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WEIGHTED PSEUDO PERIODIC SOLUTIONS OF NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we introduced and explore the properties of two sets of functions: weighted pseudo periodic functions of class r, and weighted Stepanov-like pseudo periodic functions of class r. We show the existence and uniqueness of weighted pseudo periodic solution of class r that are solutions to neutral functional differential equations. Other applications to partial differential equations and scalar reaction-diffusion equations with delay are also given.

1. Introduction

The existence periodic solutions is one of the most interesting and important topics in the qualitative theory of differential equations. Many authors have made important contributions to this theory. Recently, in [1, 19], the concept of weighted pseudo periodicity, weighted Stepanov-like pseudo periodicity, is introduced and studied, respectively. On the other hand, to study issues related to delay differential equations, Diagana [5] introduce the functions called pseudo almost periodic of class r, for more on this topic and related applications in differential equations, we refer the reader to [2, 3, 4, 10, 11].

Motivated by the above mentioned papers, in this paper, we introduce new class of functions called weighted pseudo periodic of class r, weighted Stepanov-like pseudo periodic of class r, respectively. We systematically explore the properties of these functions in general Banach space including composition results and its applications in differential equations.

In recent years, neutral functional differential equations have attracted a great deal of attention of many mathematicians due to their significance and applications in physics, mathematical biology, control theory, and so on. The general asymptotic behavior of solutions have been one of the most attracting topics in the context of neutral functional differential equations [6, 9, 12, 13, 14, 15, 18]. However, to the best of our knowledge, the studies on the weighted pseudo periodic solutions of neutral functional differential equations is quite new and an untreated topic. This is one of the key motivations of this study.

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The paper is organized as follows. In Section 2, some notations and preliminary results are presented. Next, we propose new class of functions called weighted pseudo periodic functions of class r, weighted Stepanov-like pseudo periodic functions of class r, explore the properties of these functions and establish the composition theorems. Sections 3 is devoted to the existence and uniqueness of weighted pseudo periodic solutions of class of r to neutral functional differential equations. In section 4, we present applications to partial differential equations and scalar reaction-diffusion equations with delay.

2. Preliminaries and basic results

Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ be two Banach spaces and $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ be the stand sets of natural numbers, integers, real numbers, respectively. To facilitate the discussion below, we further introduce the following notation:

- $C(\mathbb{R}, X)$ (resp. $C(\mathbb{R}\times Y, X)$): the set of continuous functions from $\mathbb{R}$ to X (resp. from $\mathbb{R}\times Y$ to X).
- $BC(\mathbb{R}, X)$ (resp. $BC(\mathbb{R}\times Y, X)$: the Banach space of bounded continuous functions from $\mathbb{R}$ to X (resp. from $\mathbb{R}\times Y$ to X) with the supremum norm;
- $L^p(\mathbb{R}, X)$: the space of all classes of equivalence (with respect to the equality almost everywhere on $\mathbb{R}$) of measurable functions $f : \mathbb{R} \to X$ such that $\|f\| \in L^p(\mathbb{R}, \mathbb{R})$;
- $L^p_{loc}(\mathbb{R}, X)$: stand for the space of all classes of equivalence of measurable functions $f : \mathbb{R} \to X$ such that the restriction of f to every bounded subinterval of $\mathbb{R}$ is in $L^p(\mathbb{R}, X)$.
- $\mathcal{C}$: the space $C([-r, 0], X)$ endowed with the sup norm $\|\psi\|_{\mathcal{C}}$ on $[-r, 0]$.
- $[D(A)]$: the domain of A when it is endowed with graph norm, $\|x\|_{[D(A)]} = \|x\| + \|Ax\|$ for each $x \in D(A)$.

2.1. **Weighted pseudo periodic of class r.** In this subsection, we introduce the new class of functions called weighted pseudo anti-periodic of class r, weighted pseudo periodic functions of class r, and investigate the properties of these functions.

Definition 2.1. A function $f \in C(\mathbb{R}, X)$ is said to be anti-periodic if there exists a $\omega \in \mathbb{R}\backslash\{0\}$ with the property that $f(t+\omega) = -f(t)$ for all $t \in \mathbb{R}$. The least positive ω with this property is called the anti-periodic of f. The collection of those functions is denoted by $P_{\omega ap}(\mathbb{R}, X)$.

Definition 2.2. A function $f \in C(\mathbb{R}, X)$ is said to be periodic if there exists a $\omega \in \mathbb{R}\backslash\{0\}$ with the property that $f(t+\omega) = f(t)$ for all $t \in \mathbb{R}$. The least positive ω with this property is called the periodic of f. The collection of those ω-periodic functions is denoted by $P_\omega(\mathbb{R}, X)$.

Note that if $f \in P_{\omega ap}(\mathbb{R}, X)$, then $f \in P_{2\omega}(\mathbb{R}, X)$.

Let U be the set of all functions $\rho : \mathbb{R} \to (0, \infty)$ which are positive and locally integrable over $\mathbb{R}$. For a given $T > 0$ and each $\rho \in U$, set

$$\mu(T, \rho) := \int_{-T}^{T} \rho(t)\mathrm{d}t.$$

Define

$$U_\infty := \{\rho \in U : \lim_{T\to\infty} \mu(T, \rho) = \infty\},$$

$$U_B := \{\rho \in U_\infty : \rho \text{ is bounded and } \inf_{x\in\mathbb{R}} \rho(x) > 0\}.$$

It is clear that $U_B \subset U_\infty \subset U$.

Definition 2.3. Let $\rho_1, \rho_2 \in U_\infty$. The function ρ_1 is said to be equivalent to ρ_2 (i.e., $\rho_1 \sim \rho_2$) if $\frac{\rho_1}{\rho_2} \in U_B$.

It is trivial to show that " $\sim$ " is a binary equivalence relation on U_∞. The equivalence class of a given weight $\rho \in U_\infty$ is denoted by $cl(\rho) = \{\varrho \in U_\infty : \rho \sim \varrho\}$. It is clear that $U_\infty = \bigcup_{\rho\in U_\infty} cl(\rho)$.

Let $\rho \in U_\infty$, $\tau \in \mathbb{R}$ be given, and defined ρ^τ by $\rho^\tau(t) = \rho(t+\tau)$ for $t \in \mathbb{R}$. Define [21]

$$U_T = \{\rho \in U_\infty : \rho \sim \rho^\tau \text{ for each } \tau \in \mathbb{R}\}.$$

It is easy to see that U_T contains many of weights, such as 1, $(1+t^2)/(2+t^2)$, e^t, and $1+|t|^n$ with $n \in \mathbb{N}$ etc. For $\rho_1, \rho_2 \in U_\infty$, define

$$WPP_0(\mathbb{R}, X, \rho_1, \rho_2) := \Big\{f \in BC(\mathbb{R}, X) : \lim_{T\to\infty} \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \|f(t)\|\rho_2(t)dt = 0\Big\},$$

$$\begin{aligned}
&WPP_0(\mathbb{R}, X, r, \rho_1, \rho_2)\\
&:= \Big\{f \in BC(\mathbb{R}, X) : \lim_{T\to\infty} \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \Big(\sup_{\theta\in[t-r,t]} \|f(\theta)\|\Big)\rho_2(t)dt = 0\Big\},
\end{aligned}$$

$$\begin{aligned}
&WPP_0(\mathbb{R}\times Y, X, r, \rho_1, \rho_2)\\
&:= \Big\{f \in BC(\mathbb{R}\times Y, X) : \lim_{T\to\infty} \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \Big(\sup_{\theta\in[t-r,t]} \|f(\theta, u)\|\Big)\rho_2(t)dt = 0\\
&\quad \text{uniformly for } u \in Y\Big\}.
\end{aligned}$$

Definition 2.4. Let $\rho_1, \rho_2 \in U_\infty$. A function $f \in C(\mathbb{R}, X)$ is called weighted pseudo anti-periodic for $\omega \in \mathbb{R}\backslash\{0\}$ if it can be decomposed as $f = g + \varphi$, where $g \in P_{\omega ap}(\mathbb{R}, X)$ and $\varphi \in WPP_0(\mathbb{R}, X, \rho_1, \rho_2)$. Denote by $WPP_{\omega ap}(\mathbb{R}, X, \rho_1, \rho_2)$ the set of such functions.

