept of the resolvent operator adapted to a class of non-autonomous partial neutral functional differential equations with unbounded delay. They gave some basic results on the existence and uniqueness of solutions. Hernandez and Henriquez [17] established some results of existence of periodic solutions for a class of partial neutral functional differential equations with unbounded delay and appropriate nonlinear functions defined on a phase space.

Ezzinbi [11] investigated (1.1) in the particular case where $F=0$, the function G is continuous from $\mathcal{B}$ with values in the extrapolation space $\mathbb{F}_1$ and the delay is unbounded. He used an approach based on the theory of the extrapolation spaces to study the existence, uniqueness, regularity and asymptotic behavior of solutions.

The theory of extrapolation spaces was introduced by Da Prato and Grisvard [8] in 1982 (see also Engel and Nagel [10]). It was used by Nagel and Sinestrari [20] for a class of Volterra Integrodifferential equations with Hille-Yosida operators, and by Maniar and Rhandi [19] for retarded differential equations in infinite dimensional spaces. The use of this theory allows to consider nonlinear perturbations belonging to a class of spaces, larger than the space in which the unperturbed system is defined.

The main tools used to investigate (1.1) are based on the variation of constants formula. The nonlinear functions F and G are not defined in the same space. Then, we cannot consider the classical variation of constants formula introduced in our previous works. We use the extrapolation methods introduced in [8] to construct a new variation of constants formula adapted to (1.1). Then, we study the existence and regularity of mild solutions of (1.1). We establish a linearization principle for the stability of the equilibria. For the regularity of mild solutions, we adapt the method developed in [24] for partial functional differential equations with finite delay and to establish the linearization principle, we use an approach developed in [22] and [23]. This work is an extension of [11] to partial neutral functional differential equations with infinite delay.

2. Extrapolation spaces and Favard class

Throughout this article, we assume that the operator $A : D(A) \subset X \to X$ satisfies the Hille-Yosida condition on a Banach space X:

(H1) there exist $N_0 \geq 1$ and $\omega_0 \in \mathbb{R}$ such that $(\omega_0, +\infty) \subset \rho(A)$ and
$$\sup\{(\lambda-\omega_0)^n|(\lambda I - A)^{-n}| : n \in \mathbb{N},\ \lambda > \omega_0\} \leq N_0.$$

Let A_0 be the part of A in $X_0 := \overline{D(A)}$ which is defined by
$$D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\},$$
$$A_0x = Ax \quad \text{for } x \in D(A_0).$$

Then, we have the following classical result.

Lemma 2.1 ([10]). *A_0 generates a C_0-semigroup $(T_0(t))_{t\geq 0}$ on X_0 with $|T_0(t)| \leq N_0e^{\omega_0 t}$, for $t \geq 0$. Moreover, $\rho(A) \subset \rho(A_0)$ and $R(\lambda, A_0) = R(\lambda, A)\big|_{X_0}$ for $\lambda \in \rho(A)$, where $R(\lambda, A)\big|_{X_0}$ is the restriction of $R(\lambda, A)$ to X_0.*

For a fixed $\lambda_0 \in \rho(A)$, we introduce on X_0 the norm
$$|x|_{-1} = |R(\lambda_0, A_0)x| \quad \text{for } x \in X_0.$$

The completion X_{-1} of $(X_0, |\cdot|_{-1})$ is called the extrapolation space of X associated with the operator A. The norm $|\cdot|_{-1}$, associated with $\lambda_0 \in \rho(A)$ and any other norm on X_0 given for $\lambda \in \rho(A)$ by $|R(\lambda, A_0)x|$ are equivalent. The operator $T_0(t)$ has a unique bounded linear extension $T_{-1}(t)$ to the Banach space X_{-1} and $(T_{-1}(t))_{t\geq 0}$ is a C_0-semigroup on X_{-1}. $(T_{-1}(t))_{t\geq 0}$ is called the extrapolated semigroup of $(T_0(t))_{t\geq 0}$. We denote by $(A_{-1}, D(A_{-1}))$ the generator of $(T_{-1}(t))_{t\geq 0}$ on the space X_{-1}.

For a Banach space Y, we denote by $L(Y)$ the space of bounded linear operators on Y. We have the following fundamental results.

Lemma 2.2 ([10]). *The following properties hold:*

(i) $|T_{-1}(t)|_{L(X_{-1})} = |T_0(t)|_{L(X_0)}$;
(ii) $D(A_{-1}) = X_0$;
(iii) $A_{-1} : X_0 \to X_{-1}$ *is the unique continuous extension of the operator* $A_0 : D(A_0) \subseteq (X_0, |.|) \to (X_{-1}, |.|_{-1})$ *and* $(\lambda_0 I - A_{-1})$ *is an isometry from* $(X_0, |\cdot|)$ *to* $(X_{-1}, |\cdot|_{-1})$;
(iv) *If* $\lambda \in \rho(A)$, *then* $(\lambda I - A_{-1})$ *is invertible and* $(\lambda I - A_{-1})^{-1} \in L(X_{-1})$. *In particular,* $\lambda \in \rho(A_{-1})$ *and* $R(\lambda, A_{-1})\big|_{X_0} = R(\lambda, A_0)$;
(v) *The space* $X_0 = \overline{D(A)}$ *is dense in* $(X_{-1}, |\cdot|_{-1})$. *Hence, the extrapolation space* X_{-1} *is also the completion of* $(X, |\cdot|_{-1})$ *and we have* $X \hookrightarrow X_{-1}$;
(vi) *The operator* A_{-1} *is an extension of the operator* A. *In particular, if* $\lambda \in \rho(A)$ *then* $R(\lambda, A_{-1})\big|_{X_0} = R(\lambda, A)$ *and* $R(\lambda, A_{-1})(X) = D(A)$.

Next we introduce the Favard class of the C_0-semigroup $(T_0(t))_{t\geq 0}$.

Definition 2.3 ([10]). Let $(S(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space Y such that $|S(t)| \leq Ne^{\nu t}$ for some $N \geq 1$ and $\nu \in \mathbb{R}$. The Favard class of $(S(t))_{t\geq 0}$ is the space

$$\mathbb{F} = \{x \in Y : \sup_{t>0}\big(\frac{1}{t}|e^{-\nu t}S(t)x - x|\big) < +\infty\}.$$

This space equipped with the norm

$$|x|_{\mathbb{F}} = |x| + \sup_{t>0}\big(\frac{1}{t}|e^{-\nu t}S(t)x - x|\big),$$

is a Banach space.

