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REGULARITY OF MILD SOLUTIONS TO FRACTIONAL CAUCHY PROBLEMS WITH RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE

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Abstract. As an extension of the fact that a sectorial operator can determine an analytic semigroup, we first show that a sectorial operator can determine a real analytic α-order fractional resolvent which is defined in terms of Mittag-Leffler function and the curve integral. Then we give some properties of real analytic α-order fractional resolvent. Finally, based on these properties, we discuss the regularity of mild solution of a class of fractional abstract Cauchy problems with Riemann-Liouville fractional derivative.

1. Introduction

Fractional differential equations are widely and efficiently used to describe many phenomena arising in viscoelasticity, fractal, porous media, economic and science. More details on this theory and its applications can be found in [2, 5, 9, 12, 13, 18, 19, 20, 21, 23, 25].

Recently, fractional abstract Cauchy problems have attracted much attention due to their wide application. Bajlekova [3] defined a solution operator which extends the classical semigroup to study the fractional abstract Cauchy problem. Under the condition that the coefficient operator is the generator of a solution operator, some authors got the existence and uniqueness of mild solution of the inhomogeneous α-order abstract Cauchy problem [10, 14, 15, 16]. Under the condition that the coefficient operator generates a C_0-semigroup, there is another tool to deal with the fractional abstract Cauchy problem, it is a new operator described by the C_0-semigroup and the probability density function. For more details, we refer to [6, 7, 8, 24, 26, 27, 28].

However, these papers considered the fractional abstract Cauchy problem only in the Cupto's sense. Heymans and Podlubny [11] showed that in some examples from the field of viscoelasticity, it is possible to attribute physical meaning to initial conditions expressed in terms of Riemann-Liouville fractional derivative or integral. Li, Peng and Jia [17] developed an operator theory to study fractional abstract Cauchy problem with Riemann-Liouville fractional derivative. They proved that a homogeneous α-order Cauchy problem is well posed if and only if its coefficient

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operator is the generator of an α-order fractional resolvent, and gave sufficient conditions to guarantee the existence and uniqueness of weak solutions and strong solutions of an inhomogeneous α-order Cauchy problem. On the other hand, it is well known that a sectorial operator can determine an analytic semigroup. Thus, it is natural to ask whether a sectorial operator can determine a real analytic α-order fractional resolvent.

Our first aim in this paper is to show that a sectorial operator of angle $\theta \in [0,(1-\frac{\alpha}{2})\pi)$ determines a real analytic α-order fractional resolvent $\{T_\alpha(t)\}_{t\geq 0}$ which is defined in terms of Mittag-Leffler function and the curve integral. We also present some properties of $\{T_\alpha(t)\}_{t\geq 0}$.

Our second purpose is to study the regularity of mild solution of an inhomogeneous α-order abstract Cauchy problem. To the best of the authors' knowledge, the regularity of mild solution of fractional abstract Cauchy problem is a subject that has not been treated in the literature. So, in this paper, we will fill the gap in this area. We discuss the regularity of mild solution of the problem

$$\begin{gathered} D_t^\alpha u(t) + Au(t) = f(t), \quad t \in (0,T], \\ (g_{2-\alpha} * u)(0) = 0, \quad (g_{2-\alpha} * u)'(0) = x, \end{gathered} \tag{1.1}$$

where $1 < \alpha < 2$, A is a sectorial operator of angle $\theta \in [0,(1-\frac{\alpha}{2})\pi)$, D_t^α is the α-order Riemann-Liouville fractional derivative operator, $g_{2-\alpha}(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}$ for $t > 0$ and $g_{2-\alpha}(t) = 0$ for $t \leq 0$, $f : [0,T] \to X$, X is a Banach space, $x \in X$. We prove that if $f \in L^p((0,T);X)$ with $p \in (\frac{1}{\alpha}, \frac{1}{\alpha-1})$ then the mild solution of (1.1) is Hölder continuous on $(\varepsilon,T]$ for every $\varepsilon > 0$. We also show that, the Hölder continuity of f ensures that the mild solution u of (1.1) is a classical solution and Au, $D_t^\alpha u$ is Hölder continuous.

The rest of this paper is organized as follows. In Section 2, we provide some preliminaries of the fractional calculus and the Mittag-Leffler function. In Section 3, we introduce an operator family $\{T_\alpha(t)\}_{t\geq 0}$ and analyze its properties. The regularity of mild solution of (1.1) is established in Section 4.

2. Preliminaries

Throughout this paper, let X be a Banach space, $B(X)$ denotes the space of all bounded linear operators from X to X. If A is a closed linear operator, $\rho(A)$ and $\sigma(A)$ denote the resolvent set and the spectral set of A respectively, $R(\lambda, A) = (\lambda I - A)^{-1}$ denotes the resolvent operator of A. $L^1(\mathbb{R}^+, X)$ denotes the Banach space of X-valued Bochner integrable functions.

For convenience, we recall the following known definitions. By $*$ we denote the convolution of functions $(f*g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau$, $t \geq 0$. Let $g_\alpha(\alpha > 0)$ denotes the function

$$g_\alpha(t) = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

and $g_0(t) = \delta_0(t)$, the Dirac delta function.

The Riemann-Liouville fractional integral of order $\alpha > 0$ of f is defined by

$$J_t^\alpha f(t) = (g_\alpha * f)(t).$$

The Riemann-Liouville fractional derivative of order $\alpha > 0$ of f can be written as

$$D_t^\alpha f(t) = \frac{d^m}{dt^m} J_t^{m-\alpha} f(t),$$

where m is the smallest integer greater than or equal to α. For more details about fractional calculus, we refer to [13, 20, 21, 25].

The Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad z, \beta \in \mathbb{C}, \ \operatorname{Re} \alpha > 0.$$

The Mittag-Leffler function has the following properties (see [13]):

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\mu t^\alpha) dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \mu}, \quad \operatorname{Re} \lambda > |\mu|^{1/\alpha}, \tag{2.1}$$

$$\frac{d^n}{dt^n}(t^{\alpha-1} E_{\alpha,\alpha}(\mu t^\alpha)) = t^{\alpha-n-1} E_{\alpha,\alpha-n}(\mu t^\alpha), \quad n \in \mathbb{Z}^+. \tag{2.2}$$

The following lemma gives asymptotic formulae for the Mittag-Leffler functions.

Lemma 2.1 ([21, Theorem 1.4]). *If $0 < \alpha < 2$, β is an arbitrary real number, then for an arbitrary integer $N > 1$,*

$$E_{\alpha,\beta}(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(|z|^{-N}), \quad \frac{\pi\alpha}{2} < |\arg z| \le \pi, \tag{2.3}$$

as $|z| \to \infty$.

Remark 2.2. Since $\frac{1}{\Gamma(-n)} = 0$, $n = 0, 1, 2, \dots$, from (2.3), we know if $\beta - \alpha = -n$, $(n = 0, 1, 2, \dots)$,

$$|E_{\alpha,\beta}(z)| \le \frac{C}{1 + |z|^2}, \quad \frac{\pi\alpha}{2} < |\arg z| \le \pi, \tag{2.4}$$

where C is a real constant.

Now, we present introduction to sectorial operators.

Definition 2.3 ([4, Definition 1.2.1]). Let A be a densely defined closed linear operator on Banach space X, then A is called a sectorial operator of angle $\omega \in [0, \pi)$ ($A \in \operatorname{Sect}(w)$, in short) if

1. $\sigma(A) \subseteq \overline{\Sigma_\omega}$, where
$$\Sigma_\omega := \begin{cases} \{z \in \mathbb{C} : z \ne 0 \text{ and } |\arg z| < \omega\}, & \omega > 0, \\ (0, \infty), & \omega = 0, \end{cases}$$
2. for every $\omega' \in (\omega, \pi)$, $\sup\{\|zR(z, A)\| : z \in \mathbb{C} \setminus \overline{\Sigma_{\omega'}}\} < \infty$.

For a closed linear operator A on a Banach space X, recall the following statement.

Lemma 2.4 ([1, Proposition 1.1.7]). *Let A be a closed linear operator on X and I be an interval in $\mathbb{R}$. Let $f : I \to X$ be Bochner integrable. Suppose that $f(t) \in D(A)$ for $t \in I$ and $Af : I \to X$ is Bochner integrable. Then $\int_I f(t)dt \in D(A)$ and $A \int_I f(t)dt = \int_I Af(t)dt$.*

The following definition is a direct consequence of [17, Definition 3.1 and Theorem 3.12].

Definition 2.5. Let A be a closed linear operator defined on X and $1 \le \alpha \le 2$. A family $\{T_\alpha(t)\}_{t \ge 0} \subset B(X)$ is called an α-order fractional resolvent generated by A, if for every $t \ge 0$, $T_\alpha(t)$ is strongly continuous and there exists $\omega \in \mathbb{R}$ such that $\{\lambda^\alpha : \operatorname{Re} \lambda > \omega\} \subset \rho(A)$ and $(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} T_\alpha(t)x \, dt$, $\operatorname{Re} \lambda > \omega$, $x \in X$.

$\{T_\alpha(t)\}_{t\geq 0}$ has the following property [17, Proposition 3.7]: If $-A$ is the generator of $\{T_\alpha(t)\}_{t\geq 0}$, then for every $t \geq 0$ and $x \in X$, $(g_\alpha * T_\alpha)(t)x \in D(A)$, and

$$T_\alpha(t)x = g_\alpha(t)x - A(g_\alpha * T_\alpha)(t)x. \tag{2.5}$$

Below the letter C denotes various positive constants, and C_α denote various positive constants depending on α.

3. The operator $T_\alpha(t)$

For the rest of this article, let $1 < \alpha < 2$, $A \in \mathrm{Sect}(\theta)$ with $\theta \in [0, (1-\frac{\alpha}{2})\pi)$ and $0 \in \rho(A)$. Inspired by the expression of an analytic semigroup determined by a sectorial operator A, we introduce an operator family $\{T_\alpha(t)\}_{t\geq 0}$ by

$$T_\alpha(t) = \frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} t^{\alpha-1}E_{\alpha,\alpha}(\mu t^\alpha)(\mu I + A)^{-1}d\mu, \tag{3.1}$$

where the integral path $\Gamma_{\pi-\theta} := \{\mathbb{R}^+ e^{i(\pi-\theta)}\} \cup \{\mathbb{R}^+ e^{-i(\pi-\theta)}\}$ is oriented counter clockwise. First, we show some basic properties of $\{T_\alpha(t)\}_{t\geq 0}$.

Theorem 3.1. *For every $t \geq 0$, $T_\alpha(t)$ is well defined and $\{T_\alpha(t)\}_{t\geq 0}$ is a real analytic α-order fractional resolvent. Moreover, there exists a constant C_α such that*

$$\|T_\alpha(t)\| \leq C_\alpha t^{\alpha-1}, \quad t \geq 0. \tag{3.2}$$

Proof. $A \in \mathrm{Sect}(\theta)$ implies that $\Sigma_{\pi-\theta} \subset \rho(-A)$ and

$$\|(\mu I + A)^{-1}\| \leq \frac{C}{|\mu|}, \quad \mu \in \Gamma_{\pi-\theta} \setminus \{0\}, \tag{3.3}$$

which combines with Remark 2.2, we can get that, for every $t \geq 0$, $T_\alpha(t)$ is well defined. For $\mu \in \Gamma_{\pi-\theta}$, since $(\mu I + A)^{-1}$ is a bounded linear operator, it is easy to see that $T_\alpha(t)$ is also a bounded linear operator.

