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HOMOGENIZATION OF GEOLOGICAL FISSURED SYSTEMS WITH CURVED NON-PERIODIC CRACKS

FERNANDO A. MORALES

Abstract. We analyze the steady fluid flow in a porous medium containing a network of thin fissures of width $\mathcal{O}(\epsilon)$, generated by the rigid translation of continuous piecewise C^1 functions in a fixed direction. The phenomenon is modeled in mixed variational formulation, using the stationary Darcy's law and coefficients of low resistance $\mathcal{O}(\epsilon)$ on the network. The singularities are removed by asymptotic analysis as $\epsilon \to 0$ which yields an analogous system hosting only tangential flow in the fissures. Finally the fissures are collapsed into two dimensional manifolds.

1. Introduction

Groundwater and oil reservoirs are frequently fissured or layered; i.e., the bedrock contains fissures of characteristic dimensions considerably higher than those of the average pore size of the rock. The modeling of saturated flow through geological structures such as these, gives rise to singular problems of partial differential equations [20]. On one hand, the singularities are caused by the drastic change of permeability from the rock matrix to the fissures. On the other hand, a geometric singularity is introduced due to the thinness of the fractures. The presence of singularities in the model has non-desirable effects in their numerical implementation; some of these are ill-conditioned matrices, high computational costs, numerical instability, etc. This subject is a very active research field, see [2, 5, 9, 11, 12] for numerical analysis aspects, see [8, 10] for modeling discussion and see [1, 3, 4, 13, 14, 15] for rigorous mathematical treatment of the phenomenon. Homogenization and asymptotic analysis techniques are a common approach for the analytical point of view. However, the remarkable achievements in the field require very restrictive hypotheses for the description of the geometry such as regular geometric shapes or periodically arrayed structures [7, 17]. In general the variational methods for partial differential equations can give successful formulations for a wide class of geometric domains. The limited treatment of the geometry is due to the notorious technical difficulties, that more general shapes introduce in the asymptotic analysis of the problem.

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In the present work, the geometric possibilities of the medium are extended to fissures that are not necessarily flat. We use the *mixed mixed formulation* and the scaling for the flow resistance coefficients presented in [16], then a careful choice of directions or "stream lines", consistent with the natural geometry of the problem permits a successful asymptotic analysis of the model. This leads to a system coupled through multiple two dimensional manifolds representing the fissures in the upscaled model. Additionally, the formulation allows remarkable generality in the fluid exchange balance conditions between the rock matrix and the channels and substantial efficiency for handling the system of equations as well as the information (coefficients, matrices, etc) describing the geometry of the fractures. This is mostly due to the fact that the formulation does not demand coupling constraints on the underlying spaces of functions. The main goal of the paper is to emphasize the geometry, consequently the study is limited to the steady case. We describe flow with Darcy's law

$$a(\cdot)\mathbf{u} + \boldsymbol{\nabla} p + \mathbf{g} = 0, \tag{1.1a}$$

together with the conservation law

$$\boldsymbol{\nabla} \cdot \mathbf{u} = F. \tag{1.1b}$$

Drained and null-flux boundary conditions will be specified on different parts of the boundary of the domain to set a boundary value problem. The balance conditions of normal stress and normal flux across the interface Γ, separating the regions Ω_1 and Ω_2 are given by

$$p^1 - p^2 = \alpha \mathbf{u}^1 \cdot \widehat{\boldsymbol{n}} \quad \text{and} \tag{1.1c}$$

$$\mathbf{u}^1 \cdot \widehat{\boldsymbol{n}} - \mathbf{u}^2 \cdot \widehat{\boldsymbol{n}} = f_\Gamma \quad \text{on } \Gamma. \tag{1.1d}$$

The superscripts denote restrictions to Ω_1 and Ω_2. The coefficient $a(\cdot)$ is the flow resistance, i.e. the fluid viscosity times the inverse of the permeability of the medium, to be scaled consistently with the fast and slow flow regions of the medium. Finally, the coefficient α indicates the fluid entry resistance of the rock matrix.

In the following section we define the geometric setting, formulate the problem in mixed mixed variational formulation and prove its well-posedness. In section three the problem is referred to a common geometric setting in order make possible the asymptotic analysis, the existence of a-priori estimates and the structure of the limiting solution are also shown. Section four studies the formulation and well-posedness of the limiting problem as well as its strong form (particularly important for boundary and interface conditions); it also proves the strong convergence of the solutions. Section five sets the limiting problem as a coupled system with two dimensional interfaces and section six discusses the possibilities and limitations of the technique as well as related future work.

2. Formulation and geometric setting

Vectors are denoted by boldface letters as are vector-valued functions and corresponding function spaces. We use $\widetilde{\mathbf{x}}$ to indicate a vector in $\mathbb{R}^2$; if $\mathbf{x} \in \mathbb{R}^3$ then the $\mathbb{R}^2 \times \{0\}$ projection is identified with $\widetilde{\mathbf{x}} := (x_1, x_2)$ so that $\mathbf{x} = (\widetilde{\mathbf{x}}, x_3)$. The symbol $\widetilde{\boldsymbol{\nabla}}$ represents the gradient in the first first two directions: $\widehat{\boldsymbol{\imath}}$, $\widehat{\boldsymbol{\jmath}}$. Given a function $f : \mathbb{R}^3 \to \mathbb{R}$ then $\int_{\mathcal{M}} f dS$ is the notation for its surface integral on the two dimensional manifold $\mathcal{M} \subseteq \mathbb{R}^3$. $\int_A f \, d\mathbf{x}$ stands for the volume integral in the set $A \subseteq \mathbb{R}^3$; whenever the context is clear we simply write $\int_A f$.

The symbol $\widehat{\boldsymbol{\nu}}$ denotes the outwards normal vector on the boundary of a given domain $\mathcal{O}$ and $\widehat{\boldsymbol{n}}$ denotes the upwards vector normal to a given surface i.e. $\widehat{\boldsymbol{n}}\cdot\widehat{\boldsymbol{k}} \geq 0$. For any $A \subseteq \mathbb{R}^3$ and $t \in \mathbb{R}$ we define its t-vertical shift by

$$A + t := \big\{\mathbf{x} + t\widehat{\boldsymbol{k}} : \mathbf{x} \in A\big\} \tag{2.1}$$

2.1. **General geometric setting.** The present work will be limited to the study of fractured media where each fissure can be described in a specific way.

Definition 2.1. Let $\mathcal{O} \subseteq \mathbb{R}^2$ be a bounded open simply connected set and $\zeta \in C(\overline{\mathcal{O}})$ be a piecewise C^1 function. Define the surface

$$\Upsilon := \{[\widetilde{\mathbf{x}}, \zeta(\widetilde{\mathbf{x}})] : \widetilde{\mathbf{x}} \in \mathcal{O}\}. \tag{2.2}$$

We assume that Υ is non-vertical with $\operatorname{ess\,inf}\{\widehat{\boldsymbol{n}}(s) \cdot \widehat{\boldsymbol{k}} : s \in \Upsilon\} > 0$. Given $h > 0$, define the fissure of height h generated by a rigid vertical translation of Υ as the domain

$$\Theta(h, \Upsilon) := \{(\widetilde{\mathbf{x}}, y) : \zeta(\widetilde{\mathbf{x}}) < y < \zeta(\widetilde{\mathbf{x}}) + h\}. \tag{2.3}$$

Remark 2.2. Notice that in the definition of $\Theta(h, \Upsilon)$ we mention h as the height and not as the width of the crack. Figure 4 shows that, depending on the gradient of the surface, the height h can become significantly different from the actual width.

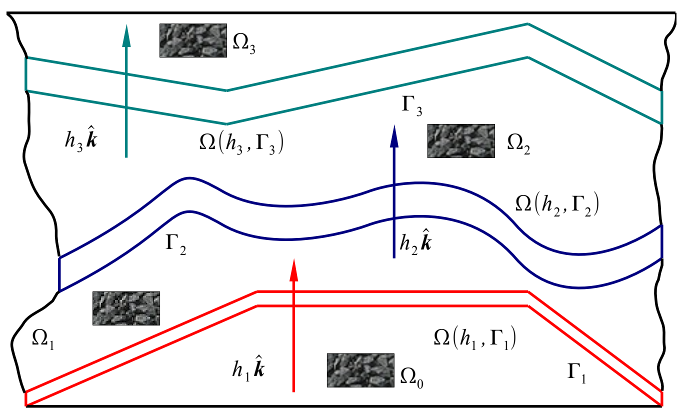

FIGURE 1. Unidirectional translation generated fissures

Figure 1 depicts a region $\Omega \subseteq \mathbb{R}^3$ containing a network of fissures generated by vertical rigid translation of continuous piecewise C^1 surfaces. For the sake of clarity we restrict the analysis to the case where only one fissure is embedded in the domain. From here, the generalization to a system with multiple fissures is a rather simple exercise. Next we define this domain

Definition 2.3. A medium Ω with a vertical translation generated fissure is composed by three connected components denoted Θ_1, Θ_2 and $\Theta(h, \Gamma_1)$ such that

(i) $\Theta(h, \Gamma_1)$ is a region generated by the vertical translation $h\widehat{\boldsymbol{k}}$ of a piecewise C^1 function Γ_1 as given in definition 2.1.

(ii) $\Theta_i \subseteq \mathbb{R}^3$ are open bounded simply connected regions for $i = 1, 2$. These stand for rock matrix regions and satisfy

$$\begin{aligned} \partial\Theta_1 \cap \partial\Theta(h, \Gamma_1) &= \Gamma_1, \\ \partial\Theta_2 \cap \partial\Theta(h, \Gamma_1) &= \Gamma_1 + h, \end{aligned} \tag{2.4a}$$

$$cl(\Theta_1) \cap cl(\Theta_2) = \emptyset. \tag{2.4b}$$

The sets Ω_1, Ω_2 are the rock matrix and the fissures regions respectively; i.e.,

$$\Omega_1 := \Theta_1 \cup \Theta_2, \quad \Omega_2 := \Theta(h, \Gamma_1), \quad \Omega := cl(\Omega_1 \cup \Omega_2). \tag{2.5}$$

The interfaces are defined by

$$\Gamma_1, \quad \Gamma_2 := \Gamma_1 + h, \quad \Gamma := \Gamma_2 \cup \Gamma_1 \tag{2.6}$$

The upwards normal vector to the surface Γ_1 is denoted $\widehat{\boldsymbol{n}}$; i.e.,

$$\widehat{\boldsymbol{n}} := \frac{(-\widetilde{\boldsymbol{\nabla}}\zeta, 1)}{|(-\widetilde{\boldsymbol{\nabla}}\zeta, 1)|}. \tag{2.7}$$

Finally, we introduce the notation $\Omega = (\Theta_1, \Theta_2, \zeta, h)$ for this type of domains.

Remark 2.4. The condition 2.4b of connectivity of the rock matrix regions only through the fissure is not required for modeling the problem in mixed formulation as it is presented in section 2.4; however it is necessary for the asymptotic analysis of the system. The same holds for the simple connectedness requirement on the domains.

2.2. **A local system of coordinates.** Several aspects of the flow through the fissure are handled more conveniently, when the velocities are expressed in a coordinate system consistent with the geometry of the surface that generates the crack. Let Γ be a surface as defined in (2.2) and $\widehat{\boldsymbol{n}}$ the upwards normal to the surface Γ i.e. $\widehat{\boldsymbol{n}} = \widehat{\boldsymbol{n}}(s) = \widehat{\boldsymbol{n}}(\widetilde{\mathbf{x}})$. Now, for each point $\widetilde{\mathbf{x}}$ we choose a local orthonormal basis in the following way,

$$\mathcal{B}(\widetilde{\mathbf{x}}) := \{\widehat{\boldsymbol{e}}_1(\widetilde{\mathbf{x}}), \widehat{\boldsymbol{e}}_2(\widetilde{\mathbf{x}}), \widehat{\boldsymbol{n}}(\widetilde{\mathbf{x}})\}. \tag{2.8}$$

Let $M = M(\widetilde{\mathbf{x}})$ be the orthogonal matrix relating the global canonical basis with the local one; i.e.,

$$M(\widetilde{\mathbf{x}})\widehat{\boldsymbol{\imath}} = \widehat{\boldsymbol{e}}_1(\widetilde{\mathbf{x}}), \tag{2.9a}$$

$$M(\widetilde{\mathbf{x}})\widehat{\boldsymbol{\jmath}} = \widehat{\boldsymbol{e}}_2(\widetilde{\mathbf{x}}), \tag{2.9b}$$

$$M(\widetilde{\mathbf{x}})\widehat{\boldsymbol{k}} = \widehat{\boldsymbol{n}}(\widetilde{\mathbf{x}}). \tag{2.9c}$$

The block matrix notation for this local matrix will be

$$M(\widetilde{\mathbf{x}}) := \begin{pmatrix} M^{T,\boldsymbol{\tau}} & M^{T,\widehat{\boldsymbol{n}}} \\ M^{\widehat{\boldsymbol{k}},\boldsymbol{\tau}} & M^{\widehat{\boldsymbol{k}},\widehat{\boldsymbol{n}}} \end{pmatrix} (\widetilde{\mathbf{x}}). \tag{2.10}$$

Here, the index T stands for the first two components in the directions $\widehat{\boldsymbol{\imath}}, \widehat{\boldsymbol{\jmath}}$ while the index τ stands for the tangential velocity, orthogonal to $\widehat{\boldsymbol{n}}$. Then $\mathbf{w} = [\mathbf{w}_\tau, \mathbf{w}_{\widehat{\boldsymbol{n}}}](\widetilde{\mathbf{x}})$ with the following relations

$$w_{\widehat{\boldsymbol{n}}} := \mathbf{w} \cdot \widehat{\boldsymbol{n}}(\widetilde{\mathbf{x}}), \tag{2.11a}$$

$$\mathbf{w}_\tau := (\mathbf{w} \cdot \widehat{\boldsymbol{e}}_1(\widetilde{\mathbf{x}}), \mathbf{w} \cdot \widehat{\boldsymbol{e}}_2(\widetilde{\mathbf{x}})). \tag{2.11b}$$