For the rest of this article, we denote by $\mathbb{F}_0 \subset X_0$ the Favard class of the C_0-semigroup $(T_0(t))_{t\geq 0}$ and by $\mathbb{F}_1$ the Favard class of the C_0-semigroup $(T_{-1}(t))_{t\geq 0}$.

Lemma 2.4 ([10]). *For the Banach spaces* $\mathbb{F}_0$ *and* $\mathbb{F}_1$ *the following properties hold:*

(i) $(\lambda_0 I - A_{-1})(\mathbb{F}_0) = \mathbb{F}_1$;
(ii) $T_{-1}(t)(\mathbb{F}_1) \subset \mathbb{F}_1$ *for* $t \geq 0$;
(iii) $D(A_0) \hookrightarrow D(A) \hookrightarrow \mathbb{F}_0 \hookrightarrow X_0 \hookrightarrow X \hookrightarrow \mathbb{F}_1 \hookrightarrow X_{-1}$, *where* $D(A)$ *is equipped with the graph norm.*

Proposition 2.5 ([20]). *For* $f \in L^1_{\rm loc}(\mathbb{R}^+, \mathbb{F}_1)$, *we define*

$$(T_{-1} * f)(t) = \int_0^t T_{-1}(t-s)f(s)ds \quad \text{for } t \geq 0.$$

Then

(i) $(T_{-1} * f)(t) \in X_0$ *for all* $t \geq 0$;
(ii) $|(T_{-1} * f)(t)| \leq Me^{\omega t}\int_0^t e^{-\omega s}|f(s)|_{\mathbb{F}_1} ds$, *where* M *is a constant independent of* f;
(iii) $\lim_{t\to 0}|(T_{-1} * f)(t)| = 0$.

Remark 2.6. Assertion (iii) in Proposition 2.5 implies that the function

$$T_{-1} * f : t \to \int_0^t T_{-1}(t-s)f(s)ds$$

is continuous from $\mathbb{R}^+$ to X_0.

3. Existence, uniqueness and regularity of solutions

Let $\mathcal{B}$ be the phase space of (1.1). That is a linear space of functions from $(-\infty, 0]$ into X satisfying the following two assumptions (see [15]).

(A1) There exist a constant $H > 0$ and functions $K, M : \mathbb{R}^+ \to \mathbb{R}^+$ with K continuous and $M \in L^\infty_{\rm loc}(\mathbb{R}^+)$ such that for all $\sigma \in \mathbb{R}$ and any $a > 0$ if $x : (-\infty, \sigma + a] \to X$ is such that $x_\sigma \in \mathcal{B}$ and $x : [\sigma, \sigma + a] \to X$ is continuous, then for all $t \in [\sigma, \sigma + a]$ we have
  - (i) $x_t \in \mathcal{B}$,
  - (ii) $|x(t)| \le H|x_t|_\mathcal{B}$,
  - (iii) $|x_t|_\mathcal{B} \le K(t - \sigma) \sup_{\sigma \le s \le t} |x(s)| + M(t - \sigma)|x_\sigma|_\mathcal{B}$.

(A2) the function $t \to x_t$ is continuous from $[\sigma, \sigma + a]$ to $\mathcal{B}$.

(B1) $\mathcal{B}$ is complete.

For the nonlinear functions F and G, we assume that they are Lipschitz continuous.

(H2) F is a Lipschitz continuous function from $\mathcal{B}$ to X:
$$|F(\varphi_1) - F(\varphi_2)| \le L_0|\varphi_1 - \varphi_2|_\mathcal{B} \quad \text{for } \varphi_1, \varphi_2 \in \mathcal{B}.$$

(H3) G is a Lipschitz continuous function from $\mathcal{B}$ to $\mathbb{F}_1$:
$$|G(\varphi_1) - G(\varphi_2)|_{\mathbb{F}_1} \le L_1|\varphi_1 - \varphi_2|_\mathcal{B} \quad \text{for } \varphi_1, \varphi_2 \in \mathcal{B}.$$

All the results in this work are obtained by assuming that the function K satisfies (A1), and the Lischitz constant L_0 in (H2) satisfies

(H4) $L_0K(0) < 1$.

This assumption implies that there exists $a > 0$ such that $L_0K_a < 1$ where $K_a = \sup_{0 \le t \le a}(K(t))$.

We need the following fundamental prior estimation.

Lemma 3.1. *Assume that* (H2) *and* (H4) *hold and let $a > 0$ be such that $L_0K_a < 1$. Let $\psi \in \mathcal{B}$ and $h \in C([0, a]; X)$ be such that $\psi(0) - F(\psi) = h(0)$. Then, there exists a unique continuous function x on $[0, a]$ such that*
$$\begin{gathered} x(t) - F(x_t) = h(t) \quad \text{for } t \in [0, a], \\ x(t) = \psi(t) \quad \text{for } t \in (-\infty, 0]. \end{gathered} \tag{3.1}$$
Moreover, there exist $\alpha_a > 0$ and $\beta_a > 0$ such that
$$|x_t|_\mathcal{B} \le \alpha_a|\psi|_\mathcal{B} + \beta_a \sup_{0 \le s \le t} |h(s)| \quad \text{for } t \in [0, a]. \tag{3.2}$$

Proof. We introduce the space
$$Y = \{x \in C([0, a]; X) : x(0) = \psi(0)\}$$
endowed with the uniform norm topology. For $x \in Y$, we define its extension $\tilde{x}$ over $(-\infty, 0]$ by
$$\tilde{x}(t) = \begin{cases} x(t) & \text{for } t \in [0, a], \\ \psi(t) & \text{for } t \in (-\infty, 0]. \end{cases}$$
Then, by (A2), the function $t \to \tilde{x}_t$ is continuous from $[0, a]$ to $\mathcal{B}$. Let us now define the operator $\mathbb{K}$ by
$$(\mathbb{K}(x))(t) = F(\tilde{x}_t) + h(t) \quad \text{for } t \ge 0.$$

We have to show that $\mathbb{K}$ has a unique fixed point on Y. Since $h \in C([0,a];X)$ and $\psi(0) - F(\psi) = h(0)$, then $\mathbb{K}(Y) \subset Y$. Moreover,

$$|(\mathbb{K}(x) - \mathbb{K}(y))(t)| \leq L_0|\tilde{x}_t - \tilde{y}_t|_{\mathcal{B}}$$