Definition 2.5. Let $\rho_1, \rho_2 \in U_\infty$. A function $f \in C(\mathbb{R}, X)$ is called weighted pseudo periodic for $\omega \in \mathbb{R}\backslash\{0\}$ if it can be decomposed as $f = g + \varphi$, where $g \in P_\omega(\mathbb{R}, X)$ and $\varphi \in WPP_0(\mathbb{R}, X, \rho_1, \rho_2)$. Denote by $WPP_\omega(\mathbb{R}, X, \rho_1, \rho_2)$ the set of such functions.

If $\rho_1 \sim \rho_2$, $WPP_{\omega ap}(\mathbb{R}, X, \rho_1, \rho_2)$, and $WPP_\omega(\mathbb{R}, X, \rho_1, \rho_2)$ coincide with the weighted pseudo anti-periodic, and the weighted pseudo periodic function respectively, introduce by [1].

Definition 2.6. Let $\rho_1, \rho_2 \in U_\infty$. A function $f \in C(\mathbb{R}, X)$ is called weighted pseudo anti-periodic of class r for $\omega \in \mathbb{R}\backslash\{0\}$ if it can be decomposed as $f = g+\varphi$, where $g \in P_{\omega ap}(\mathbb{R}, X)$ and $\varphi \in WPP_0(\mathbb{R}, X, r, \rho_1, \rho_2)$. The set of these functions is denote by $WPP_{\omega ap}(\mathbb{R}, X, r, \rho_1, \rho_2)$.

Definition 2.7. Let $\rho_1, \rho_2 \in U_\infty$. A function $f \in C(\mathbb{R}, X)$ is called weighted pseudo periodic of class r for $\omega \in \mathbb{R}\backslash\{0\}$ if it can be decomposed as $f = g+\varphi$, where $g \in P_\omega(\mathbb{R}, X)$ and $\varphi \in WPP_0(\mathbb{R}, X, r, \rho_1, \rho_2)$. Denote by $WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2)$ the set of such functions.

Remark 2.8. If $r = 0$, then the weighted pseudo anti-periodic function of class r reduces to the weighted pseudo anti-periodic function, the weighted pseudo periodic function of class r reduces to the weighted pseudo periodic function. That is, $WPP_{\omega ap}(\mathbb{R}, X, 0, \rho_1, \rho_2) = WPP_{\omega ap}(\mathbb{R}, X, \rho_1, \rho_2)$, and $WPP_\omega(\mathbb{R}, X, 0, \rho_1, \rho_2) = WPP_\omega(\mathbb{R}, X, \rho_1, \rho_2)$.

Next, we show some properties of the space $WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2)$. Similarly results hold for $WPP_{\omega ap}(\mathbb{R}, X, r, \rho_1, \rho_2)$.

Lemma 2.9. *Let $f \in BC(\mathbb{R}, X)$, then $f \in WPP_0(\mathbb{R}, X, r, \rho_1, \rho_2)$, $\rho_1, \rho_2 \in U_\infty$, $\sup_{T>0} \frac{\mu(T,\rho_2)}{\mu(T,\rho_1)} < \infty$ if and only if for every $\varepsilon > 0$,*

$$\lim_{T\to\infty} \frac{1}{\mu(T, \rho_1)} \int_{M(T,\varepsilon,f)} \rho_2(t)dt = 0,$$

where $M(T, \varepsilon, f) := \{t \in [-T, T] : \sup_{\theta\in[t-r,t]} \|f(\theta)\| \geq \varepsilon\}$.

Proof. The proof is similar as [10].

Sufficiency: From the statement of the lemma it is clear that for any $\varepsilon > 0$, there exists $T_0 > 0$ such that for $T > T_0$,

$$\frac{1}{\mu(T, \rho_1)} \int_{M(T,\varepsilon,f)} \rho_2(t)dt < \frac{\varepsilon}{\|f\|}.$$

Then

$$\begin{aligned}
&\frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \Big(\sup_{\theta\in[t-r,t]} \|f(\theta)\|\Big)\rho_2(t)dt \\
&= \frac{1}{\mu(T, \rho_1)} \int_{M(T,\varepsilon,f)} \Big(\sup_{\theta\in[t-r,t]} \|f(\theta)\|\Big)\rho_2(t)dt \\
&\quad + \frac{1}{\mu(T, \rho_1)} \int_{[-T,T]\setminus M(T,\varepsilon,f)} \Big(\sup_{\theta\in[t-r,t]} \|f(\theta)\|\Big)\rho_2(t)dt \\
&\leq \frac{\|f\|}{\mu(T, \rho_1)} \int_{M(T,\varepsilon,f)} \rho_2(t)dt + \frac{\varepsilon}{\mu(T, \rho_1)} \int_{-T}^{T} \rho_2(t)dt \\
&\leq \varepsilon + \sup_{T>0} \frac{\mu(T, \rho_2)}{\mu(T, \rho_1)}\varepsilon,
\end{aligned}$$

so

$$\lim_{T\to\infty} \frac{1}{\mu(T, \rho_1)} \int_{-T}^{T} \Big(\sup_{\theta\in[t-r,t]} \|f(\theta)\|\Big)\rho_2(t)dt = 0.$$

That is, $f \in WPP_0(\mathbb{R}, X, r, \rho_1, \rho_2)$.

Necessity: Suppose on the contrary that there exists $\varepsilon_0 > 0$ such that

$$\frac{1}{\mu(T, \rho_1)} \int_{M(T,\varepsilon_0,f)} \rho_2(t)dt$$

does not converge to 0 as $T \to \infty$. Then there exists $\delta > 0$ such that for each n,

$$\frac{1}{\mu(T_n, \rho_1)} \int_{M(T_n,\varepsilon_0,f)} \rho_2(t)dt \geq \delta \quad \text{for some } T_n \geq n.$$

Then

$$\frac{1}{\mu(T_n, \rho_1)} \int_{-T_n}^{T_n} \rho_2(t)\Big(\sup_{\theta\in[t-r,t]} \|f(\theta)\|\Big)dt$$

$$
\begin{aligned}
&\ge \frac{1}{\mu(T_n,\rho_1)}\int_{M(T_n,\varepsilon_0,f)}\rho_2(t)\Big(\sup_{\theta\in[t-r,t]}\|f(\theta)\|\Big)dt\\
&\ge \frac{\varepsilon_0}{\mu(T_n,\rho_1)}\int_{M(T_n,\varepsilon_0,f)}\rho_2(t)dt \ge \varepsilon_0\delta,
\end{aligned}
$$

which contradicts the fact that $f \in WPP_0(\mathbb{R}, X, r, \rho_1, \rho_2)$, and the proof is complete. □

Let

$$
WPP_0(\mathbb{R},\mathbb{R}^+,r,\rho_1,\rho_2) = \{f \in WPP_0(\mathbb{R},\mathbb{R},r,\rho_1,\rho_2) : f(t) \ge 0,\ \forall t \in \mathbb{R}\}.
$$

Lemma 2.10. *Let $\alpha > 0$, then $f \in WPP_0(\mathbb{R},\mathbb{R}^+,r,\rho_1,\rho_2)$ if and only if $f^\alpha \in WPP_0(\mathbb{R},\mathbb{R}^+,r,\rho_1,\rho_2)$, where $f^\alpha(t) = [f(t)]^\alpha$, $\rho_1,\rho_2 \in U_\infty$, $\sup_{T>0}\frac{\mu(T,\rho_2)}{\mu(T,\rho_1)} < \infty$.*

Proof. By Lemma 2.9, $f \in WPP_0(\mathbb{R},\mathbb{R}^+,r,\rho_1,\rho_2)$ if and only if for every $\varepsilon > 0$,

$$
\lim_{T\to\infty}\frac{1}{\mu(T,\rho_1)}\int_{M(T,\varepsilon,f)}\rho_2(t)dt = 0,
$$

where $M(T,\varepsilon,f) := \{t \in [-T,T] : \sup_{\theta\in[t-r,t]} f(\theta) \ge \varepsilon\}$. It is equivalent to for every $\varepsilon > 0$,

$$
\lim_{T\to\infty}\frac{1}{\mu(T,\rho_1)}\int_{M(T,\varepsilon,f^\alpha)}\rho_2(t)dt = 0,
$$

where

$$
M(T,\varepsilon,f^\alpha) := \{t \in [-T,T] : \sup_{\theta\in[t-r,t]} f^\alpha(\theta) \ge \varepsilon\}.
$$

