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EXISTENCE OF INFINITELY MANY SOLUTIONS FOR NONLINEAR NEUMANN PROBLEMS WITH INDEFINITE COEFFICIENTS

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Abstract. We consider the nonlinear Neumann boundary-value problem

$$\begin{gathered} -\Delta u + u = a(x)|u|^{p-2}u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda b(x)|u|^{q-2}u \quad \text{on } \partial\Omega, \end{gathered}$$

where $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. We suppose a and b are possibly sign-changing functions in $\overline{\Omega}$ and on $\partial\Omega$ respectively. Under some additional assumptions on a and b, we show that there are infinitely many solutions for sufficiently small $\lambda > 0$ if $1 < q < 2 < p \leq 2^* = 2N/(N-2)$. When $p = 2^*$, we use the concentration compactness argument to ensure the PS condition for the associated functional. We also consider a general problem including the supercritical case and obtain the existence of infinitely many solutions.

1. Introduction

In this article we study the nonlinear Neumann boundary value problem

$$\begin{gathered} -\Delta u + u = a(x)|u|^{p-2}u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda b(x)|u|^{q-2}u \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

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where $N \geq 3$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $\partial/\partial\nu$ denotes the outer normal derivative. In addition, let $1 < q < 2 < p \leq 2^* = 2N/(N-2)$ and suppose a and b are possibly sign-changing functions in $\overline{\Omega}$ and on $\partial\Omega$ respectively. Main purpose of this paper is to show the existence of infinitely many solutions for (1.1). To do that, we define the energy functional associated to (1.1),

$$\mathcal{F}(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + u^2\right)\,dx - \frac{1}{p}\int_\Omega a(x)|u|^p\,dx - \frac{\lambda}{q}\int_{\partial\Omega} b(x)|u|^q d\sigma.$$

We can easily verify that $\mathcal{F}$ is well-defined on $H^1(\Omega)$ and continuously Fréchet differentiable on that space. In this paper, we define the solutions of (1.1) as the critical points of $\mathcal{F}$.

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To state our results, we put a condition on b,

(B1) there exist an open set $D \subset \mathbb{R}^N$ with $D \cap \partial\Omega \neq \emptyset$ and a positive constant $\delta > 0$ such that $b \geq \delta$ on $D \cap \partial\Omega$.

Our main result is the following.

R1 **Theorem 1.1.** *Let $1 < q < 2 < p \leq 2^* = 2N/(N-2)$. Suppose $a \in C(\overline{\Omega})$, $b \in L^\infty(\partial\Omega)$ and further, b satisfies the condition* (B1). *Then there exists a constant $\Lambda > 0$ such that* (1.1) *has infinitely many solutions $(u_k) \subset H^1(\Omega)$ for every $0 < \lambda < \Lambda$. Moreover $\mathcal{F}(u_k) < 0$ and $\mathcal{F}(u_k) \to 0$ as $k \to \infty$.*

Remark 1.2. It is sufficient to choose $a \in L^\infty(\Omega)$ if $p < 2^*$.

Remark 1.3. If we assume $b \in C(\partial\Omega)$ and there exists a point $x_0 \in \partial\Omega$ such that $b(x_0) > 0$, then b satisfies the hypotheses in Theorem 1.1.

In 1994, Ambrosetti, Brezis and Cerami [1] considered the elliptic problem with the convex-concave nonlinearities. They obtained several existence results for the Dirichlet boundary value problem, including multiple positive solutions and infinitely many ones which may change their signs. Recently some authors have begun to consider such problems with nonlinear Neumann boundary conditions. As a pioneering work, Garcia-Azorero, Peral and Rossi [3] study problem (1.1) for the case $a \equiv 1$ and $b \equiv 1$. They obtain the Ambrosetti-Brezis-Cerami type results. One of their results shows that if $1 < q < 2 < p < 2^*$ and $\lambda > 0$ is sufficiently small, there exist infinitely many solutions for (1.1) with negative energies. Motivated by their result, we consider a general case; i.e., the indefinite coefficients a and b. Consequently we obtain Theorem 1.1. We emphasize that a and b may change their signs. Note that in Theorem 1.1, we also consider the critical case; i.e. $p = 2^*$ which is not considered in [3]. If p is critical, a typical difficulty occurs in proving the PS condition for $\mathcal{F}$ because of the lack of the compactness of the embedding $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$. We overcome this difficulty by applying the concentration compactness lemma by Lions [6] and conclude Theorem 1.1.

This paper is organized as follows. In Section 2, we give the proof of Theorem 1.1 for the subcritical case, i.e. $p < 2^*$. To this aim, we use the variational method in [3]. By careful reading of the proof in [3] and considering the conditions on the coefficients a and b, we can get the result. Especially see the proof of Lemma 2.2. Next, in Section 3, we give the proof of Theorem 1.1 for the critical case, i.e. $p = 2^*$. As we indicated before, the main difficulty arises in the proof of the PS conditions for $\mathcal{F}$. In view of this, we shall show the proof of $L^{2^*}(\Omega)$ convergence for the PS sequences. This is the key of the proof of this section, see Lemma 3.1. Lastly in Section 4, we consider a general problem. Utilizing the argument in [8], we give a result including the supercritical case. In the following sections we use the characters C_1, C_2, C_3 and so on, to denote several positive constants.