Now, we show that $\{T_\alpha(t)\}_{t\geq 0}$ is an α-order fractional resolvent generated by $-A$. We first show that, for every $t \geq 0$, $T_\alpha(t)$ is a strongly continuous operator. Fix $t_0 \geq 0$, then for $t > 0$, $x \in X$, we have

$$T_\alpha(t)x - T_\alpha(t_0)x = \frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} (t^{\alpha-1}E_{\alpha,\alpha}(\mu t^\alpha) - t_0^{\alpha-1}E_{\alpha,\alpha}(\mu t_0^\alpha))(\mu I + A)^{-1}x d\mu.$$

Then by the continuity of $t^{\alpha-1}E_{\alpha,\alpha}(\mu t^\alpha)$ and the dominated convergence theorem, we know that $\lim_{t\to t_0} T_\alpha(t)x = T_\alpha(t_0)x$.

Let $\theta_0 \in (\frac{\pi}{2}, \frac{\pi-\theta}{\alpha})$, $\varrho > 0$, and

$$l_{\theta_0} := \{re^{-i\theta_0}, \varrho \leq r < \infty\} \cup \{\varrho e^{i\varphi}, |\varphi| < \theta_0\} \cup \{re^{i\theta_0}, \varrho \leq r < \infty\} \tag{3.4}$$

be oriented counter clockwise. Then for $\lambda \in l_{\theta_0}$, $\lambda^\alpha \in \Sigma_{\pi-\theta} \subset \rho(-A)$, hence $\{\lambda^\alpha : Re\lambda > \varrho\} \subset \rho(-A)$. In view of (2.1), we know that

$$t^{\alpha-1}E_{\alpha,\alpha}(\mu t^\alpha) = \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}(\lambda^\alpha - \mu)^{-1}d\lambda, \ \mu \in \Gamma_{\pi-\theta}. \tag{3.5}$$

For $x \in X$, from Fubini's theorem, (3.5) and the Cauchy's integral formula, we see that

$$
\begin{aligned}
T_\alpha(t)x &= \frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} t^{\alpha-1}E_{\alpha,\alpha}(\mu t^\alpha)(\mu I + A)^{-1}x d\mu \\
&= \frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}(\lambda^\alpha - \mu)^{-1} d\lambda (\mu I + A)^{-1} x d\mu \\
&= \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}\frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} (\lambda^\alpha - \mu)^{-1}(\mu I + A)^{-1} x d\mu d\lambda \\
&= \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}(\lambda^\alpha I + A)^{-1} x d\lambda.
\end{aligned}
\tag{3.6}
$$

Then taking Laplace transform on both sides, we obtain

$$
(\lambda^\alpha I + A)^{-1}x = \int_0^\infty e^{-\lambda t}T_\alpha(t)x\, dt, \quad \operatorname{Re}\lambda > \varrho,\ x \in X. \tag{3.7}
$$

Next, we prove that the estimate (3.2) holds. It is clear that $T_\alpha(0) = 0$. For $t > 0$, in view of (3.6) and (3.3), we deduce

$$
\begin{aligned}
\|T_\alpha(t)\| &= \|\frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}(\lambda^\alpha I + A)^{-1} d\lambda\| \\
&= \frac{1}{2\pi}\|\int_{l'_{\theta_0}} e^{\mu}((\frac{\mu}{t})^\alpha I + A)^{-1}\frac{1}{t} d\mu\| \\
&\le \frac{C}{2\pi}\int_{l'_{\theta_0}} |e^\mu|\frac{t^{\alpha-1}}{|\mu|^\alpha}|d\mu| = C_\alpha t^{\alpha-1}.
\end{aligned}
$$

Finally, we verify that $T_\alpha(t)$ is real analytic. From the dominated convergence theorem and (2.2), we have, for $n \in \mathbb{N}^+$,

$$
\begin{aligned}
T_\alpha^{(n)}(t) &= \frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} t^{\alpha-n-1}E_{\alpha,\alpha-n}(\mu t^\alpha)(\mu I + A)^{-1} d\mu \\
&= \frac{1}{2\pi i}\int_{\Gamma'_{\pi-\theta}} t^{\alpha-n-1}E_{\alpha,\alpha-n}(\xi)(\frac{\xi}{t^\alpha} I + A)^{-1}\frac{1}{t^\alpha} d\xi.
\end{aligned}
$$

This combined with (3.3), yields

$$
\|T_\alpha^{(n)}(t)\| \le C_\alpha t^{\alpha-n-1}, \quad t \ge 0. \tag{3.8}
$$

Let $\tilde{c} := \inf_{n\in\mathbb{N}^+}\{C_\alpha^{-\frac{1}{n}}\}$, where C_α is given in (3.8). For fixed $z \in \mathbb{R}^+$, denote $\tilde{z} := \inf_{n\in\mathbb{N}^+}\{z^{1+\frac{1-\alpha}{n}}\}$. Choose $|t - z| \le K\tilde{c}\tilde{z}$, $0 < K < 1$, then $|t - z| \le KC_\alpha^{-\frac{1}{n}} z^{1+\frac{1-\alpha}{n}}$. Thus, the series

$$
T_\alpha(z) + \sum_{n=1}^\infty \frac{T_\alpha^{(n)}(z)}{n!}(t - z)^n
$$

is convergent by means of the operator topology. So $T_\alpha(t)$ is real analytic. □

Theorem 3.2. *For $t > 0$ and $x \in X$, we have $T_\alpha(t)x \in D(A)$ and $\|AT_\alpha(t)\| \le \frac{C}{t}$.*