So $f^\alpha \in WPP_0(\mathbb{R},\mathbb{R}^+,r,\rho_1,\rho_2)$. □

Lemma 2.11. *Let $\varphi_n \to \varphi$ uniformly on $\mathbb{R}$ where each $\varphi_n \in WPP_0(\mathbb{R},X,r,\rho_1,\rho_2)$, $\rho_1,\rho_2 \in U_\infty$, if $\sup_{T>0}\frac{\mu(T,\rho_2)}{\mu(T,\rho_1)} < \infty$, then $\varphi \in WPP_0(\mathbb{R},X,r,\rho_1,\rho_2)$.*

Proof. For $T > 0$,

$$
\begin{aligned}
&\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\Big(\sup_{\theta\in[t-r,t]}\|\varphi(\theta)\|\Big)\rho_2(t)dt\\
&\le \frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\Big(\sup_{\theta\in[t-r,t]}\|\varphi_n(\theta)-\varphi(\theta)\|\Big)\rho_2(t)dt\\
&\quad + \frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\Big(\sup_{\theta\in[t-r,t]}\|\varphi_n(\theta)\|\Big)\rho_2(t)dt\\
&\le \frac{\mu(T,\rho_2)}{\mu(T,\rho_1)}\|\varphi_n-\varphi\| + \frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\Big(\sup_{\theta\in[t-r,t]}\|\varphi_n(\theta)\|\Big)\rho_2(t)dt\\
&\le \sup_{T>0}\frac{\mu(T,\rho_2)}{\mu(T,\rho_1)}\|\varphi_n-\varphi\| + \frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\Big(\sup_{\theta\in[t-r,t]}\|\varphi_n(\theta)\|\Big)\rho_2(t)dt,
\end{aligned}
$$

Let $T \to \infty$ and then $n \to \infty$ in the above inequality, it follows that $\varphi \in WPP_0(\mathbb{R},X,r,\rho_1,\rho_2)$. □

By carrying out similar arguments as those in the proof of [21, Lemma 4.1], we conclude the following.

Lemma 2.12. *Let $\rho_1,\rho_2 \in U_T$, $\varphi \in WPP_0(\mathbb{R},X,r,\rho_1,\rho_2)$, then $\varphi(\cdot-\tau)$ belongs to $WPP_0(\mathbb{R},X,r,\rho_1,\rho_2)$ for $\tau\in\mathbb{R}$.*

Using similar ideas as in [7, 8], one can easily show the following result.

Lemma 2.13. *If $\rho_1,\rho_2 \in U_T$ and $\inf_{T>0}\frac{\mu(T,\rho_2)}{\mu(T,\rho_1)}=\delta_0>0$, then the decomposition of weighted pseudo periodic function of class r is unique.*

By Lemma 2.13, it is obvious that $(WPP_\omega(\mathbb{R},X,r,\rho_1,\rho_2),\|\cdot\|)$, $\rho_1,\rho_2\in U_T$ and $\inf_{T>0}\frac{\mu(T,\rho_2)}{\mu(T,\rho_1)}=\delta_0>0$ is a Banach space when endowed with the sup norm.

Lemma 2.14. *Let $\rho_1,\rho_2 \in U_T$, $u\in WPP_\omega(\mathbb{R},X,r,\rho_1,\rho_2)$, then u_t belongs to $WPP_\omega(\mathbb{R},\mathcal{C},r,\rho_1,\rho_2)$.*

Proof. Suppose that $u=\alpha+\beta$, where $\alpha\in P_\omega(\mathbb{R},X)$ and $\beta\in WPP_0(\mathbb{R},X,r,\rho_1,\rho_2)$, then $u_t=\alpha_t+\beta_t$ and $\alpha_t\in P_\omega(\mathbb{R},\mathcal{C},r,\rho_1,\rho_2)$. On the other hand, for $T>0$, we see that

$$
\begin{aligned}
&\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\Big[\sup_{\theta\in[t-r,t]}\Big(\sup_{\tau\in[-r,0]}\|\beta(\theta+\tau)\|\Big)\Big]\rho_2(t)dt\\
&\le \frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\Big(\sup_{\theta\in[t-2r,t]}\|\beta(\theta)\|\Big)\rho_2(t)dt\\
&\le \frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\Big(\sup_{\theta\in[t-2r,t-r]}\|\beta(\theta)\|+\sup_{\theta\in[t-r,t]}\|\beta(\theta)\|\Big)\rho_2(t)dt\\
&\le \frac{1}{\mu(T,\rho_1)}\int_{-T-r}^{T-r}\Big(\sup_{\theta\in[t-r,t]}\|\beta(\theta)\|\Big)\rho_2(t+r)dt\\
&\quad+\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\Big(\sup_{\theta\in[t-r,t]}\|\beta(\theta)\|\Big)\rho_2(t)dt\\
&\le \frac{\mu(T+r,\rho_1)}{\mu(T,\rho_1)}\frac{1}{\mu(T+r,\rho_1)}\int_{-T-r}^{T+r}\Big(\sup_{\theta\in[t-r,t]}\|\beta(\theta)\|\Big)\rho_2(t)\frac{\rho_2(t+r)}{\rho_2(t)}dt\\
&\quad+\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\Big(\sup_{\theta\in[t-r,t]}\|\beta(\theta)\|\Big)\rho_2(t)dt.
\end{aligned}
$$

Since $\rho_1,\rho_2\in U_T$ implies that there exists $\eta>0$ such that $\rho_1(t+r)/\rho_1(t)\le\eta$, $\rho_1(t-r)/\rho_1(t)\le\eta$, $\rho_2(t+r)/\rho_2(t)\le\eta$. For $T>r$,

$$
\begin{aligned}
\mu(T+r,\rho_1)&=\int_{-T-r}^{T-r}\rho_1(t)dt+\int_{T-r}^{T+r}\rho_1(t)dt\le\int_{-T-r}^{T-r}\rho_1(t)dt+\int_{-T+r}^{T+r}\rho_1(t)dt\\
&=\int_{-T}^{T}\rho_1(t-r)dt+\int_{-T}^{T}\rho_1(t+r)dt\le 2\eta\mu(T,\rho_1),
\end{aligned}
$$

then

$$
\begin{aligned}
&\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\Big[\sup_{\theta\in[t-r,t]}\Big(\sup_{\tau\in[-r,0]}\|\beta(\theta+\tau)\|\Big)\Big]\rho_2(t)dt\\
&\le\frac{2\eta^2}{\mu(T+r,\rho_1)}\int_{-T-r}^{T+r}\Big(\sup_{\theta\in[t-r,t]}\|\beta(\theta)\|\Big)\rho_2(t)dt\\
&\quad+\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\Big(\sup_{\theta\in[t-r,t]}\|\beta(\theta)\|\Big)\rho_2(t)dt.
\end{aligned}
$$

Note that $\beta \in WPP_0(\mathbb{R}, X, r, \rho_1, \rho_2), \rho_1, \rho_2 \in U_T$, then $\beta_t \in WPP_0(\mathbb{R}, \mathcal{C}, r, \rho_1, \rho_2)$. Therefore, $u_t \in WPP_\omega(\mathbb{R}, \mathcal{C}, r, \rho_1, \rho_2)$. □

Similarly as [3, Theorem 3.9], we have the following composition theorem for weighted pseudo periodic function of class r.

Theorem 2.15. *Assume that $\rho_1, \rho_2 \in U_\infty, r \geq 0, f \in WPP_\omega(\mathbb{R} \times Y, X, r, \rho_1, \rho_2)$ and there exists a function $L_f : \mathbb{R} \to [0, +\infty)$ satisfying*

- (A1) $\|f(t,u) - f(t,v)\| \leq L_f(t)\|u - v\|$, *for all* $t \in \mathbb{R}$, $u, v \in Y$;
- (A2) $\limsup_{T\to\infty} \frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \Big(\sup_{\theta\in[t-r,t]} L_f(\theta)\Big)\rho_2(t)dt < \infty$;
- (A3) $\lim_{T\to\infty} \frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \Big(\sup_{\theta\in[t-r,t]} L_f(\theta)\Big)\xi(t)\rho_2(t)dt = 0$ *for each function* $\xi \in WPP_0(\mathbb{R}, \mathbb{R}, \rho_1, \rho_2)$.