Clearly, the relationship between velocities is given by

$$\begin{aligned} \mathbf{w}(\widetilde{\mathbf{x}}, x_3) &= \begin{Bmatrix} \widetilde{\mathbf{w}} \\ \mathbf{w}\cdot\widehat{\boldsymbol{k}} \end{Bmatrix} (\widetilde{\mathbf{x}}, x_3) = M(\widetilde{\mathbf{x}}) \begin{Bmatrix} \mathbf{w}_\tau \\ \mathbf{w}\cdot\widehat{\boldsymbol{n}} \end{Bmatrix} (\widetilde{\mathbf{x}}, x_3) \\ &= \begin{pmatrix} M^{T,\boldsymbol{\tau}}(\widetilde{\mathbf{x}}) & M^{T,\widehat{\boldsymbol{n}}}(\widetilde{\mathbf{x}}) \\ M^{\widehat{\boldsymbol{k}},\boldsymbol{\tau}}(\widetilde{\mathbf{x}}) & M^{\widehat{\boldsymbol{k}},\widehat{\boldsymbol{n}}}(\widetilde{\mathbf{x}}) \end{pmatrix} \begin{Bmatrix} \mathbf{w}_\tau \\ \mathbf{w}\cdot\widehat{\boldsymbol{n}} \end{Bmatrix} (\widetilde{\mathbf{x}}, x_3). \end{aligned} \tag{2.12}$$

Proposition 2.5. *Let $h > 0$, Υ, $\Theta(h,\Upsilon)$ be as in definition 2.1; let $\widehat{\boldsymbol{n}}$ be the upwards normal to the surface Υ and M be the matrix defined by (2.9). Then*

(i) *The map $\mathbf{w} \mapsto M(\widetilde{\mathbf{x}})\mathbf{w}$ is an isometry in $\mathbf{L}^2(\Theta(h,\Upsilon))$. In particular if $\mathbf{w}_\tau, \mathbf{w}\cdot\widehat{\boldsymbol{n}}$ are defined as in (2.11) then $\mathbf{w} \in \mathbf{L}^2(\Theta(h,\Upsilon))$ if and only if $\mathbf{w}_\tau \in L^2(\Theta(h,\Upsilon)) \times L^2(\Theta(h,\Upsilon))$ and $\mathbf{w}\cdot\widehat{\boldsymbol{n}} \in L^2(\Theta(h,\Upsilon))$.*

(ii) *If $\mathbf{w} \in \mathbf{L}^2(\Theta(h,\Upsilon))$ is such that $\partial_z \mathbf{w} \in \mathbf{L}^2(\Theta(h,\Upsilon))$ then*

$$\partial_z \mathbf{w}(\widetilde{\mathbf{x}}, z) = M(\widetilde{\mathbf{x}}) \begin{Bmatrix} \partial_z \mathbf{w}_\tau \\ (\partial_z \mathbf{w})\cdot\widehat{\boldsymbol{n}} \end{Bmatrix} (\widetilde{\mathbf{x}}, z). \tag{2.13}$$

Proof. (i) For $\widetilde{\mathbf{x}}$ fixed the matrix $M(\widetilde{\mathbf{x}})$ is orthogonal; i.e., for arbitrary functions $\mathbf{v}, \mathbf{w} \in \mathbf{L}^2(\Theta(h,\Upsilon))$ and $\mathbf{x} \in \Theta(h,\Upsilon)$ holds $\mathbf{v}(\mathbf{x})\cdot\mathbf{w}(\mathbf{x}) = M(\widetilde{\mathbf{x}})\mathbf{v}(\mathbf{x})\cdot M(\widetilde{\mathbf{x}})\mathbf{w}(\mathbf{x})$. Hence

$$\begin{aligned} &\int_{\Theta(h,\Upsilon)} \mathbf{v}(\mathbf{x})\cdot\mathbf{w}(\mathbf{x})d\mathbf{x} \\ &= \int_{\Theta(h,\Upsilon)} M(\widetilde{\mathbf{x}})\mathbf{v}(\mathbf{x})\cdot M(\widetilde{\mathbf{x}})\mathbf{w}(\mathbf{x})d\mathbf{x} \\ &= \int_{\Theta(h,\Upsilon)} \mathbf{v}_\tau(\mathbf{x})\cdot\mathbf{w}_\tau(\mathbf{x})d\mathbf{x} + \int_{\Theta(h,\Upsilon)} (\mathbf{v}\cdot\widehat{\boldsymbol{n}})(\mathbf{x})\cdot(\mathbf{w}\cdot\widehat{\boldsymbol{n}})(\mathbf{x})d\mathbf{x} \end{aligned}$$

The equality of the second line shows the necessity and sufficiency of the tangential and normal components being square integrable in the domain $\Theta(h,\Upsilon)$.

(ii) It follows from a direct calculation of distributions with $\boldsymbol{\varphi} \in [C_0^\infty(\Theta(h,\Upsilon))]^3$ arbitrary and the fact that $\partial_z M = 0$. □

2.3. **The problem and its formulation.** In this section we define the problem in a rigorous way and give a variational formulation in which it is well-posed. Let $\Omega = (\Theta_1, \Theta_2, \zeta, h)$ be a fractured domain with a vertical translation generated fissure as in definition 2.3. We denote $\mathbf{v}^1, p^1$ the velocity and pressure in the rock matrix region Ω_1. In the same fashion $\mathbf{v}^2, p^2$ denote the velocity and pressure in the fissures region Ω_2.

Consider the problem

$$a_1\mathbf{u}^1 + \boldsymbol{\nabla} p^1 + \mathbf{g} = 0 \quad \text{and} \tag{2.14a}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u}^1 = F \quad \text{in } \Omega_1, \tag{2.14b}$$

$$p^1 = 0 \quad \text{on } \partial\Omega_1 - \Gamma, \tag{2.14c}$$

$$p^1 - p^2 = \alpha\mathbf{u}^1\cdot\widehat{\boldsymbol{n}}\mathbb{1}_{\Gamma_2} - \alpha\mathbf{u}^1\cdot\widehat{\boldsymbol{n}}\mathbb{1}_{\Gamma_1} \quad \text{and} \tag{2.14d}$$

$$(\mathbf{u}^1 - \mathbf{u}^2)\cdot\widehat{\boldsymbol{n}}\mathbb{1}_{\Gamma_1} - (\mathbf{u}^1 - \mathbf{u}^2)\cdot\widehat{\boldsymbol{n}}\mathbb{1}_{\Gamma_2} = f_\Gamma \quad \text{on } \Gamma, \tag{2.14e}$$

$$a_2\mathbf{u}^2 + \boldsymbol{\nabla} p^2 + \mathbf{g} = 0 \quad \text{and} \tag{2.14f}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u}^2 = F \quad \text{in } \Omega_2, \tag{2.14g}$$

$$\mathbf{u}^2\cdot\widehat{\boldsymbol{n}} = 0 \quad \text{on} \partial\Omega_2 - \Gamma. \tag{2.14h}$$

The flow resistance coefficients a_1, a_2 and the fluid entry resistance coefficient α are assumed to be positively bounded from below and above, see [16]. In equations (2.14d) (2.14e) the split of cases, Γ_2 and Γ_1, is made in order to be consistent with the sign of the upwards normal vector $\widehat{\boldsymbol{n}}$.

2.4. **Mixed formulation of the problem.** We start defining the spaces of velocity and pressure.

$$\mathbf{V} := \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \boldsymbol{\nabla} \cdot \mathbf{v}^1 \in \mathbf{L}^2(\Omega_1), \mathbf{v}^1 \cdot \widehat{\boldsymbol{n}}|_\Gamma \in L^2(\Gamma)\}, \tag{2.15a}$$

$$Q := \{q \in L^2(\Omega) : \boldsymbol{\nabla} q^2 \in \mathbf{L}^2(\Omega_2)\} \tag{2.15b}$$

Endowed with their natural norms

$$\|\mathbf{v}\|_{\mathbf{V}} := \{\|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\nabla} \cdot \mathbf{v}^1\|^2_{L^2(\Omega_1)} + \|\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}}\|^2_{L^2(\Gamma)}\}^{1/2}, \tag{2.15c}$$

$$\|q\|_Q := \{\|q\|^2_{L^2(\Omega)} + \|\boldsymbol{\nabla} q^2\|^2_{L^2(\Omega_2)}\}^{1/2} \tag{2.15d}$$

Remark 2.6. In the spaces above it is understood that

$$\|\mathbf{v} \cdot \widehat{\boldsymbol{n}}\|^2_{L^2(\Gamma)} = \|\mathbf{v} \cdot \widehat{\boldsymbol{n}}\|^2_{L^2(\Gamma_2)} + \|\mathbf{v} \cdot \widehat{\boldsymbol{n}}\|^2_{L^2(\Gamma_1)}\,. \tag{2.16}$$

Consider the problem: Find $p \in Q$ and $\mathbf{u} \in \mathbf{V}$ such that

$$\begin{aligned} &\int_{\Omega_1} a_1 \mathbf{u} \cdot \mathbf{v} + \int_{\Omega_2} a_2 \mathbf{u} \cdot \mathbf{v} - \int_{\Omega_1} p \boldsymbol{\nabla} \cdot \mathbf{v} + \int_{\Omega_2} \boldsymbol{\nabla} p \cdot \mathbf{v} \\ &+ \alpha \int_\Gamma (\mathbf{u}^1 \cdot \widehat{\boldsymbol{n}})(\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}})\, dS - \int_{\Gamma_1} p^2 (\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}})\, dS + \int_{\Gamma_2} p^2 (\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}})\, dS \\ &= -\int_\Omega \mathbf{g} \cdot \mathbf{v}\,, \end{aligned} \tag{2.17a}$$

$$\begin{aligned} &\int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{u} q - \int_{\Omega_2} \mathbf{u} \cdot \boldsymbol{\nabla} q + \int_{\Gamma_1} (\mathbf{u}^1 \cdot \widehat{\boldsymbol{n}}) q^2\, dS - \int_{\Gamma_2} (\mathbf{u}^1 \cdot \widehat{\boldsymbol{n}}) q^2\, dS \\ &= \int_\Omega F q + \int_\Gamma f_\Gamma q^2\, dS \quad \text{for all } q \in Q,\ \mathbf{v} \in \mathbf{V}\,. \end{aligned} \tag{2.17b}$$

Remark 2.7. In the formulation above the non-symmetric interface terms are split in two pieces in order to express everything in terms of the upwards normal vector $\widehat{\boldsymbol{n}}$. In the case of the symmetric term $\int_\Gamma (\mathbf{u}^1 \cdot \widehat{\boldsymbol{n}})(\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}}) dS$ in (2.17a) such split becomes unnecessary since the sign of the normal vector changes in both factors canceling each other.

Define the bilinear forms $\mathcal{A} : \mathbf{V} \to \mathbf{V}'$, $\mathcal{B} : V \to Q'$, $\mathcal{C} : Q \to Q'$ by

$$\mathcal{A}\mathbf{v}(\mathbf{w}) := \int_{\Omega_1} a_1 \mathbf{v} \cdot \mathbf{w} + \int_{\Omega_2} a_2 \mathbf{v} \cdot \mathbf{w} + \alpha \int_\Gamma (\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}})(\mathbf{w}^1 \cdot \widehat{\boldsymbol{n}})\, dS, \tag{2.18a}$$

$$\mathcal{B}\mathbf{v}(q) := -\int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{v} q + \int_{\Omega_2} \mathbf{v} \cdot \boldsymbol{\nabla} q - \int_{\Gamma_1} (\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}}) q^2\, dS + \int_{\Gamma_2} (\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}}) q^2\, dS\,. \tag{2.18b}$$

Then, the system (2.17) is a mixed formulation for the problem (2.14) with the abstract form

$$\begin{aligned} \mathbf{u} \in \mathbf{V},\ p \in Q : \quad \mathcal{A}\mathbf{u} + \mathcal{B}' p &= -\mathbf{g} \text{ in } \mathbf{V}', \\ -\mathcal{B}\mathbf{u} &= f \quad \text{in } Q'. \end{aligned} \tag{2.19}$$

For the sake of completeness we recall some well known results

Theorem 2.8. *Let $\mathbf{V}, Q$ be Hilbert spaces and $\|\cdot\|_{\mathbf{V}}, \|\cdot\|_Q$ be their respective norms. Let $\mathcal{A} : \mathbf{V} \to \mathbf{V}'$, $\mathcal{B} : \mathbf{V} \to Q'$ be continuous linear operators such that*

(i) *$\mathcal{A}$ is non-negative and $\mathbf{V}$-coercive on $\ker \mathcal{B}$.*
(ii) *The operator $\mathcal{B}$ satisfies the inf-sup condition*

$$\inf_{q\in Q} \sup_{\mathbf{v}\in\mathbf{V}} \frac{|\mathcal{B}\mathbf{v}(q)|}{\|\mathbf{v}\|_{\mathbf{V}}\|q\|_Q} > 0. \tag{2.20}$$

Then, for each $\mathbf{g} \in \mathbf{V}'$ and $f \in Q'$ there exists a unique solution $[\mathbf{u}, p] \in \mathbf{V} \times Q$ to the problem (2.19)*. Moreover, it satisfies the estimate*

$$\|\mathbf{u}\|_{\mathbf{V}} + \|p\|_Q \le K(\|\mathbf{g}\|_{\mathbf{V}'} + \|f\|_{Q'})\,. \tag{2.21}$$

For a proof of the above theorem, see [6] .

