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FRACTIONAL POROUS MEDIUM AND MEAN FIELD EQUATIONS IN BESOV SPACES

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Abstract. In this article, we consider the evolution model

$$\partial_t u - \nabla \cdot (u \nabla P u) = 0, \quad Pu = (-\Delta)^{-s} u, \quad 0 < s \leq 1, \ x \in \mathbb{R}^d, \ t > 0.$$

We show that when $s \in [1/2, 1)$, $\alpha > d+1$, $d \geq 2$, the equation has a unique local in time solution for any initial data in $B^{\alpha}_{1,\infty}$. Moreover, in the critical case $s = 1$, the solution exists in $B^{\alpha}_{p,\infty}$, $2 \leq p \leq \infty$, $\alpha > d/p$, $d \geq 3$.

1. Introduction

Let $0 < s \leq 1$, we consider the evolution equation

$$\partial_t u - \nabla \cdot (u \nabla P u) = 0, \quad Pu = (-\Delta)^{-s} u, \quad u(x, 0) = u_0(x). \tag{1.1}$$

where $x \in \mathbb{R}^d$, $t > 0$. $u = u(x,t)$ is a real-valued function, representing the density or concentration. P represents the pressure.

When $0 < s < 1$, we refer the equation as a fractional porous medium equation. It was first introduced by Caffarelli and Vázquez [2]. They proved the existence of a weak solution when u_0 is a bounded function and has exponential decay at infinity. Caffarelli, Soria, and Vázquez [1] studied its regularity theory of the weak solution with $u_0 \in L^1 \cap L^\infty$ and the continuity of bounded solutions.

When $s = 1$, the equation leads to a mean field equation

$$u_t = \nabla \cdot (u \nabla P u), \quad Pu = (-\Delta)^{-1} u, \quad u(x, 0) = u_0(x). \tag{1.2}$$

Equation that was first studied by Lin and Zhang [7]. They proved the existence and uniqueness of positive L^∞ solution in two dimensions. When $d \geq 3$, Vázquez and Serfaty [11] studied the existence and uniqueness of the weak solution in L^∞ spaces for (1.2) by taking the limit $s \to 1$ in the weak solutions of (1.1). Since papers in literature only addressed the weak solution of (1.2), in this article we show that the strong solution exists in $B^{\alpha}_{p,\infty}$, $\alpha > 0$.

In this article, we are interested in finding the strong solutions of (1.1) in the Besov spaces $B^{\alpha}_{p,\infty}$. We will show that when $d \geq 2$, $s \in [1/2, 1)$, $\alpha > d+1$, equation (1.1) has a unique local in time solution for any initial data in $B^{\alpha}_{1,\infty}$. When $s = 1$, the solution extends to $B^{\alpha}_{p,\infty}$, $\alpha > d/p$, $2 \leq p \leq \infty$, $d \geq 3$. The idea

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of our proof is inspired by the methods used in [3, 12], where the authors studied the quasi-geostrophic equation. In our proof, we construct two commutators, give three estimates of them, and construct a function sequence fitting our equation.

The rest of this article is divided in four parts. Section 2 recalls the definition and some properties of the Besov spaces, the Bernstein inequality involving both integer and fractional derivatives, as well as some properties of the fractional Laplacian. In Section 3 we prove three estimates about the constructed commutators and a priori estimate of the solution. Section 4 proves the existence and uniqueness of the fractional porous medium equation. Section 5 is devoted to the mean field equation. The main results are the following two theorems.

Theorem 1.1. *Let $d \geq 2, s \in [1/2, 1], \alpha > d+1$. Assume that the initial data $u_0 \in B^{\alpha}_{1,\infty}$. Then we can find $T = T(\|u_0\|_{B^{\alpha}_{1,\infty}})$, such that a unique solution u to* (1.1) *on $[0,T]\times\mathbb{R}^d$ exists. And the solution belongs to $C^1([0,T]; B^{\alpha+2s-2}_{1,\infty})\cap L^\infty([0,T]; B^{\beta}_{1,\infty})$, and $\beta \in [\alpha+2s-2, \alpha]$.*

Theorem 1.2. *Let $d \geq 3, \alpha > d/p$ when $2 \leq p \leq \infty$. Assume that the initial data $u_0 \in B^{\alpha}_{p,\infty}$. Then we can find $T = T(\|u_0\|_{B^{\alpha}_{p,\infty}})$, such that a unique solution u to the given mean field equation* (1.2) *on $[0,T] \times \mathbb{R}^d$ exists. And the solution belongs to $C^1([0,T]; B^{\beta}_{p,\infty}) \cap L^\infty([0,T]; B^{\alpha}_{p,\infty})$, $\beta \in (d/p, \alpha)$, $2 \leq p \leq \infty$.*

2. Preliminaries

In this section, we recall the definition of the Besov space. We start with a dyadic decomposition of $\mathbb{R}^d$.

Suppose $\chi \in C_0^\infty(\mathbb{R}^d), \phi \in C_0^\infty(\mathbb{R}^d \setminus \{0\})$ satisfying

$$\begin{gathered}
\operatorname{supp}\chi \subset \{\xi \in \mathbb{R}^d : |\xi| \leq \frac{4}{3}\},\\
\operatorname{supp}\phi \subset \{\xi \in \mathbb{R}^d : \frac{3}{4} < |\xi| < \frac{8}{3}\},\\
\chi(\xi) + \sum_{j\geq 0}\phi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^d,\\
\sum_{j\in\mathbb{Z}}\phi(2^{-j}\xi) = 1, \quad \xi \in \mathbb{R}^d\backslash\{0\}.
\end{gathered}$$