Then $f(\cdot, h(\cdot)) \in WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2)$ if $h \in WPP_\omega(\mathbb{R}, Y, r, \rho_1, \rho_2)$.

Remark 2.16. *Note that (A2) and (A3) are verified by many functions. Concrete examples include constant functions, and functions in $WPP_\omega(\mathbb{R}, \mathbb{R}, r, \rho_1, \rho_2)$.*

2.2. **Weighted Stepanov-like pseudo periodic of class r.** In this subsection, we introduce the new class of functions called weighted S^p-pseudo anti-periodic of class r, weighted S^p-pseudo periodic functions of class r, and investigate the properties of these functions.

Let $p \in [1, \infty)$. The space $BS^p(\mathbb{R}, X)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \to X$ such that $f^b \in L^\infty(\mathbb{R}, L^p([0,1]; X))$, where f^b is the Bochner transform of f defined by $f^b(t,s) := f(t+s)$, $t \in \mathbb{R}$, $s \in [0,1]$. $BS^p(\mathbb{R}, X)$ is a Banach space with the norm [17]

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}} \Big(\int_t^{t+1} \|f(\tau)\|^p \mathrm{d}\tau\Big)^{1/p}.$$

It is clear that $L^p(\mathbb{R}, X) \subset BS^p(\mathbb{R}, X) \subset L^p_{loc}(\mathbb{R}, X)$ and $BS^p(\mathbb{R}, X) \subset BS^q(\mathbb{R}, X)$ for $p \geq q \geq 1$.

For $\rho_1, \rho_2 \in U_\infty$, define the weighted ergodic space in $BS^p(\mathbb{R}, X)$

$$\begin{aligned} S^pWPP_0(\mathbb{R}, X, r, \rho_1, \rho_2) := \Big\{ & f \in BS^p(\mathbb{R}, X) : \lim_{T\to\infty} \frac{1}{\mu(T,\rho_1)} \\ & \times \int_{-T}^{T} \rho_2(t)\Big(\sup_{\theta\in[t-r,t]} \Big(\int_\theta^{\theta+1} \|f(s)\|^p ds\Big)^{1/p}\Big)dt = 0\Big\}. \end{aligned}$$

Definition 2.17. Let $\rho_1, \rho_2 \in U_\infty$. A function $f \in BS^p(\mathbb{R}, X)$ is said to be weighted Stepanov-like pseudo anti-periodic of class r (or weighted S^p-pseudo anti-periodic of class r) if there exists $\varphi \in S^pWPP_0(\mathbb{R}, X, r, \rho_1, \rho_2)$ such that the function $g = f - \varphi$ satisfies $g(t+\omega) + g(t) = 0$ *a.e.* $t \in \mathbb{R}$. The collection of such functions is denoted by $S^pWPP_{\omega ap}(\mathbb{R}, X, r, \rho_1, \rho_2)$

Definition 2.18. Let $\rho_1, \rho_2 \in U_\infty$. A function $f \in BS^p(\mathbb{R}, X)$ is said to be weighted Stepanov-like pseudo periodic of class r (or weighted S^p-pseudo periodic of class r) if there exists $\varphi \in S^pWPP_0(\mathbb{R}, X, r, \rho_1, \rho_2)$ such that the function $g = f - \varphi$ satisfies $g(t+\omega) - g(t) = 0$ a. e. $t \in \mathbb{R}$. Denote by $S^pWPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2)$ the collection of such functions.

Next, we show some properties of the space $S^pWPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2)$. Similarly results hold for $S^pWPP_{\omega ap}(\mathbb{R}, X, r, \rho_1, \rho_2)$.

Lemma 2.19. *Let $\rho_1,\rho_2\in U_T$, then*
$$WPP_\omega(\mathbb{R},X,r,\rho_1,\rho_2)\subset S^pWPP_\omega(\mathbb{R},X,r,\rho_1,\rho_2).$$

Proof. If $f\in WPP_\omega(\mathbb{R},X,r,\rho_1,\rho_2)$, let $f=f_1+f_2$, where $f_1\in P_\omega(\mathbb{R},X)$, $f_2\in WPP_0(\mathbb{R},X,r,\rho_1,\rho_2)$. Then $\|f_2(\cdot)\|\in WPP_0(\mathbb{R},\mathbb{R}^+,r,\rho_1,\rho_2)$. By Lemma 2.10, $\|f_2(\cdot)\|^p\in WPP_0(\mathbb{R},\mathbb{R}^+,r,\rho_1,\rho_2)$. Note that $\|f_2(\cdot+\sigma)\|^p\in WPP_0(\mathbb{R},\mathbb{R}^+,r,\rho_1,\rho_2)$ for each $\sigma\in[0,1]$, then
$$\lim_{T\to\infty}\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\Big(\sup_{\theta\in[t-r,t]}\|f_2(\theta+\sigma)\|^p\Big)\rho_2(t)dt=0.$$
by Lebesgue's dominated convergence theorem, one has
$$\int_0^1\Big(\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\Big(\sup_{\theta\in[t-r,t]}\|f_2(\theta+\sigma)\|^p\Big)\rho_2(t)dt\Big)d\sigma\to 0,\quad T\to\infty,$$
i.e.,
$$\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\rho_2(t)\Big(\int_0^1\sup_{\theta\in[t-r,t]}\|f_2(\theta+\sigma)\|^pd\sigma\Big)dt\to 0,\quad T\to\infty,$$
which means that
$$\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\rho_2(t)\sup_{\theta\in[t-r,t]}\Big(\int_0^1\|f_2(\theta+\sigma)\|^pd\sigma\Big)dt\to 0,\quad T\to\infty;$$
i.e.,
$$\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\rho_2(t)\sup_{\theta\in[t-r,t]}\Big(h_2(\theta)\Big)dt\to 0,\quad T\to\infty,$$
so $h_2\in WPP_0(\mathbb{R},\mathbb{R}^+,r,\rho_1,\rho_2)$, where
$$h_2(t)=\int_0^1\|f_2(t+\sigma)\|^pd\sigma,\quad t\in\mathbb{R}.$$
By Lemma 2.10, $h_2^{1/p}\in WPP_0(\mathbb{R},\mathbb{R}^+,r,\rho_1,\rho_2)$; i.e.,
$$\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\rho_2(t)\sup_{\theta\in[t-r,t]}\Big(\int_0^1\|f_2(\theta+\sigma)\|^pd\sigma\Big)^{1/p}dt\to 0,\quad T\to\infty,$$
which means that $f_2\in S^pWPP_0(\mathbb{R},X,r,\rho_1,\rho_2)$, then $f\in S^pWPP_\omega(\mathbb{R},X,r,\rho_1,\rho_2)$. The proof is complete. $\square$

Theorem 2.20. *Assume $\rho_1,\rho_2\in U_\infty$, $f=f_1+f_2\in S^pWPP_\omega(\mathbb{R}\times Y,X,r,\rho_1,\rho_2)$ with $f_2\in S^pWPP_0(\mathbb{R}\times Y,X,r,\rho_1,\rho_2)$, $f_1(t+\omega,u)-f_1(t,u)=0$ a. e. $t\in\mathbb{R}$, $u\in X$, and there exists a function $L_f:\mathbb{R}\to[0,+\infty)$ satisfying:*

- (A1') $\Big(\int_t^{t+1}\|f(s,u)-f(s,v)\|^pds\Big)^{1/p}\le L_f(t)\|u-v\|$, *for all $t\in\mathbb{R}$, $u,v\in Y$;*
- (A2') $\limsup_{T\to\infty}\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\Big(\sup_{\theta\in[t-r,t]}L_f(\theta)\Big)\rho_2(t)dt<\infty$;
- (A3') $\lim_{T\to\infty}\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\Big(\sup_{\theta\in[t-r,t]}L_f(\theta)\Big)\xi(t)\rho_2(t)dt=0$ *for each function $\xi^b\in WPP_0(\mathbb{R},L^p([0,1],\mathbb{R}),\rho_1,\rho_2)$;*
- (A4') *f_1 is uniform continuous on bounded set $K'\subset Y$ for any $t\in\mathbb{R}$.*

Then $f(\cdot,h(\cdot))\in S^pWPP_\omega(\mathbb{R},X,r,\rho_1,\rho_2)$ if $h\in S^pWPP_\omega(\mathbb{R},Y,r,\rho_1,\rho_2)$.