Lemma 2.9. *Let $\mathcal{O}$ be an open connected bounded set in $\mathbb{R}^N$ and $\mathcal{G} \subseteq \partial\mathcal{O}$ with non-null $\mathbb{R}^{N-1}$-Lebesgue measure, then there exists $\kappa = \kappa(\mathcal{O}) > 0$ such that*

$$\|\boldsymbol{\nabla}\eta\|_{\mathbf{L}^2(\mathcal{O})} + \|\eta\|_{L^2(\mathcal{G})} \ge \kappa\|\eta\|_{H^1(\mathcal{O})} \tag{2.22}$$

for all $\eta \in H^1(\mathcal{O})$.

For a proof of the above lemma, see [19, Proposition 5.2] or [16, Lemma 1.2].

Corollary 2.10. *There exists a constant $\kappa > 0$ such that*

$$\|\boldsymbol{\nabla} q\|^2_{\mathbf{L}^2(\Omega_2)} + \|q\|^2_{L^2(\Gamma)} \ge \kappa\|q\|^2_{L^2(\Omega_2)} \tag{2.23}$$

for all $q \in H^1(\Omega_2)$.

Lemma 2.11. *The operator $\mathcal{B}$ satisfies the inf-sup condition* (2.20)*.*

Proof. We use the same strategy presented lemma 1.3 in [16] with a slight modification in the construction of the particular test function. Fix $q \in Q$ and for $j = 1, 2$ denote ξ_j the unique solution of the problem

$$\begin{gathered} -\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\xi_j = q^1 \quad \text{in } \Theta_j, \\ \boldsymbol{\nabla}\xi_j\cdot\widehat{\boldsymbol{n}} = (-1)^{j-1}q^2 \quad \text{on } \Gamma_j, \quad \boldsymbol{\nabla}\xi_j\cdot\widehat{\boldsymbol{n}} = -q^2 \quad \text{on } \Gamma_j + h_j\,, \\ \xi_j = 0 \quad \text{on } \partial\Theta_j - \Gamma_j\,. \end{gathered} \tag{2.24}$$

Define $\mathbf{v}^1 := \boldsymbol{\nabla}\xi_1\mathbb{1}_{\Theta_1} + \boldsymbol{\nabla}\xi_2\mathbb{1}_{\Theta_2}$. Thus, $-\boldsymbol{\nabla}\cdot\mathbf{v}^1 = q^1\mathbb{1}_{\Theta_1} + q^1\mathbb{1}_{\Theta_2}$ and

$$\mathbf{v}^1\cdot\widehat{\boldsymbol{n}} = q^2\mathbb{1}_{\Gamma_1} - q^2\mathbb{1}_{\Gamma_2}\,.$$

By the Poincaré inequality $c_1\|\mathbf{v}^1\|_{H_{\rm div}(\Omega_1)} \le \|q^1\|_{L^2(\Omega_1)} + \|q^2\|_{L^2(\Gamma)}$. Hence, setting $\mathbf{v}^2 := \boldsymbol{\nabla} q^2$ we have

$$\begin{aligned} \mathcal{B}\mathbf{v}(q) &= -\int_{\Omega_1}\boldsymbol{\nabla}\cdot\mathbf{v}^1 q^1 + \int_{\Omega_2}\mathbf{v}^2\cdot\boldsymbol{\nabla} q^2 - \int_{\Gamma_1}(\mathbf{v}^1\cdot\widehat{\boldsymbol{n}})q^2\,dS + \int_{\Gamma_2}(\mathbf{v}^1\cdot\widehat{\boldsymbol{n}})q^2\,dS \\ &= \int_{\Omega_1}|q^1|^2 + \int_{\Omega_2}|\boldsymbol{\nabla} q^2|^2 + \int_{\Gamma_1}|q^2|^2\,dS + \int_{\Gamma_2}|q^2|^2\,dS \\ &\ge \int_{\Omega_1}|q^1|^2 + \frac{\kappa}{2}\int_{\Omega_2}|q^2|^2 + \frac{1}{2}\Big(\int_{\Omega_2}|q^2|^2 + \int_{\Gamma}|q^2|^2\,dS\Big) \\ &\ge c\|\mathbf{v}\|_{\mathbf{V}}\|q\|_Q\,, \end{aligned} \tag{2.25}$$

for $c := \min\{c_1, \frac{1}{2}, \frac{\kappa}{2}\}$, which gives the inf-sup condition of the operator $\mathcal{B}$. □

Theorem 2.12. *Let $a_i(\cdot) \in L^\infty(\Omega)$ and*

$$a^* := \min_{i=1,2} \operatorname{ess\,inf}\{a_i(\mathbf{x}) : \mathbf{x} \in \Omega_i\}. \tag{2.26}$$

Then, if a^* is positive and $0 < \alpha$, the mixed variational formulation (2.19) *(or equivalently, the system* (2.17)*) is well-posed.*

Proof. Clearly $\mathcal{A}$ is non-negative and $\mathbf{V}$-coercive on $\ker \mathcal{B}$. The operator $\mathcal{B}$ satisfies the inf-sup condition as seen in the preceding lemma. By theorem 2.8 the result follows. □

3. Scaling the problem and convergence statements

The successful asymptotic analysis of the problem (2.17) demands scaling of the heights and resistance coefficients of the medium. We have the following definition (see figure 2).

Definition 3.1. Let $\Omega = (\Theta_1, \Theta_2, \zeta, h)$ be a fractured medium with a vertical translation generated fissure. For $\epsilon \in (0, 1)$ we define its associated ϵ-scaled fissured system $\{(\Theta_1^\epsilon, \Theta_2^\epsilon, \zeta, \epsilon h) : \epsilon > 0\}$ by

$$\Theta_1^\epsilon := \Theta_1, \quad \Theta_2^\epsilon := \Theta_2 - (1-\epsilon)h, \tag{3.1a}$$

$$\Gamma_1^\epsilon = \Gamma_1, \quad \Gamma_2^\epsilon = \Gamma_2 - (1-\epsilon)h. \tag{3.1b}$$

The domains $\Omega_1^\epsilon, \Omega_2^\epsilon, \Omega^\epsilon$ and $\Gamma_1^\epsilon, \Gamma^\epsilon, \Gamma^\epsilon$ are defined as in (2.5) and (2.6) respectively.

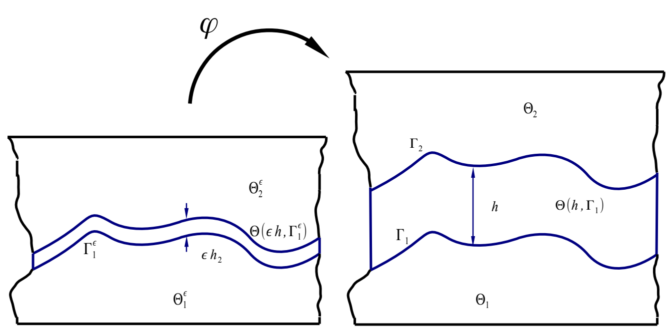

Figure 2. Domains mapping

Remark 3.2. Clearly the system $(\Theta_1^\epsilon, \Theta_2^\epsilon, \zeta, \epsilon h)$, satisfies the conditions of definition 2.3 for every $\epsilon > 0$.

3.1. Isomorphisms of spaces and formulation.

Let $\Omega_1^\epsilon, \Omega_2^\epsilon$, Ω^ϵ and $\Gamma_1^\epsilon, \Gamma_2^\epsilon$, Γ^ϵ be the domains and surfaces associated to the family $\{(\Theta_1^\epsilon, \Theta_2^\epsilon, \zeta, \epsilon h) : \epsilon > 0\}$ as in definition 3.1. Define the spaces

$$\mathbf{V}^\epsilon := \{\mathbf{v} \in \mathbf{L}^2(\Omega^\epsilon) : \boldsymbol{\nabla} \cdot \mathbf{v}^1 \in \mathbf{L}^2(\Omega_1^\epsilon), \mathbf{v}^1 \cdot \widehat{\boldsymbol{n}}|_{\Gamma^\epsilon} \in L^2(\Gamma^\epsilon)\}, \tag{3.2a}$$

$$Q^\epsilon := \{q \in L^2(\Omega^\epsilon) : \boldsymbol{\nabla} q^2 \in \mathbf{L}^2(\Omega_2^\epsilon)\}. \tag{3.2b}$$

We endow these spaces with the norms coming from the natural inner product

$$\|\mathbf{v}\|_{\mathbf{V}^\epsilon} := \{\|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega^\epsilon)} + \|\boldsymbol{\nabla} \cdot \mathbf{v}^1\|^2_{L^2(\Omega_1^\epsilon)} + \|\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}}\|^2_{L^2(\Gamma^\epsilon)}\}^{1/2}, \tag{3.2c}$$

$$\|q\|_{Q^\epsilon} := \{\|q\|^2_{L^2(\Omega^\epsilon)} + \|\boldsymbol{\nabla} q^2\|^2_{L^2(\Omega_2^\epsilon)}\}^{1/2}\,. \tag{3.2d}$$

Consider the scaled problem: Find $p^\epsilon \in Q^\epsilon$ and $\mathbf{u}^\epsilon \in \mathbf{V}^\epsilon$ such that

$$\begin{aligned}&\int_{\Omega_1^\epsilon} a_1\mathbf{u}^\epsilon\cdot\mathbf{v}d\mathbf{y} + \epsilon\int_{\Omega_2^\epsilon} a_2\mathbf{u}^\epsilon\cdot\mathbf{v}d\mathbf{y} - \int_{\Omega_1^\epsilon} p^\epsilon\boldsymbol{\nabla}\cdot\mathbf{v}d\mathbf{y} + \int_{\Omega_2^\epsilon}\boldsymbol{\nabla} p^\epsilon\cdot\mathbf{v}d\mathbf{y}\\ &+ \alpha\int_{\Gamma^\epsilon}(\mathbf{u}^{\epsilon,1}\cdot\widehat{\boldsymbol{n}})(\mathbf{v}^1\cdot\widehat{\boldsymbol{n}})\,dS - \int_{\Gamma_1^\epsilon} p^{\epsilon,2}(\mathbf{v}^1\cdot\widehat{\boldsymbol{n}})\,dS + \int_{\Gamma_2^\epsilon} p^{\epsilon,2}(\mathbf{v}^1\cdot\widehat{\boldsymbol{n}})\,dS\\ &= -\int_{\Omega^\epsilon}\mathbf{g}^\epsilon\cdot\mathbf{v}d\mathbf{y}\,,\end{aligned} \tag{3.3a}$$

$$\begin{aligned}&\int_{\Omega_1^\epsilon}\boldsymbol{\nabla}\cdot\mathbf{u}^\epsilon q d\mathbf{y} - \int_{\Omega_2^\epsilon}\mathbf{u}^\epsilon\cdot\boldsymbol{\nabla} q d\mathbf{y} + \int_{\Gamma_1^\epsilon}(\mathbf{u}^{\epsilon,1}\cdot\widehat{\boldsymbol{n}})q^2\,dS - \int_{\Gamma_2^\epsilon}(\mathbf{u}^{\epsilon,1}\cdot\widehat{\boldsymbol{n}})q^2\,dS\\ &= \int_{\Omega^\epsilon} F^\epsilon q d\mathbf{y} + \int_{\Gamma^\epsilon} f^\epsilon_{\Gamma^\epsilon} q^2\,dS \quad \text{for all } q\in Q^\epsilon, \mathbf{v}\in\mathbf{V}^\epsilon.\end{aligned} \tag{3.3b}$$

Clearly, the problem (3.3) is well-posed since it verifies all the hypothesis of theorem 2.12. In order to analyze the asymptotic behavior of the solution $(\mathbf{u}^\epsilon, p^\epsilon)$ as $\epsilon \downarrow 0$ the geometry of the ϵ-domains must be mapped to a common domain of reference.

3.2. **The ϵ-problems in a reference domain.** We introduce the change of variable (see figure 2) $\boldsymbol{\varphi} : \Omega^\epsilon \to \Omega$ defined by

$$\begin{aligned}\boldsymbol{\varphi}(\mathbf{y}) &:= \mathbf{y}\mathbb{1}_{\Theta_1^\epsilon}(\mathbf{y}) + (\widetilde{\mathbf{y}}, y_3 + (1-\epsilon)h)\mathbb{1}_{\Theta_j^\epsilon}(\mathbf{y})\\ &\quad + (\widetilde{\mathbf{y}}, \frac{1}{\epsilon}(y_3 - \zeta^\epsilon(\widetilde{\mathbf{y}})) + \zeta^\epsilon(\widetilde{\mathbf{y}}) + (1-\epsilon)h)\mathbb{1}_{\Theta(\epsilon h,\Gamma_1^\epsilon)}(\mathbf{y})\end{aligned} \tag{3.4}$$

Defining $(\widetilde{\mathbf{x}}, z) := \boldsymbol{\varphi}(\mathbf{y})$ the gradients are related as follows

$$\boldsymbol{\nabla}_{\mathbf{y}} = \begin{Bmatrix}\widetilde{\boldsymbol{\nabla}}_{\mathbf{x}}\\ \partial_z\end{Bmatrix}\mathbb{1}_{\Omega_1^\epsilon} + \sum_i\begin{bmatrix} I & (1-\frac{1}{\epsilon})\widetilde{\boldsymbol{\nabla}}_{\mathbf{x}}\zeta(\widetilde{\mathbf{x}})\\ 0 & \frac{1}{\epsilon}\end{bmatrix}\begin{Bmatrix}\widetilde{\boldsymbol{\nabla}}_{\mathbf{x}}\\ \partial_z\end{Bmatrix}\mathbb{1}_{\Omega_2^\epsilon} \tag{3.5}$$

Here, it is understood that I is the identity matrix in $\mathbb{R}^{2\times 2}$. We write ζ instead of ζ^ϵ for the sake of simplicity, recalling that both surfaces differ only by a constant of vertical translation.