2. Subcritical case

In this section, we consider the subcritical case. Let $1 < q < 2 < p < 2^*$. Here we use the variational argument in [3]. First of all, since in general, $\mathcal{F}$ is not bounded from below, we perform the appropriate truncation for the functional $\mathcal{F}$. To do that, first notice that by the Sobolev embedding and the trace theorem,

$$\mathcal{F}(u) \geq \frac{1}{2}\|u\|^2_{H^1(\Omega)} - \frac{C_1}{p}\|u\|^p_{H^1(\Omega)} - \frac{\lambda C_2}{q}\|u\|^q_{H^1(\Omega)} = f_\lambda(\|u\|_{H^1(\Omega)})$$

where $f_\lambda(x) := \frac{1}{2}x^2 - \frac{C_1}{p}x^p - \frac{\lambda C_2}{q}x^q$. Take $\Lambda_0 > 0$ so small that $\max_{[0,\infty)} f_\lambda$ is positive for all $0 < \lambda < \Lambda_0$. Choose $0 < m < x_0 < x_1 < M$ so that $f(m) < 0 < f(x_0) < f(x_1) < f(M)$ where m and M are local minimum and maximum points of f respectively. Now consider a cut off function $\tau \in C^1(\mathbb{R})$ defined by

$$\tau(\xi) = \begin{cases} 1 & \text{if } 0 \le \xi < x_0, \\ 0 & \text{if } \xi > x_1, \end{cases}$$
$$0 \le \tau(\xi) \le 1 \quad \text{if } x_0 \le \xi \le x_1,$$

and a C^1 functional on $H^1(\Omega)$,

$$\Phi(u) = \tau(\|u\|_{H^1(\Omega)}).$$

Finally we give the truncated functional,

$$\tilde{\mathcal{F}}(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + u^2\right)\, dx - \frac{1}{p}\int_\Omega a(x)\Phi(u)|u|^p\, dx - \frac{\lambda}{q}\int_{\partial\Omega} b(x)|u|^q d\sigma.$$

We can easily check that $\tilde{\mathcal{F}}$ is well-defined and continuously Fréchet differentiable on $H^1(\Omega)$. Notice also that $\tilde{\mathcal{F}} = \mathcal{F}$ on some neighborhood of u satisfying $\tilde{\mathcal{F}}(u) < 0$. In addition observe that $\tilde{\mathcal{F}}(u)$ is even in u and $\tilde{\mathcal{F}}(0) = 0$. Now we can get the following lemma.

bddbelow **Lemma 2.1.** *Assume $a \in L^\infty(\Omega)$ and $b \in L^\infty(\partial\Omega)$. Then $\tilde{\mathcal{F}}$ is bounded from below and satisfies the (PS)$_c$ condition if $c < 0$.*

Proof. Let us first show that $\tilde{\mathcal{F}}$ is bounded from below. In fact, by the definition of $\Phi(u)$, if $\|u\|_{H^1(\Omega)} < x_1$, $0 \le \Phi(u) \le 1$ and if $\|u\|_{H^1(\Omega)} > x_1$, $\Phi(u) = 0$. So $\Phi(u)\|u\|^p_{H^1(\Omega)} \le x_1^p$. Hence by the Sobolev embedding and the trace theorem,

$$\begin{aligned} \tilde{\mathcal{F}}(u) &\ge \frac{1}{2}\|u\|^2_{H^1(\Omega)} - \frac{C_1}{p}\Phi(u)\|u\|^p_{H^1(\Omega)} - \frac{\lambda C_2}{q}\|u\|^q_{H^1(\Omega)} \\ &\ge \frac{1}{2}\|u\|^2_{H^1(\Omega)} - \frac{C_1 x_1^p}{p} - \frac{\lambda C_2}{q}\|u\|^q_{H^1(\Omega)}. \end{aligned}$$

Since $q < 2$, $\tilde{\mathcal{F}}$ is bounded from below. We next prove that $\tilde{\mathcal{F}}$ satisfies the (PS)$_c$ condition if $c < 0$. To do that, let (u_j) be a (PS)$_c$ sequence for $\tilde{\mathcal{F}}$ at level $c < 0$. By the property of $\tilde{\mathcal{F}}$, $\tilde{\mathcal{F}}(u_j) = \mathcal{F}(u_j)$ for large j since $c < 0$. Therefore (u_j) is also a (PS)$_c$ sequence for $\mathcal{F}$; i.e., $\mathcal{F}(u_j) \to c$ and $\mathcal{F}'(u_j) \to 0$ in $H^{-1}(\Omega)$. Now we claim that (u_j) is bounded in $H^1(\Omega)$. Actually

$$\begin{aligned} c + 1 &\ge \mathcal{F}(u_j) - \frac{1}{p}\langle \mathcal{F}'(u_j), u_j\rangle + \frac{1}{p}\langle \mathcal{F}'(u_j), u_j\rangle \\ &\ge \big(\frac{1}{2} - \frac{1}{p}\big)\|u_j\|^2_{H^1(\Omega)} - \lambda\big(\frac{1}{q} - \frac{1}{p}\big)b_\infty\|u_j\|^q_{H^1(\Omega)} - \|u\|_{H^1(\Omega)} \end{aligned}$$

for large j, where $b_\infty := \|b\|_{L^\infty(\partial\Omega)}$. Since $1 < q < 2 < p$, (u_j) is bounded in $H^1(\Omega)$. Therefore we can assume there exists a function $u \in H^1(\Omega)$ such that $u_j \rightharpoonup u$ weakly in $H^1(\Omega)$. Moreover noting that $p < 2^*$ and $q < 2$, by the Rellich Theorem, we can also assume

$$\begin{aligned} u_j &\to u \quad \text{in } L^p(\Omega), \\ u_j &\to u \quad \text{in } L^q(\partial\Omega). \end{aligned} \tag{2.1}$$ conv