The proof of the above theorem is similar to the proof of [20, Theorem 3.2] and it is omitted here. It is not difficult to see that Theorem 2.15, Theorem 2.20 hold for $WPP_{\omega ap}(\mathbb{R}, X, r, \rho_1, \rho_2)$, $S^pWPP_{\omega ap}(\mathbb{R}, X, r, \rho_1, \rho_2)$ respectively.

3. Neutral functional differential equations

As an application, the main goal of this section is to establish sufficient criteria for the existence, uniqueness of the weighted pseudo periodic solution to a class of neutral functional differential equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}[u(t) + f(t, u_t)] = Au(t) + g(t, u_t), \quad t \in \mathbb{R}, \tag{3.1}$$

where A is the infinitesimal generator of a semigroup of linear operators on X, $u_t \in \mathcal{C}$ is defined by $u_t(\theta) = u(t + \theta)$ for $\theta \in [-r, 0]$, where r is a nonnegative constant.

First, we recall the definition of the so called exponential dichotomy of a semigroup.

Definition 3.1. [16] A semigroup $(T(t))_{t\geq 0}$ is said to be exponential dichotomy if there exist projection P and constants $M, \delta > 0$ such that each $T(t)$ commutes with P, $KerP$ is invariant with respect to $T(t)$, $T(t) : ImQ \to ImQ$ is invertible and

$$\|T(t)Px\| \leq Me^{-\delta t}\|x\| \quad \text{for } t \geq 0, \tag{3.2}$$

$$\|T(t)Qx\| \leq Me^{\delta t}\|x\| \quad \text{for } t \leq 0, \tag{3.3}$$

where $Q := I - P$ and $T(t) := (T(-t))^{-1}$ for $t \leq 0$.

To study (3.1), we make the following assumptions:

(H1) The operator $A : D(A) \subset X \to X$ is the infinitesimal generator of semigroup $(T(t))_{t\geq 0}$ which has an exponential dichotomy.

(H2) $f \in WPP_\omega(\mathbb{R} \times \mathcal{C}, X, r, \rho_1, \rho_2)$, f is $D(A)$-valued, there exists a positive constant L_f such that

$$|f(t, \psi_1) - f(t, \psi_2)\|_{[D(A)]} \leq L_f\|\psi_1 - \psi_2\|_{\mathcal{C}}, \quad \text{for all } t \in \mathbb{R},\ \psi_i \in \mathcal{C},\ i = 1, 2.$$

(H3) $g \in WPP_\omega(\mathbb{R} \times \mathcal{C}, X, r, \rho_1, \rho_2)$ and there exists a continuous function $L_g : \mathbb{R} \to \mathbb{R}^+$ such that

$$\|g(t, \psi_1) - g(t, \psi_2)\| \leq L_g(t)\|\psi_1 - \psi_2\|_{\mathcal{C}}, \quad \text{for all } t \in \mathbb{R},\ \psi_i \in \mathcal{C},\ i = 1, 2.$$

(H3') $g = g_1 + g_2 \in S^pWPP_\omega(\mathbb{R} \times \mathcal{C}, X, r, \rho_1, \rho_2)$ with $g_2 \in S^pWPP_0(\mathbb{R} \times \mathcal{C}, X, r, \rho_1, \rho_2)$, $g_1(t + \omega, \psi) - g(t, \psi) = 0$ *a.e.* $t \in \mathbb{R}$, $\psi \in \mathcal{C}$, satisfying
 (i) g_1 is uniform continuous on bounded set $K \subset \mathcal{C}$ for any $t \in \mathbb{R}$.
 (ii) there exists a positive constant L_g such that

$$\Big(\int_t^{t+1} \|g(t, \psi_1) - g(t, \psi_2)\|^p ds\Big)^{1/p} \leq L_g\|\psi_1 - \psi_2\|_{\mathcal{C}}, \quad \text{for all } t \in \mathbb{R}, \psi_i \in \mathcal{C}, i = 1, 2.$$

(H4) $\rho_1, \rho_2 \in U_T$, $\inf_{T>0} \frac{\mu(T,\rho_2)}{\mu(T,\rho_1)} = \delta_0 > 0$ and $\sup_{T>0} \frac{\mu(T,\rho_2)}{\mu(T,\rho_1)} < \infty$.

Definition 3.2 ([6])**.** A continuous function u is said to be a mild solution of (3.1) provided that the function $s \to AT(t-s)Pf(s,u_s)$ is integrable on $(-\infty,t)$, $s \to AT(t-s)Qf(s,u_s)$ is integrable on (t,∞) for $t \in \mathbb{R}$ and

$$\begin{aligned} u(t) = -f(t,u_t) - \int_{-\infty}^{t} AT(t-s)Pf(s,u_s)ds + \int_{t}^{\infty} AT(t-s)Qf(s,u_s)ds \\ + \int_{-\infty}^{t} T(t-s)Pg(s,u_s)ds - \int_{t}^{\infty} T(t-s)Qg(s,u_s)ds, \quad t \in \mathbb{R}. \end{aligned}$$

Lemma 3.3. *Assume that* (H1), (H2), (H4) *hold, if* $u \in WPP_\omega(\mathbb{R},X,r,\rho_1,\rho_2)$, *then*

$$\begin{aligned} (\Lambda_1 f)(t) &= \int_{-\infty}^{t} AT(t-s)Pf(s,u_s)ds \in WPP_\omega(\mathbb{R},X,r,\rho_1,\rho_2). \\ (\Lambda_2 f)(t) &= \int_{t}^{\infty} AT(t-s)Qf(s,u_s)ds \in WPP_\omega(\mathbb{R},X,r,\rho_1,\rho_2). \end{aligned}$$

Proof. By Lemma 2.14 and Theorem 2.15, $f(s,u_s) := h(s) \in WPP_\omega(\mathbb{R},X,r,\rho_1,\rho_2)$ and h is $D(A)$-valued. Let $h(s) = h_1(s) + h_2(s)$ where $h_1 \in P_\omega(\mathbb{R},X)$ and $h_2 \in WPP_0(\mathbb{R},X,r,\rho_1,\rho_2)$. Then

$$\begin{aligned} (\Lambda_1 f)(t) &= \int_{-\infty}^{t} AT(t-s)Ph_1(s)ds + \int_{-\infty}^{t} AT(t-s)Ph_2(s)ds \\ &:= (\Lambda_{11}h_1)(t) + (\Lambda_{12}h_2)(t), \end{aligned}$$

where

$$(\Lambda_{11}h_1)(t) = \int_{-\infty}^{t} AT(t-s)Ph_1(s)ds, \quad (\Lambda_{12}h_2)(t) = \int_{-\infty}^{t} AT(t-s)Ph_2(s)ds,$$

for $t \in \mathbb{R}$. From $h_1 \in P_\omega(\mathbb{R},X)$,

$$(\Lambda_{11}h_1)(t+\omega) = \int_{-\infty}^{t+\omega} AT(t+\omega-s)Ph_1(s)ds = (\Lambda_{11}h_1)(t);$$

then $\Lambda_{11}h_1 \in P_\omega(\mathbb{R},X)$.