Theorem 3.3. *Let $\boldsymbol{\varphi} : \Omega^\epsilon \to \Omega$ be the change of variable defined in equation (3.4). Then, the maps defined $\Phi_1 : \mathbf{V} \to \mathbf{V}^\epsilon$, $\Phi_2 : Q \to Q^\epsilon$ defined respectively by $(\Phi_1\mathbf{v})(\mathbf{y}) := \mathbf{v}(\boldsymbol{\varphi}(\mathbf{y}))$ and $(\Phi_2 q)(\mathbf{y}) := q(\boldsymbol{\varphi}(\mathbf{y}))$ are isomorphisms.*

Proof. First notice for $\mathbf{v} \in \mathbf{V}$ and $q \in Q$ the functions $\Phi_1\mathbf{v}$ and $\Phi_2 q$ are defined on Ω^ϵ. Moreover, for $\ell = 1, 2$ the restriction of the change of variable is a bijection; i.e., $\boldsymbol{\varphi} : \Omega_\ell^\epsilon \to \Omega_\ell$ is a bijection. Therefore $\mathbf{v}(\cdot) \in \mathbf{L}^2(\Omega_\ell)$ if and only if $\mathbf{v}(\boldsymbol{\varphi}(\cdot)) \in \mathbf{L}^2(\Omega_\ell^\epsilon)$ and $q(\cdot) \in L^2(\Omega_\ell)$ if and only if $q(\boldsymbol{\varphi}(\cdot)) \in L^2(\Omega_\ell^\epsilon)$. Even more, $\boldsymbol{\varphi} : \Gamma_1^\epsilon \to \Gamma_1$ and $\boldsymbol{\varphi} : \Gamma_1^\epsilon + \epsilon h \to \Gamma_1 + h$ are bijective rigid translations. Therefore, the isomorphisms $L^2(\Gamma_1^\epsilon) \simeq L^2(\Gamma_1)$, $L^2(\Gamma_1^\epsilon + \epsilon h) \simeq L^2(\Gamma_1 + h)$ follow.

For the isomorphism Φ_1 take $\mathbf{v} \in \mathbf{V}$ which is equivalent to $\mathbf{v}(\mathbf{y}) \in \mathbf{L}^2(\Omega)$ and $\boldsymbol{\nabla}_{\mathbf{y}}\cdot\mathbf{v}(\mathbf{y}) \in L^2(\Omega_1)$. By the previous discussion these two conditions are equivalent to $\mathbf{v}(\boldsymbol{\varphi}(\mathbf{y})) \in \mathbf{L}^2(\Omega^\epsilon)$ and $\boldsymbol{\nabla}_{\mathbf{y}}\cdot\mathbf{v}(\boldsymbol{\varphi}(\mathbf{y})) = \boldsymbol{\nabla}_{\mathbf{y}}\cdot\mathbf{v}(\mathbf{x}) \in L^2(\Omega_1^\epsilon)$. However, equation (3.4) yields $\boldsymbol{\nabla}_{\mathbf{y}}\cdot\mathbf{v}(\boldsymbol{\varphi}(\mathbf{y})) = \boldsymbol{\nabla}_{\mathbf{y}}\cdot\mathbf{v}(\mathbf{x}) = \boldsymbol{\nabla}_{\mathbf{x}}\cdot\mathbf{v}(\mathbf{x})$ whenever $\mathbf{x} \in \Omega_1$; i.e., $\boldsymbol{\nabla}_{\mathbf{y}}\cdot\mathbf{v}(\mathbf{y}) \in L^2(\Omega_1^\epsilon)$ if and only if $\boldsymbol{\nabla}_{\mathbf{x}}\cdot\mathbf{v}(\mathbf{x}) = \boldsymbol{\nabla}_{\mathbf{x}}\cdot\mathbf{v}(\boldsymbol{\varphi}(\mathbf{y})) \in L^2(\Omega_1)$ as desired.

For the map Φ_2, the L^2-integrability condition between spaces Q and Q^ϵ is shown using the same arguments of the first paragraph. It remains to show the L^2-integrability condition on the gradient. First observe that the last row in the matrix equation (3.5) implies that $\frac{\partial}{\partial y_3} q(\mathbf{y}) \in L^2(\Omega_2^\epsilon)$ if and only if $\frac{\partial}{\partial z} q(\mathbf{x}) \in L^2(\Omega_2)$. Second, for the derivatives in the first two directions, the equation (3.5) yields

$$\frac{\partial}{\partial y_\ell} q(\mathbf{y}) = \frac{\partial}{\partial x_\ell} q(\mathbf{x}) + (1 - \frac{1}{\epsilon}) \frac{\partial}{\partial x_\ell} \zeta(\mathbf{x}) \, \frac{\partial}{\partial z} q(\mathbf{x}), \quad \ell = 1, 2.$$

Recalling the gradient of ζ is bounded, we conclude $\frac{\partial}{\partial y_\ell} q(\mathbf{y}) \in L^2(\Omega_2^\epsilon)$ if an only if $\frac{\partial}{\partial x_\ell} q(\mathbf{x}) \in L^2(\Omega_2)$ for $\ell = 1, 2$. Since $\frac{\partial}{\partial z} q(\mathbf{x}) \in L^2(\Omega_2)$ is immediate, the proof is complete. □

We are to apply the change of variable $\boldsymbol{\varphi} : \Omega^\epsilon \to \Omega$ in the problem (3.3), to this end, it is more convenient to write the system in terms of the quantities and directions which yield estimates agreeable with the asymptotic analysis. Hence, recalling the definition of the upwards normal vector (2.7) the following relationships hold

$$|(-\widetilde{\boldsymbol{\nabla}}\zeta, 1)| \mathbf{v} \cdot \widehat{\boldsymbol{n}} = -\widetilde{\mathbf{v}} \cdot \widetilde{\boldsymbol{\nabla}}\zeta + v_3, \tag{3.6a}$$

$$(\widetilde{\mathbf{v}}, \widetilde{\mathbf{v}} \cdot \widetilde{\boldsymbol{\nabla}}\zeta) \cdot \widehat{\boldsymbol{n}} = 0 \quad \text{in } \Theta(h, \Gamma_1). \tag{3.6b}$$

Applying the change of variable (3.4) to problem (3.3) and combining with relation (3.6a) we obtain the following variational statement: Find $p^\epsilon \in Q$ and $\mathbf{u}^\epsilon \in \mathbf{V}$ such that

$$\begin{aligned}
&\int_{\Omega_1} a_1 \mathbf{u}^\epsilon \cdot \mathbf{v} + \epsilon^2 \int_{\Omega_2} a_2 \mathbf{u}^\epsilon \cdot \mathbf{v} - \int_{\Omega_1} p^\epsilon \boldsymbol{\nabla} \cdot \mathbf{v} \\
&+ \int_{\Omega_2} \epsilon (\widetilde{\boldsymbol{\nabla}} p^\epsilon + \partial_z p^\epsilon \widetilde{\boldsymbol{\nabla}}\zeta) \cdot \widetilde{\mathbf{v}} + \int_{\Omega_2} |(-\widetilde{\boldsymbol{\nabla}}\zeta, 1)| \partial_z p^\epsilon (\mathbf{v} \cdot \widehat{\boldsymbol{n}}) \\
&- \int_{\Gamma_1} p^{\epsilon,2} (\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}}) \, dS + \int_{\Gamma_2} p^{\epsilon,2} (\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}}) \, dS + \alpha \int_{\Gamma} (\mathbf{u}^{\epsilon,1} \cdot \widehat{\boldsymbol{n}}) (\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}}) \, dS \\
&= - \int_{\Omega_1} \mathbf{g}^\epsilon \cdot \mathbf{v} - \epsilon \int_{\Omega_2} \mathbf{g}^\epsilon \cdot \mathbf{v} \, .
\end{aligned} \tag{3.7a}$$

$$\begin{aligned}
&\int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{u}^\epsilon q - \int_{\Omega_2} \epsilon \widetilde{\mathbf{u}}^{\epsilon,2} \cdot (\widetilde{\boldsymbol{\nabla}} q + \partial_z q \widetilde{\boldsymbol{\nabla}}\zeta) - \int_{\Omega_2} |(-\widetilde{\boldsymbol{\nabla}}\zeta, 1)| (\mathbf{u}^{\epsilon,2} \cdot \widehat{\boldsymbol{n}}) \partial_z q \\
&+ \int_{\Gamma_1} (\mathbf{u}^{\epsilon,1} \cdot \widehat{\boldsymbol{n}}) q^2 \, dS - \int_{\Gamma_2} (\mathbf{u}^{\epsilon,1} \cdot \widehat{\boldsymbol{n}}) q^2 \, dS \\
&= \int_{\Omega_1} F^{\epsilon,1} q + \epsilon \int_{\Omega_2} F^{\epsilon,2} q + \int_{\Gamma} f_\Gamma^\epsilon q^2 \, dS \quad \text{for all } q \in Q, \mathbf{v} \in \mathbf{V} \, .
\end{aligned} \tag{3.7b}$$

Finally, by the theorem 3.3 on isomorphisms of function spaces, we conclude that the problems (3.7) and (3.3) are equivalent.

3.2.1. *The strong rescaled problem.* The solution of the problem (3.7) is the weak solution of the system of equations

$$a_1 \mathbf{u}^{\epsilon,1} + \boldsymbol{\nabla} p^{\epsilon,1} + \mathbf{g} = 0 \quad \text{and} \tag{3.8a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u}^{\epsilon,1} = f^{\epsilon,1} \quad \text{in } \Omega_1 \, . \tag{3.8b}$$

$$p^{\epsilon,1} = 0 \quad \text{on } \partial\Omega_1 - \Gamma \, . \tag{3.8c}$$

$$p^{\epsilon,1} - p^{\epsilon,2} = \alpha \mathbf{u}^1 \cdot \widehat{\boldsymbol{n}} \mathbb{1}_{\Gamma_2} - \alpha \mathbf{u}^1 \cdot \widehat{\boldsymbol{n}} \mathbb{1}_{\Gamma_1} \quad \text{and} \tag{3.8d}$$

$$(\mathbf{u}^{\epsilon,1} - \mathbf{u}^{\epsilon,2}) \cdot \widehat{\boldsymbol{n}} \mathbb{1}_{\Gamma_1} - (\mathbf{u}^{\epsilon,1} - \mathbf{u}^{\epsilon,2}) \cdot \widehat{\boldsymbol{n}} \mathbb{1}_{\Gamma_2} = f_\Gamma^\epsilon \quad \text{on } \Gamma \, . \tag{3.8e}$$

$$\begin{bmatrix} \epsilon a_2 \widetilde{\mathbf{u}}^{\epsilon,2} + \widetilde{\boldsymbol{\nabla}} p^{\epsilon,2} + (1-\frac{1}{\epsilon})\partial_z p^{\epsilon,2}\widetilde{\boldsymbol{\nabla}}\zeta + \widetilde{\mathbf{g}}^{\epsilon} \\ \epsilon^2 a_2 {u_3^{\epsilon,2}}^2 + \partial_z p^{\epsilon,2} + \epsilon g_3^{\epsilon} \end{bmatrix}^T = \mathbf{0}\,, \tag{3.8f}$$

$$\boldsymbol{\nabla}\cdot(\epsilon\widetilde{\mathbf{u}}^{\epsilon,2}, \epsilon\widetilde{\mathbf{u}}^{\epsilon,2}\cdot\widetilde{\boldsymbol{\nabla}}\zeta + |(-\widetilde{\boldsymbol{\nabla}}\zeta,1)|(\mathbf{u}^{\epsilon,2}\cdot\widehat{\boldsymbol{n}})) = \epsilon F^{\epsilon,2} \quad \text{in } \Omega_2\,, \tag{3.8g}$$

$$\widetilde{\mathbf{u}}^{\epsilon,2}\cdot\widetilde{\boldsymbol{\nu}} = 0 \quad \text{on } \partial\Omega_2 - \Gamma\,. \tag{3.8h}$$