Define the operators

$$\begin{gathered}
\Delta_j u = \phi(2^{-j}D)u = 2^{jd}\int h(2^j y)u(x-y)\,dy,\\
S_j f = \sum_{k\leq j-1}\Delta_k f = \chi(2^{-j}D)u = 2^{jd}\int g(2^j y)u(x-y)\,dy,
\end{gathered} \tag{2.1}$$

where $g = \chi^\vee$ and $h = \phi^\vee$ are the inverse Fourier transform of χ and ϕ, respectively. It can be easily verified that with our choice of ϕ,

$$\Delta_j\Delta_k f \equiv 0, \ if \ |j-k| \geq 2, \quad \Delta_j(S_{k-1}f\Delta_k f) \equiv 0, \quad \text{if } |j-k| \geq 5. \tag{2.2}$$

Definition 2.1. For any $\alpha \in \mathbb{R}$, and $p, q \in [1,\infty]$, the homogeneous Besov spaces $\dot{B}^r_{p,q}$ are defined as

$$\dot{B}^\alpha_{p,q} = \{f \in \mathcal{Z}'(\mathbb{R}^d) : \|f\|_{\dot{B}^\alpha_{p,q}} < \infty\}.$$

Here,

$$\|f\|_{\dot{B}^{\alpha}_{p,q}} = \Big[\sum_{j\in\mathbb{Z}} 2^{j\alpha q}\|\Delta_j f\|^q_{L^p}\Big]^{1/q}, \quad \text{where } q<\infty.$$

When $q=\infty$,

$$\|f\|_{\dot{B}^{\alpha}_{p,\infty}} = \sup_{j\in\mathbb{Z}} 2^{j\alpha}\|\Delta_j f\|_{L^p}.$$

$\mathcal{Z}'(\mathbb{R}^d)$ denotes the dual space of $\mathcal{Z}(\mathbb{R}^d) = \{f\in\mathcal{S}(\mathbb{R}^d) : \partial^\gamma \hat{f}(0)=0, \forall\gamma\in\mathbb{N}^d\}$ and can be identified by the quotient space of $\mathcal{S}'/\mathcal{P}$ with the polynomials space $\mathcal{P}$.

Definition 2.2. For any $\alpha\in\mathbb{R}$, and $p,q\in[1,\infty]$, the inhomogeneous Besov space $B^r_{p,q}$ is defined as

$$B^{\alpha}_{p,q} = \{f\in\mathcal{S}'(\mathbb{R}^d) : \|f\|_{B^{\alpha}_{p,q}}<\infty\}.$$

Here,

$$\|f\|_{B^{\alpha}_{p,q}} = \Big(\sum_{j\geq 0}^{\infty} 2^{j\alpha q}\|\Delta_j f\|^q_{L^p}\Big)^{1/q} + \|S_0(f)\|_{L^p}, \quad \text{when } q<\infty.$$

When $q=\infty$,

$$\|f\|_{B^{\alpha}_{p,\infty}} = \sup_{j\geq 0} 2^{j\alpha}\|\Delta_j f\|_{L^p} + \|S_0(f)\|_{L^p}.$$

Let us state some basic properties of the Besov spaces.

Proposition 2.3. *Let $s\in\mathbb{R}$, $1\leq p\leq\infty$, $1\leq q\leq\infty$.*

(i) *If $\alpha>0$, then $B^{\alpha}_{p,q}=\dot{B}^{\alpha}_{p,q}\cap L^p$, and $\|f\|_{B^{\alpha}_{p,q}} = \|f\|_{\dot{B}^{\alpha}_{p,q}} + \|f\|_{L^p}$;*

(ii) *If $\alpha_1\leq\alpha_2$, then $B^{\alpha_2}_{p,q}\subset B^{\alpha_1}_{p,q}$. If $1\leq q_1\leq q_2\leq\infty$, then $\dot{B}^{\alpha}_{p,q_1}\subset\dot{B}^{\alpha}_{p,q_2}$ and $B^{\alpha}_{p,q_1}\subset B^{\alpha}_{p,q_2}$;*

(iii) *If $\alpha>\frac{d}{p}$, then $B^{\alpha}_{p,q}\hookrightarrow L^\infty$. If $p_1\leq p_2$, $\alpha_1-\frac{d}{p_1}>\alpha_2-\frac{d}{p_2}$, then $B^{\alpha_1}_{p_1,q_1}\hookrightarrow B^{\alpha_2}_{p_2,q_2}$, $B^{\alpha}_{p,\min(p,2)}\hookrightarrow H^{\alpha}_p\hookrightarrow B^{\alpha}_{p,\max(p,2)}$;*

(iv) *If $\alpha>0$, $p\geq 1$, then $\|uv\|_{B^{\alpha}_{p,\infty}}\leq C\|u\|_{L^\infty}\|v\|_{B^{\alpha}_{p,\infty}} + C\|u\|_{B^{\alpha}_{p,\infty}}\|v\|_{L^\infty}$.*

We now turn to Bernstein's inequalities. When the Fourier transform of a function is supported on a ball or an annulus, the L^p-norms of the derivatives of the function can be bounded in terms of the L^p-norms of the function itself. And it also exists when one replaces the derivatives by the fractional derivatives (see [6, 14]).

Proposition 2.4. *Let $1\leq p\leq q\leq\infty$, $\gamma\in\mathbb{N}^d$. (1) If $\alpha\geq 0$ and $\operatorname{supp}\hat{f}\subset\{\xi\in\mathbb{R}^d : |\xi|\leq K2^j\}$ for some $K>0$ and integer j, then*

$$\|(-\Delta)^{\alpha}D^{\gamma}f\|_{L^q}\leq C2^{j(2\alpha+|\gamma|)+jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p}.$$

(2) If $\alpha\in\mathbb{R}$ and $\operatorname{supp}\hat{f}\subset\{\xi\in\mathbb{R}^d : K_1 2^j\leq|\xi|\leq K_2 2^j\}$ for some $K_1,K_2>0$ and integer j, then

$$C2^{j(2\alpha+|\gamma|)+jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p}\leq\|(-\Delta)^{\alpha}D^{\gamma}f\|_{L^q}\leq\widetilde{C}2^{j(2\alpha+|\gamma|)+jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p},$$

where C and $\widetilde{C}$ are positive constants independent of j.

Next we state two pointwise inequalities which were proved in [5, 13].