Proof. From $A(\lambda^\alpha I + A)^{-1} = I - \lambda^\alpha(\lambda^\alpha I + A)^{-1}$, for $t > 0$ and $x \in X$, we have

$$
\int_{l_{\theta_0}} e^{\lambda t}A(\lambda^\alpha I + A)^{-1}x\, d\lambda
$$

$$
\begin{aligned}
&= \int_{l_{\theta_0}} e^{\lambda t} x d\lambda - \int_{l_{\theta_0}} e^{\lambda t} \lambda^\alpha (\lambda^\alpha I + A)^{-1} x \, d\lambda \\
&= \int_{l'_{\theta_0}} e^{\mu} \frac{1}{t} x d\mu - \int_{l'_{\theta_0}} e^{\mu} (\frac{\mu}{t})^\alpha ((\frac{\mu}{t})^\alpha I + A)^{-1} \frac{1}{t} x \, d\mu,
\end{aligned}
$$

where l_{θ_0} is given by (3.4). Since $\theta_0 < \frac{\pi-\theta}{\alpha}$, for $\mu \in l'_{\theta_0}$, we have $\mu^\alpha \in \Sigma_{\pi-\theta} \subset \rho(-A)$, and

$$
\|((\mu/t)^\alpha I + A)^{-1}\| \le \frac{Ct^\alpha}{|\mu|^\alpha}. \tag{3.9}
$$

Consequently,

$$
\| \int_{l_{\theta_0}} e^{\lambda t} A (\lambda^\alpha I + A)^{-1} x d\lambda \| \le \frac{C}{t} \int_{l'_{\theta_0}} |e^{\mu}| |d\mu| \le \frac{C}{t}. \tag{3.10}
$$

Thus, by (3.6), (3.10), the closeness of A and Lemma 2.4, we conclude that for every $x \in X$ and $t > 0$, $T_\alpha(t)x \in D(A)$ and $\|AT_\alpha(t)\| \le \frac{C}{t}$. ☐

4. Main results

In this section, we apply the theory developed in Section 3 to discuss the regularity of mild solution of the following linear inhomogeneous fractional Cauchy problem

$$
\begin{gathered}
D_t^\alpha u(t) + Au(t) = f(t), \quad t \in (0, T], \\
(g_{2-\alpha} * u)(0) = 0, \quad (g_{2-\alpha} * u)'(0) = x,
\end{gathered} \tag{4.1}
$$

where $f \in L^1((0,T); X)$ and $x \in X$.

To present definition of mild solution of problem (4.1), we give the following lemmas.

Lemma 4.1. *Suppose $u \in C([0,T]; X)$ such that $(g_{2-\alpha} * u) \in C^2((0,T]; X), u(t) \in D(A)$ for $t \in [0,T], Au \in L^1((0,T); X)$ and u satisfies (4.1). Then*

$$
u(t) = T_\alpha(t)x + \int_0^t T_\alpha(t-s) f(s) ds. \tag{4.2}
$$

Proof. If u satisfies the assumptions, we can write u as

$$
u(t) = g_\alpha(t)x - A(g_\alpha * u)(t) + (g_\alpha * f)(t), \quad t \in [0, T]. \tag{4.3}
$$

Applying the Laplace transform to (4.3), then, for $\lambda > 0$,

$$
\hat{u}(\lambda) = \lambda^{-\alpha} x - \lambda^{-\alpha} A \hat{u}(\lambda) + \lambda^{-\alpha} \hat{f}(\lambda);
$$

that is

$$
\hat{u}(\lambda) = (\lambda^\alpha I + A)^{-1} x + (\lambda^\alpha I + A)^{-1} \hat{f}(\lambda), \quad \lambda > 0. \tag{4.4}
$$

Then taking inverse Laplace transform to (4.4) and by (3.7), we obtain the conclusion. ☐

Lemma 4.2. *If $f \in L^1((0,T); X)$, then the integral $\int_0^t T_\alpha(t-s) f(s) \, ds$ exists and defines a continuous function.*

Proof. Since $f \in L^1((0,T); X)$, $T_\alpha(t) \in B(X)$ for $t \in (0,T)$, by [22, Theorem 1.3.4], we know that $(T_\alpha * f)(t) = \int_0^t T_\alpha(t-s) f(s) \, ds$ exists and defines a continuous function. ☐

Definition 4.3. The function $u \in C([0,T],X)$ given by

$$u(t) = T_\alpha(t)x + \int_0^t T_\alpha(t-s)f(s)ds$$

is called a mild solution of the Cauchy problem (4.1).

By Definition 4.3 and Lemma 4.1, for $f \in L^1((0,T);X)$, we know the Cauchy problem (4.1) has a unique mild solution.

Definition 4.4. A function $u \in C([0,T],X)$ is called a classical solution of (4.1) if $D_t^\alpha u \in C((0,T],X)$, and for all $t \in (0,T]$, $u(t) \in D(A)$ and satisfies (4.1).

Theorem 4.5. *Let u be the mild solution of (4.1). If $f \in L^p((0,T);X)$ with $\frac{1}{\alpha} < p < \frac{1}{\alpha-1}$, then u is Hölder continuous with exponent $\frac{\alpha p-1}{p}$ on $[\varepsilon,T]$ for every $\varepsilon > 0$.*

Proof. By (3.8), we have $\|T_\alpha'(t)\| \le C_\alpha t^{\alpha-2}$, then from the mean value theorem, we know that $T_\alpha(t)x$ is Lipschitz continuous on $[\varepsilon,T]$ for every $\varepsilon > 0$. If $\frac{1}{\alpha} < p < 1$, we show the Hölder continuity of $T_\alpha(t)x$ at 0, $\frac{1}{\alpha} < p < 1$ implies that $\alpha - 1 \ge \frac{\alpha p-1}{p}$, thus $\|T_\alpha(t)x\| \le C_\alpha\|x\|t^{\alpha-1} \le C_\alpha\|x\|t^{\frac{\alpha p-1}{p}}$.