As before equations (3.8d), (3.8e) have the separation of cases $\mathbb{1}_{\Gamma_2}, \mathbb{1}_{\Gamma_1}$ in order to be consistent with the upwards normal vector $\widehat{\boldsymbol{n}}$. However, the equations (3.8e) and (3.8h) need further clarification. Reordering and integrating by parts the second and third summands of equation (3.7b), we have

$$\begin{aligned}
&-\int_{\Omega_2}\epsilon\widetilde{\mathbf{u}}^{\epsilon,2}\cdot(\widetilde{\boldsymbol{\nabla}}q + \partial_z q\widetilde{\boldsymbol{\nabla}}\zeta) - \int_{\Omega_2}|(-\widetilde{\boldsymbol{\nabla}}\zeta,1)|(\mathbf{u}^{\epsilon,2}\cdot\widehat{\boldsymbol{n}})\partial_z q\\
&= -\int_{\Omega_2}(\epsilon\widetilde{\mathbf{u}}^{\epsilon,2}, \epsilon\widetilde{\mathbf{u}}^{\epsilon,2}\cdot\widetilde{\boldsymbol{\nabla}}\zeta + |(-\widetilde{\boldsymbol{\nabla}}\zeta,1)|(\mathbf{u}^{\epsilon,2}\cdot\widehat{\boldsymbol{n}}))\cdot\boldsymbol{\nabla}q\\
&= \int_{\Omega_2}\boldsymbol{\nabla}\cdot(\epsilon\widetilde{\mathbf{u}}^{\epsilon,2}, \epsilon\widetilde{\mathbf{u}}^{\epsilon,2}\cdot\widetilde{\boldsymbol{\nabla}}\zeta + |(-\widetilde{\boldsymbol{\nabla}}\zeta,1)|(\mathbf{u}^{\epsilon,2}\cdot\widehat{\boldsymbol{n}}))q\\
&\quad - \int_{\partial\Omega_2} q(\epsilon\widetilde{\mathbf{u}}^{\epsilon,2}, \epsilon\widetilde{\mathbf{u}}^{\epsilon,2}\cdot\widetilde{\boldsymbol{\nabla}}\zeta + |(-\widetilde{\boldsymbol{\nabla}}\zeta,1)|(\mathbf{u}^{\epsilon,2}\cdot\widehat{\boldsymbol{n}}))\cdot\widehat{\boldsymbol{\nu}}\,dS\,,
\end{aligned}$$

where $\widehat{\boldsymbol{\nu}}$ is the outwards unit normal vector of the boundary Ω_2. We focus on the boundary term

$$\begin{aligned}
&\int_{\partial\Omega_2} q(\epsilon\widetilde{\mathbf{u}}^{\epsilon,2}, \epsilon\widetilde{\mathbf{u}}^{\epsilon,2}\cdot\widetilde{\boldsymbol{\nabla}}\zeta + |(-\widetilde{\boldsymbol{\nabla}}\zeta,1)|(\mathbf{u}^{\epsilon,2}\cdot\widehat{\boldsymbol{n}}))\cdot\widehat{\boldsymbol{\nu}}\,dS\\
&= \int_{\partial\Omega_2-(\Gamma_1\cup\Gamma_2)} q(\epsilon\widetilde{\mathbf{u}}^{\epsilon,2}, \epsilon\widetilde{\mathbf{u}}^{\epsilon,2}\cdot\widetilde{\boldsymbol{\nabla}}\zeta + |(-\widetilde{\boldsymbol{\nabla}}\zeta,1)|(\mathbf{u}^{\epsilon,2}\cdot\widehat{\boldsymbol{n}}))\cdot\widehat{\boldsymbol{\nu}}\,dS\\
&\quad + \sum_{\ell=0,1}\int_{\Gamma_\ell} q(\epsilon\widetilde{\mathbf{u}}^{\epsilon,2}, \epsilon\widetilde{\mathbf{u}}^{\epsilon,2}\cdot\widetilde{\boldsymbol{\nabla}}\zeta + |(-\widetilde{\boldsymbol{\nabla}}\zeta,1)|(\mathbf{u}^{\epsilon,2}\cdot\widehat{\boldsymbol{n}}))\cdot\widehat{\boldsymbol{\nu}}\,dS.
\end{aligned}$$

The equality $\widehat{\boldsymbol{\nu}}\cdot\widehat{\boldsymbol{k}} = 0$ holds on the portion of the vertical wall $\partial\Omega_2 - (\Gamma_1\cup\Gamma_2)$; i.e., the equation (3.8h) follows. For the remaining pieces of the boundary recall $\widehat{\boldsymbol{n}} = \widehat{\boldsymbol{\nu}}$ on Γ_2 and $\widehat{\boldsymbol{n}} = -\widehat{\boldsymbol{\nu}}$ on Γ_1; together with the equation (2.7), we obtain

$$\begin{aligned}
&-\int_{\Gamma_\ell} q(\epsilon\widetilde{\mathbf{u}}^{\epsilon,2}, \epsilon\widetilde{\mathbf{u}}^{\epsilon,2}\cdot\widetilde{\boldsymbol{\nabla}}\zeta + |(-\widetilde{\boldsymbol{\nabla}}\zeta,1)|(\mathbf{u}^{\epsilon,2}\cdot\widehat{\boldsymbol{n}}))\cdot\widehat{\boldsymbol{\nu}}\,dS\\
&= (-1)^{\ell-1}\int_{\Gamma_\ell} q(\epsilon\widetilde{\mathbf{u}}^{\epsilon,2}, \epsilon\widetilde{\mathbf{u}}^{\epsilon,2}\cdot\widetilde{\boldsymbol{\nabla}}\zeta + |(-\widetilde{\boldsymbol{\nabla}}\zeta,1)|(\mathbf{u}^{\epsilon,2}\cdot\widehat{\boldsymbol{n}}))\cdot\frac{(-\widetilde{\boldsymbol{\nabla}}\zeta,1)}{|(-\widetilde{\boldsymbol{\nabla}}\zeta,1)|}\,dS\\
&= (-1)^{\ell-1}\int_{\Gamma_\ell} q(\mathbf{u}^{\epsilon,2}\cdot\widehat{\boldsymbol{n}})\,dS \quad \text{for } \ell = 1,2.
\end{aligned}$$

Combining this last identity with the interface terms in equation (3.7b), the strong normal flux balance condition (3.8e) follows.

3.3. **A-priori estimates and convergence statements.** To obtain a-priori estimates on the norm of the solutions the following hypothesis are assumed:

$$\|F^{\epsilon}\|_{L^2(\Omega)} \text{ is bounded and } F^{1,\epsilon} \overset{w}{\rightharpoonup} F^1 \text{ in } L^2(\Omega_1), \tag{3.9a}$$

$$\mathbf{g}^{\epsilon} \overset{w}{\rightharpoonup} \mathbf{g} \text{ in } \mathbf{L}^2(\Omega_1), \quad \mathbf{g}^{2,\epsilon}(\widetilde{\mathbf{x}},\epsilon z) \overset{w}{\rightharpoonup} \mathbf{g}(\widetilde{\mathbf{x}}) \text{ in } \mathbf{L}^2(\Omega_2), \tag{3.9b}$$

$$\text{and } f_\Gamma^{\epsilon} \overset{w}{\rightharpoonup} f_\Gamma \text{ in } L^2(\Gamma). \tag{3.9c}$$

Now test equation (3.7a) with $\mathbf{u}^\epsilon$ and equation (3.7b) with p^ϵ, add them together and obtain

$$\begin{aligned}
&a^*(\|\mathbf{u}^{\epsilon,1}\|_{0,\Omega_1}^2 + \|\epsilon\mathbf{u}^{\epsilon,2}\|_{0,\Omega_2}^2) + \alpha\|\mathbf{u}^{\epsilon,1}\cdot\widehat{\boldsymbol{n}}\|_{L^2(\Gamma)}^2 \\
&= \int_{\Omega_1} F^{\epsilon,1}p^\epsilon + \epsilon\int_{\Omega_2} F^{\epsilon,2}p^\epsilon + \int_\Gamma f_\Gamma^\epsilon p^{\epsilon,2}\,dS - \int_{\Omega_1}\mathbf{g}^1\cdot\mathbf{u}^\epsilon - \int_{\Omega_2}\mathbf{g}^2\cdot\epsilon\mathbf{u}^\epsilon \\
&\le C(\|F^\epsilon\|_{0,\Omega} + \|f_\Gamma^\epsilon\|_{0,\Gamma})\|p^\epsilon\|_Q + \|\mathbf{g}^\epsilon\|_{0,\Omega}(\|\mathbf{u}^{\epsilon,1}\|_{0,\Omega_1} + \|\epsilon\mathbf{u}^{\epsilon,2}\|_{0,\Omega_2}).
\end{aligned} \tag{3.10}$$

Here, the constant $C > 0$ is independent from $\epsilon > 0$. Next, the term $\|p^\epsilon\|_Q$ must be bounded in terms of the flux $\mathbf{u}^{\epsilon,1}\mathbb{1}_{\Omega_1} + \epsilon\mathbf{u}^{\epsilon,2}\mathbb{1}_{\Omega_2}$ and the forcing terms. By the third component of the vector equation (3.8f) we have

$$\|\frac{1}{\epsilon}\partial_z p^{\epsilon,2}\|_{0,\Omega_2} \le \epsilon\|a_2\|_{L^\infty(\Omega_2)}\|u_3^{\epsilon,2}\|_{0,\Omega_2} + \|g_3^\epsilon\|_{0,\Omega_2}. \tag{3.11a}$$

Combined with the first two components of the vector equation (3.8f) yields

$$\|\widetilde{\boldsymbol{\nabla}} p^{\epsilon,2}\|_{0,\Omega_2} \le C(\|a_2\|_{L^\infty(\Omega_2)}\|\epsilon\mathbf{u}^{\epsilon,2}\|_{0,\Omega_2} + \|\mathbf{g}^\epsilon\|_{0,\Omega_2}), \tag{3.11b}$$

for an adequate constant $C > 0$. Thus

$$\|\boldsymbol{\nabla} p^{\epsilon,2}\|_{0,\Omega_2} \le C(\|a_2\|_{L^\infty(\Omega_2)}\|\epsilon\mathbf{u}^{\epsilon,2}\|_{0,\Omega_2} + \|\mathbf{g}\|_{0,\Omega_2}). \tag{3.12}$$

With $C > 0$ a constant independent from $\epsilon > 0$. Additionally, the equation (3.8a) yields

$$\|\boldsymbol{\nabla} p^{\epsilon,2}\|_{0,\Omega_1} \le \|a_1\|_{L^\infty(\Omega_2)}\|\mathbf{u}^{\epsilon,2}\|_{0,\Omega_1} + \|\mathbf{g}\|_{0,\Omega_1}. \tag{3.13}$$

The boundary condition (3.8c) together with Poincaré inequality give the control $\|p^{\epsilon,1}\|_{1,\Omega_1} \le C\|\boldsymbol{\nabla} p^\epsilon\|_{0,\Omega_1}$. On the other hand, the inequality (2.23) implies $\|p^\epsilon\|_{1,\Omega_2} \le C(\|p^\epsilon\|_{0,\Gamma} + \|\boldsymbol{\nabla} p^\epsilon\|_{0,\Omega_2})$; combined with the normal stress balance conditions (3.8d) we conclude:

$$\|p^\epsilon\|_Q \le \|p^\epsilon\|_{1,\Omega} \le C\|\boldsymbol{\nabla} p^\epsilon\|_{0,\Omega}. \tag{3.14}$$

And $C > 0$ is independent from $\epsilon > 0$. Finally, a combination of inequalities (3.14), (3.13) and (3.12) imply that the left hand side of inequality (3.10) is bounded. From the observations above we conclude that the following sequences are bounded

$$\|\mathbf{u}^{\epsilon,1}\|_{0,\Omega_1}, \quad \|\epsilon\mathbf{u}^{\epsilon,2}\|_{0,\Omega_2}, \quad \sqrt{\alpha}\|\mathbf{u}^{\epsilon,1}\cdot\widehat{\boldsymbol{n}}\|_{L^2(\Gamma)}, \tag{3.15a}$$

$$\|p^{\epsilon,1}\|_{H^1(\Omega_1)}, \quad \|p^{\epsilon,2}\|_{H^1(\Omega_2)}, \quad \|\frac{1}{\epsilon}\partial_z p^\epsilon\|_{0,\Omega_2}, \quad \|\boldsymbol{\nabla}\cdot\mathbf{u}^{\epsilon,1}\|_{L^2(\Omega_1)}. \tag{3.15b}$$

Remark 3.4. The change of variable $\boldsymbol{\varphi}$ modifies the structure of the divergence on the domain Ω_2, therefore it can only be claimed that the linear combination $\epsilon\widetilde{\boldsymbol{\nabla}}\cdot\widetilde{\mathbf{u}}^{\epsilon,2} + \epsilon(1-\frac{1}{\epsilon})\partial_z(\widetilde{\boldsymbol{\nabla}}\zeta\cdot\widetilde{\mathbf{u}}^{\epsilon,2}) + \partial_z u_3^{\epsilon,2}$ is bounded in $L^2(\Omega_2)$.

3.4. **Weak limits.** In the previous section bounds independent from $\epsilon > 0$ are obtained for $[\mathbf{u}^{\epsilon,1}, \epsilon\mathbf{u}^{\epsilon,2}] \in \mathbf{V}$ and $p^\epsilon = [p^{\epsilon,1}, p^{\epsilon,2}] \in H^1(\Omega_1)\times H^1(\Omega_2)$, consequently in Q. Then, there must exist $\mathbf{u}\in\mathbf{V}$, $p\in Q$, $\eta\in L^2(\Omega_2)$ and a subsequence, from now on denoted the same, such that

$$p^\epsilon \overset{w}{\rightharpoonup} p \quad \text{in } Q \text{ and strongly in } L^2(\Omega), \tag{3.16a}$$

$$\mathbf{u}^{\epsilon,1} \overset{w}{\rightharpoonup} \mathbf{u}^1 \text{ in } \mathbf{L}^2(\Omega_1) \quad\text{and}\quad \boldsymbol{\nabla}\cdot\mathbf{u}^{\epsilon,1} \overset{w}{\rightharpoonup} \boldsymbol{\nabla}\cdot\mathbf{u}^1 \text{ in } L^2(\Omega_1), \tag{3.16b}$$

$$\sqrt{\alpha}\mathbf{u}^{\epsilon,1}\cdot\widehat{\boldsymbol{n}} \overset{w}{\rightharpoonup} \sqrt{\alpha}\mathbf{u}^1\cdot\widehat{\boldsymbol{n}} \quad \text{in } L^2(\Gamma), \tag{3.16c}$$

$$\epsilon\mathbf{u}^{\epsilon,1} \overset{w}{\rightharpoonup} \mathbf{u}^2 \quad \text{in } \mathbf{L}^2(\Omega_2), \tag{3.16d}$$

$$\frac{1}{\epsilon}\partial_z p^{\epsilon,2} \xrightarrow{w} \eta \text{ in } L^2(\Omega_2) \quad \text{and} \quad \partial_z p^{\epsilon,2} \to 0 \text{ strongly in } L^2(\Omega_2). \tag{3.16e}$$