Proposition 2.5. *Let $0\leq\alpha\leq 1$, $f\in C^2(\mathbb{R}^d)$ decay sufficiently fast at infinity. Then for any $x\in\mathbb{R}^d$,*

$$2f(x)(-\Delta)^{\alpha}f(x)\geq(-\Delta)^{\alpha}f^2(x).$$

Proposition 2.6. *Let $0 \le \alpha \le 1$, $p_1 = \frac{k_1}{l_1} \ge 0$, $p_2 = \frac{k_2}{l_2} \ge 1$ be rational numbers with l_1, l_2 odd, and $k_1 l_1 + k_2 l_2$ even. Then for any $f \in C^2(\mathbb{R}^d)$ that decays sufficiently fast at infinity, and for any $x \in \mathbb{R}^d$,*

$$(p_1 + p_2) f^{p_1}(x)(-\Delta)^\alpha f^{p_2}(x) \ge p_2 (-\Delta)^\alpha f^{p_1+p_2}(x).$$

3. A priori estimate

Proposition 3.1. *Let $\alpha > 0$, $s \in (0,1)$, $p \in [1,\infty]$ be given. Assume $r > d/p$. Then there exists some constant C such that*

$$2^{j\alpha} \|[\Delta_j, \partial_i(-\Delta)^{-s} u]\partial_i v\|_{L^p} \le C\|v\|_{B^{r+1}_{p,\infty}} \|u\|_{B^{\alpha+1-2s}_{p,\infty}} + C\|u\|_{B^{r+2-2s}_{p,\infty}} \|v\|_{B^{\alpha}_{p,\infty}}, \quad (3.1)$$

where the brackets $[,]$ represents the commutator, namely

$$[\Delta_j, \partial_i(-\Delta)^{-s} u]\partial_i v = \Delta_j(\partial_i(-\Delta)^{-s} u)\partial_i v) - \partial_i(-\Delta)^{-s} u \Delta_j(\partial_i v).$$

Proof. Using Bony's para-product decomposition, we have

$$[\Delta_j, \partial_i(-\Delta)^{-s} u]\partial_i v = L_1 + L_2 + L_3 + L_4 + L_5,$$

where

$$L_1 = \sum_{|k-j|\le 4} \Delta_j[S_{k-1}(\partial_i(-\Delta)^{-s} u)\Delta_k(\partial_i v)] - S_{k-1}(\partial_i(-\Delta)^{-s} u)\Delta_k(\Delta_j(\partial_i v)),$$

$$L_2 = \sum_{|k-j|\le 4} \Delta_j[S_{k-1}(\partial_i v)\Delta_k(\partial_i(-\Delta)^{-s} u)],$$

$$L_3 = \sum_{k\ge j-2} \Delta_j(\Delta_k(\partial_i(-\Delta)^{-s} u)\widetilde{\Delta}_k(\partial_i v)),$$

$$L_4 = \sum_{k} S_{k-1}(\Delta_j(\partial_i v))\Delta_k(\partial_i(-\Delta)^{-s} u),$$

$$L_5 = \sum_{|j'-j''|\le 1} \Delta_{j'}(\Delta_j(\partial_i v))\Delta_{j''}(\partial_i(-\Delta)^{-s} u).$$

We shall estimate the above terms separately. First observe

$$L_1 = \sum_{|k-j|\le 4} 2^{jd} \int h(2^j(x-y)) \big[S_{k-1}(\partial_i(-\Delta)^{-s} u)(y) - S_{k-1}(\partial_i(-\Delta)^{-s} u)(x)\big] \Delta_k(\Delta_j(\partial_i v))(y)\, dy.$$

By Young's inequality and Bernstein's inequality,

$$\begin{aligned}
\|L_1\|_{L^p} &\le C \sum_{|k-j|\le 4} 2^{-j} \|\nabla \partial_i(-\Delta)^{-s} u\|_{L^\infty} \|\Delta_j(\partial_i v)\|_{L^p} \int |y||h(y)|\, dy \\
&\le C 2^{-j} 2^j \|(-\Delta)^{1-s} u\|_{L^\infty} \|\Delta_j v\|_{L^p} \\
&\le C\|\Delta_j v\|_{L^p} \|u\|_{B^{r+2-2s}_{p,\infty}}.
\end{aligned}$$

Similarly,

$$\|L_2\|_{L^p} \le C 2^{j(1-2s)} \|\nabla v\|_{L^\infty} \|\Delta_j u\|_{L^p} \le C 2^{-j\alpha} \|v\|_{B^{r+1}_{p,\infty}} \|u\|_{B^{\alpha+1-2s}_{p,\infty}}.$$

We can also estimate,

$$\|L_3\|_{L^p} \le C \sum_{k\ge j-2} \|\Delta_k(\partial_i(-\Delta)^{-s} u)\|_{L^p} \|\nabla v\|_{L^\infty}$$

$$
\begin{aligned}
&\le C2^{-j\alpha}\sum_{k\ge j-2}2^{(j-k)\alpha}2^{k(\alpha+1-2s)}\|\Delta_k u\|_{L^p}\|v\|_{B^{r+1}_{p,\infty}}\\
&\le C2^{-j\alpha}\|u\|_{B^{\alpha+1-2s}_{p,\infty}}\|v\|_{B^{r+1}_{p,\infty}}.
\end{aligned}
$$

To estimate L_4, by the definition of S_j and Δ_j, we can observe that only k satisfying $k\ge j$ survive. Thus

$$
\begin{aligned}
\|L_4\|_{L^p} &\le \sum_{k\ge j} C\|\nabla v\|_{L^\infty}2^{k(1-2s)}\|\Delta_k u\|_{L^p}\\
&\le C2^{-j\alpha}\sum_{k\ge j}2^{(j-k)\alpha}\|v\|_{B^{r+1}_{p,\infty}}\|u\|_{B^{\alpha+1-2s}_{p,\infty}}\\
&\le C2^{-j\alpha}\|v\|_{B^{r+1}_{p,\infty}}\|u\|_{B^{\alpha+1-2s}_{p,\infty}}.
\end{aligned}
$$

