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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO HIGHER ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study the oscillation and asymptotic behavior of solutions to the nonlinear delay differential equation

$$x^{(n+3)}(t) + p(t)x^{(n)}(t) + q(t)f(x(g(t))) = 0.$$

By using a generalized Riccati transformation and an integral averaging technique, we establish sufficient conditions for all solutions to oscillate, or to converge to zero. Especially when the delay has the form $g(t) = at - \tau$, we provide two convenient oscillatory criteria. Some examples are given to illustrate our results.

1. Introduction

In this article, we study the oscillation and the asymptotic behavior of solutions to the $n+3$-order nonlinear delay differential equation

$$x^{(n+3)}(t) + p(t)x^{(n)}(t) + q(t)f(x(g(t))) = 0, \quad t \in I := [a, +\infty) \tag{1.1}$$

where $q \in C(I, \mathbb{R}^+), p \in C^1(I, \mathbb{R})$ with $p(t) \geq 0$ and it does not vanish identically on any $[T, \infty) \subset I$, $g \in C^1(I, \mathbb{R})$ with $0 < g(t) < t, g'(t) \geq 0$ and $\lim_{t\to+\infty} g(t) = +\infty$, $f \in C(\mathbb{R}, \mathbb{R})$ and $f(u)/u \geq K (u \neq 0)$ for some positive constant K.

Our attention is restricted to those solutions of (1.1) which exist on I and satisfy $\sup_{t\geq T} |x(t)| > 0$ for any $T \geq a$. We make a standing hypothesis that (1.1) possess such solutions. As usual, a solution of (1.1) is called oscillatory if it has arbitrarily large zeros, and non-oscillatory otherwise. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

The oscillation and asymptotic behavior have extensive applications in the real world. See the monographs [1] for more details. The problem of obtaining the oscillation and asymptotic behavior of certain higher-order nonlinear functional differential equations has been studied by a number of authors, see [1, 2, 3, 5, 6, 8, 12, 13, 14, 16] and the references cited therein.

In 1971 and 1977, [10, 11] discussed the oscillation of solutions of the equation

$$x^{(n)}(t) + a(t)f(x(g(t))) = 0,$$

where $0 < g(t) < t, g(t) \to \infty$ as $t \to +\infty$ and $a(t) > 0$.

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Recently, the authors in [3] studies the $2n$-order nonlinear functional differential equation

$$\frac{d^n}{dt^n}\Big(a(t)\big(\frac{d^n x(t)}{dt^n}\big)^{\alpha}\Big) + q(t)f(x(g(t))),$$

where α is the ratio of two positive odd integers. The oscillation theorems established here extend a number of existing results.

On the other hand, there are many publications about nonlinear functional differential equations with damping. For example, the authors in [17] investigated the third-order nonlinear functional differential equation

$$(r_2(t)(r_1(t)y')')' + q(t)y' + f(y(g(t))) = 0.$$

Using a generalized Riccati transformation and integral averaging technique, they establish some new sufficient conditions which insure that any solution of this equation oscillates or converges to zero.

The authors in [9] studied the nonlinear functional differential equation

$$y^{(4)}(t) + p(t)y'(t) + q(t)f(y(g(t))) = 0. \tag{1.2}$$

By applying the generalized Riccati transformation, it was shown that all solutions of (1.2) oscillate or converge to zero under some conditions.

The goal of the present paper is to study the oscillation and asymptotic behavior of solutions of the nonlinear delay differential equation (1.1). We note that equation (1.1) with $n = 1$ is exactly (1.2). The authors in [9] showed that the oscillation and asymptotic behavior of (1.2) may yield useful information in real problems. Therefore, we think that it is interesting to study the oscillation of (1.1) since it extends the former studies. The main idea in the proof of our results comes from [9, 17]. This paper is organized as follows: In Section 2, we present some lemmas which are useful in the proof of our main results. Section 3 will provide several oscillatory and asymptotic criteria for system (1.1). We note that, in many applications the delay $g(t)$ has the form $g(t) = t - \tau$ or the form $g(t) = at$. As the corollary of our main results, we give two convenient oscillatory and asymptotic criterions for system (1.1) having such a common delay; see Corollaries 3.4 and 3.8, respectively. In Section 4, some examples illustrate our main results.

2. Some preliminary lemmas

In this section we state and prove some lemmas which we will use in the proof of our main results.