Choose $\phi \in C_0^\infty(\Omega_2)$ arbitrary, test the equation (3.7b) with $q := \epsilon\phi$ and let $\epsilon \downarrow 0$. Recalling (3.16d) this gives

$$\begin{aligned} 0 &= \lim_{\epsilon\downarrow 0} \int_{\Omega_2} |(-\widetilde{\boldsymbol{\nabla}}\zeta, 1)|(\epsilon \mathbf{u}^{\epsilon,2} \cdot \widehat{\boldsymbol{n}})\partial_z\phi \\ &= \int_{\Omega_2} |(-\widetilde{\boldsymbol{\nabla}}\zeta, 1)|(\mathbf{u}^2 \cdot \widehat{\boldsymbol{n}})\partial_z\phi \\ &= -\langle \partial_z |(-\widetilde{\boldsymbol{\nabla}}\zeta, 1)|(\mathbf{u}^2 \cdot \widehat{\boldsymbol{n}}), \phi\rangle_{D'(\Omega_2), D(\Omega_2)}. \end{aligned}$$

Since $(-\widetilde{\boldsymbol{\nabla}}\zeta, 1)$ does not depend on the vertical variable z and it is the non-zero vector almost everywhere, we conclude that $\partial_z(\mathbf{u}^2 \cdot \widehat{\boldsymbol{n}}) = 0$ i.e. the component of the velocity normal to the surface Γ_1 is independent from z in Ω_2. Now choose $q \in Q$ arbitrary, test (3.7b) with ϵq and let $\epsilon \downarrow 0$ to get

$$\begin{aligned} 0 &= \int_{\Omega_2} |(-\widetilde{\boldsymbol{\nabla}}\zeta, 1)|(\mathbf{u}^2 \cdot \widehat{\boldsymbol{n}})\partial_z q \, d\mathbf{x} \\ &= \int_G \int_{\zeta(\widetilde{\mathbf{x}})}^{\zeta(\widetilde{\mathbf{x}})+h} |(-\widetilde{\boldsymbol{\nabla}}\zeta, 1)|(\mathbf{u}^2 \cdot \widehat{\boldsymbol{n}})\partial_z q \, dz d\widetilde{\mathbf{x}} \\ &= \int_G |(-\widetilde{\boldsymbol{\nabla}}\zeta, 1)|(\mathbf{u}^2 \cdot \widehat{\boldsymbol{n}})[q(\widetilde{\mathbf{x}}, \zeta(\widetilde{\mathbf{x}}) + h) - q(\widetilde{\mathbf{x}}, \zeta(\widetilde{\mathbf{x}}))] \, d\widetilde{\mathbf{x}}. \end{aligned}$$

The above holds for all $q \in Q$, in particular choosing $q(\widetilde{\mathbf{x}}, \zeta(\widetilde{\mathbf{x}})) = \phi(\widetilde{\mathbf{x}})$ for $\phi \in C_0^\infty(G)$ arbitrary and $q(\widetilde{\mathbf{x}}, \zeta(\widetilde{\mathbf{x}}) + h) = 0$ the statement above transforms in

$$\int_G |(-\widetilde{\boldsymbol{\nabla}}\zeta, 1)|(\mathbf{u}^2 \cdot \widehat{\boldsymbol{n}})(\widetilde{\mathbf{x}}, \zeta(\widetilde{\mathbf{x}}))\phi(\widetilde{\mathbf{x}}, \zeta(\widetilde{\mathbf{x}})) \, d\widetilde{\mathbf{x}} \quad \forall \phi \in C_0^\infty(G).$$

Therefore, $|(-\widetilde{\boldsymbol{\nabla}}\zeta, 1)|(\mathbf{u}^2 \cdot \widehat{\boldsymbol{n}})$ must be null and since $|(-\widetilde{\boldsymbol{\nabla}}\zeta, 1)|$ is non-zero almost everywhere we conclude

$$\mathbf{u}^2 \cdot \widehat{\boldsymbol{n}} = 0 \text{ in } \Omega_2. \tag{3.17}$$

This implies that the Cartesian coordinates of $\mathbf{u}^2$ satisfy the relation

$$\mathbf{u}^2 = \begin{Bmatrix} \widetilde{\mathbf{u}}^2 \\ u_3^2 \end{Bmatrix} = \begin{Bmatrix} \widetilde{\mathbf{u}}^2 \\ \widetilde{\mathbf{u}}^2 \cdot \widetilde{\boldsymbol{\nabla}}\zeta \end{Bmatrix} \quad \text{in } \Omega_2. \tag{3.18}$$

Now take a function $\mathbf{v}_\tau^2 \in (C_0^\infty(\Omega_2))^2$. Recalling (2.12) define $\widetilde{\mathbf{v}} := M^{T,\boldsymbol{\tau}}\mathbf{v}_\tau^2$ and $v_3 := M^{\widehat{\boldsymbol{k}},\boldsymbol{\tau}}\mathbf{v}_\tau^2$. Then, the function $\mathbf{v}^2 := \frac{1}{\epsilon}(\widetilde{\mathbf{v}}, v_3)$ has the structure (3.18) or equivalently $\mathbf{v}^2 \cdot \widehat{\boldsymbol{n}} = 0$ inside Ω_2. Define $\mathbf{v}$ as the trivial extension of $\mathbf{v}^2$ to the whole domain Ω, therefore $\mathbf{v} \in \mathbf{V}$. Test (3.7a) with $\mathbf{v}$ and let $\epsilon \downarrow 0$, this gives

$$\int_{\Omega_2} a_2(\mathbf{x})\mathbf{u}^2 \cdot (\widetilde{\mathbf{v}}, v_3) + \int_{\Omega_2} \widetilde{\boldsymbol{\nabla}} p^2 \cdot \widetilde{\mathbf{v}} + \int_{\Omega_2} \mathbf{g} \cdot (\widetilde{\mathbf{v}}, v_3) = 0.$$

Consequently,

$$\int_{\Omega_2} a_2(\mathbf{x})\mathbf{u}_\tau^2 \cdot \mathbf{v}_\tau + \int_{\Omega_2} \widetilde{\boldsymbol{\nabla}} p^2 \cdot M^{T,\boldsymbol{\tau}}\mathbf{v}_\tau + \int_{\Omega_2} \mathbf{g}_\tau \cdot \mathbf{v}_\tau = 0.$$

The equation above holds for all $\mathbf{v}_\tau \in (C_0^\infty(\Omega_2)^2$ and due to the isomorphism of proposition 2.5 we conclude

$$a_2(\mathbf{x})\mathbf{u}_\tau^2 + (M^{T,\boldsymbol{\tau}})'\widetilde{\boldsymbol{\nabla}} p^2 + \mathbf{g}_\tau = 0 \quad \text{in } \Omega_2. \tag{3.19}$$

Equation (3.16e) implies that p^2 does not depend on the variable z on Ω_2; i.e., $p^2 = p^2(\widetilde{\mathbf{x}})$. Therefore assuming

$$a_2 = a_2(\widetilde{\mathbf{x}}), \quad \widetilde{\mathbf{g}} = \widetilde{\mathbf{g}}(\widetilde{\mathbf{x}}) \quad \text{in } \Omega_2 \tag{3.20}$$

the equation (3.19) gives $\mathbf{u}^2_\tau = \mathbf{u}^2_\tau(\widetilde{\mathbf{x}})$ i.e. $\mathbf{u}^2_\tau$ is independent from z in Ω_2. Together with the fact $\mathbf{u}^2 \cdot \widehat{\boldsymbol{n}} = 0$ in Ω_2 we conclude that the whole vector velocity $\mathbf{u}^2$ is independent from z in Ω_2.

Remark 3.5. Observe that by the assumptions for the data (3.20) the equation (3.19) is independent from z, becoming a lower-dimensional Darcy-type constitutive law on the stream lines parallel to ζ.

4. The limit problem

Define the subspaces

$$\mathbf{V}_0 := \{\mathbf{v} \in \mathbf{V} : \partial_z \mathbf{v}^2 = 0,\ \mathbf{v}^2 \cdot \widehat{\boldsymbol{n}} = 0 \text{ in } \Omega_2\}; \tag{4.1a}$$

$$Q_0 := \{q \in Q : \partial_z q = 0 \text{ in } \Omega_2\}. \tag{4.1b}$$

From the structure of the space, if $\mathbf{v} = [\mathbf{v}^1, \mathbf{v}^2] \in \mathbf{V}_0$ then the function $[\mathbf{v}^1, \frac{1}{\epsilon}\mathbf{v}^2]$ is also in $\mathbf{V}_0$. Hence, we use $[\mathbf{v}^1, \frac{1}{\epsilon}\mathbf{v}^2]$ to test (3.7a) and $q \in Q_0$ to test (3.7b), then we let $\epsilon \downarrow 0$ and conclude that the limits $[\mathbf{u}^{\epsilon,1}, \epsilon\mathbf{u}^{\epsilon,2}] \to \mathbf{u}$ and $p^\epsilon \to p$ are a solution of the *limit problem*: Find $p \in Q_0$ and $\mathbf{u} \in \mathbf{V}_0$ such that

$$\begin{aligned}
&\int_{\Omega_1} a_1 \mathbf{u} \cdot \mathbf{v} - \int_{\Omega_1} p \boldsymbol{\nabla} \cdot \mathbf{v} + \int_{\Omega_2} a_2 \mathbf{u}^2_\tau \cdot \mathbf{v}^2_\tau + \int_{\Omega_2} \widetilde{\boldsymbol{\nabla}} p^\epsilon \cdot \widetilde{\mathbf{v}} \\
&- \int_{\Gamma_1} p^2 (\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}})\, dS + \int_{\Gamma_2} p^2 (\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}})\, dS + \alpha \int_\Gamma (\mathbf{u}^1 \cdot \widehat{\boldsymbol{n}})(\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}})\, dS \qquad (4.2a)\\
&= - \int_{\Omega_1} \mathbf{g} \cdot \mathbf{v} - \int_{\Omega_2} \mathbf{g}_\tau \cdot \mathbf{v}^2_\tau\,,
\end{aligned}$$

$$\begin{aligned}
&\int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{u}\, q - \int_{\Omega_2} \widetilde{\mathbf{u}}^2 \cdot \widetilde{\boldsymbol{\nabla}} q + \int_{\Gamma_1} (\mathbf{u}^1 \cdot \widehat{\boldsymbol{n}}) q^2\, dS - \int_{\Gamma_2} (\mathbf{u}^1 \cdot \widehat{\boldsymbol{n}}) q^2\, dS \\
&= \int_{\Omega_1} F^1 q + \int_\Gamma f_\Gamma q^2\, dS \quad \text{for all } q \in Q_0, \mathbf{v} \in \mathbf{V}_0\,.
\end{aligned} \tag{4.2b}$$

4.1. **Well-Posedness of the limit problem.** Problem (4.2) is a mixed formulation of the type (2.19) with the operators $\mathcal{A}^0 : \mathbf{V}_0 \to \mathbf{V}_0'$ and $\mathcal{B}^0 : \mathbf{V}_0 \to Q_0'$ defined by

$$\mathcal{A}^0 \mathbf{v}(\mathbf{w}) := \int_{\Omega_1} a_1 \mathbf{v} \cdot \mathbf{w} + \int_{\Omega_2} a_2 \mathbf{v}^2_\tau \cdot \mathbf{w}^2_\tau + \alpha \int_\Gamma (\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}})(\mathbf{w}^1 \cdot \widehat{\boldsymbol{n}})\, dS, \tag{4.3a}$$

$$\mathcal{B}^0 \mathbf{v}(q) := - \int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{v}\, q + \int_{\Omega_2} \widetilde{\mathbf{v}} \cdot \widetilde{\boldsymbol{\nabla}} q - \int_{\Gamma_1} (\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}}) q^2 + \int_{\Gamma_2} (\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}}) q^2\,. \tag{4.3b}$$

Theorem 4.1. *The operator $\mathcal{B}^0$ satisfies the inf-sup condition.*

Proof. The proof has the same structure as lemma 2.11, there is only one detail to be examined in the construction of the test functions. Fix $q = [q^1, q^2] \in Q_0$, construct $\mathbf{v}^1$ in the same way it is done in problem (2.24). On the other hand since $q^2 \in H^1(\Omega_2)$ and $\partial_z q^2 = 0$, define

$$\mathbf{v}^2 := (\widetilde{\boldsymbol{\nabla}} q^2, \widetilde{\mathbf{v}} \cdot \widetilde{\boldsymbol{\nabla}} \zeta) \quad \text{in } \Omega_2.$$

Then $\mathbf{v}^2 \cdot \widehat{\boldsymbol{n}} = 0$ and $\partial_z \mathbf{v}^2 = 0$ in Ω_2; i.e., $\mathbf{v}^2 \in \mathbf{V}_0$ and $\|\mathbf{v}^2\|_{0,\Omega_2} \le C\|q^2\|_{1,\Omega_2}$ as desired. Repeating the inequalities presented in (2.25) the proof follows. □

Since the inf-sup condition holds, the theorem 2.12 applies to the operators (4.3) on the spaces $\mathbf{V}_0, Q_0$ and the limit problem (4.2) is well-posed. By the uniqueness of the solution of the limit problem, it follows that the original sequence converges weakly to the limit $\mathbf{u} \in \mathbf{V}_0, p \in Q_0$.