Now we show that $v(t) := \int_0^t T_\alpha(t-s)f(s)ds$ is Hölder continuous with exponent $\frac{\alpha p-1}{p}$. For $h > 0$ and $t \in [0, T-h]$, we have

$$\begin{aligned}
v(t+h) - v(t) &= \int_0^{t+h} T_\alpha(t+h-s)f(s)ds - \int_0^t T_\alpha(t-s)f(s)ds \\
&= \int_t^{t+h} T_\alpha(t+h-s)f(s)ds + \int_0^t (T_\alpha(t+h-s) - T_\alpha(t-s))f(s)ds \\
&= I_1 + I_2.
\end{aligned}$$

By (3.2) and $p > 1/\alpha$, we have

$$\begin{aligned}
\|I_1\| &\le C_\alpha \int_t^{t+h} (t+h-s)^{\alpha-1}\|f(s)\|ds \\
&\le C_\alpha \Big(\int_t^{t+h} (t+h-s)^{\frac{p(\alpha-1)}{p-1}} ds\Big)^{\frac{p-1}{p}} \|f\|_{L^p} \\
&\le C_\alpha \|f\|_{L^p} h^{\frac{\alpha p-1}{p}}.
\end{aligned}$$

To estimate I_2, we use that (3.2) implies

$$\|T_\alpha(t+h) - T_\alpha(t)\| \le C_\alpha T^{\alpha-1}.$$

On the other hand, from the mean value theorem and (3.8), we obtain

$$\|T_\alpha(t+h) - T_\alpha(t)\| \le C_\alpha t^{\alpha-2}h.$$

Therefore,

$$\|T_\alpha(t+h) - T_\alpha(t)\| \le \mu(h,t) := C_\alpha \min\{T^{\alpha-1},\ t^{\alpha-2}h\}. \tag{4.5}$$

Using (4.5) and the Hölder's inequality, we have

$$\begin{aligned}
\|I_2\| &\le C_\alpha \int_0^t \mu(h, t-s)\|f(s)\|ds \\
&\le C_\alpha \|f\|_{L^p} \Big(\int_0^t \mu(h,t-s)^{\frac{p}{p-1}} ds\Big)^{\frac{p-1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&= C_\alpha \|f\|_{L^p} \Big(\int_0^t \mu(h,\tau)^{\frac{p}{p-1}} d\tau \Big)^{\frac{p-1}{p}} \\
&\le C_\alpha \|f\|_{L^p} \Big(\int_0^\infty \mu(h,\tau)^{\frac{p}{p-1}} d\tau \Big)^{\frac{p-1}{p}} \\
&= C_\alpha \|f\|_{L^p} T^{\alpha-1} h + C_\alpha \|f\|_{L^p} \Big(\int_h^\infty \tau^{\frac{p(\alpha-2)}{p-1}} ds \Big)^{\frac{p-1}{p}} h \\
&= C_\alpha \|f\|_{L^p} T^{\alpha-1} h + C_\alpha \|f\|_{L^p} h^{\frac{p\alpha-1}{p}} \\
&\le C_\alpha \|f\|_{L^p} h^{\frac{\alpha p-1}{p}}.
\end{aligned}
$$

□

Theorem 4.6. *Suppose $f \in C^\gamma([0,T];X)$ for $\gamma \in (0,1)$; that is, there is a constant $k > 0$ such that*

$$
\|f(t) - f(s)\| \le k|t-s|^\gamma, \ 0 < t, s \le T.
$$

Then for every $x \in X$, the mild solution of (4.1) is a classical solution.

Proof. We first show that, for $x \in X$, $T_\alpha(t)x$ is a classical solution of (4.1) with $f = 0$ and $x \in X$. By (2.5) and Theorem 3.2, we have

$$
T_\alpha(t)x = g_\alpha(t)x - A(g_\alpha * T_\alpha)(t)x = g_\alpha(t)x - (g_\alpha * AT_\alpha)(t)x, \quad t \ge 0, \ x \in X. \tag{4.6}
$$

Then

$$
\begin{aligned}
D_t^\alpha T_\alpha(t)x &= \frac{d^2}{dt^2} g_{2-\alpha} * (g_\alpha(t)x - (g_\alpha * AT_\alpha)(t)x) \\
&= \frac{d^2}{dt^2}(g_{2-\alpha} * g_\alpha)(t)x - \frac{d^2}{dt^2}(g_{2-\alpha} * g_\alpha * AT_\alpha)(t)x \\
&= \frac{d^2}{dt^2} g_2(t)x - \frac{d^2}{dt^2}(g_2 * AT_\alpha)(t)x \\
&= -AT_\alpha(t)x,
\end{aligned}
$$

and it is clear that $(g_{2-\alpha} * T_\alpha)(0)x = 0$, $(g_{2-\alpha} * T_\alpha)'(0)x = x$.

Now, we verify that $v(t) := \int_0^t T_\alpha(t-s)f(s)ds$ is a classical solution of the problem

$$
\begin{gathered}
D_t^\alpha u(t) + Au(t) = f(t), \quad t \in (0,T], \\
(g_{2-\alpha} * u)(0) = 0, \quad (g_{2-\alpha} * u)'(0) = 0.
\end{gathered} \tag{4.7}
$$

Lemma 4.2 implies $v \in C([0,T];X)$. It is clear that $v(t) = I_1(t) + I_2(t)$, where

$$
\begin{gathered}
I_1(t) = \int_0^t T_\alpha(t-s)(f(s) - f(t))\, ds, \quad 0 < t \le T, \\
I_2(t) = \int_0^t T_\alpha(t-s)f(t)\, ds, \ 0 < t \le T.
\end{gathered}
$$

Firstly, we show that $v(t) \in D(A)$ for $t \in (0,T]$.