Lemma 2.1. *Suppose the linear third-order differential equation*

$$u'''(t) + p(t)u(t) = 0, \quad t \ge a \tag{2.1}$$

has an eventually positive increasing solution. If x is a non-oscillatory solution of (1.1)*, then there exists a constant T such that $|x^{(n)}(t)| > 0$ for $t \ge T$.*

Proof. Without loss of generality that $x(t) > 0$ for $t \ge a$. It is easy to see that $y = -x^{(n)}$ is a solution of

$$y'''(t) + p(t)y(t) = q(t)f(x(g(t))). \tag{2.2}$$

By using a similar argument as that in the proof of [9, Lemma 1.2], we conclude that all solutions of (2.2) are non-oscillatory. Thus $x^{(n)}(t)$ is eventually positive or eventually negative. □

Lemma 2.2. *Let x be a non-oscillatory solution of* (1.1)*. If there exists a constant $T_1 \geq a$ such that $x(t)x^{(n)}(t) > 0$ for $t \geq T_1$, then $x(t)x^{(n+2)}(t)$ is eventually positive.*

Proof. Suppose firstly that $x(t) > 0$ for $t \geq T_1$. Since $g'(t) > 0$ and $g(t) \to +\infty$ as $t \to +\infty$, it follows that $x(g(t)) > 0$ for $t > T_1'$ for some constant $T_1' \geq T_1$. For the sake of brevity we assume that $T_1 = T_1' = a$ without loss of generality.

By (1.1) we find that $x^{(n+3)}(t) < 0$ for $t \geq a$. Thus there exists a $\mu \in \mathbb{R} \cup \{-\infty\}$ such that $\lim_{t\to+\infty} x^{(n+2)}(t) = \mu$. In view of $x^{(n)}(t) > 0, t \in I$, it turns out that $\mu \geq 0$. Thus $x^{(n+2)}(t)$ is eventually positive.

The case that $x(t) < 0$ for $t \geq T_1$ can be discussed in a way completely analogous to the previous one, and hence it is omitted. This completes the proof. □

By a careful check of the proof of [9, Lemma 1.1], we obtain the following result.

Lemma 2.3. *Assume that $x \in C^n(I, \mathbb{R})$ such that $x(t) > 0, x^{(n)}(t) \leq 0$ for $t \in I$ and $x^{(n)}(t)$ does not vanish identically on any $[T, \infty) \subset I$. If n is even (or odd), then there exists $l \in \{1, 3, \dots, n-1\}$ (resp. $l \in \{0, 2, \dots, n-1\}$) such that for all sufficiently large t, $x(t)x^{(j)}(t) > 0$ for $j = 0, 1, \dots, l$ and $(-1)^{n+j-1}x(t)x^{(j)}(t) > 0$ for $j = l+1, l+2, \dots, n-1$. Furthermore, if $l \geq 1$, then*

$$|x'(g(t))| \geq \frac{g^{l-1}(t)(t-g(t))^{n-l-1}}{2^{l-1}(l-1)!(n-l-1)!}|x^{(n-1)}(t)| \tag{2.3}$$

for all sufficiently large t.

Remark 2.4. Lemma 2.3 is different from [9, Lemma 1.1] by pointing out that inequality (2.3) is invalid for $l = 0$. And the case $l = 0$ needs a separate treatment in the proof of our main results.

3. Asymptotic Dichotomy

In this section we present some sufficient conditions which guarantee that every solution of (1.1) oscillates or converges to zero. Throughout this section we will impose the following condition:

$$\lim_{t\to\infty} \int_a^t [q(\tau) - Mp_+'(\tau)| d\tau = +\infty, \tag{3.1}$$

for any $M > 0$, where

$$p_+'(t) = \begin{cases} p'(t), & \text{if } p'(t) > 0, \\ 0, & \text{if } p'(t) \leq 0. \end{cases}$$

Theorem 3.1. *Suppose that* (2.1) *has an eventually positive increasing solution and that* (3.1) *holds. Assume further that there exists a $\rho \in C^1(I, \mathbb{R}^+)$ such that*

$$\limsup_{t\to\infty} \int_T^t \Big[K\rho(s)q(s) - \frac{2^{l-3}(l-1)!(n-l+2)!(\rho'(s))^2}{g^{l-1}(s)(s-g(s))^{n-l+2}g'(s)\rho(s)}\Big] ds = +\infty \tag{3.2}$$

holds for every $T \geq a$ and for all $l = 2, 4, \dots, n+2$ when n is even and for all $l = 1, 3, \dots, n+2$ when n is odd. Then every solution x of (1.1) *is oscillatory, or satisfies $x(t) \to 0$ as $t \to \infty$.*

Proof. Let x be a non-oscillatory solution of (1.1). Without loss of generality, we may assume that $x(t) > 0$ and $x(g(t)) > 0$ for $t \geq a$. By Lemma 2.1, there exists a constant $T \geq a$ such that $x^{(n)}(t) > 0$ or $x^{(n)}(t) < 0$ for $t \geq T$.