Since $\Delta_k\Delta_j=0$, for $|j-k|\ge 2$, we have

$$
\|L_5\|_{L^p}\le C\|\nabla v\|_{L^\infty}2^{j(1-2s)}\|\Delta_j u\|_{L^p}\le C2^{-j\alpha}\|v\|_{B^{r+1}_{p,\infty}}\|u\|_{B^{\alpha+1-2s}_{p,\infty}}.
$$

Collecting the estimates above, we obtain

$$
\begin{aligned}
&2^{j\alpha}\|[\Delta_j,\partial_i(-\Delta)^{-s}u]\partial_i v\|_{L^p}\\
&\le C2^{j\alpha}\|\Delta_j v\|_{L^p}\|u\|_{B^{r+2-2s}_{p,\infty}}+2^{j(\alpha+1-2s)}\|v\|_{B^{r+1}_{p,\infty}}\|\Delta_j u\|_{L^p}+\|u\|_{B^{\alpha+1-2s}_{p,\infty}}\|v\|_{B^{r+1}_{p,\infty}}\\
&\le \|v\|_{B^{\alpha}_{p,\infty}}\|u\|_{B^{r+2-2s}_{p,\infty}}+\|u\|_{B^{\alpha+1-2s}_{p,\infty}}\|v\|_{B^{r+1}_{p,\infty}}.
\end{aligned}
$$

This completes the proof. □

Proposition 3.2. *Let α,s,p,r be as in Proposition 3.1. Then there exists a constant C such that*

$$
2^{j\alpha}\|[\Delta_j,\partial_i(-\Delta)^{-s}u]\partial_i v\|_{L^p}\le C\|v\|_{B^{r}_{p,\infty}}\|u\|_{B^{\alpha+2-2s}_{p,\infty}}+C\|u\|_{B^{r+2-2s}_{p,\infty}}\|v\|_{B^{\alpha}_{p,\infty}}. \tag{3.2}
$$

When $p=\infty$, this inequality becomes: for any $r>0$,

$$
2^{j\alpha}\|[\Delta_j,\partial_i(-\Delta)^{-s}u]\partial_i v\|_{L^\infty}\le C\|v\|_{B^{r}_{\infty,\infty}}\|u\|_{B^{\alpha+2-2s}_{\infty,\infty}}+C\|u\|_{B^{r+2-2s}_{\infty,\infty}}\|v\|_{B^{\alpha}_{\infty,\infty}}. \tag{3.3}
$$

Proof. We want to give a new estimate of the commutator in Proposition 3.1. Following the above proof, the estimate of L_1 unchanged, we give different bounds for L_2,L_3,L_4,L_5. First,

$$
\begin{aligned}
L_2 &= \sum_{|k-j|\le 4}2^{jd}\int h(2^j(x-y))(S_{k-1}\partial_i v)(y)\Delta_k(\partial_i(-\Delta)^{-s}u)(y)\,dy\\
&= \sum_{|k-j|\le 4}2^{jd}\int \partial_i h(2^j(x-y))2^j(S_{k-1}v)(y)\Delta_k(\partial_i(-\Delta)^{-s}u)(y)\,dy\\
&\quad -\sum_{|k-j|\le 4}2^{jd}\int h(2^j(x-y))(S_{k-1}v)(y)\Delta_k(\partial_{ii}(-\Delta)^{-s}u)(y)\,dy.
\end{aligned}
$$

So we obtain

$$
\begin{aligned}
\|L_2\|_{L^p} &\le C2^{2-2s}(\|\partial_i h\|_{L^1}+\|h\|_{L^1})\|v\|_{L^\infty}\|\Delta_j u\|_{L^p}\\
&\le C2^{-j\alpha}\|u\|_{B^{\alpha+2-2s}_{p,\infty}}\|v\|_{B^{r}_{p,\infty}}.
\end{aligned}
$$

Similarly,

$$\begin{aligned}\|L_3\|_{L^p} \le C \sum_{k\ge j-2} & 2^{jd}\int \partial_i h(2^j(x-y))2^j\Delta_k(\partial_i(-\Delta)^{-s}u)(y)\Delta_k v(y)\,dy \\ & - 2^{jd}\int h(2^j(x-y))\Delta_k(\partial_{ii}(-\Delta)^{-s}u)(y)\Delta_k v(y)\,dy.\end{aligned}$$

Hence we obtain,

$$\begin{aligned}\|L_3\|_{L^p} &\le C\sum_{k\ge j-2}(2^{(j-k)(\alpha+1)}+2^{(j-k)\alpha})2^{k(\alpha+2-2s)}\|v\|_{L^\infty}\|\Delta_k u\|_{L^p} \\ &\le C2^{-j\alpha}\|u\|_{B^{\alpha+2-2s}_{p,\infty}}\|v\|_{B^r_{p,\infty}}.\end{aligned}$$

Also,

$$\begin{aligned}\|L_4\|_{L^p} &\le \sum_{k\ge j} C2^j\|\Delta_j v\|_{L^\infty}2^{k(1-2s)}\|\Delta_k u\|_{L^p} \\ &\le C2^{-j\alpha}\sum_{k\ge j}2^{(j-k)(\alpha+1)}2^{k(\alpha+2-2s)}\|v\|_{B^r_{p,\infty}}\|\Delta_k u\|_{L^p} \\ &\le C2^{-j\alpha}\|v\|_{B^r_{p,\infty}}\|u\|_{B^{\alpha+2-2s}_{p,\infty}}.\end{aligned}$$

Finally,

$$\|L_5\|_{L^p}\le C\|\Delta_j v\|_{L^\infty}2^{j(2-2s)}\|\Delta_j u\|_{L^p}\le C2^{-j\alpha}\|v\|_{B^r_{p,\infty}}\|u\|_{B^{\alpha+2-2s}_{p,\infty}}.$$