For fixed $t \in (0,T]$, from Theorem 3.2 and Hölder continuity of f, we have

$$
\|AT_\alpha(t-s)(f(s) - f(t))\| \le \frac{C}{t-s}(t-s)^\gamma \in L^1(0,t).
$$

According to the closeness of A and Lemma 2.4, we see $I_1(t) \in D(A)$. To prove the same conclusion for $I_2(t)$, from (3.6) and the Laplace transform property of convolution, we see that

$$I_2(t) = \int_0^t T_\alpha(t-s)f(t)ds = (1 * T_\alpha)(t)f(t) = \frac{1}{2\pi i}\int_{l_{\theta_0}} e^{\lambda t}\lambda^{-1}(\lambda^\alpha I + A)^{-1}d\lambda.$$

On the other hand,

$$\begin{aligned}\int_{l_{\theta_0}} e^{\lambda t}\lambda^{-1}A(\lambda^\alpha I + A)^{-1}d\lambda &= \int_{l_{\theta_0}} e^{\lambda t}\lambda^{-1}d\lambda - \int_{l_{\theta_0}} e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha I + A)^{-1}d\lambda \\ &= \int_{l'_{\theta_0}} e^{\mu}\frac{1}{\mu}d\mu - \int_{l'_{\theta_0}} e^{\mu}(\frac{\mu}{t})^{\alpha-1}((\frac{\mu}{t})^\alpha I + A)^{-1}\frac{1}{t}d\mu.\end{aligned}$$

Thus, by (3.3), we have

$$\|\int_{l_{\theta_0}} e^{\lambda t}\lambda^{-1}A(\lambda^\alpha I + A)^{-1}d\lambda\| \le C\int_{l'_{\theta_0}} |e^{\mu}|\frac{1}{|\mu|}|d\mu| \le C,$$

which implies that the integral $\int_{l_{\theta_0}} e^{\lambda t}\lambda^{-1}A(\lambda^\alpha I + A)^{-1}d\lambda$ is convergent. Then the closeness of A and Lemma 2.4 conclude that

$$(1 * T_\alpha)(t)x \in D(A), \quad x \in X, \quad \text{and} \quad \|A(1 * T_\alpha)(t)\| \le C. \tag{4.8}$$

Thus $I_2(t) \in D(A)$.

Next, we show that $D_t^\alpha v \in C((0,T];X)$. Equality (4.6) implies

$$\begin{aligned}D_t^\alpha v(t) &= \frac{d^2}{dt^2}(g_{2-\alpha} * T_\alpha * f)(t) \\ &= \frac{d^2}{dt^2}\big((g_2 * f)(t) + (g_2 * AT_\alpha * f)(t)\big) \\ &= f(t) + A(T_\alpha * f)(t) \\ &= f(t) + Av(t).\end{aligned}$$

Therefore, it remains to prove $Av = AI_1(t) + AI_2(t) \in C((0,T];X)$. Since $AI_2(t) = (1 * T_\alpha)(t)f(t)$, and from the assumption on f and Theorem 3.1, we see that $AI_2(t)$ is continuous on $(0,T]$.

For $AI_1(t)$, if $h > 0$ and $t \in (0, T-h]$, we have

$$\begin{aligned}AI_1(t+h) - AI_1(t) &= \int_0^t A[T_\alpha(t+h-s) - T_\alpha(t-s)](f(s) - f(t))ds \\ &\quad + \int_0^t AT_\alpha(t+h-s)(f(t) - f(t+h))ds \\ &\quad + \int_t^{t+h} AT_\alpha(t+h-s)(f(s) - f(t+h))ds \\ &= h_1 + h_2 + h_3.\end{aligned} \tag{4.9}$$

For h_1,

$$\lim_{h\to 0} AT_\alpha(t+h-s)(f(s) - f(t)) = AT_\alpha(t-s)(f(s) - f(t)),$$

and from Theorem 3.2, we know that

$$\|AT_\alpha(t+h-s)(f(s) - f(t))\| \le C(t+h-s)^{-1}(t-s)^\gamma \le C(t-s)^{\gamma-1} \in L^1(0,t).$$

Thus, by means of the dominated convergence theorem, we obtain that $h_1 \to 0$ as $h \to 0$.

For h_2, we have

$$\begin{aligned}
\|h_2\| &= \| \int_0^t AT_\alpha(t+h-s)(f(t)-f(t+h))\,ds\| \\
&\le C\int_0^t (t+h-s)^{-1}h^\gamma\,ds \\
&= C(\ln(t+h)-\ln h)h^\gamma,
\end{aligned}$$

so, $h_2 \to 0$ as $h \to 0$. Also

$$\|h_3\| \le C\int_t^{t+h}(t+h-s)^{-1}(t+h-s)^\gamma ds = \frac{Ch^\gamma}{\gamma} \to 0 \quad \text{as } h \to 0.$$

Consequently, $Av \in C((0,T];X)$. It is easy to see that $(g_{2-\alpha} * v)(0) = 0$, $(g_{2-\alpha} * v)'(0) = 0$. ∎

Lemma 4.7. *Suppose $f \in C^\gamma([0,T];X)$ for $\gamma \in (0,1)$, denote*

$$I_1(t) := \int_0^t T_\alpha(t-s)(f(s)-f(t))ds, \quad t \in (0,T],$$

then $I_1(t) \in D(A)$ for $0 \le t \le T$ and $AI_1 \in C^\gamma([0,T];X)$.

Proof. The fact that $I_1(t) \in D(A)$ for $0 \le t \le T$ is an immediate consequence of the proof of Theorem 4.6, so we only need to prove the Hölder continuity of $AI_1(t)$.

From the dominated convergence theorem and (2.2), we have

$$\begin{aligned}
&\frac{d}{dt}AT_\alpha(t) \\
&= \frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} t^{\alpha-2}E_{\alpha,\alpha-1}(\mu t^\alpha)A(\mu I + A)^{-1}d\mu \\
&= \frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} t^{\alpha-2}E_{\alpha,\alpha-1}(\mu t^\alpha)d\mu - \frac{1}{2\pi i}\int_{\Gamma_{\pi-\theta}} t^{\alpha-2}E_{\alpha,\alpha-1}(\mu t^\alpha)\mu(\mu I + A)^{-1}d\mu \\
&= \frac{1}{2\pi i}\int_{\Gamma'_{\pi-\theta}} t^{\alpha-2}E_{\alpha,\alpha-1}(\xi)\frac{1}{t^\alpha}d\xi - \frac{1}{2\pi i}\int_{\Gamma'_{\pi-\theta}} t^{\alpha-2}E_{\alpha,\alpha-1}(\xi)\frac{\xi}{t^\alpha}(\frac{\xi}{t^\alpha}I + A)^{-1}\frac{1}{t^\alpha}d\xi.
\end{aligned}$$