By the property $L_0K_a < 1$, we obtain that $\mathbb{K}$ is a strict contraction. By a Banach fixed point theorem, we deduce the existence and uniqueness of x, solution of Problem (3.1) on the interval $(-\infty, a]$. By (A1)-(iii), for $t \in [0,a]$ we have

$$|x_t|_{\mathcal{B}} \leq K_a \sup_{0\leq s\leq t} |x(s)| + M_a|x_0|_{\mathcal{B}}, \quad \text{where } M_a = \sup_{0\leq s\leq a} M(s).$$

It follows that

$$\begin{aligned} |x_t|_{\mathcal{B}} &\leq K_a \sup_{0\leq s\leq t} (|F(x_s)| + |h(s)|) + M_a|\psi|_{\mathcal{B}}, \\ &\leq K_a(L_0 \sup_{0\leq s\leq t} (|x_s - \psi|_{\mathcal{B}}) + |F(\psi)| + \sup_{0\leq s\leq t} |h(s)|) + M_a|\psi|_{\mathcal{B}}. \end{aligned}$$

Since $\psi(0) - F(\psi) = h(0)$, we deduce that

$$(1 - K_aL_0)|x_t|_{\mathcal{B}} \leq (K_aL_0 + M_a)|\psi|_{\mathcal{B}} + K_a \sup_{0\leq s\leq t} |h(s)| + K_a(|\psi(0)| + |h(0)|).$$

By (A1)-(ii), we obtain

$$(1 - K_aL_0)|x_t|_{\mathcal{B}} \leq (K_aL_0 + M_a + K_aH)|\psi|_{\mathcal{B}} + 2K_a \sup_{0\leq s\leq t} |h(s)|.$$

Finally, we arrive to

$$|x_t|_{\mathcal{B}} \leq \alpha_a|\psi|_{\mathcal{B}} + \beta_a \sup_{0\leq s\leq t} |h(s)| \quad \text{for } t \in [0,a],$$

where

$$\alpha_a = \frac{K_a(L_0 + H) + M_a}{1 - K_aL_0} \quad \text{and} \quad \beta_a = \frac{2K_a}{1 - K_aL_0}.$$

□

Definition 3.2. Let $a > 0$. A function $x : (-\infty, a] \to X$ is called a mild solution of (1.1) on $(-\infty, a]$ if x is continuous and satisfies

$$\begin{gathered} x(t) - F(x_t) = T_0(t)[\varphi(0) - F(\varphi)] + \int_0^t T_{-1}(t-s)G(x_s)ds \quad \text{for } t \in [0,a], \\ x(t) = \varphi(t) \quad \text{for } t \in (-\infty, 0]. \end{gathered} \tag{3.3}$$

Theorem 3.3. *Assume that* (H1), (H2), (H3), (H4) *hold. Let $a > 0$ be fixed such that $L_0K_a < 1$. Then, for $\varphi \in \mathcal{B}$ such that*

$$\varphi(0) - F(\varphi) \in X_0,$$

Equation (1.1) *has a unique mild solution $x(.,\varphi)$ defined on $(-\infty, a]$. Moreover, if $L_0K_\infty < 1$, where $K_\infty = \sup_{t\geq 0}(K(t))$, then the unique mild solution $x(.,\varphi)$ is defined on $(-\infty, \infty)$.*

Proof. As in the proof of Lemma 3.1, consider the set

$$Y = \{x \in C([0,a];X) : x(0) = \varphi(0)\},$$

and the extension $\tilde{x}$ of $x \in Y$ over $(-\infty, 0]$ defined by

$$\tilde{x}(t) = \begin{cases} x(t) & \text{for } t \in [0,a], \\ \varphi(t) & \text{for } t \in (-\infty, 0]. \end{cases}$$

Then by (A2), the function $t \to \tilde{x}_t$ is continuous. Let us now define the operator $\mathbb{H}$ by

$$(\mathbb{H}(x))(t) = F(\tilde{x}_t) + T_0(t)[\varphi(0) - F(\varphi)] + \int_0^t T_{-1}(t-s)G(\tilde{x}_s)ds \quad \text{for } t \in [0,a].$$

We have to show that $\mathbb{H}$ has a unique fixed point on Y. In fact, by Proposition 2.5, $\mathbb{H}(Y) \subset Y$. Moreover, for $t \in [0,a]$, we have

$$|(\mathbb{H}(x) - \mathbb{H}(y))(t)| \le L_0|\tilde{x}_t - \tilde{y}_t|_{\mathcal{B}} + N_0 \int_0^t e^{\omega_0(t-s)}|G(\tilde{x}_s) - G(\tilde{y}_s)|_{\mathbb{F}_1} ds.$$

Hence

$$|(\mathbb{H}(x) - \mathbb{H}(y))(t)| \le L_0|\tilde{x}_t - \tilde{y}_t|_{\mathcal{B}} + N_0 L_1 \int_0^t e^{\omega_0(t-s)}|\tilde{x}_s - \tilde{y}_s|_{\mathcal{B}} ds.$$

Without loss of generality, we suppose that $\omega_0 > 0$. Let $b \in (0,a]$. Then, for $t \in [0,b]$

$$|(\mathbb{H}(x) - \mathbb{H}(y))(t)| \le L_0 K_a \sup_{0\le s\le a} |x(s) - y(s)| + N_0 L_1 b K_a e^{\omega_0 a} \sup_{0\le s\le a} |x(s) - y(s)|.$$

Consequently,

$$|(\mathbb{H}(x) - \mathbb{H}(y))(t)| \le (L_0 K_a + N_0 L_1 b K_a e^{\omega_0 a}) \sup_{0\le s\le a} |x(s) - y(s)|.$$

We choose $b \in (0,a]$ such that

$$K_a L_0 + N_0 L_1 b K_a e^{\omega_0 a} < 1.$$

Then, $\mathbb{H}$ is a strict contraction for $t \in [0,b]$. By the Banach fixed point theorem, we have the existence and uniqueness of a mild solution of (1.1) on the interval $(-\infty, b]$. We proceed by steps on each interval $[kb, (k+1)b]$, $k = 0, 1, \dots$ to extend the solution continuously on $(-\infty, a]$. Furthermore, if we suppose $L_0 K_\infty < 1$ then, we can use the same method to extend the solution continuously on $(-\infty, +\infty)$. $\square$

We study now the regularity of the solution. We give a sufficient condition for the mild solution of (1.1) to be continuously differentiable and to satisfy an abstract differential equation. We need some preliminary results on the space $\mathcal{B}$. To this end, we suppose the additionally assumption.