Collecting the estimates above, we can obtain

$$\begin{aligned}2^{j\alpha}\|[\Delta_j,\partial_i(-\Delta)^{-s}u]\partial_i v\|_{L^p} &\le C2^{j\alpha}\|\Delta_j v\|_{L^p}\|u\|_{B^{r+2-2s}_{p,\infty}}+\|v\|_{B^r_{p,\infty}}\|u\|_{B^{\alpha+2-2s}_{p,\infty}} \\ &\le \|v\|_{B^\alpha_{p,\infty}}\|u\|_{B^{r+2-2s}_{p,\infty}}+\|u\|_{B^{\alpha+2-2s}_{p,\infty}}\|v\|_{B^r_{p,\infty}}.\end{aligned}$$

This completes the proof. □

Proposition 3.3. *Let $\alpha>0$, $s\in(0,1)$, $p\in[1,\infty]$. Assume $r>\frac{d}{p}$. Then there exists a constant C such that*

$$2^{j\alpha}\|[\Delta_j,v](-\Delta)^{1-s}u\|_{L^p}\le C\|v\|_{B^{r+1}_{p,\infty}}\|u\|_{B^{\alpha+1-2s}_{p,\infty}}+C\|u\|_{B^{r+2-2s}_{p,\infty}}\|v\|_{B^\alpha_{p,\infty}}, \quad (3.4)$$

where the brackets $[,]$ represents the commutator,

$$[\Delta_j,v](-\Delta)^{1-s}u=\Delta_j(v(-\Delta)^{1-s}u)-v\Delta_j((-\Delta)^{1-s}u). \quad (3.5)$$

Proof. This proposition is proved similarly to Proposition 3.1. We start the proof by writing

$$[\Delta_j,v](-\Delta)^{1-s}u=I_1+I_2+I_3+I_4+I_5,$$

where

$$\begin{gathered}I_1=\sum_{|k-j|\le 4}\Delta_j(S_{k-1}v\Delta_k(-\Delta)^{1-s}u)-S_{k-1}v\Delta_k(\Delta_j(-\Delta)^{1-s}u), \\ I_2=\sum_{|k-j|\le 4}\Delta_j(S_{k-1}(-\Delta)^{1-s}u)\Delta_k v), \\ I_3=\sum_{k\ge j-2}\Delta_j(\Delta_k v)\widetilde{\Delta}_k(-\Delta)^{1-s}u), \\ I_4=\sum_k S_{k-1}(\Delta_j(-\Delta)^{1-s}u)\Delta_k v,\end{gathered}$$

$$I_5 = \sum_{|j'-j''|\le 1} \Delta_{j'} v \Delta_{j''}(\Delta_k(-\Delta)^{1-s}u).$$

Similar to the proof for Proposition 3.1, first we observe that

$$\begin{aligned}
\|I_1\|_{L^p} &= \|\sum_{|k-j|\le 4} 2^{jd}\int h(2^j(x-y))(S_{k-1}v(y)-S_{k-1}v(x))\Delta_k(-\Delta)^{1-s}u)(y)dy\|_{L^p}\\
&\le C\sum_{|k-j|\le 4} 2^{-j}\|\nabla v\|_{L^\infty}\|\Delta_j(-\Delta)^{1-s}u\|_{L^p}\int |y||h(y)|\,dy\\
&\le C2^{-j}2^{2j(1-s)}\|\nabla v\|_{L^\infty}\|\Delta_j u\|_{L^p}\\
&\le C2^{j(1-2s)}\|\Delta_j u\|_{L^p}\|v\|_{B^{r+1}_{p,\infty}}.
\end{aligned}$$

Also we obtain

$$\begin{aligned}
\|I_2\|_{L^p} &\le C\|(-\Delta)^{1-s}u\|_{L^\infty}\|\Delta_j v\|_{L^p} \le C\|u\|_{B^{r+2-2s}_{p,\infty}}\|\Delta_j v\|_{L^p},\\
\|I_3\|_{L^p} &\le C\sum_{k\ge j-2}\|\Delta_k v\|_{L^p}2^{k(2-2s)}\|\Delta_k u\|_{L^\infty}\\
&\le C2^{-j\alpha}\sum_{k\ge j-2}2^{(j-k)\alpha}2^{k\alpha}\|\Delta_k v\|_{L^p}2^{k(2-2s)}\|\Delta_k u\|_{L^\infty}\\
&\le C2^{-j\alpha}\|v\|_{B^{\alpha}_{p,\infty}}\|u\|_{B^{r+2-2s}_{p,\infty}}.
\end{aligned}$$

Similarly, we estimate

$$\begin{aligned}
\|I_4\|_{L^p} &\le C\sum_{k\ge j}\|(-\Delta)^{1-s}u\|_{L^\infty}\|\Delta_k v\|_{L^p}\\
&\le C2^{-j\alpha}\sum_{k\ge j}2^{(j-k)\alpha}\|(-\Delta)^{1-s}u\|_{L^\infty}2^{k\alpha}\|\Delta_j v\|_{L^p}\\
&\le C2^{-j\alpha}\|u\|_{B^{r+2-2s}_{p,\infty}}\|v\|_{B^{\alpha}_{p,\infty}},\\
\|I_5\|_{L^p} &\le C\|(-\Delta)^{1-s}u\|_{L^\infty}\|\Delta_j v\|_{L^p} \le C\|u\|_{B^{r+2-2s}_{p,\infty}}\|\Delta_j v\|_{L^p}.
\end{aligned}$$

Collecting the estimates above, we obtain

$$\begin{aligned}
2^{j\alpha}\|[\Delta_j, v](-\Delta)^{1-s}u\|_{L^p} &\le C2^{j(\alpha+1-2s)}\|\Delta_j u\|_{L^p}\|v\|_{B^{r+1}_{p,\infty}}\\
&\quad + C2^{j\alpha}\|u\|_{B^{r+2-2s}_{p,\infty}}\|\Delta_j v\|_{L^p} + C\|v\|_{B^{\alpha}_{p,\infty}}\|u\|_{B^{r+2-2s}_{p,\infty}}\\
&\le C\|v\|_{B^{r+1}_{p,\infty}}\|u\|_{B^{\alpha+1-2s}_{p,\infty}} + C\|v\|_{B^{\alpha}_{p,\infty}}\|u\|_{B^{r+2-2s}_{p,\infty}}.
\end{aligned}$$