Then we obtain

$$\begin{aligned} a|u_j|^{p-2}u_j &\to a|u|^{p-2}u \quad \text{in } H^{-1}(\Omega),\\ b|u_j|^{q-2}u_j &\to b|u|^{q-2}u \quad \text{in } H^{-1}(\Omega). \end{aligned}$$

By the Lax-Milgram Theorem, we conclude

$$u_j \to u \quad \text{in } H^1(\Omega).$$

This completes the proof. □

The condition (B1) on the indefinite function b in Theorem 1.1 is essential for the following lemma.

keyof2 **Lemma 2.2.** *Suppose $a \in L^\infty(\Omega)$, $b \in L^\infty(\partial\Omega)$ and further, b satisfies condition (B1). Then for every $n \in \mathbb{N}$, there exist an n-dimensional subspace $E_n \subset H^1(\Omega)$, and constants $\rho > 0$ and $\varepsilon > 0$ such that*

$$\tilde{\mathcal{F}}(u) \le -\varepsilon$$

for all $u \in E_n$ with $\|u\|_{H^1(\Omega)} = \rho$.

Proof. From condition (B1) on $b \in L^\infty(\partial\Omega)$, for every $n \in \mathbb{N}$, we can construct an n-dimensional subspace E_n in $\{u \in C^\infty(\overline{\Omega}) : u \equiv 0 \text{ on } \partial\Omega \setminus D\}$ such that if $u \in E_n$, $u \equiv 0$ on $\partial\Omega$ if and only if $u = 0$. Then we take a nonzero function $u \in E_n$ with $\|u\|_{H^1(\Omega)} = \rho$. By the Sobolev embedding, we obtain

$$\begin{aligned} \tilde{\mathcal{F}}(u) &= \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + u^2\right) dx - \frac{1}{p}\int_\Omega a(x)\Phi(u)|u|^p\, dx - \frac{\lambda}{q}\int_{\partial\Omega\cap D} b(x)|u|^q d\sigma\\ &\le \frac{1}{2}\rho^2 + \frac{a_\infty C_3}{p}\rho^p - \frac{\lambda\delta}{q}\int_{\partial\Omega}|u|^q d\sigma, \end{aligned}$$

where $a_\infty := \|a\|_{L^\infty(\Omega)}$. Since E_n is finite dimensional, we obtain

$$\tilde{\mathcal{F}}(u) \le \frac{1}{2}\rho^2 + \frac{a_\infty C_3}{p}\rho^p - \frac{\lambda\delta C_4}{q}\rho^q.$$

As $q < 2 < p$, there exist constants $\rho > 0$ and $\varepsilon > 0$ such that

$$\tilde{\mathcal{F}}(u) \le -\varepsilon$$

for all $u \in E_n$ with $\|u\|_{H^1(\Omega)} = \rho$. This concludes the proof. □

Now we introduce the *genus* as a topological tool [5, 2, 7]. We give the following definition according to [7]: Consider the class

$$\Sigma = \{A \subset H^1(\Omega) \setminus \{0\} : A \text{ is closed, } A = -A\}.$$

Then we define the genus, $\gamma : \Sigma \to \{0\} \cup \mathbb{N} \cup \{\infty\}$ so that

$$\gamma(A) = \min\{k \in \mathbb{N} : \text{ there exists an odd map } \phi \in C(A, \mathbb{R}^k \setminus \{0\})\}.$$

If there exists no such a minimum, we put $\gamma(A) = \infty$. In addition we define $\gamma(\emptyset) = 0$. Consequently we obtain the following properties of the genus([7]). Let $A, B \in \Sigma$ then

- (g1) Normalization: If $x \ne 0$, $\gamma(\{x\} \cup \{-x\}) = 1$.
- (g2) Mapping property: If there exists an odd map $f \in C(A, B)$ then $\gamma(A) \le \gamma(B)$.
- (g3) Monotonicity property: If $A \subset B$, $\gamma(A) \le \gamma(B)$.
- (g4) Subadditivity: $\gamma(A \cup B) \le \gamma(A) + \gamma(B)$.

(g5) Continuity property: If A is compact, then $\gamma(A) < \infty$ and there exists $d > 0$ such that $N_d = \{u \in H^1(\Omega) : \operatorname{dist}(u, A) \le d\} \in \Sigma$ and $\gamma(N_d) = \gamma(A)$.

Here we prove the following lemma.

gen **Lemma 2.3.** *Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be as given by Lemma 2.2. Then*
$$\gamma(\tilde{\mathcal{F}}^{-\varepsilon}) \ge n.$$
where $\tilde{\mathcal{F}}^c = \{u \in H^1(\Omega) : \tilde{\mathcal{F}}(u) \le c\}$.

Proof. We define $S_{\rho,n} = \{u \in E_n : \|u\|_{H^1(\Omega)} = \rho\}$ where the n-dimensional subspace E_n and a constant $\rho > 0$ are given by Lemma 2.2. Then we have $S_{\rho,n} \subset \tilde{\mathcal{F}}^{-\varepsilon}$. By the monotonicity of the genus, we conclude that
$$\gamma(\tilde{\mathcal{F}}^{-\varepsilon}) \ge \gamma(S_{\rho,n}) = n.$$
□

Finally we prove the main result of this section.

def **Theorem 2.4.** *Let*
$$\Sigma = \{A \subset H^1(\Omega) \setminus \{0\} : A \text{ is closed, } A = -A\}, \quad \Sigma_k = \{A \in \Sigma : \gamma(A) \ge k\},$$
and put
$$c_k = \inf_{A \in \Sigma_k} \sup_{u \in A} \tilde{\mathcal{F}}(u),$$
then c_k is a negative critical value of $\mathcal{F}$. Moreover if $c := c_k = c_{k+1} = \dots = c_{k+r}$
$$\gamma(K_c) \ge r + 1$$
where $K_c = \{u \in H^1(\Omega) : \tilde{\mathcal{F}}(u) = c, \tilde{\mathcal{F}}'(u) = 0\}$.