- (C1) If $(\varphi_n)_{n\ge 0}$ is a Cauchy sequence in $\mathcal{B}$ and if $(\varphi_n)_{n\ge 0}$ converges compactly to φ on $(-\infty, 0]$, then φ is in $\mathcal{B}$ and $|\varphi_n - \varphi|_{\mathcal{B}} \to 0$ as $n \to +\infty$.

We recall the following result.

Lemma 3.4 ([21]). *Let $\mathcal{B}$ be satisfy* (C1) *and $f : [0,a] \to \mathcal{B}$ be a continuous function such that the function $(t,\theta) \to f(t)(\theta)$ is continuous on $[0,a] \times (-\infty, 0]$. Then*

$$\Big(\int_0^a f(t)dt\Big)(\theta) = \int_0^a f(t)(\theta)dt \quad \textit{for } \theta \in (-\infty, 0].$$

For the regularity of the mild solutions, we add the following hypotheses on F, G and the initial condition.

- (H5) $F : \mathcal{B} \to X$ is continuously differentiable and F' is locally Lipschitz continuous.
- (H6) $G : \mathcal{B} \to \mathbb{F}_1$ is continuously differentiable and G' is locally Lipschitz continuous.

(H7) $\varphi \in C^1((-\infty,0];X)\cap\mathcal{B}$, $\varphi' \in \mathcal{B}$, $\varphi(0)-F(\varphi) \in \mathbb{F}_0$, $\varphi'(0)-F'(\varphi)\varphi' \in \overline{D(A)}$ and $\varphi'(0) - F'(\varphi)\varphi' = A_{-1}[\varphi(0) - F(\varphi)] + G(\varphi)$.

We have the following theorem.

Theorem 3.5. *Assume that* (H1)–(H7) *hold and let $a > 0$ be such that $L_0K_a < 1$. Then, the mild solution x of* (1.1) *on $(-\infty, a]$ with $x_0 = \varphi \in \mathcal{B}$, belongs to $C^1([0,a],X) \cap C([0,a],\mathbb{F}_0)$ and satisfies*

$$\frac{d}{dt}[x(t) - F(x_t)] = A_{-1}[x(t) - F(x_t)] + G(x_t) \quad \text{for } t \in [0,a]. \tag{3.4}$$

The proof of this theorem is based on the following fundamental lemma.

Lemma 3.6 ([20, Corollary 3.5])**.** *Let $u : \mathbb{R}^+ \to X$ be defined by*

$$u(t) = T_0(t)u_0 + \int_0^t T_{-1}(t-s)f(s)ds \quad \text{for } t \geq 0. \tag{3.5}$$

If $u_0 \in \mathbb{F}_0$ and $f \in W^{1,1}(\mathbb{R}^+,\mathbb{F}_1)$ such that $A_{-1}u_0 + f(0) \in \overline{D(A)}$, then $u \in C^1(\mathbb{R}^+,X) \cap C(\mathbb{R}^+,\mathbb{F}_0)$ and satisfies

$$\frac{d}{dt}u(t) = A_{-1}u(t) + f(t) \quad \text{for } t \geq 0.$$

Proof of Theorem 3.5. Let x be the mild solution of (1.1) on $[0,a]$. Consider the function

$$y(t) = \begin{cases} F'(x_t)y_t + T_0(t)[\varphi'(0) - F'(\varphi)\varphi'] \\ + \int_0^t T_{-1}(t-s)G'(x_s)y_s ds & \text{for } t \in [0,a], \\ \varphi'(t), & \text{for } t \in (-\infty,0]. \end{cases} \tag{3.6}$$

Using the strict contraction principle, we show that (3.6) has a unique solution y on $(-\infty,a]$. Let $z : (-\infty,a] \to X$ be defined by

$$z(t) = \begin{cases} \varphi(0) + \int_0^t y(s)\,ds & \text{for } t \in [0,a], \\ \varphi(t) & \text{for } t \in (-\infty,0]. \end{cases}$$

As a consequence of Lemma 3.4, we see that

$$z_t = \varphi + \int_0^t y_s ds \quad \text{for } t \in [0,a].$$

To complete the proof, we have to show that $x = z$ on $[0,a]$. Since $s \to G(z_s)$ is continuously differentiable with the $X_{-1}-$norm,

$$\frac{d}{dt}\int_0^t T_{-1}(t-s)G(z_s)ds = T_{-1}(t)G(\varphi) + \int_0^t T_{-1}(t-s)G'(z_s)y_s\,ds.$$

This implies that

$$\int_0^t T_{-1}(s)G(\varphi)ds = \int_0^t T_{-1}(t-s)G(z_s)ds - \int_0^t \int_0^s T_{-1}(s-\tau)G'(z_\tau)y_\tau\,d\tau\,ds. \tag{3.7}$$

Let $t \in [0,a]$ and define

$$z_1(t) = x(t) - F(x_t), \quad z_2(t) = z(t) - F(z_t).$$

We have

$$z_2(t) - z_2(0) = \int_0^t z_2'(s)ds = \int_0^t (z'(s) - F'(z_s)y_s)ds,$$

$$= \int_0^t y(s)ds - \int_0^t F'(z_s)y_s ds.$$

It follows that

$$z_2(t) = \varphi(0) - F(\varphi) + \int_0^t F'(x_s)y_s ds + \int_0^t T_{-1}(s)(A_{-1}[\varphi(0) - F(\varphi)] + G(\varphi))ds \\ + \int_0^t \int_0^s T_{-1}(s-\tau)G'(x_\tau)y_\tau \, d\tau \, ds - \int_0^t F'(z_s)y_s ds.$$

Since

$$\int_0^t T_{-1}(s)(A_{-1}[\varphi(0) - F(\varphi)])ds = T_{-1}(t)(\varphi(0) - F(\varphi)) - (\varphi(0) - F(\varphi)),$$

we obtain

$$z_2(t) = \varphi(0) - F(\varphi) + \int_0^t (F'(x_s) - F'(z_s))y_s ds + T_{-1}(t)(\varphi(0) \\ - F(\varphi)) - (\varphi(0) - F(\varphi)) + \int_0^t T_{-1}(s)G(\varphi)ds \\ + \int_0^t \int_0^s T_{-1}(s-\tau)G'(x_\tau)y_\tau \, d\tau \, ds.$$