Next, we show that $\Lambda_{12}h_2 \in WPP_0(\mathbb{R},X,r,\rho_1,\rho_2)$; that is,

$$\lim_{T\to\infty} \frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \Big(\sup_{\theta\in[t-r,t]} \|(\Lambda_{12}h_2)(\theta)\| \Big) \rho_2(t)dt = 0.$$

In fact, for $T > 0$, one has

$$\begin{aligned} &\frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \Big(\sup_{\theta\in[t-r,t]} \|(\Lambda_{12}h_2)(\theta)\| \Big) \rho_2(t)dt \\ &= \frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \Big(\sup_{\theta\in[t-r,t]} \| \int_{-\infty}^{\theta} AT(\theta-s)Ph_2(s)ds \| \Big) \rho_2(t)dt \\ &= \frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \Big(\sup_{\theta\in[t-r,t]} \| \int_{0}^{\infty} AT(s)Ph_2(\theta-s)ds \| \Big) \rho_2(t)dt \\ &\le \frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \Big(\sup_{\theta\in[t-r,t]} \int_{0}^{\infty} Me^{-\delta s} \|Ah_2(\theta-s)\| ds \Big) \rho_2(t)dt \\ &\le \frac{1}{\mu(T,\rho_1)} \int_{-T}^{T} \Big(\sup_{\theta\in[t-r,t]} \int_{0}^{\infty} Me^{-\delta s} \|h_2(\theta-s)\|_{[D(A)]} ds \Big) \rho_2(t)dt \end{aligned}$$

$$\leq \int_0^\infty Me^{-\delta s}\Phi_T(s)ds,$$

where

$$\Phi_T(s) = \frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\Big(\sup_{\theta\in[t-r,t]}\|h_2(\theta-s)\|_{[D(A)]}\Big)\rho_2(t)dt.$$

Since $\rho_1,\rho_2\in U_T$, by Lemma 2.12, we have $h_2(\cdot-s)\in WPP_0(\mathbb{R},[D(A)],r,\rho_1,\rho_2)$ for each $s\in\mathbb{R}$; hence $\lim_{T\to\infty}\Phi_T(s)=0$ for all $s\in\mathbb{R}$. Then $\Lambda_{12}h_2$ belongs to $WPP_0(\mathbb{R},X,r,\rho_1,\rho_2)$ by using the Lebesgue dominated convergence theorem, so $\Lambda_1 f\in WPP_\omega(\mathbb{R},X,r,\rho_1,\rho_2)$.

The proof of $\Lambda_2 f$ is similar to that of $\Lambda_1 f$, one makes use of (3.3) rather than (3.2). This completes the proof. $\square$

Lemma 3.4. *Assume that* (H1), (H4) *hold, if $\phi\in S^pWPP_\omega(\mathbb{R},X,r,\rho_1,\rho_2)$, then*

$$(\Gamma_1\phi)(t)=\int_{-\infty}^{t}T(t-s)P\phi(s)ds\in WPP_\omega(\mathbb{R},X,r,\rho_1,\rho_2),$$

$$(\Gamma_2\phi)(t)=\int_{t}^{\infty}T(t-s)Q\phi(s)ds\in WPP_\omega(\mathbb{R},X,r,\rho_1,\rho_2).$$

Proof. By $\phi\in S^pWPP_\omega(\mathbb{R},X,r,\rho_1,\rho_2)$, we let $\phi(s)=\phi_1(s)+\phi_2(s)$, where $\phi_2\in S^pWPP_0(\mathbb{R},X,r,\rho_1,\rho_2)$ and $\phi_1(t+\omega)-\phi_1(t)=0$ a.e. $t\in\mathbb{R}$, then

$$(\Gamma_1\phi)(t)=\int_{-\infty}^{t}T(t-s)P\phi_1(s)ds+\int_{-\infty}^{t}T(t-s)P\phi_2(s)ds:=(\Gamma_{11}\phi_1)(t)+(\Gamma_{12}\phi_2)(t).$$

First, we show that $\Gamma_{12}\phi_2\in WPP_0(\mathbb{R},X,r,\rho_1,\rho_2)$. Consider the integrals

$$Y_n(t)=\int_{t-n}^{t-n+1}T(t-s)P\phi_2(s)ds.$$

Fix $n\in\mathbb{N}$ and $t\in\mathbb{R}$, we have

$$\begin{aligned}
\|Y_n(t+h)-Y_n(t)\| &\leq \int_{n-1}^{n}\|T(s)P(\phi_2(t+h-s)-\phi_2(t-s))\|ds\\
&\leq M\int_{t-n}^{t-n+1}\|\phi_2(s+h)-\phi_2(s)\|ds\\
&\leq M\Big(\int_{t-n}^{t-n+1}\|\phi_2(s+h)-\phi_2(s)\|^pds\Big)^{1/p}.
\end{aligned}$$

In view of $\phi_2\in L^p_{loc}(\mathbb{R},X)$, we get

$$\lim_{h\to0}\int_{t-n}^{t-n+1}\|\phi_2(s+h)-\phi_2(s)\|^pds=0,$$

which yields $\lim_{h\to0}\|Y_n(t+h)-Y_n(t)\|=0$. This means that $Y_n(t)$ is continuous.

By Hölder's inequality, one has

$$\begin{aligned}
\|Y_n(t)\| &\leq \int_{n-1}^{n}\|T(s)P\phi_2(t-s)\|ds\\
&\leq \int_{n-1}^{n}Me^{-\delta s}\|\phi_2(t-s)\|ds\\
&\leq Me^{-\delta(n-1)}\int_{n-1}^{n}\|\phi_2(t-s)\|ds
\end{aligned}$$

$$
\begin{aligned}
&\le Me^{-\delta(n-1)}\int_{t-n}^{t-n+1}\|\phi_2(s)\|ds\\
&\le Me^{-\delta(n-1)}\Big(\int_{t-n}^{t-n+1}\|\phi_2(s)\|^p ds\Big)^{1/p}\\
&\le Me^{-\delta(n-1)}\|\phi_2\|_{S^p}.
\end{aligned}
$$

Since

$$
\sum_{n=1}^{\infty} Me^{-\delta(n-1)}\|\phi_2\|_{S^p} \le \frac{M}{1-e^{-\delta}}\|\phi_2\|_{S^p} < +\infty,
$$

it follows that $\sum_{n=1}^{\infty} Y_n(t)$ converges uniformly on $\mathbb{R}$. Let $Y(t)=\sum_{n=1}^{\infty} Y_n(t)$ for $t\in\mathbb{R}$. Then

$$
Y(t)=(\Gamma_{12}\phi_2)(t)=\int_{-\infty}^{t} T(t-s)P\phi_2(s)ds, \quad t\in\mathbb{R}.
$$

It is obvious that $Y(t)\in BC(\mathbb{R},X)$. So, we only need to show that

$$
\lim_{T\to\infty}\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\rho_2(t)\Big(\sup_{\theta\in[t-r,t]}\|Y(\theta)\|\Big)dt=0. \tag{3.4}
$$

In fact, by Hölder inequality,

$$
\begin{aligned}
\|Y_n(t)\| &\le \int_{n-1}^{n} Me^{-\delta s}\|\phi_2(t-s)\|ds\\
&\le \widetilde{M}\int_{t-n}^{t-n+1}\|\phi_2(s)\|ds\\
&\le \widetilde{M}\Big(\int_{t-n}^{t-n+1}\|\phi_2(s)\|^p ds\Big)^{1/p},
\end{aligned}
$$

for some constant $\widetilde{M}>0$; then

$$
\begin{aligned}
&\frac{1}{\mu(T,\rho_1)}\int_{-T}^{T}\rho_2(t)\Big(\sup_{\theta\in[t-r,t]}\|Y_n(\theta)\|\Big)dt\\
&\le \frac{\widetilde{M}}{\mu(T,\rho_1)}\int_{-T}^{T}\rho_2(t)\Big(\sup_{\theta\in[t-r,t]}\Big(\int_{\theta-n}^{\theta-n+1}\|\phi_2(s)\|^p ds\Big)^{1/p}\Big)dt,
\end{aligned}
$$

and hence $Y_n\in WPP_0(\mathbb{R},X,r,\rho_1,\rho_2)$ since $\phi_2\in S^pWPP_0(\mathbb{R},X,r,\rho_1,\rho_2)$. By Lemma 2.11, (3.4) holds, whence $\Gamma_{12}\phi_2\in WPP_0(\mathbb{R},X,r,\rho_1,\rho_2)$.

From $\phi_1(t+\omega)-\phi_1(t)=0$ a.e. $t\in\mathbb{R}$, one has

$$
(\Gamma_{11}\phi_1)(t+\omega)=\int_{-\infty}^{t+\omega}T(t+\omega-s)P\phi_1(s)ds=(\Gamma_{11}\phi_1)(t), \quad \text{a.e. } t\in\mathbb{R}.
$$

Hence $\Gamma_1\phi\in WPP_\omega(\mathbb{R},X,r,\rho_1,\rho_2)$.