4.2. **The strong form.** To describe the *strong limit problem* corresponding to (4.2), two properties have to be exploited. First, the structure $\mathbf{v} \cdot \widehat{\boldsymbol{n}} = 0$ in Ω_2 for all $\mathbf{v} \in \mathbf{V}_0$, this implies that $\widetilde{\mathbf{v}} = M^{T,\boldsymbol{\tau}} \mathbf{v}^2_\tau$, with $M^{T,\boldsymbol{\tau}}$ the matrix defined in (2.12). Second, the independence of the velocities and pressures with respect to z in Ω_2. This last property allows to write the integrals over Ω_2 as surface integrals on Γ_1. Hence, the system (4.2) transforms into: Find $p \in Q_0$ and $\mathbf{u} \in \mathbf{V}_0$ such that

$$\begin{aligned}
&\int_{\Omega_1} a_1 \mathbf{u} \cdot \mathbf{v} - \int_{\Omega_1} p \boldsymbol{\nabla} \cdot \mathbf{v} + h \int_{\Gamma_1} (\widehat{\boldsymbol{n}} \cdot \widehat{\boldsymbol{k}})(a_2 \mathbf{u}^2_\tau + (M^{T,\boldsymbol{\tau}})' \widetilde{\boldsymbol{\nabla}} p^\epsilon + \mathbf{g}_\tau) \cdot \mathbf{v}^2_\tau \, dS \\
&- \int_{\Gamma_1} p^2 (\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}}) \, dS + \int_{\Gamma_2} p^2 (\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}}) \, dS + \alpha \int_{\Gamma} (\mathbf{u}^1 \cdot \widehat{\boldsymbol{n}})(\mathbf{v}^1 \cdot \widehat{\boldsymbol{n}}) \, dS \qquad (4.4a)\\
&= - \int_{\Omega_1} \mathbf{g} \cdot \mathbf{v} \,,
\end{aligned}$$

$$\begin{aligned}
&\int_{\Omega_1} \boldsymbol{\nabla} \cdot \mathbf{u} q - h \int_{\Gamma_1} (\widehat{\boldsymbol{n}} \cdot \widehat{\boldsymbol{k}}) M^{T,\boldsymbol{\tau}} \mathbf{u}^2_\tau \cdot \widetilde{\boldsymbol{\nabla}} q^2 \, dS + \int_{\Gamma_1} (\mathbf{u}^1 \cdot \widehat{\boldsymbol{n}}) q^2 \, dS - \int_{\Gamma_2} (\mathbf{u}^1 \cdot \widehat{\boldsymbol{n}}) q^2 \, dS \\
&= \int_{\Omega_1} F^1 q + \int_{\Gamma} f_\Gamma q^2 \, dS \quad \text{for all } q \in Q_0, \ \mathbf{v} \in \mathbf{V}_0 \,. \qquad (4.4b)
\end{aligned}$$

Integrating by parts the statement above, we obtain the strong problem with piecewise C^1 surface interfaces:

$$a_1 \mathbf{u} + \boldsymbol{\nabla} p^1 + \mathbf{g}^1 = 0, \qquad (4.5a)$$
$$\boldsymbol{\nabla} \cdot \mathbf{u} = F^1 \quad \text{in } \Omega_1 \,, \qquad (4.5b)$$
$$p^1 = 0 \quad \text{on } \partial\Omega_1 - \Gamma \,, \qquad (4.5c)$$
$$\mathbf{u}^2 \cdot \widehat{\boldsymbol{n}} = 0, \quad \partial_z p^2 = 0 \,, \qquad (4.5d)$$
$$[a_2(s) \mathbf{u}^2_\tau + (M^{T,\boldsymbol{\tau}})' \widetilde{\boldsymbol{\nabla}} p^2 + \mathbf{g}^2_\tau(s)] \mathbb{1}_{\Gamma_1} = 0 \,, \qquad (4.5e)$$
$$[(\mathbf{u}^1 \cdot \widehat{\boldsymbol{n}}|_{\Gamma_2} - \mathbf{u}^1 \cdot \widehat{\boldsymbol{n}}|_{\Gamma_1})] \mathbb{1}_{\Gamma_1} + h (\widehat{\boldsymbol{n}} \cdot \widehat{\boldsymbol{k}}) \widetilde{\boldsymbol{\nabla}} \cdot (M^{T,\boldsymbol{\tau}} \mathbf{u}^2_\tau) \mathbb{1}_{\Gamma_1} = f_\Gamma \quad \text{in } \Gamma \,, \qquad (4.5f)$$
$$p^1 - p^2 = \alpha \mathbf{u}^1 \cdot \widehat{\boldsymbol{n}} \mathbb{1}_{\Gamma_2} - \alpha \mathbf{u}^1 \cdot \widehat{\boldsymbol{n}} \mathbb{1}_{\Gamma_1} \,, \qquad (4.5g)$$
$$\mathbf{u}^2 \cdot \widehat{\boldsymbol{\nu}} = 0 \quad \text{on } \partial G \,. \qquad (4.5h)$$

The statement of equation (4.5e) was already shown in (3.19), however the statements (4.5f) and (4.5h) need further discussion.

4.3. **The interface integrals.**

Definition 4.2. Let G, ζ and Υ be as in definition 2.1, define the spaces

$$L^2(\Upsilon) := \{h : \Upsilon \to \mathbb{R} : \int_\Upsilon h^2(s) \, dS < +\infty\}, \qquad (4.6a)$$
$$H^1(\Upsilon) := \{h \in L^2(\Upsilon) : \widetilde{\boldsymbol{\nabla}} h \in L^2(\Upsilon) \times L^2(\Upsilon)\}, \qquad (4.6b)$$

$$H_0^1(\Upsilon) := \{h \in H^1(\Upsilon) : h|_{\partial G} = 0\}. \tag{4.6c}$$

Here $\widetilde{\boldsymbol{\nabla}}$ indicates the gradient with respect to the variables (x_1, x_2) contained in G.

We have the following isomorphism result.

Theorem 4.3. *Let $\mathcal{O}$ and Υ be as in definition 2.1. Consider the natural embedding $\jmath : \Upsilon \to \mathcal{O}$ defined by $\jmath(\widetilde{\mathbf{x}}, \zeta(\widetilde{\mathbf{x}})) := \widetilde{\mathbf{x}}$ and the map*

$$\varphi \mapsto \varphi \circ \jmath \tag{4.7}$$

Then

(i) *The embedding (4.7) is an isomorphism between $L^2(\mathcal{O})$ and $L^2(\Upsilon)$;*
(ii) *The embedding (4.7) is an isomorphism between $H^1(\mathcal{O})$ and $H^1(\Upsilon)$;*
(iii) *The embedding (4.7) is an isomorphism between $H_0^1(\mathcal{O})$ and $H_0^1(\Upsilon)$.*

Proof. By definition $\jmath : \Upsilon \to \mathcal{O}$ is linear and bijective, therefore the map (4.7) is bijective between spaces of functions.

(i) By the hypothesis that ζ satisfies $C_1 = \operatorname{ess\,inf}\{\widehat{\boldsymbol{n}}(s) \cdot \widehat{\boldsymbol{k}} : s \in \Upsilon\} > 0$ then, for any $\phi \in L^2(\Upsilon)$

$$\int_\Upsilon (\phi \circ \jmath)^2 \, dS = \int_{\mathcal{O}} (\widehat{\boldsymbol{n}}(\widetilde{\mathbf{x}}) \cdot \widehat{\boldsymbol{k}})^{-1} \phi^2(\widetilde{\mathbf{x}}) d\widetilde{\mathbf{x}} \le \frac{1}{C_1} \int_{\mathcal{O}} \phi^2(\widetilde{\mathbf{x}}) d\widetilde{\mathbf{x}}.$$

The inequality above gives the continuity of the application $\varphi \mapsto \varphi \circ j$. By Banach's inversion theorem the map is an isomorphism.

(ii) By definition $\widetilde{\boldsymbol{\nabla}}(\phi \circ \jmath) = \widetilde{\boldsymbol{\nabla}} \phi(\widetilde{\mathbf{x}})$ holds for any $\phi \in H^1(\mathcal{O})$.

(iii) Is immediate from (ii). □

Choose $q \in Q_0$ supported inside Ω_2 and test equation (4.4b); hence

$$-\int_{\Omega_2} \widetilde{\mathbf{u}}^2 \cdot \widetilde{\boldsymbol{\nabla}} q - \int_{\Gamma_1} (\mathbf{u}^1 \cdot \widehat{\boldsymbol{n}}) q^2 \, dS + \int_{\Gamma_2} (\mathbf{u}^1 \cdot \widehat{\boldsymbol{n}}) q^2 \, dS = \int_{\Gamma_1 \cup \Gamma_2} f_{\Gamma_1} q^2 \, dS. \tag{4.8}$$

We focus on the first term of the left-hand side. First $\partial_z q^2 = 0$ implies $\widetilde{\mathbf{u}}^2 \cdot \widetilde{\boldsymbol{\nabla}} q^2 = \mathbf{u}^2 \cdot \boldsymbol{\nabla} q^2$, then

$$-\int_{\Omega_2} \mathbf{u}^2 \cdot \boldsymbol{\nabla} q = \int_{\Omega_2} \boldsymbol{\nabla} \cdot \mathbf{u}^2 q^2 - \int_{\partial\Omega_2} q^2 \mathbf{u}^2 \cdot \widehat{\boldsymbol{\nu}} \, dS. \tag{4.9}$$

The two summands on the right-hand side are treated separately. For the first summand the independence from the variable z implies that $\boldsymbol{\nabla} \cdot \mathbf{u}^2 = \widetilde{\boldsymbol{\nabla}} \cdot \widetilde{\mathbf{u}}^2$. The fact $\mathbf{u}^2 \cdot \widehat{\boldsymbol{n}} = 0$ in Ω_2 gives $\widetilde{\mathbf{u}}^2 = M^{T,\boldsymbol{\tau}} \mathbf{u}_\tau^2$. Thus

$$\int_{\Omega_2} \widetilde{\boldsymbol{\nabla}} \cdot \widetilde{\mathbf{u}}^2 q^2 \, d\mathbf{x} = h \int_G \widetilde{\boldsymbol{\nabla}} \cdot (M^{T,\boldsymbol{\tau}} \mathbf{u}_\tau^2) q^2 \, d\widetilde{\mathbf{x}} = h \int_{\Gamma_1} (\widehat{\boldsymbol{n}} \cdot \widehat{\boldsymbol{k}}) \widetilde{\boldsymbol{\nabla}} \cdot (M^{T,\boldsymbol{\tau}} \mathbf{u}_\tau^2) q^2 \, dS. \tag{4.10}$$

The boundary term in (4.9) can be written as

$$-\int_{\Gamma_1 \cup \Gamma_2} q^2 \mathbf{u}^2 \cdot \widehat{\boldsymbol{\nu}} \, dS - \int_{\partial\Omega_2 - (\Gamma_1 \cup \Gamma_2)} q^2 \mathbf{u}^2 \cdot \widehat{\boldsymbol{\nu}} \, dS.$$

The first summand vanishes since $\mathbf{u}^2 \cdot \widehat{\boldsymbol{n}} = 0$ in Ω_2. The boundary piece described in the second summand is a vertical wall, then $\widehat{\boldsymbol{\nu}} \cdot \widehat{\boldsymbol{k}} = 0$ and it can be identified with the outwards normal vector to the set $G \subseteq \mathbb{R}^2$. Moreover, from the independence of the integrand with respect to the variable z, the surface integral can be collapsed to a line integral over ∂G. Combining these observations with (4.10) and (4.9), the equation (4.8) transforms into

$$\begin{aligned} & h\int_{\Gamma_1} (\widehat{\boldsymbol{n}}\cdot\widehat{\boldsymbol{k}})\widetilde{\boldsymbol{\nabla}}\cdot(M^{T,\boldsymbol{\tau}}\mathbf{u}^2_\tau)q^2\,dS - h\int_{\partial G} q^2\mathbf{u}\cdot\widehat{\boldsymbol{\nu}}\,dC \\ & - \int_{\Gamma_1}(\mathbf{u}^1\cdot\widehat{\boldsymbol{n}})q^2\,dS + \int_{\Gamma_2}(\mathbf{u}^1\cdot\widehat{\boldsymbol{n}})q^2\,dS \\ & = \int_{\Gamma_1\cup\Gamma_2} f_\Gamma q^2 dS, \end{aligned}$$

where dC is the arc-length measure on ∂G. The isomorphisms provided by theorem 4.3 imply that the quantifier $q^2|_{\Gamma_1}$ can hit any function in the space $H^1_0(\Gamma_1)$. Therefore, the equation (4.5f) follows. Finally, using again theorem 4.3, the trace of the test function $q^2|_{\Gamma_1}$ can hit any function in the space $H^1_0(\Gamma_1)$, and combined with equation (4.5f) we obtain (4.5h).

4.4. **Strong convergence of solutions.**

Theorem 4.4. *Under the hypotheses*

$$\|F^{\epsilon,1}-F^1\|_{0,\Omega_1}\to 0,\quad \|f^\epsilon_\Gamma - f_\Gamma\|_{0,\Gamma}\to 0,\quad \|\mathbf{g}^\epsilon-\mathbf{g}\|_{0,\Omega}\to 0, \tag{4.11}$$

the solutions $\mathbf{u}^\epsilon, p^\epsilon$ *satisfy the following strong convergence statements*

$$\begin{gathered} \|\mathbf{u}^{\epsilon,1}-\mathbf{u}^1\|_{0,\Omega_1}\to 0,\quad \|\epsilon\mathbf{u}^{\epsilon,2}-\mathbf{u}^2\|_{0,\Omega_2}\to 0, \\ \|p^{\epsilon,1}-p^1\|_{1,\Omega_1}\to 0,\quad \|p^{\epsilon,2}-p^2\|_{1,\Omega_2}\to 0. \end{gathered} \tag{4.12}$$

The proof of the above theorem uses exactly the same arguments presented in [16, Theorem 3.2], and it is omitted.