Using (3.7), we obtain

$$z_2(t) = T_0(t)(\varphi(0) - F(\varphi)) + \int_0^t (F'(x_s) - F'(z_s))y_s ds \\ + \int_0^t T_{-1}(s)G(\varphi)ds + \int_0^t \int_0^s T_{-1}(s-\tau)G'(x_\tau)y_\tau \, d\tau \, ds,$$

and

$$z_2(t) = T_0(t)(\varphi(0) - F(\varphi)) + \int_0^t (F'(x_s) - F'(z_s))y_s ds \\ + \int_0^t T_{-1}(t-s)G(z_s)ds + \int_0^t \int_0^s T_{-1}(s-\tau)(G'(x_\tau) - G'(z_\tau))y_\tau \, d\tau \, ds.$$

Since

$$z_1(t) = x(t) - F(x_t) = T_0(t)[\varphi(0) - F(\varphi)] + \int_0^t T_{-1}(t-s)G(x_s)ds,$$

we deduce that

$$z_2(t) - z_1(t) = \int_0^t T_{-1}(t-s)(G(z_s) - G(x_s))ds + \int_0^t (F'(x_s) - F'(z_s))y_s ds \\ + \int_0^t \int_0^s T_{-1}(s-\tau)(G'(x_\tau) - G'(z_\tau))y_\tau \, d\tau \, ds.$$

The local Lipschitz conditions on F' and G' imply that there is a positive constant k_0 such that

$$|z_2(t) - z_1(t)| \le k_0 \int_0^t |x_s - z_s|_{\mathcal{B}} ds.$$

Consequently,

$$|x(t) - z(t)| \le L_0 |x_t - z_t|_{\mathcal{B}} + k_0 \int_0^t |x_s - z_s|_{\mathcal{B}} ds,$$

$$\leq L_0 K_a \sup_{0\leq s\leq t} |x(s) - z(s)| + k_0 \int_0^t |x_s - z_s|_{\mathcal{B}} ds.$$

It follows that

$$\sup_{0\leq s\leq t} |x(s) - z(s)| \leq \frac{k_0}{1 - L_0 K_a} \int_0^t |x_s - z_s|_{\mathcal{B}} ds.$$

Then, we obtain

$$|x_t - z_t|_{\mathcal{B}} \leq K_a \sup_{0\leq s\leq t} |x(s) - z(s)| \leq \frac{k_0 K_a}{1 - L_0 K_a} \int_0^t |x_s - z_s|_{\mathcal{B}} ds.$$

By Gronwall's lemma, we deduce that $x_t = z_t$ on $[0, a]$. Consequently, $x(t) = z(t)$ for all $t \in (-\infty, a]$. Hence, $t \to x_t$ is continuously differentiable on $[0, a]$. We deduce that $t \to G(x_t) \in C^1([0, a], \mathbb{F}_1)$. By Lemma 3.6, we conclude that the function $t \to x(t) - F(x_t)$ belongs to $C^1([0, a], X) \cap C([0, a], \mathbb{F}_0)$ and satisfies (3.4) for all $t \in [0, a]$. □

4. The solution semigroup and the principle of linearized stability

In this section, we assume that

(H2') the function F is a bounded linear operator from $\mathcal{B}$ to X with L_0 its norm, and

(H4') $L_0 K_\infty < 1$.

Let $\mathcal{H}$ be the phase space of (1.1) given by

$$\mathcal{H} = \{\varphi \in \mathcal{B} : \varphi(0) - F(\varphi) \in X_0\}.$$

Define the operator $U(t)$ on $\mathcal{H}$, for $t \geq 0$, by $U(t)(\varphi) = x_t(., \varphi)$, where $x(., \varphi)$ is the mild solution of (1.1) on $\mathbb{R}$. Then, we have the following proposition.

Proposition 4.1. *The family $(U(t))_{t\geq 0}$ is a nonlinear strongly continuous semigroup on $\mathcal{H}$; that is,*

(i) *$U(0) = I$;*
(ii) *$U(t + s) = U(t)U(s)$, for $t, s \geq 0$;*
(iii) *for all $\varphi \in \mathcal{H}$, $U(t)(\varphi)$ is a continuous function of $t \geq 0$ with values in $\mathcal{H}$;*
(iv) *$U(t)$ satisfies, for $t \geq 0$, $\theta \in (-\infty, 0]$ and $\varphi \in \mathcal{H}$, the translation property*

$$U(t)(\varphi)(\theta) = \begin{cases} (U(t+\theta)(\varphi))(0) & \text{if } t + \theta \geq 0, \\ \varphi(t + \theta) & \text{if } t + \theta \leq 0, \end{cases}$$

(v) *for each $a > 0$, there exists a function $m \in L^\infty((0, a), \mathbb{R}^+)$ such that*

$$|U(t)\varphi - U(t)\psi|_{\mathcal{B}} \leq m(t)|\varphi - \psi|_{\mathcal{B}} \quad \text{for } t \in [0, a] \text{ and } \varphi, \psi \in \mathcal{H}.$$

Proof. (i) and (ii) are a consequence of the uniqueness of the solution. (iii) comes from the fact that the solution is continuous for every $t \geq 0$. (iv) is a consequence of the definition of U. To prove (v), consider $\varphi, \psi \in \mathcal{H}$ and their associated solutions x and y. Then, for $t \geq 0$, we have

$$x(t) = F(x_t) + T_0(t)[\varphi(0) - F(\varphi)] + \int_0^t T_{-1}(t - s)G(x_s) ds,$$

$$y(t) = F(y_t) + T_0(t)[\psi(0) - F(\psi)] + \int_0^t T_{-1}(t - s)G(y_s) ds.$$

It follows that

$$|x(t)-y(t)| \le L_0|x_t-y_t|_{\mathcal{B}} + N_0e^{\omega_0 t}(H+L_0)|\varphi-\psi|_{\mathcal{B}} + N_0L_1e^{\omega_0 t}\int_0^t |x_s-y_s|_{\mathcal{B}}ds.$$

Then

$$(1-K_\infty L_0)|x_t-y_t|_{\mathcal{B}} \le K_\infty N_0e^{\omega_0 t}(H+L_0)|\varphi-\psi|_{\mathcal{B}} + N_0K_\infty L_1e^{\omega_0 t}\int_0^t |x_s-y_s|_{\mathcal{B}}ds.$$

By Gronwall's lemma, we conclude that, for every $t \ge 0$, $U(t)$ is a Lipschitz continuous function. □

By an equilibrium, we mean a constant solution x^* of (1.1). Without loss of generality, we suppose that $x^* = 0$. Then, we assume that

(H8) $G(0) = 0$ and G is continuously differentiable at zero.