The proof of $\Gamma_2\phi$ is similar to that of $\Gamma_1\phi$, one uses (3.3) rather than (3.2). This completes the proof. □

Theorem 3.5. *Assume that* (H1)–(H4) *hold and g satisfy the conditions* (A2)–(A3), *if*

$$
\vartheta := \Big(L_f+\frac{2ML_f}{\delta}+M\sup_{t\in\mathbb{R}}\int_{-\infty}^{t}e^{-\delta(t-s)}L_g(s)ds+M\sup_{t\in\mathbb{R}}\int_{t}^{\infty}e^{\delta(t-s)}L_g(s)ds\Big)<1,
$$

then (3.1) *has a unique mild solution of* WPP_ω *type.*

Proof. Define $\mathcal{F} : WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2) \to WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2)$ as

$$\begin{aligned}(\mathcal{F}u)(t) &= -f(t, u_t) - \int_{-\infty}^{t} AT(t-s)Pf(s, u_s)ds + \int_{t}^{\infty} AT(t-s)Qf(s, u_s)ds \\ &\quad + \int_{-\infty}^{t} T(t-s)Pg(s, u_s)ds - \int_{t}^{\infty} T(t-s)Qg(s, u_s)ds, \quad t \in \mathbb{R}.\end{aligned} \tag{3.5}$$

If $u \in WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2)$, then $u_t \in WPP_\omega(\mathbb{R}, \mathcal{C}, r, \rho_1, \rho_2)$ by Lemma 2.14; therefore by Theorem 2.15,

$$g(s, u_s) \in WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2) \subset S^p WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2).$$

By Lemmas 3.3 and 3.4, it is not difficult to see that $\mathcal{F}$ is well defined.

For any $u, v \in WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2)$, we have

$$\begin{aligned}
&\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\| \\
&\le \|f(t, u_t) - f(t, v_t)\| + \int_{-\infty}^{t} \|AT(t-s)P(f(s, u_s) - f(s, v_s))\|ds \\
&\quad + \int_{t}^{\infty} \|AT(t-s)Q(f(s, u_s) - f(s, v_s))\|ds \\
&\quad + \int_{-\infty}^{t} \|T(t-s)P(g(s, u_s) - g(s, v_s))\|ds \\
&\quad + \int_{t}^{\infty} \|T(t-s)Q(g(s, u_s) - g(s, v_s))\|ds \\
&\le L_f \|u_t - u_t\|_{\mathcal{C}} + ML_f \int_{-\infty}^{t} e^{-\delta(t-s)} \|u_s - u_s\|_{\mathcal{C}} ds \\
&\quad + ML_f \int_{t}^{\infty} e^{\delta(t-s)} \|u_s - u_s\|_{\mathcal{C}} ds + M \int_{-\infty}^{t} e^{-\delta(t-s)} L_g(s) \|u_s - u_s\|_{\mathcal{C}} ds \\
&\quad + M \int_{t}^{\infty} e^{\delta(t-s)} L_g(s) \|u_s - u_s\|_{\mathcal{C}} ds \\
&\le \Big(L_f + \frac{2ML_f}{\delta} + M \sup_{t\in\mathbb{R}} \int_{-\infty}^{t} e^{-\delta(t-s)} L_g(s) ds \\
&\quad + M \sup_{t\in\mathbb{R}} \int_{t}^{\infty} e^{\delta(t-s)} L_g(s) ds \Big) \|u - v\| \\
&\le \vartheta \|u - v\|,
\end{aligned}$$

then $\mathcal{F}$ is a contraction since $\vartheta < 1$. By the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point in $WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2)$, which is the unique WPP_ω solution to (3.1). □

In (H3), if $L_g(t) \equiv L_g$, It is not difficult to see that g satisfy the conditions (A2)–(A3) and $\vartheta = L_f + 2M(L_f + L_g)/\delta$.

Theorem 3.6. *Assume that* (H1), (H2), (H3'), (H4) *hold. If*

$$\Theta := \Big(L_f + \frac{2ML_f}{\delta} + \frac{2ML_g}{1 - e^{-\delta}} \Big) < 1,$$

then (3.1) *has a unique mild solution of* WPP_ω *type.*

Proof. Define the operator $\mathcal{F}$ as in (3.5). Let $u \in WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2)$, then it is not difficult to see that $g(s, u_s) \in S^p WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2)$ by Theorem 2.20, so Γ is well defined by Lemma 3.3 and Lemma 3.4.

Let $u, v \in WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2)$, one has

$$
\begin{aligned}
&\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\| \\
&\leq L_f \|u_t - v_t\|_{\mathcal{C}} + ML_f \int_{-\infty}^{t} e^{-\delta(t-s)} \|u_s - v_s\|_{\mathcal{C}} ds \\
&\quad + ML_f \int_t^{\infty} e^{\delta(t-s)} \|u_s - v_s\|_{\mathcal{C}} ds + M \int_{-\infty}^{t} e^{-\delta(t-s)} \|g(s, u_s) - g(s, v_s)\| ds \\
&\quad + M \int_t^{\infty} e^{\delta(t-s)} \|g(s, u_s) - g(s, v_s)\| ds \\
&\leq L_f \|u_t - v_t\|_{\mathcal{C}} + ML_f \int_{-\infty}^{t} e^{-\delta(t-s)} \|u_s - v_s\|_{\mathcal{C}} ds \\
&\quad + ML_f \int_t^{\infty} e^{\delta(t-s)} \|u_s - v_s\|_{\mathcal{C}} ds + M \int_0^{\infty} e^{-\delta s} \|g(t-s, u_{t-s}) - g(t-s, v_{t-s})\| ds \\
&\quad + M \int_{-\infty}^{0} e^{\delta s} \|g(t-s, u_{t-s}) - g(t-s, v_{t-s})\| ds \\
&\leq \Big(L_f + \frac{2ML_f}{\delta}\Big) \|u - v\| + M \sum_{k=0}^{\infty} \int_k^{k+1} e^{-\delta s} \|g(t-s, u_{t-s}) - g(t-s, v_{t-s})\| ds \\
&\quad + M \sum_{k=-\infty}^{0} \int_{k-1}^{k} e^{\delta s} \|g(t-s, u_{t-s}) - g(t-s, v_{t-s})\| ds \\
&\leq \Big(L_f + \frac{2ML_f}{\delta}\Big) \|u - v\| + M \sum_{k=0}^{\infty} e^{-\delta k} \Big(\int_{t-k-1}^{t-k} \|g(s, u_s) - g(s, v_s)\|^p ds \Big)^{1/p} \\
&\quad + M \sum_{k=-\infty}^{0} e^{\delta k} \Big(\int_{t-k}^{t-k+1} \|g(s, u_s) - g(s, v_s)\|^p ds \Big)^{1/p} \\
&\leq \Big(L_f + \frac{2ML_f}{\delta} + ML_g \sum_{k=0}^{\infty} e^{-\delta k} + M \sum_{k=-\infty}^{0} e^{\delta k}\Big) \|u - v\| \\
&\leq \Big(L_f + \frac{2ML_f}{\delta} + \frac{2ML_g}{1 - e^{-\delta}}\Big) \|u - v\| \\
&\leq \Theta \|u - v\|.
\end{aligned}
$$

By the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point in $WPP_\omega(\mathbb{R}, X, r, \rho_1, \rho_2)$, which is the unique WPP_ω solution to (3.1). $\square$

Remark 3.7. It is easy to see that similar results of Theorem 3.5 and Theorem 3.6 hold for $WPP_{\omega ap}(\mathbb{R}, X, r, \rho_1, \rho_2)$ mild solution, that is (3.1) has a unique $WPP_{\omega ap}$ mild solution, in this case, $f \in WPP_{\omega ap}(\mathbb{R} \times \mathcal{C}, X, r, \rho_1, \rho_2)$ in (H2), $g \in WPP_{\omega ap}(\mathbb{R} \times \mathcal{C}, X, r, \rho_1, \rho_2)$ in (H3), $g \in S^p WPP_{\omega ap}(\mathbb{R} \times \mathcal{C}, X, r, \rho_1, \rho_2)$ in (H3').

4. Examples

In this section, we provide some examples to illustrate our main results.