Consider firstly the case that $x^{(n)}(t) > 0, t \geq T$. By (1.1) we know that $x^{(n+3)}(t) < 0, t \geq T$. Therefore, it follows form Lemma 2.3 (it worth mentioning here that n is replaced with $n+3$) that there exists $l \in \{1, 3, \dots, n+2\}$ (resp. $l \in \{0, 2, \dots, n+2\}$) when n is odd (resp. n is even) such that for all sufficiently large t, $x^{(j)}(t) > 0$ for $j = 0, 1, \dots, l$ and $(-1)^{n+j}x^{(j)}(t) > 0$ for $j = l+1, l+2, \dots, n+2$.

If $l \geq 1$, then we consider the function w defined by

$$w(t) = \frac{\rho(t)x^{(n+2)}(t)}{x(g(t))}, \quad t \in I. \tag{3.3}$$

According to Lemma 2.2, $w(t)$ is eventually positive. It follows from (1.1) and Lemma 2.3 that

$$\begin{aligned}
&w'(t)\\
&= \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\rho(t)q(t)f(x(g(t)))}{x(g(t))} - \frac{\rho(t)p(t)x^{(n)}(t)}{x(g(t))} - \frac{\rho(t)x^{(n+2)}(t)x'(g(t))g'(t)}{x^2(g(t))}\\
&\leq \frac{\rho'(t)}{\rho(t)}w(t) - K\rho(t)q(t) - \frac{g^{l-1}(t)(t-g(t))^{n-l+2}g'(t)\rho(t)(x^{(n+2)}(t))^2}{2^{l-1}(l-1)!(n-l+2)!x^2(g(t))}\\
&= \frac{\rho'(t)}{\rho(t)}w(t) - K\rho(t)q(t) - w^2(t)\frac{g^{l-1}(t)(t-g(t))^{n-l+2}g'(t)}{2^{l-1}(l-1)!(n-l+2)!\rho(t)}\\
&= -K\rho(t)q(t) - \frac{g^{l-1}(t)(t-g(t))^{n-l+2}g'(t)}{2^{l-1}(l-1)!(n-l+2)!\rho(t)}\Big(w(t)\\
&\quad - \frac{2^{l-1}(l-1)!(n-l+2)!\rho(t)\rho'(t)}{2\rho(t)g^{l-1}(t)(t-g(t))^{n-l+2}g'(t)}\Big)^2 + \frac{2^{l-3}(l-1)!(n-l+2)!\rho'^2(t)}{\rho(t)g^{l-1}(t)(t-g(t))^{n-l+2}g'(t)}.
\end{aligned} \tag{3.4}$$

Thus

$$w'(t) \leq -K\rho(t)q(t) + \frac{2^{l-3}(l-1)!(n-l+2)!\rho'^2(t)}{\rho(t)g^{l-1}(t)(t-g(t))^{n-l+2}g'(t)}.$$

Integration yields

$$\int_T^t \Big(K\rho(s)q(s) - \frac{2^{l-3}(l-1)!(n-l+2)!\rho'^2(s)}{\rho(s)g^{l-1}(s)(s-g(s))^{n-l+2}g'(s)}\Big)ds \leq w(T) - w(t), \quad t > T,$$

which contradicts (3.2).

If $l = 0$ (which means that n is even), then

$$\begin{gathered}
x'(t) < 0, \quad x''(t) > 0, \quad x'''(t) < 0, \quad \dots,\\
x^{(n)}(t) > 0, \quad x^{(n+1)}(t) < 0, \quad x^{(n+2)}(t) > 0
\end{gathered} \tag{3.5}$$

for sufficiently large t, namely, for $t \geq T_1$. Let $\lim_{t\to\infty} x(t) = \mu$. If $\mu \neq 0$, then there exists a constant $T_2 \geq T_1$ such that $x(g(t)) \geq x(t) > \mu > 0, t \geq T_2$. From (1.1) we obtain

$$x^{(n+2)}(t) \leq x^{(n+2)}(T_2) - K\int_{T_2}^t x(g(u))q(u)du \leq x^{(n+2)}(T_2) - K\mu\int_{T_2}^t q(u)du, \tag{3.6}$$

for $t \geq T_2$. By (3.1) we know that $\int_{T_2}^\infty q(u)du = +\infty$. Thus inequality (3.6) implies that $x^{(n+2)}(t)$ is eventually negative, a contradiction to (3.5).

Consider next the case that $x^{(n)}(t) < 0$ for $t \geq T$. By Lemma 2.3, $x(t)$ is eventually monotonous and $x^{(n-1)}(t)$ is eventually positive. Let

$$\lim_{t\to+\infty} x(t) = \alpha_1, \quad \lim_{t\to+\infty} x^{(n-1)}(t) = \alpha_2.$$

We claim that $\alpha_1 = 0$. If this is not true, then there exist constants $\beta_1, \beta_2 > 0$ such that

$$x(g(t)) > \beta_1, \quad 0 < x^{(n-1)}(t) < \beta_2, \quad t \geq T_3 \tag{3.7}$$

for some constant $T_3 > 0$.