In view of (3.3), we deduce that

$$\|\frac{d}{dt}AT_\alpha(t)\| \le C_\alpha t^{-2}, \quad 0 < t \le T. \tag{4.10}$$

Thus, for every $0 < s < t \le T$, we obtain

$$\begin{aligned}
\|AT_\alpha(t) - AT_\alpha(s)\| &= \| \int_s^t \frac{d}{d\tau}AT_\alpha(\tau)d\tau\| \\
&\le \int_s^t \|\frac{d}{d\tau}AT_\alpha(\tau)\|d\tau \\
&\le C_\alpha\int_s^t \tau^{-2}d\tau = C_\alpha t^{-1}s^{-1}(t-s).
\end{aligned}\tag{4.11}$$

For $h > 0$ and $t \in [0, T-h]$, from (4.9), we know that

$$AI_1(t+h) - AI_1(t) = h_1 + h_2 + h_3. \tag{4.12}$$

From $f \in C^\gamma([0,T];X)$ and (4.11), it follows that

$$\begin{aligned} \|h_1\| &\le \int_0^t \|AT_\alpha(t+h-s) - AT_\alpha(t-s)\| \|f(s)-f(t)\|\, ds \\ &\le C_\alpha h \int_0^t (t+h-s)^{-1}(t-s)^{\gamma-1}\, ds \\ &= C_\alpha h \int_0^t (s+h)^{-1} s^{\gamma-1}\, ds \\ &\le C_\alpha \int_0^h \frac{h}{s+h} s^{\gamma-1} ds + C_\alpha \int_h^\infty \frac{s^{\gamma-1}}{s+h} h\, ds \\ &\le C_\alpha \int_0^h s^{\gamma-1}\, ds + C_\alpha \int_h^\infty s^{\gamma-2} h\, ds = C_\alpha h^\gamma. \end{aligned} \tag{4.13}$$

For h_2, by Theorem 3.2 and the mean value theorem, we have

$$\begin{aligned} \|h_2\| &\le \int_0^t \|AT_\alpha(t+h-s)\| \|f(t)-f(t+h)\|\, ds \\ &\le C \int_0^t (t+h-s)^{-1}\, ds\, h^\gamma = C \int_h^{t+h} s^{-1}\, ds\, h^\gamma \\ &= C \frac{t}{\theta t + h} h^\gamma \le \frac{C}{\theta} h^\gamma, \end{aligned} \tag{4.14}$$

where $\theta \in (0,1)$.

For h_3, it follows from Theorem 3.2 and the assumption on f, we see that

$$\begin{aligned} \|h_3\| &\le \int_t^{t+h} \|AT_\alpha(t+h-s)\| \|f(s)-f(t+h)\|\, ds \\ &\le C \int_t^{t+h} (t+h-s)^{\gamma-1}\, ds \le Ch^\gamma. \end{aligned} \tag{4.15}$$

Combining (4.12) with the estimates (4.13), (4.14) and (4.15), we obtain that AI_1 is Hölder continuous with exponent γ on [0,T]. □

Theorem 4.8. *Suppose $f \in C^\gamma([0,T];X)$ for $\gamma \in (0,1)$. If u is a classical solution of the problem* (4.1) *on $[0,T]$, then*

- (i) *For every $\varepsilon > 0$, $Au \in C^\gamma([\varepsilon,T];X)$ and $D_t^\alpha u(t) \in C^\gamma([\varepsilon,T];X)$.*
- (ii) *If $x \in D(A), f(0) = 0$, then Au and $D_t^\alpha u(t)$ are continuous on $[0,T]$.*
- (iii) *If $x = 0, f(0) = 0$, then $Au, D_t^\alpha u(t) \in C^\gamma([0,T];X)$.*

Proof. (i) If u is a classical solution of the initial value problem (4.1) on $[0,T]$, then

$$u(t) = T_\alpha(t)x + \int_0^t T_\alpha(t-s)f(s)ds = T_\alpha(t)x + v(t).$$

By (4.10), we know that $AT_\alpha(t)x$ is Lipschitz continuous on $[\varepsilon,T]$ for every $\varepsilon > 0$. So, it suffices to show that $Av \in C^\gamma([\varepsilon,T];X)$. As in Theorem 4.6, we write $v(t)$ as

$$v(t) = I_1(t) + I_2(t) = \int_0^t T_\alpha(t-s)(f(s)-f(t))ds + \int_0^t T_\alpha(t-s)f(t)ds,$$

for $0 < t \le T$. It follows from Lemma 4.7 that $AI_1 \in C^\gamma([0,T];X)$. So it remains to verify that $AI_2 \in C^\gamma([\varepsilon,T];X)$ for every $\varepsilon > 0$. To this end, let $h > 0$ and

$t \in [\varepsilon, T-h]$, then

$$\begin{aligned} AI_2(t+h) - AI_2(t) &= \int_0^{t+h} AT_\alpha(t+h-s) f(t+h)\, ds - \int_0^t AT_\alpha(t-s) f(t)\, ds \\ &= \int_0^{t+h} AT_\alpha(s) f(t+h)\, ds - \int_0^t AT_\alpha(s) f(t)\, ds \\ &= \int_0^{t+h} AT_\alpha(s)(f(t+h) - f(t))\, ds + \int_t^{t+h} AT_\alpha(s) f(t)\, ds. \end{aligned}$$

This combined with (4.8) yield

$$\begin{aligned} \|AI_2(t+h) - AI_2(t)\| &\le C\|A(1 * T_\alpha)(t+h)\| h^\gamma + C\int_0^h s^{-1} ds \|f\|_\infty \\ &\le Ch^\gamma + \frac{C}{\varepsilon} h \le Ch^\gamma, \end{aligned}$$

where $\|f\|_\infty = \max_{0\le t\le T} \|f(t)\|$.