This completes the proof. □

Proposition 3.4. *Let $s\in[1/2,1]$, $p=k/l$ be a rational number with k even, l odd, and $\alpha>\frac{d}{p}+1$. Assume that $u(x,t)\in B^{\alpha}_{p,\infty}$ is a solution of (1.1) with $u_0\in B^{\alpha}_{p,\infty}$ for $t\in[0,T]$. Then, when $u(x,t)\ge 0$, we can find some $C=C(p,\alpha)$, that for any $t\le T$,*

$$\|u\|_{B^{\alpha}_{p,\infty}} \le C\|u_0\|_{B^{\alpha}_{p,\infty}}\exp\{C\int_0^t\|u\|_{B^{\alpha}_{p,\infty}}d\tau\}. \tag{3.6}$$

Proof. Applying Δ_j on (1.1), we obtain

$$\begin{aligned}
\partial_t\Delta_j u &= \sum[\Delta_j,\partial_i(-\Delta)^{-s}u]\partial_i u + \nabla(-\Delta)^{-s}u\Delta_j(\nabla u)\\
&\quad - [\Delta_j, u](-\Delta)^{1-s}u - u\Delta_j((-\Delta)^{1-s}u.
\end{aligned}$$

Multiplying both sides by $p\Delta_j u|\Delta_j u|^{p-2}$ and integrating over $\mathbb{R}^d$, the equation becomes

$$
\begin{aligned}
&\frac{d}{dt}\|\Delta_j u\|_{L^p}^p\\
&=\sum\int p\Delta_j u|\Delta_j u|^{p-2}[\Delta_j,\partial_i(-\Delta)^{-s}u]\partial_i u+\int p\Delta_j u|\Delta_j u|^{p-2}\nabla(-\Delta)^{-s}u\Delta_j(\nabla u)\\
&\quad-\int p\Delta_j u|\Delta_j u|^{p-2}[\Delta_j,u](-\Delta)^{1-s}u-\int p\Delta_j u|\Delta_j u|^{p-2}u\Delta_j((-\Delta)^{1-s}u\\
&=J_1+J_2+J_3+J_4.
\end{aligned}
$$

From Propositions 3.1 and 3.3, we obtain the estimates

$$
\begin{aligned}
J_1&\le C2^{-j\alpha}\|\Delta_j u\|_{L^p}^{p-1}\|u\|_{B^\alpha_{p,\infty}}\|u\|_{B^{\alpha+1-2s}_{p,\infty}}\le C2^{-j\alpha}\|\Delta_j u\|_{L^p}^{p-1}\|u\|^2_{B^\alpha_{p,\infty}},\\
J_3&\le C2^{-j\alpha}\|\Delta_j u\|_{L^p}^{p-1}\|u\|_{B^\alpha_{p,\infty}}\|u\|_{B^{\alpha+1-2s}_{p,\infty}}\le C2^{-j\alpha}\|\Delta_j u\|_{L^p}^{p-1}\|u\|^2_{B^\alpha_{p,\infty}}.
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
J_2&=\int\nabla(-\Delta)^{-s}u)\nabla(|\Delta_j(u)|^p)=\int(-\Delta)^{1-s}u|\Delta_j u|^p dx\\
&\le C\|(-\Delta)^{1-s}u\|_{B^{\alpha-1}_{p,\infty}}\|\Delta_j u\|_{L^p}^p\le C\|u\|_{B^{\alpha+1-2s}_{p,\infty}}\|\Delta_j u\|_{L^p}^{p-1}\|\Delta_j u\|_{L^p}\\
&\le C2^{-j\alpha}\|u\|_{B^{\alpha+1-2s}_{p,\infty}}\|u\|_{B^\alpha_{p,\infty}}\|\Delta_j u\|_{L^p}^{p-1}.
\end{aligned}
$$

Using the fact that $u\ge 0$ and Propositions 2.5 and 2.6, we estimate

$$
\begin{aligned}
J_4&\le-\int p|\Delta_j u|^{p-2}u(-\Delta)^{1-s}|\Delta_j u|^2\le-\int u(-\Delta)^{1-s}|\Delta_j u|^p\\
&=-\int(-\Delta)^{1-s}u|\Delta_j u|^p\le C\|(-\Delta)^{1-s}u\|_{B^{\alpha-1}_{p,\infty}}\|\Delta_j u\|_{L^p}^p\\
&\le C2^{-j\alpha}\|u\|_{B^{\alpha+1-2s}_{p,\infty}}\|u\|_{B^\alpha_{p,\infty}}\|\Delta_j u\|_{L^p}^{p-1}.
\end{aligned}
$$

Combining the above four estimates, we set $\frac{d}{dt}\|u\|_{\dot B^\alpha_{p,\infty}}\le C\|u\|^2_{B^\alpha_{p,\infty}}$. Since $u\ge 0$, it follows

$$
\begin{aligned}
\frac{d}{dt}\|u\|_{L^p}&=\frac{d}{dt}\int u^p dx=p\int u^{p-1}u_t dx\\
&=p\int u^{p-1}\nabla\cdot(u\nabla(-\Delta)^{-s}u)=-(p-1)\int\nabla u^p\cdot(\nabla(-\Delta)^{-s}u)\\
&=-(p-1)\int u^p((-\Delta)^{1-s}u)\\
&\le C\|u\|_{L^p}^{p-1}\|u\|_{B^\alpha_{p,\infty}}\|u\|_{B^{\alpha+1-2s}_{p,\infty}}.
\end{aligned}
$$