Integrating (1.1) from T_3 to t yields

$$\begin{aligned} &x^{(n+2)}(t) + \int_{T_3}^{t} [(p(u)x^{(n-1)}(u))' - p'(u)x^{(n-1)}(u)]du \\ &+ \int_{T_3}^{t} x(g(u))q(u)\frac{f(x(g(u)))}{x(g(u))}du \\ &= x^{(n+2)}(T_3). \end{aligned}$$

Thus by (3.7) we obtain

$$\begin{aligned} &x^{(n+2)}(t) \\ &\leq x^{(n+2)}(T_3) + p(T_3)x^{(n-1)}(T_3) + \int_{T_3}^{t} p'(u)x^{(n-1)}(u)du - \int_{T_3}^{t} \beta_1 K q(u)du \\ &\leq x^{(n+2)}(T_3) + p(T_3)x^{(n-1)}(T_3) + \int_{T_3}^{t} x^{(n-1)}p'_+(u)du - \int_{T_3}^{t} \beta_1 K q(u)du \\ &\leq x^{(n+2)}(T_3) + p(T_3)x^{(n-1)}(T_3) + \int_{T_3}^{t} \beta_2 p'_+(u)du - \int_{T_3}^{t} \beta_1 K q(u)du \\ &= x^{(n+2)}(T_3) + p(T_3)x^{(n-1)}(T_3) - \beta_1 K \int_{T_3}^{t} [q(u) - \frac{\beta_2}{\beta_1 K}p'_+(u)]du. \end{aligned} \tag{3.8}$$

By letting $t \to +\infty$, we get from (3.1) that $x^{(n+2)}(t) \to -\infty$. Consequently, there is a constant $T_4 \geq T_3$ such that $x^{(n+2)}(t) \leq -1$ for $t \geq T_4$. Hence $x^{(n+1)}(t) \leq x^{(n+1)}(T_4) - (t - T_4) \to -\infty$ as $t \to +\infty$. By the same way, it follows that $x^{(n)}(t), x^{(n-1)}(t), \dots, x'(t), x(t) \to -\infty$ as $t \to +\infty$. This contradict the assumption that $x(t)$ is eventually positive. $\square$

Remark 3.2. Conditions (3.1) are not equivalent to [9, Condition (2.2)]. We would like to point out here that, unfortunately, the proof of the main theorem in [9] contains an error. In fact, in the first paragraph of Page 6, under the assumption $y(t) > 0, y'(t) < 0$, the authors conclude that $y'(t) \to 0$ as $t \to \infty$. Obviously, this is not necessarily true.

In what follows we give two interesting criteria for the oscillatory and asymptotic behavior of the solutions to (1.1).

Corollary 3.3. *Suppose that* (2.1) *has an eventually positive increasing solution and that* (3.1) *holds. Assume further that*

$$\limsup_{t\to\infty} \int_{T}^{t} \Big[Kq(s) - \frac{2^{l-3}(l-1)!(n-l+2)!g'(s)}{g^{l+1}(s)(s-g(s))^{n-l+2}}\Big]g(s)ds = +\infty \tag{3.9}$$

holds for all $l = 2, 4, \dots, n+2$ when n is even and for all $l = 1, 3, \dots, n+2$ when n is odd. Then every solution x of (1.1) *is oscillatory, or satisfies $x(t) \to 0$ as $t \to \infty$.*

The conclusion of the above corollary follows from Theorem 3.1 by letting $\rho(t) = g(t)$.

Corollary 3.4. *Suppose that* (2.1) *has an eventually positive increasing solution and that* (3.1) *holds. If* $\lim_{t\to\infty} t/g(t) \geq \alpha > 1$*, then every solution* x *of* (1.1) *is oscillatory, or satisfies* $x(t) \to 0$ *as* $t \to \infty$.

Proof. Since $\lim_{t\to\infty} t/g(t) \geq \alpha > 1$, there exists a constant $\bar{\alpha} > 1$ such that $t/g(t) > \bar{\alpha}$ for $t \geq T_1$, where $T_1 \geq a$. Hence

$$
\begin{aligned}
&\int_{T_1}^{t} \frac{g'(s)g(s)}{g^{l+1}(s)(s-g(s))^{n-l+2}}ds \\
&= \int_{T_1}^{t} \frac{g'(s)}{g^{n+2}(s)(s/g(s)-1)^{n-l+2}}ds \\
&\leq \frac{1}{(\bar{\alpha}-1)^{n-l+2}} \int_{T_1}^{t} \frac{g'(s)}{g^{n+2}(s)}ds \\
&= \frac{1}{(n+1)(\bar{\alpha}-1)^{n-l+2}}\Big(\frac{1}{g^{n+1}(T_1)} - \frac{1}{g^{n+1}(t)}\Big) \\
&< \frac{1}{g^{n+1}(T_1)(n+1)(\bar{\alpha}-1)^{n-l+2}}, \quad \text{for all } t > T_1.
\end{aligned}
\tag{3.10}
$$