(ii) If $x \in D(A)$, then $AT_\alpha(t)x \in C([0,T];X)$. By Lemma 4.7 and (i), we know that $AI_1 \in C^\gamma([0,T];X)$, $AI_2 \in C^\gamma([\varepsilon,T];X)$. We need to show that AI_2 is continuous at $t = 0$. Since $f(0) = 0$ and (4.8), we have $\|AI_2(t)\| \le \|(1 * T_\alpha)(t)\| \|f(t)\| \le C\|f(t)\| \to 0$ as $t \to 0$. This completes (ii).

(iii) We only to show that $AI_2 \in C^\gamma([0,T];X)$.

$$\begin{aligned} &\|AI_2(t+h) - AI_2(t)\| \\ &\le \| \int_0^{t+h} AT_\alpha(s)(f(t+h) - f(t))\, ds\| + \| \int_t^{t+h} AT_\alpha(s) f(t)\, ds\| \\ &\le \|(1 * AT_\alpha)(t+h)\| \|f(t+h) - f(t)\|\, ds + \int_t^{t+h} \|AT_\alpha(s)\| \|f(t) - f(0)\| ds \\ &\le Ch^\gamma + \int_t^{t+h} s^{-1} t^\gamma\, ds \le Ch^\gamma + \int_t^{t+h} s^{\gamma-1}\, ds \\ &\le Ch^\gamma + \int_0^h (t+s)^{\gamma-1}\, ds \le Ch^\gamma + \int_0^h s^{\gamma-1}\, ds \\ &\le Ch^\gamma. \end{aligned}$$

□

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References

[1] W. Arendt, C. Batty, M. Hieber, F. Neubrander; *Vector-Valued Laplace Transforms and Cauchy Problems*, Monographs in Mathematics, 96. Birkhäuser Verlag, Basel, 2001.

[2] V. V. Anh, N. N. Leonenko; *Spectral analysis of fractional kinetic equations with random data*, J. Statist. Phys., 104 (2001) 1349-1387.

[3] E. Bajlekova; *Fractional evolution equations in Banach spaces*, PhD Thesis, Eindhoven University of Technology (2001).

[4] C. M. Carracedo, M. S. Alix; *The Theory of Fractional Powers of Operators*, North-Holland Mathematics Studies, 187 Elsevier, 2001.

[5] T. E. Duncan, Y. Z. Hu, B. Pasik-Duncan; *Stochastic calculus for fractional Brownian motion I. Theory*, SIAM J. Control Optim., 38 (2000) 582-612.

[6] M. M. El-Borai; *Semigroup and some nonlinear fractional differential equations*, Appl. Math. Comput., 149 (2004) 823-831.
[7] M. M. El-Borai; *Some probability densities and fundamental solutions of fractional evolution equations*, Chaos Solitons Fractals, 14 (2002) 433-440.
[8] M. M. El-Borai, K. E. El-Nadi, E. G. El-Akabawy; *On some fractional evolution equations*, Comput. Math. Anal., 59 (2010) 1352-1355.
[9] L. Gaul, P. Klein, S. Kempfle; *Damping description involving fractional operators*, Mech. Sys. Signal Process., 5 (1991) 81-88.
[10] E. Hernández, D. O'Regan, K. Balachandran; *On recent developments in the theory of abstract differential equations with fractional derivatives*, Nonlinear Anal., 73 (2010) 3462-3471.
[11] N. Heymans, I. Podlubny; *Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives*. Rheologica Acta, 45 (2006) 765-772.
[12] R. Hilfer; *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
[13] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.
[14] M. Li, C. Chen, F. B. Li; *On fractional powers of generators of fractional resolvent families*, J. Funct. Anal., 259 (2010) 2702-2726.
[15] K. Li, J. Jia; *Existence and uniqueness of mild solutions for abstract delay fractional differential equations*, Comput. Math. Anal., 62 (2011) 1398-1404.
[16] K. Li, J. Peng; *Fractional abstract Cauchy problem*, Integr. Equ. Oper. Theory, 70 (2011) 333-361.
[17] K. Li, J. Peng, J. Jia; *Cauchy problems for fractional differential equations with Riemann-Liouville fractional derivatives*, J. Funct. Anal., 263 (2012) 476-510.
[18] F. Mainardi; *Fractional calculus, Some basic problems in continuum and statistical mechanics*, Fractals and Fractional Calculus in Continuum Mechanics, 378 (1997) 291-348.
[19] F. Metzler, W. Schick, H. G. Kilian, T. F. Nonnenmache; *Relaxation in filled polymers: a fractional calculus approach*, J. Chem. Phys., 103 (1995) 7180-7186.
[20] K. S. Miller, B. Ross; *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, 1993.
[21] I. Podlubny; *Fractional Differential Equations*, Math. Sci. Eng., vol. 198, Academic Press, San Diego, 1999.
[22] J. Prüss; *Evolutionary Integral Equations and Applications*, Birkhäuser, Basel, 1993.
[23] V. Tarasov; *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer-Verlag, New York, 2011.
[24] J. R. Wang, Y. Zhou; *A class of fractional evolution equations and optimal controls*, Nonlinear Anal. Real World Appl., 12 (2011) 262-272.
[25] Y. Zhou; *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.
[26] Y. Zhou, F. Jiao; *Nonlocal Cauchy problem for fractional evolution equations*, Nonlinear Anal. Real World Appl., 11 (2010) 4465-4475.
[27] Y. Zhou, F. Jiao; *Existence of mild solutions for fractional neutral evolution equations*, Comput. Math. Anal., 59 (2010) 1063-1077.
[28] Y. Zhou, F. Jiao, J. Pecaric; *Abstract Cauchy problem for fractional functional differential equations*, Topological Methods in Nonlinear Anal. 42 (2013), 119-136.

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