Then the linearized equation at zero of (1.1) is given by

$$\begin{gathered}\frac{d}{dt}[y(t)-F(y_t)] = A[y(t)-F(y_t)] + G'(0)y_t \quad \text{for } t \ge 0,\\ y_0 = \varphi \in \mathcal{H}.\end{gathered} \tag{4.1}$$

Let $(V(t))_{t\ge 0}$ be the C_0-semigroup solution on $\mathcal{H}$ of (4.1).

Theorem 4.2. *Assume that* (H1), (H2'), (H3), (H4'), (H8) *hold. Then, for* $t \ge 0$, *the derivative at zero of* $U(t)$ *is* $V(t)$.

Proof. Let $\varphi \in \mathcal{H}$. Consider the unique solution x (resp. y) on $\mathbb{R}$ of (1.1) (resp. (4.1)). Then, for $t \ge 0$, we have

$$\begin{gathered}x(t) = F(x_t) + T_0(t)[\varphi(0)-F(\varphi)] + \int_0^t T_{-1}(t-s)G(x_s)ds,\\ y(t) = F(y_t) + T_0(t)[\varphi(0)-F(\varphi)] + \int_0^t T_{-1}(t-s)G'(0)y_sds,\end{gathered}$$

and, for $t \le 0$, $x(t) = y(t) = \varphi(t)$. Let $t \ge 0$. Then

$$x(t)-y(t) = F(x_t)-F(y_t) + \int_0^t T_{-1}(t-s)(G(x_s)-G'(0)y_s)ds.$$

Hence

$$\begin{aligned}x(t)-y(t) = F(x_t)-F(y_t) &+ \int_0^t T_{-1}(t-s)(G(y_s)-G'(0)y_s)ds\\ &+ \int_0^t T_{-1}(t-s)(G(x_s)-G(y_s))ds.\end{aligned}$$

Using (v) of Proposition 4.1 and thanks to the differentiability property of the function G at 0, we see that for $\varepsilon > 0$, there exists $\eta > 0$ such that

$$|G(y_t)-G'(0)y_t|_{\mathbb{F}_1} \le \varepsilon|\varphi|_{\mathcal{B}} \quad \text{for } |\varphi|_{\mathcal{B}} \le \eta \text{ and } t \ge 0.$$

This implies that, by (A1)-(iii), there exist constants k_0 and $\tilde{k}$ such that for $t \ge 0$

$$|x_t-y_t|_{\mathcal{B}} \le K_\infty|x(t)-y(t)| \le L_0K_\infty|x_t-y_t|_{\mathcal{B}} + k_0K_\infty\varepsilon|\varphi|_{\mathcal{B}} + \tilde{k}K_\infty\int_0^t |x_s-y_s|_{\mathcal{B}}.$$

Then

$$|x_t-y_t|_{\mathcal{B}} \le \frac{k_0K_\infty}{1-K_\infty L_0}\varepsilon|\varphi|_{\mathcal{B}} + \tilde{k}K_\infty\int_0^t |x_s-y_s|_{\mathcal{B}}.$$

By Gronwall's lemma, we obtain

$$|x_t - y_t|_{\mathcal{B}} \le \tilde{\varepsilon}|\varphi|_{\mathcal{B}} \quad \text{for } |\varphi|_{\mathcal{B}} \le \eta.$$

We conclude that $U(t)$ is differentiable at 0 and $D_\varphi U(t)(0) = V(t)$ for $t \ge 0$. □

Finally, we obtain the important result.

Theorem 4.3. *Assume that* (H1), (H2'), (H3), (H4'), (H8) *hold. If the semigroup* $(V(t))_{t\ge0}$ *on* $\mathcal{H}$ *is exponentially stable, then the zero equilibrium of* $(U(t))_{t\ge0}$ *is locally exponentially stable in the sense that there exist* $\delta > 0$, $\mu > 0$ *and* $k \ge 1$ *such that*

$$|U(t)(\varphi)| \le ke^{-\mu t}|\varphi| \quad \text{for } \varphi \in \mathcal{H} \text{ with } |\varphi| \le \delta \text{ and } t \ge 0.$$

Moreover, if $\mathcal{H}$ *can be decomposed as* $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, *where* $\mathcal{H}_i$ *are* V*-invariant subspaces of* $\mathcal{H}$, $\mathcal{H}_1$ *is finite-dimensional and*

$$\inf\{|\lambda| : \lambda \in \sigma(V(t)/\mathcal{H}_1)\} > e^{\omega t}, \quad \text{where } \omega = \lim_{h\to\infty}\frac{1}{h}\log|V(h)/\mathcal{H}_2|,$$

then, the zero equilibrium of $(U(t))_{t\ge0}$ *is not stable in the sense that there exist* $\varepsilon > 0$, *a sequence* $(\varphi_n)_n$ *converging to* 0, *and a sequence* $(t_n)_n$ *of positive real numbers such that* $|U(t_n)\varphi_n| > \varepsilon$.

The proof of this theorem is based on Theorem 4.2 and on the following result.

Theorem 4.4 ([9]). *Let* $(W(t))_{t\ge0}$ *be a nonlinear* C_0*-semigroup on a subset* Ω *of a Banach space* Y. *Assume that* $w \in \Omega$ *is an equilibrium of* $(W(t))_{t\ge0}$ *and* $W(t)$ *is differentiable at* w *for each* $t \ge 0$. *Let* $Z(t)$ *be the derivative at* w *of* $W(t)$, $t \ge 0$. *Then* $(Z(t))_{t\ge0}$ *is a* C_0*-semigroup of bounded linear operators on* Y. *If the semigroup* $(Z(t))_{t\ge0}$ *is exponentially stable, then the equilibrium* w *of* $(W(t))_{t\ge0}$ *is locally exponentially stable. Moreover, if* Y *can be decomposed as* $Y = Y_1 \oplus Y_2$, *where* Y_i *are* Z*-invariant subspaces of* Y, Y_1 *is finite-dimensional and*

$$\inf\{|\lambda| : \lambda \in \sigma(Z(t)/Y_1)\} > e^{\omega t} \quad \text{with } \omega = \lim_{h\to\infty}\frac{1}{h}\log|Z(h)/Y_2|,$$

then, the equilibrium w *is not stable in the sense that there exist* $\varepsilon > 0$ *and sequences* $(x_n)_n$ *converging to* w *and* $(t_n)_n$ *of positive real numbers such that* $|W(t_n)x_n - w| > \varepsilon$.