By (3.1) we obtain that $\int_a^\infty q(t)dt = +\infty$. Note that $\lim_{t\to\infty} g(t) = +\infty$, it turns out that $\int_a^\infty q(t)g(t)dt = +\infty$. Using this result and the inequality (3.10), the required conclusion follows from Corollary 3.3. □

In applications there are many models in which the delay $g(t)$ satisfies the condition in Corollary 3.4. As an example, $g(t) = at - \tau$ for $a \in (0,1)$, but not for $a = 1$. The case $a = 1$ will be discussed later.

Our next goal is to present some new oscillation results for (1.1), by using the so-called integral averages condition of Philos-type. Following the literature [13], we introduce a class of functions $\Re$. Let

$$
D_0 = \{(t,s) : t > s \geq a\}, \quad D = \{(t,s) : t \geq s \geq a\}.
$$

If the function $H \in C(D, \mathbb{R})$ satisfies

(i) $H(t,t) = 0$ for $t \geq a$ and $H(t,s) > 0$ for $(t,s) \in D_0$,
(ii) H has a continuous and non-positive partial derivative on D_0 with respect to the second variable such that

$$
\frac{\partial H(t,s)}{\partial s} = -h(t,s)\sqrt{H(t,s)} \quad \text{for all } (t,s) \in D_0,
$$

then H is said to belong to the class $\Re$.

Theorem 3.5. *Suppose that equation* (2.1) *has an eventually positive increasing solution and that* (3.1) *holds. Assume further that there exist functions* $H \in \Re$ *and* $\rho \in C^1(I, \mathbb{R}^+)$ *such that*

$$
\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \Big[K\rho(s)H(t,s)q(s) - \frac{(\rho(s)h(t,s) - \sqrt{H(t,s)}\rho'(s))^2}{\rho^2(s)G_l(s)}\Big]ds = +\infty,
\tag{3.11}
$$

where

$$
G_l(t) = \frac{g^{l-1}(t)(t-g(t))^{n-l+2}g'(t)}{a(l)\rho(t)} \quad \text{with } a(l) = 2^{l-3}(l-1)!(n-l+2)!,
$$

where $l = 2, 4, \dots, n+2$ *when* n *is even, and* $l = 1, 3, \dots, n+2$ *when* n *is odd. Then every solution* x *of* (1.1) *is oscillatory, or satisfies* $x(t) \to 0$ *as* $t \to \infty$.

Proof. Let x be a non-oscillatory solution of (1.1). Without loss of generality, we may assume that $x(t) > 0$ and $x(g(t)) > 0$ for $t \geq a$. By Lemma 2.1, there exists a constant $T \geq a$ such that $x^{(n)}(t) > 0$ or $x^{(n)}(t) < 0$ for $t \geq T$.

Assume firstly that $x^{(n)}(t) > 0$ for $t \geq T$. It follows from (1.1) that $x^{(n+3)}(t) < 0$ and hence there exists $l \in \{1, 3, \dots, n+2\}$ (resp. $l \in \{0, 2, \dots, n+2\}$) when n is odd (resp. n is even) such that for all sufficiently large t, $x^{(j)}(t) > 0$ for $j = 0, 1, \dots, l$ and $(-1)^{n+j}x^{(j)}(t) > 0$ for $j = l+1, l+2, \dots, n+2$.

Defining again the function w as in (3.3). If $l \neq 0$, then we get from (3.4) that

$$K\rho(t)q(t) \leq -w'(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{1}{4}w^2(t)G_l(t). \tag{3.12}$$

Thus

$$\begin{aligned} &K\int_T^t H(t,s)\rho(s)q(s)ds \\ &\leq \int_T^t \Big[-w'(s)H(t,s) + \Big(\frac{\rho'(s)}{\rho(s)}w(s) - \frac{1}{4}w^2(s)G_l(s)\Big)H(t,s)\Big]ds. \end{aligned}$$

Using integration by parts and noting that $H \in \Re$, we find

$$\begin{aligned} -\int_T^t w'(s)H(t,s)ds &= w(T)H(t,T) + \int_T^t w(s)\frac{\partial H(t,s)}{\partial s}ds \\ &= w(T)H(t,T) - \int_T^t w(s)h(t,s)\sqrt{H(t,s)}ds. \end{aligned}$$