For a proof of the above theorem, see [3, Theorem 2.1]. Using Theorem 2.4, we show the following corollary.

cor **Corollary 2.5.** *Let c_k be defined as in Theorem 2.4. Then $c_k \to 0$.*

Proof. Since c_k is negative and nondecreasing, there exists a constant $c_0 \le 0$ such that $c_k \to c_0$ as $k \to \infty$. Let us assume $c_0 < 0$ on the contrary. First notice that $c_k < c_0$ for all $k \in \mathbb{N}$. In fact, if $c_k = c_0$ for large $k \in \mathbb{N}$, $\gamma(K_{c_0}) = \infty$ by Theorem 2.4. On the other hand, the $(\mathrm{PS})_{c_0}$ condition for $\tilde{\mathcal{F}}$ implies that K_{c_0} is compact. Thus, the continuity of genus shows that $\gamma(K_{c_0}) < \infty$. This is a contradiction. We set $\gamma(K_{c_0}) = r$. Now, take $\varepsilon > 0$ so that $c_0 + \varepsilon < 0$. For large $k \in \mathbb{N}$, we also have
$$c_0 - \varepsilon < c_k. \tag{2.2}$$ c25-1

For such a k, there exists a set $A_{k+r} \in \Sigma_{k+r}$ such that $\sup_{u \in A_{k+r}} \tilde{\mathcal{F}}(u) \le c_0 + \varepsilon$. Here as K_{c_0} is compact, there exists a neighborhood $N_\delta(K_{c_0})$ such that $\gamma(K_{c_0}) = \gamma(N_\delta(K_{c_0}))$. Using the odd homeomorphism $\eta : [0,1] \times H_0^1(\Omega) \to H_0^1(\Omega)$ (constructed in [7, Appendix A] for example), we conclude that
$$\eta(1, \overline{A_{k+r} \setminus N_\delta(K_{c_0})}) \subset \eta(1, \overline{\tilde{\mathcal{F}}^{c_0+\varepsilon} \setminus N_\delta(K_{c_0})}) \subset \tilde{\mathcal{F}}^{c_0-\varepsilon}. \tag{2.3}$$ c25-2

On the other hand, from the mapping property and subadditivity of the genus, we obtain
$$\gamma(\eta(1, \overline{A_{k+r} \setminus N_\delta(K_{c_0})})) \ge \gamma(\overline{A_{k+r} \setminus N_\delta(K_{c_0})}) \ge \gamma(A_{k+r}) - \gamma(N_\delta(K_{c_0})) \ge k.$$

It follows that $\eta(1,\overline{A_{k+r}\setminus N_\delta(K_{c_0})})\in\Sigma_k$. Therefore, recalling (2.3), we obtain

$$c_k\le\sup_{u\in\eta(1,\overline{A_{k+r}\setminus N_\delta(K_{c_0})})}\tilde{\mathcal{F}}(u)\le c_0-\varepsilon.$$

This contradicts (2.2). The proof is complete. □

Proof of Theorem 1.1 for the subcritical case. We suppose $a\in L^\infty(\Omega)$, $b\in L^\infty(\partial\Omega)$ and further, b satisfies the condition (B1). Choose $\Lambda=\Lambda_0$ and take $0<\lambda<\Lambda$ as in the first paragraph of this section. Then by Theorem 2.4, we have the negative critical values $c_1,c_2,\dots$ of $\mathcal{F}$. In addition from Corollary 2.5, we conclude that the set $\{c_k:k\in\mathbb{N}\}$ has infinitely many distinct elements. This completes the proof. □

3. Critical case

In this section we prove Theorem 1.1 for the critical case, i.e., $p=2^*$. Let $1<q<2$ and consider the functional

$$\mathcal{F}(u)=\frac{1}{2}\int_\Omega\left(|\nabla u|^2+u^2\right)\,dx-\frac{1}{2^*}\int_\Omega a(x)|u|^{2^*}\,dx-\frac{\lambda}{q}\int_{\partial\Omega}b(x)|u|^q d\sigma.$$

The organization of the proof is same with that for the subcritical case, once we ensure the strong $L^{2^*}(\Omega)$ convergence for PS sequences. We begin with the following lemma.

keyof3 **Lemma 3.1.** *Assume $a\in C(\overline{\Omega})$ and $b\in L^\infty(\partial\Omega)$. Let $c<0$ and $(u_j)\subset H^1(\Omega)$ be a $(PS)_c$ sequence for $\mathcal{F}$. Then there exists a constant $\Lambda_1>0$ such that for every $0<\lambda<\Lambda_1$, (u_j) strongly converges in $L^{2^*}(\Omega)$ up to subsequences.*