This implies

$$
\frac{d}{dt}\|u\|_{B^\alpha_{p,\infty}}\le C\|u\|_{B^\alpha_{p,\infty}}\|u\|_{B^\alpha_{p,\infty}},
$$

which with the Gronwall's inequality yield (3.6). □

4. Proof of Theorem 1.1

From the definition of Riesz potentials $(-\Delta)^{-s}u=c(n,s)|x|^{-d+2s}*u$, $0<2s<d$. So when $d\ge 2$, $\frac{1}{2}\le s<1$, we define $P_\epsilon u=c(n,s)(|x|^{-d+2s}*\sigma_\epsilon)*u=(-\Delta)^{-s}(\sigma_\epsilon*u)=(-\Delta)^{-s}u_\epsilon$, $u_\epsilon=\sigma_\epsilon*u$. Here $\sigma\in C_c^\infty$, is nonnegative, radially symmetric and decreasing, $\int \sigma = 1$, $\sigma_\epsilon = \varepsilon^{-d}\sigma(x/\epsilon)$. Then we construct a sequence $\{u^{(n)}\}$, defined recursively by solving the following equations

$$\begin{aligned} u^{(1)} &= S_2(u_0) \\ \partial_t u^{(n+1)} &= \nabla\cdot(u^{(n+1)}\nabla(P_\epsilon u^{(n)})) \\ u^{(n+1)}(x,0) &= u_0^{n+1} = S_{n+2}u_0. \end{aligned} \tag{4.1}$$

Since $u^{(n)}$ solves the linear system, we can always find the sequence. Similarly to the proof of Proposition 3.4, taking Δ_j on (4.1), we obtain

$$\begin{aligned} \partial_t\Delta_j u^{(n+1)} &= \sum[\Delta_j,\partial_i(-\Delta)^{-s}u_\epsilon^{(n)}]\partial_i u^{(n+1)} + \sum\partial_i(-\Delta)^{-s}u_\epsilon^{(n)}\Delta_j(\partial_i u^{(n+1)}) \\ &\quad -[\Delta_j,u^{(n+1)}](-\Delta)^{1-s}u_\epsilon^{(n)} - u^{(n+1)}\Delta_j((-\Delta)^{1-s}u_\epsilon^{(n)}). \end{aligned} \tag{4.2}$$

Multiplying both sides by $\frac{\Delta_j u^{(n+1)}}{|\Delta_j u^{(n+1)}|}$, integrating over $\mathbb{R}^d$, we denote each corresponding part in the right side by J_1', J_2', J_3', J_4'. Now we obtain the estimates

$$\begin{aligned} J_1' &\le C2^{-j\alpha}\|u^{(n+1)}\|_{B^\alpha_{1,\infty}}\|u^{(n)}\|_{B^{\alpha+1-2s}_{1,\infty}}, \\ J_3' &\le C2^{-j\alpha}\|u^{(n+1)}\|_{B^\alpha_{1,\infty}}\|u^{(n)}\|_{B^{\alpha+1-2s}_{1,\infty}}, \end{aligned} \tag{4.3}$$

where we used the fact that $\|u_\epsilon^{(n)}\|_{B^{\alpha+1-2s}_{1,\infty}} \le \|u^{(n)}\|_{B^{\alpha+1-2s}_{1,\infty}}$. Also, we have

$$\begin{aligned} J_2' &= \int \nabla(-\Delta)^{-s}u_\epsilon^{(n)}\nabla(|\Delta_j(u^{(n+1)})|) \\ &= \int(-\Delta)^{1-s}u_\epsilon^{(n)}|\Delta_j u^{(n+1)}|dx \\ &\le C2^{-j\alpha}\|u^{(n+1)}\|_{B^\alpha_{1,\infty}}\|u^{(n)}\|_{B^{\alpha+1-2s}_{1,\infty}}, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} J_4' &= -\int\frac{\Delta_j u^{(n+1)}}{|\Delta_j u^{(n+1)}|}u^{(n+1)}\Delta_j((-\Delta)^{1-s}u_\epsilon^{(n)}) \\ &= \int\frac{\Delta_j u^{(n+1)}}{|\Delta_j u^{(n+1)}|}u^{(n+1)}\Delta_j(\Delta(-\Delta)^{-s}u_\epsilon^{(n)}) \\ &= -\int\frac{\Delta_j u^{(n+1)}}{|\Delta_j u^{(n+1)}|}\nabla u^{(n+1)}\Delta_j(\nabla(-\Delta)^{-s}u_\epsilon^{(n)}) \\ &\le C2^{-j\alpha}\|u^{(n)}\|_{B^{\alpha+1-2s}_{1,\infty}}\|u^{(n+1)}\|_{B^\alpha_{1,\infty}}. \end{aligned} \tag{4.5}$$

On the other hand,

$$\frac{d}{dt}\int|u^{(n+1)}|dx = \int\frac{u^{(n+1)}}{|u^{(n+1)}|}u_t^{(n+1)}dx = \int\nabla\cdot(|u^{(n+1)}|\nabla(-\Delta)^{-s}u_\epsilon^{(n)}) = 0.$$

Collecting the estimates above, when $s \ge 1/2$, we obtain

$$\begin{aligned} \frac{d}{dt}\|u^{(n+1)}\|_{B^\alpha_{1,\infty}} &\le C\|u^{(n+1)}\|_{B^\alpha_{1,\infty}}\|u^{(n)}\|_{B^{\alpha+1-2s}_{1,\infty}} \\ &\le C\|u^{(n+1)}\|_{B^\alpha_{1,\infty}}\|u^{(n)}\|_{B^\alpha_{1,\infty}}. \end{aligned} \tag{4.6}$$

Now from Gronwall's inequality we obtain

$$\begin{aligned}\|u^{(n+1)}\|_{B^{\alpha}_{1,\infty}} &\le \|u^{(n+1)}_0\|_{B^{\alpha}_{1,\infty}} \exp\{C\int_0^t \|u^{(n)}\|_{B^{\alpha}_{1,\infty}})d\tau\} \\ &\le C\|u_0\|_{B^{\alpha}_{1,\infty}} \exp\{C\int_0^t \|u^{(n)}\|_{B^{\alpha}_{1,\infty}})d\tau\},\end{aligned}$$

with C independent of n. Defining $X_T := C([0,T]; B^{\alpha}_{1,\infty})$, we have

$$\|u^{(n+1)}\|_{X_T} \le C\|u_0\|_{B^{\alpha}_{1,\infty}} \exp(CT\|u^{(n)}\|_{X_T}).$$