Let

$$Q(t,s) = h(t,s) - \sqrt{H(t,s)}\frac{\rho'(s)}{\rho(s)},$$

then

$$\begin{aligned} &K\int_T^t H(t,s)\rho(s)q(s)ds \\ &\leq w(T)H(t,T) - \int_T^t \Big[w(s)\sqrt{H(t,s)}Q(t,s) + \frac{1}{4}G_l(s)H(t,s)w^2(s)\Big]ds \\ &= w(T)H(t,T) - \frac{1}{4}\int_T^t G_l(s)H(t,s)\Big(w(s) + \frac{2Q(t,s)}{G_l(s)\sqrt{H(t,s)}}\Big)^2 ds + \int_T^t \frac{Q^2(t,s)}{G_l(s)}ds \\ &\leq w(T)H(t,T) + \int_T^t \frac{Q^2(t,s)}{G_l(s)}ds. \end{aligned}$$

It turns out that

$$\frac{1}{H(t,T)}\int_T^t \Big[KH(t,s)\rho(s)q(s) - \frac{Q^2(t,s)}{G_l(s)}\Big]ds \leq w(T). \tag{3.13}$$

This contradicts (3.11). The rest of the proof is the same as in Theorem 3.1, and hence it is omitted. □

By letting $\rho(t) = g(t)$ in (3.11), from Theorem 3.5 we obtain the following result.

Corollary 3.6. *Suppose that* (2.1) *has an eventually positive increasing solution and that* (3.1) *holds. Assume further that there exists function $H \in \Re$ such that*

$$
\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \Big[K g(s) H(t,s) q(s) \\
&- \frac{a(l)(g(s)h(t,s) - \sqrt{H(t,s)} g'(s))^2}{g^l(s)(s-g(s))^{n-l+2} g'(s)} \Big] ds = +\infty,
\end{aligned}
\tag{3.14}
$$

where $l = 2, 4, \dots, n+2$ when n is even and $l = 1, 3, \dots, n+2$ when n is odd. Then every solution x of (1.1) *is oscillatory, or satisfies $x(t) \to 0$ as $t \to \infty$.*

Corollary 3.7. *Suppose that* (2.1) *has an eventually positive increasing solution and that* (3.1) *holds. If $g'(t) > 0$ and there is a real number $m \neq 0$ such that*

$$
\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{[g(t)-g(T)]^m} \int_T^t \Big[K \big(\frac{g(t)}{g(s)} - 1 \big)^m q(s) \\
&- \frac{m^2 a(l)(g(t)-g(s))^{m-2} g^2(t) g'(s)}{g^{l+m+1}(s)(s-g(s))^{n-l+2}} \Big] ds = +\infty,
\end{aligned}
\tag{3.15}
$$

where $l = 2, 4, \dots, n+2$ when n is even and $l = 1, 3, \dots, n+2$ when n is odd. Then every solution x of (1.1) *is oscillatory, or satisfies $x(t) \to 0$ as $t \to \infty$.*

Proof. Let $H(t,s) = [g(t)-g(s)]^m$, $\rho(t) = 1/g^m(t)$, then $H \in \Re, \rho \in C^1(I, \mathbb{R}^+)$. Moreover, $h(t,s) = mg'(s)(g(t)-g(s))^{m/2-1}$. Consequently,

$$
\begin{aligned}
&\frac{(\rho(s)h(t,s) - \sqrt{H(t,s)}\rho'(s))^2}{\rho^2(s)G_l(s)} \\
&= \frac{(g(t)-g(s))^{m-2}(m\rho(s)g'(s) - (g(t)-g(s))\rho'(s))^2}{\rho^2(s)G_l(s)} \\
&= \frac{m^2(g(t)-g(s))^{m-2} g^2(t) g'^2(s)}{G_l(s) g^2(s)} \\
&= \frac{m^2 a(l)(g(t)-g(s))^{m-2} g^2(t) g'(s)}{g^{l+m+1}(s)(s-g(s))^{n-l+2}}.
\end{aligned}
\tag{3.16}
$$

The required conclusion follows from (3.15) and (3.16). □

In some applications, the delay $g(t)$ has the form $g(t) = t - \tau$ with $\tau > 0$ which does not satisfy the condition $\lim_{t\to\infty} t/g(t) \geq \alpha > 1$ of Corollary 3.4. Next we give a convenient criterion for system (1.1) having such a delay.