Proof. By the same argument in the proof of Lemma 2.1, we ensure that (u_j) is bounded in $H^1(\Omega)$. Hence we can assume there exists a function $u\in H^1(\Omega)$ such that $u_j\rightharpoonup u$ weakly in $H^1(\Omega)$. Furthermore, by the Rellich Theorem, we can also assume that

$$\begin{gathered}u_j\to u\quad\text{in } L^2(\Omega),\\ u_j\to u\quad\text{in } L^q(\partial\Omega),\\ u_j\to u\quad\text{a.e. on }\Omega.\end{gathered}\tag{3.1}$$ 30

We now apply the concentration compactness lemma by Lions [6]. By that, we can assume there exist some at most countable set J, distinct points $(x_k)_{k\in J}\subset\overline{\Omega}$ and positive constants $(\nu_k)_{k\in J}$, $(\mu_k)_{k\in J}$ such that

$$\begin{gathered}|\nabla u_j|^2\rightharpoonup d\mu\ge|\nabla u|^2+\sum_{k\in J}\mu_k\delta_{x_k},\\ |u_j|^{2^*}\rightharpoonup d\nu=|u|^{2^*}+\sum_{k\in J}\nu_k\delta_{x_k},\end{gathered}\tag{3.2}$$ 31

in the measure sense, where δ_x denotes the Dirac measure with mass 1 concentrated at $x\in\mathbb{R}^N$. In addition by the result in the proof of [3, Lemma 7.1],

$$\mu_k\ge S\nu_k^{2/2^*}\ (k\in J),\tag{3.3}$$ 32

where

$$S=\inf_{u\in H^1(\Omega)\setminus\{0\}}\frac{\int_\Omega\left(|\nabla u|^2+u^2\right)dx}{\left(\int_\Omega|u|^{2^*}\,dx\right)^{2/2^*}}.$$

Let us show that there exists a constant $\Lambda_1 > 0$ such that $J = \emptyset$ for all $0 < \lambda < \Lambda_1$ if $c < 0$. To do that, assume $c < 0$ and take $0 < \lambda < \Lambda_1$ where $\Lambda_1 > 0$ is determined later. Now we suppose on the contrary, $J \neq \emptyset$. Then for all $k \in J$ we introduce a cut off function $\phi \in C^\infty(\mathbb{R}^N)$ with $0 \le \phi \le 1$ such that

$$\phi(x) = \begin{cases} 1 & \text{if } x \in B(x_k, \varepsilon), \\ 0 & \text{if } x \in B(x_k, 2\varepsilon)^c. \end{cases}$$

Furthermore we can assume that $|\nabla \phi| \le 2/\varepsilon$. Since (u_j) is bounded in $H^1(\Omega)$ and $\mathcal{F}'(u_j) \to 0$ in $H^{-1}(\Omega)$, recalling (3.1), (3.2) and the assumption $a \in C(\overline{\Omega})$, we obtain

$$\begin{aligned} 0 &= \lim_{j\to\infty} \langle \mathcal{F}'(u_j), u_j\phi\rangle \\ &= \lim_{j\to\infty} \Big\{ \int_\Omega \nabla u_j \cdot \nabla(u_j\phi)\,dx + \int_\Omega u_j^2\phi\,dx - \int_\Omega a(x)|u_j|^{2^*}\phi\,dx \\ &\quad - \lambda\int_{\partial\Omega} b(x)|u_j|^q\phi d\sigma\Big\} \qquad (3.4) \\ &= \lim_{j\to\infty} \int_\Omega (\nabla u_j\cdot\nabla\phi)u_j\,dx + \int_{\overline{\Omega}}\phi d\mu + \int_\Omega u^2\phi\,dx - \int_{\overline{\Omega}} a(x)\phi d\nu \\ &\quad - \lambda\int_{\partial\Omega} b(x)|u|^q\phi d\sigma. \end{aligned}$$ 33

Here using the Schwartz inequality, the boundedness and $L^2(\Omega)$ convergence in (3.1) of (u_j) and further, applying the Hölder inequality, we obtain

$$\begin{aligned} 0 &\le \lim_{j\to\infty} \Big|\int_\Omega (\nabla u_j\cdot\nabla\phi)u_j\,dx\Big| \\ &\le \lim_{j\to\infty} \Big(\int_{\Omega\cap B(x_k,2\varepsilon)} |\nabla u_j|^2\,dx\Big)^{1/2}\Big(\int_{\Omega\cap B(x_k,2\varepsilon)} u^2|\nabla\phi|^2\,dx\Big)^{1/2} \\ &\le C_5\Big(\int_{\Omega\cap B(x_k,2\varepsilon)} |u|^{2^*}\,dx\Big)^{1/2^*}\Big(\int_{\Omega\cap B(x_k,2\varepsilon)} |\nabla\phi|^N\,dx\Big)^{1/N} \\ &\le C_6\Big(\int_{\Omega\cap B(x_k,2\varepsilon)} |u|^{2^*}\,dx\Big)^{1/2^*} \to 0 \quad \text{as } \varepsilon\to 0, \end{aligned}$$

where for the above inequality we use the assumption $|\nabla\phi| \le 2/\varepsilon$. Taking $\varepsilon \to 0$ for (3.4), we obtain

$$\mu_k - a(x_k)\nu_k \le 0. \tag{3.5}$$ 34

Since μ_k and ν_k are positive, we can assume $a(x_k) > 0$. Considering (3.3) and (3.5) together, we have

$$\nu_k \ge \Big(\frac{S}{a(x_k)}\Big)^{N/2}.$$

So using this inequality and (3.2) again, we have for all $k \in J$,

$$\begin{aligned} c &= \lim_{j\to\infty} \big\{\mathcal{F}(u_j) - \frac{1}{2}\langle \mathcal{F}'(u_j), u_j\rangle\big\} \\ &= \lim_{j\to\infty} \big\{\frac{1}{N}\int_\Omega a(x)|u_j|^{2^*}\,dx - \lambda\big(\frac{1}{q}-\frac{1}{2}\big)\int_{\partial\Omega} b(x)|u_j|^q d\sigma\big\} \qquad (3.6) \\ &\ge \frac{1}{N}\int_\Omega a(x)|u|^{2^*}\,dx + a(x_k)\big(\frac{S}{a(x_k)}\big)^{N/2} - \lambda\big(\frac{1}{q}-\frac{1}{2}\big)\int_{\partial\Omega} b(x)|u|^q d\sigma. \end{aligned}$$ 35