Example 4.1. Consider the partial differential equation

$$\begin{gathered}\frac{\partial}{\partial t}\Big[u(t,\xi)+\int_{-r}^{0}\int_{0}^{\pi}b(s,\eta,\xi)u(t+s,\eta)d\eta ds\Big]\\ =\frac{\partial^2}{\partial\xi^2}u(t,\xi)+a_0(t)u(t,\xi)+\int_{-r}^{0}a_1(s)u(t+s,\xi)ds,\quad (t,\xi)\in\mathbb{R}\times[0,\pi],\\ u(t,0)=u(t,\pi)=0,\end{gathered}\tag{4.1}$$

where $a_0\in WPP_\omega(\mathbb{R},\mathbb{R},r,\rho_1,\rho_2)$, $\rho_1=e^t,\rho_2=1+t^2$ and the following conditions hold:

The functions $b(\cdot)$, $\frac{\partial^i}{\partial\zeta^i}b(\tau,\eta,\zeta)$, $i=1,2$ are (Lebesgue) measurable, $b(\tau,\eta,0)=b(\tau,\eta,\pi)=0$ for every (τ,η) and

$$N_1:=\max\Big\{\int_0^{\pi}\int_{-r}^{0}\int_0^{\pi}\Big(\frac{\partial^i}{\partial\zeta^i}b(\tau,\eta,\zeta)\Big)^2 d\eta d\tau d\zeta,: i=0,1,2\Big\}<\infty.$$

Under these conditions, let $X=(L^2([0,\pi],\mathbb{R}),\|\cdot\|_{L^2})$ and define the operator A on X by $Au=u''$ with

$$D(A)=\{u\in X: u''\in X, u(0)=u(\pi)=0\}.$$

It is well known that A is the infinitesimal generator of a semigroup $(T(t))_{t\geq 0}$ on X such that $\|T(t)\|\leq e^{-t}$ for every $t\geq 0$.

Define the functions $f,g:\mathbb{R}\times\mathcal{C}\to X$ by

$$\begin{gathered}f(t,\psi)(\xi):=\int_{-r}^{0}\int_{0}^{\pi}b(s,\eta,\xi)\psi(s,\eta)\,d\eta\,ds,\\ g(t,\psi)(\xi):=a_0(t)\psi(0,\xi)+\int_{-r}^{0}a_1(s)\psi(s,\xi)\,ds,\end{gathered}$$

then (4.1) can be rewritten as an abstract system of the form (3.1), where $u(t)=u(t,\cdot)$. By a straightforward estimation that uses (i), one can show that f is $D(A)$-valued, and the following hold:

$$\begin{gathered}\|Af(t,\cdot)\|\leq (N_1 r)^{1/2},\quad t\in\mathbb{R},\\ \|g(t,\cdot)\|\leq\|a_0\|+r^{1/2}\Big(\int_{-r}^{0}a_1^2(s)ds\Big)^{1/2},\quad t\in\mathbb{R}.\end{gathered}$$

See [3, 5] for more details. The next result is a consequence of Theorem 3.5.

Theorem 4.2. *Under the previous assumptions,* (4.1) *as a unique weighted pseudo periodic solution whenever*

$$3\sqrt{N_1 r}+2\|a_0\|+2r^{1/2}\Big(\int_{-r}^{0}a_1^2(s)ds\Big)^{1/2}<1.$$

Example 4.3. Consider the following scalar reaction-diffusion equation with delay

$$\begin{gathered}\frac{\partial}{\partial t}u(t,x)=\frac{\partial^2}{\partial x^2}u(t,x)+g(t,u(t-r,x)),\\ u(t,0)=u(t,\pi)=0,\\ u(\tau,x)=\varphi(\tau,x),\quad \tau\in[-r,0],\ x\in[0,\pi],\end{gathered}\tag{4.2}$$

where $g=g_1+g_2\in S^pWPP_\omega(\mathbb{R}\times\mathcal{C},\mathbb{R},r,\rho_1,\rho_2)$ with $g_1\in P_\omega(\mathbb{R}\times\mathcal{C},\mathbb{R})$, $g_2\in S^pWPP_0(\mathbb{R}\times\mathcal{C},\mathbb{R},r,\rho_1,\rho_2)$, $\rho_1=e^t,\rho_2=1+t^2$.

By Theorem 3.6, one has the following result.

Theorem 4.4. *Assume that there exists a positive constant L_g such that for $i = 1, 2$,*

$$\Big(\int_t^{t+1} \|g(t,\psi_1) - g(t,\psi_2)\|^p ds \Big)^{1/p} \le L_g \|\psi_1 - \psi_2\|_{\mathcal{C}}, \quad \textit{for all } t \in \mathbb{R},\ \psi_i \in \mathcal{C}\,.$$

Then there exists a unique WPP_ω solution of (4.2) if $2L_g < 1 - e^{-1}$.

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References

[1] N. S. Al-lslam, S. M. Alsulami, T. Diagana; *Existence of weighted pseudo anti-periodic solutions to some non-autonomous differential equations*, Appl. Math. Comput. 218 (2012), 6536-6548.
[2] R. P. Agarwal, C. Cuevas, H. Soto, M. El-Gebeily; *Asymptotic periodicity for some evolution equations in Banach spaces*, Nonlinear Anal. 74 (2011), 1769-1798.
[3] R. P. Agarwal, T. Diagana, E. M. Hernández; *Weighted pseudo almost periodic solutions to some partial neutral functional differential equations*, J. Nonlinear Convex Anal. 8 (2007), 397-415.
[4] C. Cuevasa, E. M. Hernández; *Pseudo-almost periodic solutions for abstract partial functional differential equations*, Appl. Math. Lett. 22 (2009), 534-538.
[5] T. Diagana, E. M. Hernández; *Existence and uniqueness of pseudo almost periodic solutions to some abstract partial neutral functional-differential equations and applications*, J. Math. Anal. Appl. 327 (2007), 776-791.
[6] T. Diagana, R. P. Agarwal; *Existence of pseudo almost automorphic solutions for the heat equation with S^p-pseudo almost automorphic coefficients*, Bound. Value Probl. 2009 (2009), 1-19.
[7] T. Diagana; *Double-weighted pseudo almost periodic functions*, Afr. Diaspora J. Math. 12 (2011), 121-136.
[8] T. Diagana; *Existence of doubly-weighted pseudo almost periodic solutions to non-autonomous differential equations*, Electron. J. Differential Equations 28 (2011), 1-15.
[9] W. Dimbour, G. M. N'Guérékata; *S-asymptotically ω-periodic solutions to some classes of partial evolution equations*, Appl. Math. Comput. 218 (2012) 7622-7628.
[10] H. S. Ding, W. Long, G. M. N'Guérékata; *Existence of pseudo almost periodic solutions for a class of partial functional differential equations*, Electron. J. Differential Equations 104 (2013), 1-14.
[11] K. Ezzinbi, H. Toure, I. Zabsonre; *Pseudo almost periodic solutions of class r for neutral partial functional differential equations*, Afrika Mat. 24 (2013), 691-704.
[12] E. M. Hernández, H. Henríquez; *Existence of periodic solutions of partial neutral functional-differential equations with unbounded delay*, J. Math. Anal. Appl. 221 (2) (1998) 499-522.
[13] E. Hernández, H. Henríquez; *Pseudo-almost periodic solutions for non-autonomous neutral differential equations with unbounded delay*, Nonlinear Anal. Real World Appl. 9 (2008) 430-437.
[14] E. M. Hernández, M. L. Pelicer; *Asymptotically almost periodic and almost periodic solutions for partial neutral differential equations*, Appl. Math. Lett. 18 (11) (2005) 1265-1272.
[15] H. Henríquez, M. Pierri, P. Táboas; *Existence of S-asymptotically ω-periodic solutions for abstract neutral equations*, Bull. Austral. Math. Soc. 78 (2008), 365-382.
[16] A. Lunardi; *Analytic Semigroups and Optimal Regularity in Parabolic Problems,* vol.16 of Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser, Basel, Seitzerland, 1995.
[17] A. Pankov; *Bounded and Almost Periodic Solutions of Nonlinear Operator Differential Equations*, Kluwer, Dordrecht, 1990.
[18] M. Pierri, V. Rolnik; *On pseudo S-asymptotically priodic functions*, Bull. Aust. Math. Soc. 87 (2013), 238-254.

[19] Z. N. Xia; *Weighted Stepanov-like pseudoperiodicity and applications*, Abstr. Appl. Anal. Volume 2014, Article ID 980869, 14 pages.
[20] R. Zhang, Y. K. Chang, G. M. N'Guérékata; *Weighted pseudo almost automorphic solutions for non-autonomous neutral functional differential equations with infinite delay (in Chinese)*, Sci. Sin. Math. 43 (2013), 273-292.
[21] L. Zhang, H. Li; *Weighted pseudo-almost periodic solutions for some abstract differential equations with uniform continuity*, Bull. Aust. Math. Soc. 82 (2010), 424-436.

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