Corollary 3.8. *Suppose the following conditions hold:*

- (i) *Equation* (2.1) *has an eventually positive increasing solution;*
- (ii) *Condition* (3.1) *holds and there are integer $m > 1$ and constant $\alpha > 0$ such that $\lim_{t\to\infty} q(t)/t^{m-1} \geq \alpha$;*
- (iii) *$g(t) = at - \tau$ with $0 < a \leq 1$ and $\tau > 0$.*

Then every solution x of (1.1) *is oscillatory, or satisfies $x(t) \to 0$ as $t \to \infty$.*

Proof. By Corollary 3.7, it suffices to show that (3.15) holds. For the sake of brevity, we only give the proof of the case that $a = 1$. The proof of the other cases is similar and hence is omitted. Obviously, condition (ii) implies that $q(t)/(t-\tau)^{m-1} > \alpha/2, t \geq T_1$ for some constant $T_1 > a$. Hence

$$
\int_{T_1}^t \Big(\frac{g(t)}{g(s)} - 1 \Big)^m q(s) ds
$$

$$
\begin{aligned}
&= \int_{T_1}^{t} \frac{(t-s)^m}{s-\tau} \cdot \frac{q(s)}{(s-\tau)^{m-1}} ds \\
&\geq \frac{\alpha}{2} \int_{T_1}^{t} \frac{(t-s)^m}{s-\tau} ds \\
&= \frac{\alpha}{2} \int_{T_1}^{t} \frac{((t-\tau)-(s-\tau)))^m}{s-\tau} ds \\
&= \frac{\alpha}{2} \sum_{k=0}^{m} C_m^k (-1)^k (t-\tau)^{m-k} \int_{T_1}^{t} (s-\tau)^{k-1} ds \\
&= \frac{\alpha}{2} \Big((t-\tau)^m ln \frac{t-\tau}{T_1-\tau} + \sum_{k=1}^{m} C_m^k (-1)^k \frac{(t-\tau)^m - (t-\tau)^{m-k}(T_1-\tau)^k}{k} \Big),
\end{aligned}
$$

where $C_m^k = \frac{m!}{(m-k)!k!}$. It turns out that

$$
\begin{aligned}
&\lim_{t\to\infty} \frac{1}{(g(t)-g(T))^m} \int_T^t \Big(\frac{g(t)}{g(s)} - 1\Big)^m q(s) ds \\
&\geq \lim_{t\to\infty} \frac{\alpha}{2} \Big(\frac{(t-\tau)^m}{(t-T)^m} ln \frac{t-\tau}{T_1-\tau} + \sum_{k=1}^{m} C_m^k (-1)^k \frac{(t-\tau)^m - (t-\tau)^{m-k}(T_1-\tau)^k}{k(t-T)^m} \Big) \\
&= +\infty.
\end{aligned} \tag{3.17}
$$

On the other hand,

$$
\begin{aligned}
&\frac{1}{(g(t)-g(T))^m} \int_T^t \frac{(g(t)-g(s))^{m-2} g^2(t) g'(s)}{g^{l+m+1}(s)(s-g(s))^{n-l+2}} ds \\
&= \frac{(t-\tau)^2}{(t-T)^m} \int_T^t \frac{((t-\tau)-(s-\tau))^{m-2}}{(s-\tau)^{l+m+1} \tau^{n-l+2}} ds \\
&= \frac{1}{\tau^{n-l+2}} I_l(t),
\end{aligned} \tag{3.18}
$$

where

$$
I_l(t) = \frac{(t-\tau)^2}{(t-T)^m} \int_T^t \frac{((t-\tau)-(s-\tau))^{m-2}}{(s-\tau)^{l+m+1}} ds.
$$

If $m = 2$, then

$$
I_l(t) = \big(\frac{t-\tau}{t-T}\big)^2 \int_T^t \frac{1}{(s-\tau)^{l+3}} ds < M_1, \tag{3.19}
$$

where M_1 is a constant.

If $m > 2$, then

$$
\begin{aligned}
I_l(t) &= \frac{(t-\tau)^2}{(t-T)^m} \sum_{k=0}^{m-2} C_{m-2}^k (-1)^k (t-\tau)^{m-2-k} \int_T^t (s-\tau)^{k-l-m-1} ds \\
&= \big(\frac{t-\tau}{t-T}\big)^m \sum_{k=0}^{m-2} C_{m-2}^k (-1)^k (t-\tau)^{-k} \frac{(T-\tau)^{k-l-m} - (t-\tau)^{k-l-m}}{m+l-k} \\
&= \big(\frac{t-\tau}{t-T}\big)^m \sum_{k=0}^{m-2} C_{m-2}^k (-1)^k \frac{(T-\tau)^{k-l-m}(t-\tau)^{-k} - (t-\tau)^{-l-m}}{m+l-k} \\
&< M_2,
\end{aligned} \tag{3.20}
$$

where M_2 is a constant.