Now since u is a critical point of $\mathcal{F}$, we have

$$\frac{1}{N}\int_\Omega a(x)|u|^{2^*}\,dx = \frac{1}{N}\|u\|^2_{H^1(\Omega)} - \frac{1}{N}\int_{\partial\Omega} b(x)|u|^q d\sigma.$$

Substituting this equality into (3.6), noting $q < 2$ and using the trace theorem, we have

$$c \ge \frac{S^{N/2}}{a(x_k)^{\frac{N-2}{2}}} + \frac{1}{N}\|u\|^2_{H^1(\Omega)} - \lambda\big(\frac{1}{q} + \frac{1}{N} - \frac{1}{2}\big)C_7\|u\|^q_{H^1(\Omega)}.$$

Hence noting $N/2 - 1 > 0$, we obtain

$$c \ge \frac{S^{N/2}}{a_0^{(N-2)/2}} - \lambda^{\frac{2}{2-q}}K$$

where $a_0 := \max_{x\in\overline{\Omega}} a(x) > 0$ and $K > 0$ is some constant which is independent of $\lambda > 0$. Now we take $\Lambda_1 > 0$ so small that the right-hand side of the above inequality is greater than 0 for all $0 < \lambda < \Lambda_1$. Then we obtain the contradiction since $c < 0$. Here observe that we can choose $\Lambda_1 > 0$ uniformly for $k \in J$. It follows that

$$\int_\Omega |u_j|^{2^*}\,dx \to \int_\Omega |u|^{2^*}\,dx.$$

This completes the proof. □

The above lemma enable us to ensure the $(\mathrm{PS})_c$ condition for $\mathcal{F}$. Now we can prove Theorem 1.1 for the critical case by the same argument in Section 2.

Proof of Theorem 1.1 for the critical case. We suppose $a \in C(\overline{\Omega})$, $b \in L^\infty(\partial\Omega)$ and further, b satisfies the condition (B1). As we already said, the organization of the proof for the critical case is same with that for the subcritical case. So we give only a comment for the choice of $\Lambda > 0$. To perform the appropriate truncation for the functional $\mathcal{F}$, we first choose $\Lambda_0 > 0$ by the same argument with that in Section 2. Next we take $\Lambda_1 > 0$ from Lemma 3.1. Then it is enough to select $\Lambda = \min\{\Lambda_0, \Lambda_1\}$. □

4. General case

As we can observe from the proof of Theorem 1.1, the concave term in (1.1) is essential for the existence of infinitely many solutions with negative energies. Here let us generalize the convex term in (1.1). To this aim, we consider the problem

$$\begin{gathered} -\Delta u + u = f(x,u) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda b(x)|u|^{q-2}u \quad \text{on } \partial\Omega, \end{gathered} \tag{4.1}$$

gpde

where $f(x,u) : \Omega \times \mathbb{R} \mapsto \mathbb{R}$. Now we can ask that

> Under what conditions on $f(x,u)$, can we ensure the existence of infinitely many solutions with negative energies?

To answer this question, we put two conditions on f:

(F1) there exists a constant $\sigma > 0$ such that $f(x,t)$ is a continuous function on $\Omega \times [-\sigma,\sigma]$ and odd in t for all $x \in \Omega$ if $t \in [-\sigma,\sigma]$,

(F2) $f(x,t) = o(|t|)$ as $t \to 0$.

Under the conditions (F1) and (F2) on f, we formally define the (weak) solutions for (4.1) and the associated functional. We call $u \in H^1(\Omega) \cap L^\infty(\Omega)$ is a (weak) solution of (4.1) if and only if u satisfies

$$\int_\Omega \nabla u \cdot \nabla h \, dx - \int_\Omega f(x,u) h \, dx - \int_{\partial\Omega} b(x) |u|^{q-2} u h \, d\sigma = 0 \tag{4.2}$$ weaksolofgpde

for all $h \in H^1(\Omega)$. The associated energy functional is given by

$$I(u) = \frac{1}{2} \int_\Omega \left(|\nabla u|^2 + u^2 \right) - \int_\Omega F(x,u) \, dx - \frac{\lambda}{q} \int_{\partial\Omega} |u|^q \, d\sigma, \tag{4.3}$$ gfunc

where $F(x,u) = \int_0^u f(x,t) dt$. Note that thanks to (F1), (4.2) and (4.3) have meanings for all $u \in H^1(\Omega) \cap L^\infty(\Omega)$ with $\|u\|_{L^\infty(\Omega)} \le \sigma$. We give the following Theorem. A comparable result for Dirichlet boundary value problem is found in [4].