5. An example

Consider the equation

$$\begin{gathered}
\begin{aligned}
&\frac{\partial}{\partial t}[u(t,x) - \int_{-\infty}^{0}\alpha(\theta)u(t+\theta,x)d\theta] \\
&= -\frac{\partial}{\partial x}[u(t,x) - \int_{-\infty}^{0}\alpha(\theta)u(t+\theta,x)d\theta] + \int_{-\infty}^{0}H(x,\theta,u(t+\theta,x))d\theta \\
&\quad \text{for } t > 0 \text{ and } x \in [0,1],
\end{aligned} \\
u(t,0) - \int_{-\infty}^{0}\alpha(\theta)u(t+\theta,0)d\theta = 0 \quad \text{for } t > 0, \\
u(\theta,x) = u_0(\theta,x) \quad \text{for } (\theta,x) \in (-\infty,0]\times[0,1],
\end{gathered} \tag{5.1}$$

where $u_0 \in C((-\infty,0]\times[0,1];\mathbb{R})$, $\alpha:(-\infty,0]\to\mathbb{R}$ is a continuous function and $H:[0,1]\times(-\infty,0]\times\mathbb{R}\to\mathbb{R}$ is a function satisfying (E1) and (E2) below. We put $X=C([0,1];\mathbb{R})$.

We use the extrapolation method to prove the well-posedness of (5.1). Let A be the operator defined on X by

$$D(A)=\{h\in C^1([0,1];\mathbb{R}):h(0)=0\},\quad Ah=-h'.$$

Then,

$$\overline{D(A)}=C_0([0,1];\mathbb{R})=\{h\in C([0,1];\mathbb{R}):h(0)=0\}.$$

Lemma 5.1 ([20]). *The operator A satisfies the Hille-Yosida condition* (H1) *on X. The C_0-semigroup $(T_0(t))_{t\geq 0}$ on the space $\overline{D(A)}=C_0([0,1];\mathbb{R})$ generated by the part A_0 of A is given for $u\in C_0([0,1];\mathbb{R})$, by*

$$(T_0(t)u)(x)=\begin{cases}u(x-t) & \text{for } t\leq x,\\ 0 & \text{for } t>x.\end{cases}$$

Let $\mathrm{Lip}_0[0,1]$ be the space of Lipschitz continuous functions on $[0,1]$ vanishing at zero, with the norm

$$|g|_{\mathrm{Lip}}=\sup_{0\leq x_1<x_2\leq 1}\frac{|g(x_2)-g(x_1)|}{x_2-x_1}.$$

Lemma 5.2. [20] *The Favard class of the semigroup $(T_0(t))_{t\geq 0}$ is given by $\mathbb{F}_0=\mathrm{Lip}_0[0,1]$ and the Favard class of the extrapolated semigroup $(T_{-1}(t))_{t\geq 0}$ is given by $\mathbb{F}_1=L^\infty(0,1)$. The extrapolated operator A_{-1} coincides on $\mathbb{F}_0$ almost everywhere with the derivative operation.*

Let $\gamma>0$. Consider the phase space

$$\mathcal{B}=C_\gamma=\{\varphi\in C((-\infty,0];X):\sup_{\theta\leq 0}(e^{\gamma\theta}|\varphi(\theta)|)<+\infty\},$$

endowed with the norm

$$|\varphi|_{C_\gamma}=\sup_{\theta\leq 0}(e^{\gamma\theta}|\varphi(\theta)|),\quad \text{where } |\varphi(\theta)|=\sup_{x\in[0,1]}|\varphi(\theta)(x)| \text{ for } \theta\leq 0.$$

Lemma 5.3 ([15]). *The space C_γ satisfies the* (A1), (A2), (B1) (C1).

Assume that:

- (E1) $\operatorname{ess\,sup}_{x\in[0,1]}(\int_{-\infty}^0|H(x,\theta,0)|d\theta)<+\infty$;
- (E2) $|H(x,\theta,z_1)-H(x,\theta,z_2)|\leq\beta(\theta,x)|z_1-z_2|$ for $x\in[0,1]$, $\theta\in(-\infty,0]$ and $z_1,z_2\in\mathbb{R}$, with $\operatorname{ess\,sup}_{x\in[0,1]}(\int_{-\infty}^0 e^{-\gamma\theta}\beta(\theta,x)d\theta)<+\infty$;
- (E3) $\int_{-\infty}^0 e^{-\gamma\theta}|\alpha(\theta)|d\theta<1$;

Let F be the linear operator from C_γ to X defined by

$$F(\varphi)(x)=\int_{-\infty}^0\alpha(\theta)\varphi(\theta)(x)d\theta\quad\text{for } \varphi\in C_\gamma \text{ and } x\in[0,1].$$

Then

$$\begin{aligned}\sup_{x\in[0,1]}|F(\varphi)(x)|&\leq\sup_{x\in[0,1]}\Big(\int_{-\infty}^0|\alpha(\theta)|\,|\varphi(\theta)(x)|d\theta\Big)\\&\leq\Big(\int_{-\infty}^0 e^{-\gamma\theta}|\alpha(\theta)|d\theta\Big)|\varphi|_{C_\gamma}\leq L_1|\varphi|_{C_\gamma},\end{aligned}$$

where

$$L_1 = \int_{-\infty}^{0} e^{-\gamma\theta}|\alpha(\theta)|d\theta < +\infty.$$

Hence $|F(\varphi)| \le L_1|\varphi|_{C_\gamma}$. Then, F is a bounded linear operator from C_γ to X.