By (3.18), (3.19) and (3.20), it is easy to see that

$$\limsup_{t\to\infty} \frac{1}{[g(t)-g(T)]^m} \int_T^t \frac{m^2 a(l)(g(t)-g(s))^{m-2} g^2(t) g'(s)}{g^{l+m+1}(s)(s-g(s))^{n-l+2}} ds < +\infty. \tag{3.21}$$

Finally, combining (3.17) with (3.21), we find that (3.15) holds. This completes the proof. □

4. Examples

In this section, we give examples that illustrate our main results.

Example 4.1. Consider the eighth-order delay differential equation

$$x^{(8)}(t) + \frac{1}{(1+2t)^2}\Big(\frac{t^2+t-2}{(1+t)^3\ln(1+t)} + \frac{3}{(1+2t)}\Big)x^{(5)}(t) + \frac{3t+\sin t}{t^2-2} x(t-\ln t) = 0, \tag{4.1}$$

for $t \geq 2$. Here $n = 5$,

$$p(t) = \frac{1}{(1+2t)^2}\Big(\frac{t^2+t-2}{(1+t)^3\ln(1+t)} + \frac{3}{(1+2t)}\Big), \quad q(t) = \frac{3t+\sin t}{t^2-2}$$

and $f(x) = x$ with $K = 1$.

The equation $u''' + p(t)u = 0$ has a positive and strictly increasing solution $u(t) = (2t+1)^{3/2}\ln(1+t)$. It is easy to see that $\int_2^\infty q(t)dt = +\infty$, $p'(t)$ is eventually negative and hence that (3.1) is true. Let $\rho(t) = t$, then it is easy to see that for $l = 1, 3, 5, 7$,

$$\begin{aligned}
&\limsup_{t\to\infty} \int_2^t \Big[K\rho(s)q(s) - \frac{2^{l-3}(l-1)!(n-l+2)!(\rho'(s))^2}{g^{l-1}(s)(s-g(s))^{n-l+2} g'(s)\rho(s)}\Big] ds \\
&= \limsup_{t\to\infty} \int_2^t \Big[\frac{3s^2+s\sin s}{s^2-2} - \frac{2^{l-3}(l-1)!(7-l)!}{(s-\ln s)^{l-1}(\ln s)^{7-l}(s-1)}\Big] ds = +\infty.
\end{aligned}$$

Consequently, by Theorem 3.1, any solution of (4.1) is oscillatory, or satisfies $x(t) \to 0$ as $t \to \infty$.

Example 4.2. Consider the fourth-order delay differential equation

$$x^{(4)}(t) + \frac{3(\ln^2 t - 2)}{t^3\ln^3 t} x'(t) + \frac{t+1}{t^2+1} x\Big((1+\sin\frac{1}{t^2+1})\frac{t}{2}\Big) = 0, \quad t \geq 1. \tag{4.2}$$

The delay function $g(t) = (1+\sin\frac{1}{t^2+1})\frac{t}{2}$ satisfies $0 < g(t) < t$, $\lim_{t\to+\infty} g(t) = +\infty$ and $t/g(t) \geq 2/(1+\sin(1/2)) > 1$. It is not hard to check that the equation $u''' + p(t)u = 0$, with $p(t) = \frac{3(\ln^2 t-2)}{t^3\ln^3 t}$, has a positive and strictly increasing solution $u(t) = t\ln^3 t$. Moreover, since

$$p'(t) = \frac{3}{t^4\ln^4 t}(6 + 6\ln t - ln^2 t - 3\ln^3 t),$$

$p'_+(t) = 0$ for sufficiently large t. Clearly, $\int_1^\infty q(t)dt \geq \int_1^\infty \frac{t+1}{2t^2}dt = +\infty$, which implies that (3.1) is true. Thus, by Corollary 3.4, any solution of (4.2) is oscillatory, or satisfies $x(t) \to 0$ as $t \to \infty$.

Example 4.3. Consider the fifth-order delay differential equation

$$x^{(5)}(t)+\frac{2}{t^3(1+2\ln t)}x''(t)+(5+e^{-t}\cos t)tx(at-\tau)(2+\exp[-x(at-\tau)])=0, \quad (4.3)$$

for $t \geq 1$, where $a \in (0,1], \tau > 0$. Obviously, the function $f(x) = x(2+e^{-x})$ satisfies that $f(x)/x \geq 2$ for $x \neq 0$. It is easy to check that the equation $u''' + p(t)u = 0$ has a positive and strictly increasing solution $u(t) = t(2\ln t + 1)$. Moreover, since $p'(t) \leq 0$ and $\int_1^\infty q(t)dt = \int_1^\infty (5 + e^{-t}\cos t)tdt = +\infty$, it follows that (3.1) is satisfied. Clearly, $\lim_{t\to\infty} q(t)/t = 5$. Thus, by Corollary 3.8, any solution of (4.3) is oscillatory, or satisfies $x(t) \to 0$ as $t \to \infty$.

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