GR1 **Theorem 4.1.** *Suppose* $1 < q < 2$ *and* f *satisfies the conditions* (F1) *and* (F2). *Assume further,* $b \in L^\infty(\partial\Omega)$ *and satisfies the condition* (B1). *Then* (4.1) *has infinitely many solutions* $(u_k) \subset H^1(\Omega) \cap L^\infty(\Omega)$ *for every* $\lambda > 0$. *Moreover* $I(u_k) < 0$, $I(u_k) \to 0$ *and* $\|u_k\|_{L^\infty(\Omega)} \to 0$ *as* $k \to \infty$.

Remark 4.2. We need no restriction for $\lambda > 0$ to be sufficiently small for the existence.

As a consequence of Theorem 4.1, we obtain a similar conclusion to Theorem 1.1 including the supercritical case.

GR2 **Corollary 4.3.** *Let* $1 < q < 2 < p < \infty$. *We suppose* $a \in L^\infty(\Omega)$, $b \in L^\infty(\partial\Omega)$ *and further, b satisfies the condition (B1). Then* (4.1) *with* $f(x,u) = a(x)|u|^{p-2}u$ *has infinitely many solutions* $(u_k) \subset H^1(\Omega) \cap L^\infty(\Omega)$ *for every* $\lambda > 0$. *Moreover* $I(u_k) < 0$, $I(u_k) \to 0$ *and* $\|u_k\|_{L^\infty(\Omega)} \to 0$ *as* $k \to \infty$.

Remark 4.4. We point out the delicate difference between theorems above and Theorem 1.1. In the theorems above, the solutions must converge to zero (as long as obtaining from our method below). But the solutions we got in Theorem 1.1 may not do that. Thus the solutions we can get here seem to be more restricted. But this is reasonable, since we are considering the general case including supercritical case. We need to utilize more careful cut off techniques and regularity arguments. See the details below.

Next, we shall prove Theorem 4.1. To this aim, we utilize the argument in [8]. Let $1 < q < 2$ as previous sections and $f(x,u)$ satisfy (F1) and (F2). To the beginning we construct a modified function $\tilde f(x,u) \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ using $f(x,u)$ so that

- ($\tilde{\mathrm{F}}$1) $|\tilde F(x,u)| \le u^2/4$ where $\tilde F(x,t) = \int_0^t \tilde f(x,s) ds$,
- ($\tilde{\mathrm{F}}$2) there exists a constant $0 < \theta < (2-q)/2$ such that $\tilde f(x,u)u - q\tilde F(u) \le \theta u^2$,
- ($\tilde{\mathrm{F}}$3) there exists a constant $0 < a < \sigma/2$ such that $\tilde f(x,u) = f(x,u)$ if $|u| < a$.

mod **Lemma 4.5.** *Let* $f : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ *satisfy conditions* (F1) *and* (F2). *Then there exists a continuous function* $\tilde f(x,t)$ *in* $\Omega \times \mathbb{R}$ *which is odd in t and satisfies the conditions* ($\tilde{\mathrm{F}}$1), ($\tilde{\mathrm{F}}$2), ($\tilde{\mathrm{F}}$3).

Proof. For fixed $0 < \theta < (2-q)/2$, take $0 < \varepsilon < \theta/14$. From (F2) there exists a constant $0 < a < \sigma/2$ such that $|f(x,u)u| \le \varepsilon u^2$ and $|F(x,u)| \le \varepsilon u^2$ if $|u| \le 2a$. Now define a cut off function $\rho \in C^1(\mathbb{R})$ such that $\rho(t) = 1$ if $|t| \le a$, $\rho(t) = 0$

if $|t| > 2a$ and $0 \le \rho \le 1$ otherwise. Furthermore, we can assume $|\rho'(t)| \le 2/a$. Firstly, we define

$$\tilde{F}(x,u) = \rho(u)F(x,u) + (1-\rho(u))F_\infty(u)$$

where $F_\infty(u) = \beta u^2$ for some $0 < \beta < \theta/16$. Then we have

$$|\tilde{F}(x,u)| \le \frac{1}{4}u^2. \tag{4.4}$$

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Indeed, if $|u| \le 2a$, we obtain

$$|\tilde{F}(x,u)| \le |F(x,u)| + F_\infty(x,u) \le (\varepsilon + \beta)u^2 \le \frac{1}{4}u^2,$$

and if $|u| > 2a$, we obtain

$$|\tilde{F}(x,u)| \le F_\infty(x,u) \le \frac{1}{4}u^2.$$

Next we put

$$\tilde{f}(x,u) = \frac{\partial \tilde{F}}{\partial u}(x,u).$$

Then we obtain

$$\tilde{f}(x,u) = \rho'(u)F(x,u) + \rho(u)f(x,u) - \rho'(u)F_\infty(u) + (1-\rho(u))F'_\infty(u).$$

By (F1), clearly $\tilde{f}(x,u)$ is a continuous function in $\Omega \times \mathbb{R}$, odd in u and

$$f(x,u) = \tilde{f}(x,u) \quad \text{if } |u| < a. \tag{4.5}$$

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In addition, a direct calculation implies

$$\begin{aligned} \tilde{f}(x,u)u - q\tilde{F}(x,u) &= (\rho'(u)u - q\rho(u))\, F(x,u) + \rho(u)f(x,u)u \\ &\quad - (\rho'(u)u + q(1-\rho(u)))\, F_\infty(u) + (1-\rho(u))F'_\infty(u)u. \end{aligned}$$