Finally, assume that $\mathbf{u}^2_\tau \neq 0$ and consider the quotients:

$$\frac{\|\mathbf{u}^{\epsilon,2}_\tau\|_{0,\Omega_2}}{\|\mathbf{u}^{\epsilon,2}\cdot\widehat{\boldsymbol{n}}\|_{0,\Omega_2}} = \frac{\|\epsilon\mathbf{u}^{\epsilon,2}_\tau\|_{0,\Omega_2}}{\|\epsilon\mathbf{u}^{\epsilon,2}\cdot\widehat{\boldsymbol{n}}\|_{0,\Omega_2}} > \frac{\|\mathbf{u}^2_\tau\|_{0,\Omega_2}-\delta}{\|\epsilon\mathbf{u}^{\epsilon,2}\cdot\widehat{\boldsymbol{n}}\|_{0,\Omega_2}} > 0\,. \tag{4.13}$$

The lower bound holds true for $\epsilon > 0$ small enough and adequate $\delta > 0$. Therefore, we conclude that the magnitudes' ratio of the tangential over the normal components of the flux blows-up to infinity; i.e., the flow in the thin channel is predominantly tangential. Finally if $\mathbf{u}^2_\tau = 0$, unlike the analysis for flat interfaces presented in [16], no conclusions can be obtained because of the complexity introduced by the geometry of the fissure.

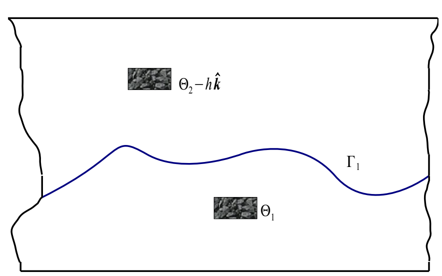

FIGURE 3. System of 2-D manifold fissures

5. A problem with two dimensional manifolds

In this section, using the independence of the limit functions with respect to z in Ω_2, it will be shown that the limiting problem (4.5) can be formulated as a system coupling Darcy flow in three dimensions, with tangential flow hosted in a piecewise C^1 surface, as depicted in figure 3. First we introduce the geometry, recall that $\Theta_2 - h\widehat{\boldsymbol{k}} = \{\omega - k\widehat{\boldsymbol{k}} : \omega \in \Omega_2\}$ and consider the domain

$$\vartheta := \Theta_1 \cup (\Theta_2 - h\widehat{\boldsymbol{k}}), \quad \vartheta_{FR} := \vartheta \cup \Gamma_1. \tag{5.1}$$

Additionally we introduce the notation Γ_1^+, Γ_1^- for the upper and lower faces of the piecewise surface Γ_1.

5.1. Spaces of functions and isomorphisms.

Definition 5.1. We define the following spaces for velocity and pressure

$$\begin{aligned}\mathbf{V}_f := \big\{\mathbf{v} \in \mathbf{L}^2(\vartheta_{FR}) : \boldsymbol{\nabla}\cdot \mathbf{v}^1|_{\Theta_1} \in \mathbf{L}^2(\Theta_1), \boldsymbol{\nabla}\cdot \mathbf{v}^1|_{\Theta_2 - h\widehat{\boldsymbol{k}}} \in \mathbf{L}^2(\Theta_2 - h\widehat{\boldsymbol{k}}),\\ \mathbf{v}^1\cdot\widehat{\boldsymbol{n}}|_{\Gamma_1^+}, \mathbf{v}^1\cdot\widehat{\boldsymbol{n}}|_{\Gamma_1^-} \in L^2(\Gamma_1), \mathbf{v}^2|_{\Gamma_1} \in \mathbf{L}^2(\Gamma_1)\big\},\end{aligned} \tag{5.2a}$$

$$Q_f := \{q \in L^2(\vartheta_{FR}) : q|_{\Gamma_1} \in H^1(\Gamma_1)\}. \tag{5.2b}$$

Endowed with the norms coming from the natural inner products

$$\begin{aligned}\|\mathbf{v}\|_{\mathbf{V}_f} := \big\{\|\mathbf{v}\|^2_{\mathbf{L}^2(\vartheta_{FR})} + \|\boldsymbol{\nabla}\cdot\mathbf{v}\|^2_{L^2(\vartheta_{FR})} + \|\mathbf{v}\cdot\widehat{\boldsymbol{n}}|_{\Gamma_1^+}\|^2_{L^2(\Gamma_1)}\\ + \|\mathbf{v}\cdot\widehat{\boldsymbol{n}}|_{\Gamma_1^-}\|^2_{L^2(\Gamma_1)} + \|\mathbf{v}\|^2_{\mathbf{L}^2(\Gamma_1)}\big\}^{1/2},\end{aligned} \tag{5.2c}$$

$$\|q\|_{Q_f} := \{\|q\|^2_{L^2(\vartheta_{FR})} + \|q\|^2_{H^1(\Gamma_1)}\}^{1/2}. \tag{5.2d}$$

Remark 5.2. Note that definition (5.2a) requires only $\mathbf{v} \in H_{\text{div}}(\Theta_1)$ and $\mathbf{v} \in H_{\text{div}}(\Theta_2 - h\widehat{\boldsymbol{k}})$; i.e., the divergence is a square integrable function only on these subdomains. Therefore, both normal traces $\mathbf{v}\cdot\widehat{\boldsymbol{n}}|_{\Gamma_1^+}$ and $\mathbf{v}\cdot\widehat{\boldsymbol{n}}|_{\Gamma_1^-}$ make sense in $H^{-1/2}(\Gamma_1)$, but we require the extra condition of been in $L^2(\Gamma_1)$. We do not demand the global condition $\mathbf{v} \in H_{\text{div}}(\vartheta_{FR})$ because this would imply the continuity of the normal traces across a surface; i.e., $\mathbf{u}^1\cdot\widehat{\boldsymbol{n}}|_{\Gamma_1^+} = \mathbf{u}^1\cdot\widehat{\boldsymbol{n}}|_{\Gamma_1^-}$. Such condition can not model jumps across the fissures as the normal stress balance interface (4.5g) and the limit equation (4.5f).

Next define a change of variable based on piecewise rigid translations

Definition 5.3. Let $\mathbf{x} = (\widetilde{\mathbf{x}}, x_3)$ and define the map $T : \Omega \to \mathbb{R}^3$

$$T\mathbf{x} := (\widetilde{\mathbf{x}}, x_3)\mathbb{1}_{\Theta_1}(\widetilde{\mathbf{x}}, x_3) + (\widetilde{\mathbf{x}}, x_3 - h)\mathbb{1}_{\Theta_2}(\widetilde{\mathbf{x}}, x_3) + (\widetilde{\mathbf{x}}, \zeta(\widetilde{\mathbf{x}}))\mathbb{1}_{\Theta(h,\Gamma_1)}(\widetilde{\mathbf{x}}, x_3) \tag{5.3}$$

Theorem 5.4. (i) *The application $\mathbf{v} \mapsto \mathbf{v}\circ T$ is an isometric isomorphism from $\mathbf{V}_0$ to $\mathbf{V}_f$.*
(ii) *The application $q \mapsto q\circ T$ is an isometric isomorphism from Q_0 to Q_f.*

Proof. (i) The proof is a direct application of part (i) in theorem 4.3. The only detail that needs further clarification is to observe that

$$\begin{gathered}\mathbf{v}^1\cdot\widehat{\boldsymbol{n}}|_{\Gamma_1} \mapsto (\mathbf{v}^1\circ T)\cdot\widehat{\boldsymbol{n}}|_{\Gamma_1^-},\\ \mathbf{v}^1\cdot\widehat{\boldsymbol{n}}|_{\Gamma_2} = \mathbf{v}^1\cdot\widehat{\boldsymbol{n}}|_{\Gamma_1+h} \mapsto (\mathbf{v}^1\circ T)\cdot\widehat{\boldsymbol{n}}|_{\Gamma_1^+}.\end{gathered}$$

(ii) It is a direct application of parts (i) and (ii) in theorem 4.3. □

5.2. **The lower dimensional mixed problem.** By the previous theorem, problem (4.2) is equivalent to the following mixed problem with a piecewise C^1 coupling interface: Find $p \in Q_f$ and $\mathbf{u} \in \mathbf{V}_f$ such that

$$\begin{aligned} &\int_{\vartheta} a_1 \mathbf{u}\cdot\mathbf{v} - \int_{\vartheta} p\boldsymbol{\nabla}\cdot\mathbf{v} + h\int_{\Gamma_1}(\widehat{\boldsymbol{n}}\cdot\widehat{\boldsymbol{k}})(a_2\mathbf{u}^2_\tau + (M^{T,\boldsymbol{\tau}})'\widetilde{\boldsymbol{\nabla}}p + \mathbf{g}_\tau)\cdot\mathbf{v}^2_\tau\,dS \\ &+ \alpha\int_{\Gamma_1}\big[(\mathbf{u}^1\cdot\widehat{\boldsymbol{n}}|_{\Gamma_1^+})(\mathbf{v}^1\cdot\widehat{\boldsymbol{n}}|_{\Gamma_1^+}) + (\mathbf{u}^1\cdot\widehat{\boldsymbol{n}}|_{\Gamma_1^-})(\mathbf{v}^1\cdot\widehat{\boldsymbol{n}}|_{\Gamma_1^-})\big]\,dS \\ &- \int_{\Gamma_1} p^2\big[(\mathbf{v}^1\cdot\widehat{\boldsymbol{n}}|_{\Gamma_1^+}) - (\mathbf{v}^1\cdot\widehat{\boldsymbol{n}}|_{\Gamma_1^-})\big]\,dS \\ &= -\int_{\vartheta}\mathbf{g}\cdot\mathbf{v} - \int_{\Gamma_1}\mathbf{g}_\tau\cdot\mathbf{v}^2_\tau\,dS\,, \end{aligned} \tag{5.4a}$$

$$\begin{aligned} &\int_{\vartheta}\boldsymbol{\nabla}\cdot\mathbf{u}q - h\int_{\Gamma_1}(\widehat{\boldsymbol{n}}\cdot\widehat{\boldsymbol{k}})M^{T,\boldsymbol{\tau}}\mathbf{u}^2_\tau\cdot\widetilde{\boldsymbol{\nabla}}q^2\,dS + \int_{\Gamma_1}\big[(\mathbf{u}^1\cdot\widehat{\boldsymbol{n}}|_{\Gamma_1^+}) - (\mathbf{u}^1\cdot\widehat{\boldsymbol{n}}|_{\Gamma_1^-})\big]q^2\,dS \\ &= \int_{\vartheta}F^1 q + \int_{\Gamma_1} f_\Gamma q^2\,dS \quad \text{for all } q\in Q_f,\ \mathbf{v}\in\mathbf{V}_f. \end{aligned} \tag{5.4b}$$

Finally the equivalence of problems (4.2) and (4.4) gives the well-posedness of the system above.

6. Final remarks and future work

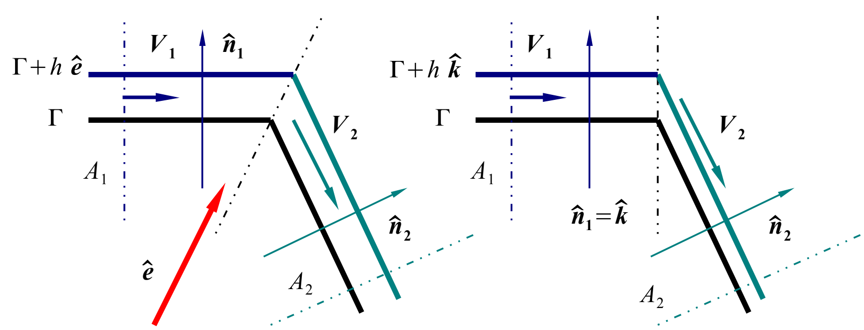

Figure 4. Translation generated fissures

(i) Giving the adequate definitions, the results can be generalized immediately to a system of multiple fissures, such as the one depicted in figure 1. The formulation presented in this work can handle large amounts of information in a remarkably efficient way. One of the main reasons is the notation introduced by Showalter in [16] for the description of function spaces.

(ii) Our results can be generalized immediately to the $\mathbb{R}^N$-setting using the same arguments presented here. The problems have analogous structure.

(iii) The approach based on analytic semigroups theory presented in section [16], can be directly applied to this case, in order to model the time dependent problem for totally fissured systems with singularities.

(iv) Although the mathematical analysis is solid, the approach used throughout the paper stops been suitable for surfaces with high gradients, such as the one depicted in the right-hand side of figure 4 where $\widehat{\boldsymbol{n}}_2 \cdot \widehat{\boldsymbol{k}} \ll \widehat{\boldsymbol{n}}_1 \cdot \widehat{\boldsymbol{k}}$. In this case, the translation in the direction $\widehat{\boldsymbol{k}}$ generates a fissure whose cross section areas can be very different from one piece to another; i.e., $A_2 \ll A_1$. Such a fissure is not realistic. On the other hand, consider a fissure such as the one depicted in the left-hand side of figure 4. Here the translation is made in the bisector vector direction

$$\widehat{\boldsymbol{e}} \equiv \frac{1}{|\frac{\widehat{\boldsymbol{n}}_1 + \widehat{\boldsymbol{n}}_2}{2}|} \frac{\widehat{\boldsymbol{n}}_1 + \widehat{\boldsymbol{n}}_2}{2}$$

This process generates a more realistic fissure.

(v) Demanding the fissures to be defined by the parallel translation of a surface in a fixed direction, is definitely a step forward with respect to previous achievements, however it is still a restrictive hypothesis for modeling the phenomenon in natural geological formations.

(vi) Setting the problem in the mixed variational formulation used here, can be easily extended to systems with fissures described by a very general type of geometry. However, the difficulty of the asymptotic analysis, for upscaling purposes, increases substantially.

(vii) Such questions will be addressed in future work by the introduction of correction factors, obtained comparing the flow energy dissipation in a real fissure and an artificial one e.g. replacing the presence of the fissure in the left-hand side of Figure 4 with the one on the right-hand side affected by a correction factor. In the same way, fissures defined by walls which are not rigid translations of a common surface, will be compared to a fissure generated by the vertical translation of an "average surface" having the same "average width".

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Fernando A. Morales
Escuela de Matemáticas, Universidad Nacional de Colombia, Sede Medellín, Colombia
E-mail address: famoralesj@unal.edu.co