We introduce the function G defined on C_γ, by

$$(G(\varphi))(x) = \int_{-\infty}^{0} H(x,\theta,\varphi(\theta)(x))d\theta \quad \text{a.e. } x \in [0,1],\ \varphi \in C_\gamma.$$

Lemma 5.4. *Assume that the conditions* (E1) *and* (E2) *are satisfied. Then, for all* $\varphi \in C_\gamma$, $G(\varphi) \in L^\infty(0,1)$ *and* $G : C_\gamma \to L^\infty(0,1)$ *is Lipschitz continuous.*

Proof. By (E1), we have

$$\operatorname{ess\,sup}_{x\in[0,1]}(|G(0)(x)|) \le \operatorname{ess\,sup}_{x\in[0,1]}\Big(\int_{-\infty}^{0}|H(x,\theta,0)|d\theta\Big) < +\infty.$$

Consequently, $G(0) \in \mathbb{F}_1 = L^\infty(0,1)$. Let $\varphi,\psi \in C_\gamma$. Then

$$\begin{aligned}
|G(\varphi) - G(\psi)|_{\mathbb{F}_1} &= \operatorname{ess\,sup}_{x\in[0,1]}(|G(\varphi)(x) - G(\psi)(x)|),\\
&\le \operatorname{ess\,sup}_{x\in[0,1]}\Big(\int_{-\infty}^{0}|H(x,\theta,\varphi(\theta)(x)) - H(x,\theta,\psi(\theta)(x))|d\theta\Big),\\
&\le \operatorname{ess\,sup}_{x\in[0,1]}\Big(\int_{-\infty}^{0}\beta(\theta)|\varphi(\theta)(x) - \psi(\theta)(x)|d\theta\Big),\\
&\le \operatorname{ess\,sup}_{x\in[0,1]}\Big(\int_{-\infty}^{0}e^{-\gamma\theta}\beta(\theta,x)d\theta\Big)\sup_{\theta\le 0}\\
&\quad\times\Big[e^{\gamma\theta}\Big(\operatorname{ess\,sup}_{x\in[0,1]}(|\varphi(\theta)(x) - \psi(\theta)(x)|)\Big)\Big],\\
&\le \operatorname{ess\,sup}_{x\in[0,1]}\Big(\int_{-\infty}^{0}e^{-\gamma\theta}\beta(\theta,x)d\theta\Big)|\varphi-\psi|_{C_\gamma}.
\end{aligned}$$

Since $G(0) \in L^\infty(0,1)$, we have $G(\varphi) \in L^\infty(0,1)$. On the other hand, we conclude that $G : C_\gamma \to L^\infty(0,1)$ is Lipschitz continuous. □

For $t \ge 0$, $x \in [0,1]$ and $\theta \le 0$, we make the following change of variables

$$v(t)(x) = u(t,x), \quad \varphi(\theta)(x) = u_0(\theta,x).$$

Then, (5.1) takes the abstract form

$$\begin{gathered}
\frac{\partial}{\partial t}[v(t) - F(v_t)] = A[v(t) - F(v_t)] + G(v_t),\\
v_0 = \varphi \in C_\gamma.
\end{gathered} \tag{5.2}$$

Proposition 5.5. *Assume that* (E1), (E2), (E3) *hold. Let* $\varphi \in C_\gamma$ *be such that* $\varphi(0) - F(\varphi) \in \overline{D(A)}$. *Then* (5.2) *has a unique mild solution* v *on an interval* $(-\infty, a]$, *with* $a > 0$.

Proof. Lemma 5.1 and Lemma 5.4 imply that the hypotheses (H1), (H2') and (H3) hold. For the space C_γ, one can see that $K(0) = 1$ and $L_1 < 1$. It follows that the hypothesis (H4) is true. □

To obtain the regularity of mild solutions of (5.2), we assume that the function $z \to H(x,\theta,z)$ is differentiable and satisfies the following hypothesis.

(E4) $|\frac{\partial}{\partial z}H(x,\theta,z)| \le \mu(x,\theta)$ for $x \in [0,1]$, $\theta \in (-\infty,0]$ and $z \in \mathbb{R}$, with $\operatorname{ess\,sup}_{x\in[0,1]}\big(\int_{-\infty}^0 e^{-\gamma\theta}\mu(x,\theta)d\theta\big) < +\infty$.

(E5) $|\frac{\partial}{\partial z}H(x,\theta,z_1) - \frac{\partial}{\partial z}H(x,\theta,z_2)| \le \vartheta(\theta,x)|z_1 - z_2|$ for $x \in [0,1]$, $\theta \in (-\infty,0]$ and $z_1, z_2 \in \mathbb{R}$, with $\operatorname{ess\,sup}_{x\in[0,1]}(\int_{-\infty}^0 e^{-\gamma\theta}\vartheta(\theta,x)d\theta) < +\infty$.

It is not difficult to see that assumptions $(E4)$ and $(E5)$ imply that the function G satisfies $(H6)$.

We add the following hypothesis on the regularity of the initial condition.

(E6) $u_0 \in C^1((-\infty;0]\times[0,1];\mathbb{R})$ such that
  (i) $\sup_{\theta\le 0}\big(e^{-\gamma\theta}[\operatorname{ess\,sup}_{x\in[0,1]}|\frac{\partial u_0}{\partial\theta}(\theta,x)|]\big) < \infty$,
  (ii) $u_0(0,.) - \int_{-\infty}^0 \alpha(\theta)u_0(\theta,.)d\theta \in \mathrm{Lip}_0[0,1]$,
  (iii) $\frac{\partial u_0}{\partial\theta}(0,.) - \int_{-\infty}^0 \alpha(\theta)\frac{\partial u_0}{\partial\theta}(\theta,.)d\theta \in C_0([0,1];\mathbb{R})$,
  (iv) for a.e. $x \in [0,1]$ we have

$$\begin{aligned}
&\frac{\partial u_0}{\partial\theta}(0,x) - \int_{-\infty}^0 \alpha(\theta)\frac{\partial u_0}{\partial\theta}(\theta,x)d\theta\\
&= -\frac{\partial}{\partial x}(u_0(0,x) - \int_{-\infty}^0 \alpha(\theta)u_0(\theta,x)d\theta) + \int_{-\infty}^0 H(x,\theta,u_0(\theta,x))d\theta\,.
\end{aligned}$$

Then we obtain

$$\begin{gathered}
\varphi \in C^1((-\infty,0];X)\cap\mathcal{B}, \quad \varphi' \in \mathcal{B}, \quad \varphi(0) - F(\varphi) \in \mathbb{F}_0,\\
\varphi'(0) - F(\varphi') \in \overline{D(A)} \quad \varphi'(0) - F(\varphi') = A_{-1}[\varphi(0) - F(\varphi)] + G(\varphi).
\end{gathered}$$

Consequently, the hypothesis (H7) is satisfied.

We conclude with the following proposition.

Proposition 5.6. *Let* (E1), (E2), (E3), (E4), (E5), (E6) *be satisfied. Then the mild solution v of* (5.2) *belongs to* $C^1([0,a];C([0,1],\mathbb{R}))\cap C([0,a];\mathrm{Lip}_0[0,1])$ *and the function* $u:[0,a]\times[0,1]\to\mathbb{R}$ *defined by*

$$u(t,x) = v(t)(x),$$

satisfies (5.1)*, for $t \in [0,a]$ and a.e $x \in [0,1]$.*

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