Here we claim

$$\tilde{f}(x,u) - q\tilde{F}(x,u) \le \theta u^2. \tag{4.6}$$

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In fact, if $|u| \le 2a$, we have

$$\tilde{f}(x,u) - q\tilde{F}(x,u) \le (7\varepsilon + 8\beta)u^2 \le \theta u^2,$$

and if $|u| > 2a$, we obtain

$$\tilde{f}(x,u) - q\tilde{F}(x,u) \le 4\beta u^2 \le \theta u^2.$$

Inequalities (4.4), (4.5) and (4.6) conclude the proof. $\square$

Let $\tilde{f}(x,u)$ be the function constructed in Lemma 4.5. We consider the modified problem

$$\begin{gathered} -\Delta u + u = \tilde{f}(x,u) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda b(x)|u|^{q-2}u \quad \text{on } \partial\Omega, \end{gathered} \tag{4.7}$$

gpde1

and the associated functional

$$\tilde{I}(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + u^2\right)\, dx - \int_\Omega \tilde{F}(x,u)\, dx - \lambda \int_{\partial\Omega} b(x)|u|^q d\sigma,$$

where $\tilde{F}(x,t) = \int_0^t \tilde{f}(x,s)ds$. Noting condition ($\tilde{\text{F}}$1), we can easily check that $\tilde{\text{I}}$ is well-defined on $H^1(\Omega)$ and continuously Fréchet differentiable on that space. Furthermore from the condition ($\tilde{\text{F}}$3), we can conclude that every critical point

$u \in H^1(\Omega) \cap L^\infty(\Omega)$ with $\|u\|_{L^\infty(\Omega)} \le a$ is a weak solution of (4.1). Next we show an important property of the modified functional.

keyof4 **Lemma 4.6.** *$\langle \tilde{I}'(u), u\rangle = 0$ and $\tilde{I}(u) = 0$ if and only if $u = 0$.*

Proof. Suppose $\langle \tilde{I}'(u), u\rangle = 0$ and $\tilde{I}(u) = 0$. Then we have

$$0 = \langle \tilde{I}'(u), u\rangle = \int_\Omega \left(|\nabla u|^2 + u^2\right)\, dx - \lambda \int_{\partial\Omega} b(x)|u|^q \, d\sigma - \int_\Omega \tilde{f}(x,u)u\, dx, \quad (4.8)$$ 44

and

$$0 = qI(u) = \frac{q}{2}\int_\Omega \left(|\nabla u|^2 + u^2\right)\, dx - \lambda \int_{\partial\Omega} b(x)|u|^q \, d\sigma - q\int_\Omega \tilde{F}(x,u)u\, dx. \quad (4.9)$$ 45

Substituting (4.9) in (4.8) and noting the condition ($\tilde{\text{F}}$2), we obtain

$$\Big(\frac{2-q}{2}\Big)\|u\|^2_{H^1(\Omega)} = \int_\Omega \left(\tilde{f}(x,u)u - q\tilde{F}(x,u)\right)\, dx \le \theta\|u\|^2_{H^1(\Omega)}.$$

Hence $u = 0$. This concludes the proof. $\square$

Considering the oddness of $\tilde{f}(x,u)$ and the condition ($\tilde{\text{F}}$1) on $\tilde{f}(x,u)$, we can check the following properties of $\tilde{\text{I}}$,

- ($\tilde{\text{I}}$1) $\tilde{I}(u)$ is even in u,
- ($\tilde{\text{I}}$2) $\tilde{I}(0) = 0$,
- ($\tilde{\text{I}}$3) $\tilde{\text{I}}$ is bounded from below,
- ($\tilde{\text{I}}$4) $\tilde{\text{I}}$ satisfies $(\text{PS})_c$ conditions for $c \le 0$,
- ($\tilde{\text{I}}$5) for every $n \in \mathbb{N}$, there exist an n-dimensional subspace $E_n \subset H^1(\Omega)$ and positive constants $\rho > 0$ and $\varepsilon > 0$ such that $\tilde{I}(u) \le -\varepsilon$ for all $u \in E_n$ with $\|u\|_{H^1(\Omega)} = \rho$.

Most parts of the proof are analogous to the ones in Section 2. So we omit it. The above properties of $\tilde{\text{I}}$ are enough to obtain the existence of infinitely many solutions $(u_k) \subset H^1(\Omega)$ for (4.7) with $\tilde{I}(u_k) < 0$ and $\tilde{I}(u_k) \to 0$ as $k \to \infty$ as in Section 2. Finally we come to the proof of Theorem 4.1.

The proof of Theorem 4.1. Firstly assume $b \in L^\infty(\partial\Omega)$ and satisfies the condition (B1). Since $\tilde{I}(u_k) \to 0$ and $\tilde{I}'(u_k) = 0$, the sequence of solutions (u_k) is $(\text{PS})_0$ sequence for $\tilde{\text{I}}$. Then by the $(\text{PS})_0$ condition for $\tilde{\text{I}}$, we can assume u_k converges to some function $u \in H^1(\Omega)$. We claim $u = 0$. In fact, from the continuity of $\tilde{\text{I}}$, $\tilde{I}(u) = 0$. Thus Lemma 4.6 confirm the claim. Considering a priori estimate (see Section 4 in [3]), we obtain for all $\beta \ge 1$, $(u_k) \subset W^{1,\beta}(\Omega)$ and $u_k \to 0$ in $W^{1,\beta}(\Omega)$. Consequently, by the Morrey inequality, we also have $\|u_k\|_{C(\overline{\Omega})} \to 0$. Thus we obtain $\|u_k\|_{L^\infty(\Omega)} \le a$ for large $k \in \mathbb{N}$. This completes the proof. $\square$

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