Thus, by the standard induction argument we find

$$\sup_{0\le t\le T_0} \|u^{(n)}\|_{B^{\alpha}_{1,\infty}} \le 2C\|u_0\|_{B^{\alpha}_{1,\infty}}, \tag{4.7}$$

for all $n \ge 1$, if $\exp(2CT_0\|u_0\|_{B^{\alpha}_{1,\infty}}) \le 2$; namely, $T_0 \le \frac{\ln 2}{2C_0(1+\|u_0\|_{B^{\alpha}_{1,\infty}})}$. Using Proposition 3.2, 3.3, and (4.4), (4.5), we see

$$\frac{d}{dt}\|u^{(n+1)}\|_{B^{\alpha+2s-2}_{1,\infty}} \le C\|u^{(n+1)}\|_{B^{\alpha+2s-2}_{1,\infty}}\|u^{(n)}\|_{B^{\alpha}_{1,\infty}}. \tag{4.8}$$

Thus, from the uniform estimate (4.7), we have

$$\sup_{0\le t\le T_0} \|\frac{\partial}{\partial t}u^{(n)}\|_{B^{\alpha+2s-2}_{1,\infty}} \le C\|u_0\|^2_{B^{\alpha}_{1,\infty}}. \tag{4.9}$$

Let $Y_T := C([0,T]; B^{\alpha+2s-2}_{1,\infty})$. We will prove that the sequence $\{u^{(n)}\}$ is Cauchy in Y_{T_1} for some $T_1 \in (0, T_0)$. Considering the difference $u^{(n+1)} - u^{(n)}$,

$$\begin{aligned}&\partial_t(u^{(n+1)} - u^{(n)}) \\ &= \nabla u^{(n+1)} \cdot \nabla(-\Delta)^{-s}u^{(n)}_{\epsilon} - u^{(n+1)}(-\Delta)^{1-s}u^{(n)}_{\epsilon} - \nabla u^{(n)} \cdot \nabla(-\Delta)^{-s}u^{(n-1)}_{\epsilon} \\ &\quad + u^{(n)}(-\Delta)^{1-s}u^{(n-1)}_{\epsilon} \\ &= \nabla\cdot((u^{(n+1)} - u^{(n)})\nabla(-\Delta)^{-s}u^{(n)}_{\epsilon}) + \nabla\cdot(u^{(n)}\nabla(-\Delta)^{-s}(u^{(n)}_{\epsilon} - u^{(n-1)}_{\epsilon})),\end{aligned}$$

with initial datum $(u^{(n+1)} - u^{(n)})(x,0) = \Delta_{n+1}u_0$. Proceeding as in the proof of (4.6), we obtain the estimate

$$\|\nabla\cdot(u^{(n)}\nabla(-\Delta)^{-s}(u^{(n)}_{\epsilon} - u^{(n-1)}_{\epsilon}))\|_{B^{\alpha+2s-2}_{1,\infty}} \le C\|u^{(n)} - u^{(n-1)}\|_{B^{\alpha-1}_{1,\infty}}\|u^{(n)}\|_{B^{\alpha}_{1,\infty}}.$$

Proceeding as in the proof of (4.8), we obtain the estimate

$$\|\nabla\cdot((u^{(n+1)} - u^{(n)})\nabla(-\Delta)^{-s}u^{(n)}_{\epsilon})\|_{B^{\alpha+2s-2}_{1,\infty}} \le C\|u^{(n+1)} - u^{(n)}\|_{B^{\alpha+2s-2}_{1,\infty}}\|u^{(n)}\|_{B^{\alpha}_{1,\infty}}.$$

Then we obtain

$$\begin{aligned}\frac{d}{dt}\|u^{(n+1)} - u^{(n)}\|_{B^{\alpha+2s-2}_{1,\infty}} &\le C\|u^{(n+1)} - u^{(n)}\|_{B^{\alpha+2s-2}_{1,\infty}}\|u^{(n)}\|_{B^{\alpha}_{1,\infty}} \\ &\quad + C\|u^{(n)} - u^{(n-1)}\|_{B^{\alpha+2s-2}_{1,\infty}}\|u^{(n)}\|_{B^{\alpha}_{1,\infty}},\end{aligned}$$

and

$$\begin{aligned}&\|(u^{(n+1)} - u^{(n)})(x,0)\|_{B^{\alpha+2s-2}_{1,\infty}} \\ &= \|\Delta_{n+1}u_0\|_{B^{\alpha+2s-2}_{1,\infty}} = \sup_j 2^{j(\alpha+2s-2)}\|\Delta_j\Delta_{n+1}u_0\|_{L^1} \\ &\le C2^{n(2s-2)} \sup_{n\le j\le n+2} 2^{j\alpha}\|\Delta_j u_0\|_{L^1}\end{aligned}$$

$$\le C2^{n(2s-2)}\|u_0\|_{B^{\alpha}_{1,\infty}}.$$

The above inequalities and Gronwall's inequality imply that

$$\begin{aligned}
&\|u^{(n+1)}(t)-u^{(n)}(t)\|_{B^{\alpha+2s-2}_{1,\infty}}\\
&\le \|(u^{(n+1)}-u^{(n)})(x,0)\|_{B^{\alpha+2s-2}_{1,\infty}}\exp(C\|u_0\|_{B^{\alpha}_{1,\infty}})\\
&\quad+\int_0^t \exp(C\|u_0\|_{B^{\alpha}_{1,\infty}}(t-s))\|u^{(n)}-u^{(n-1)}\|_{B^{\alpha+2s-2}_{1,\infty}}(s)ds\\
&\le C'2^{n(2s-2)}+C'\|u^{(n)}-u^{(n-1)}\|_{Y_{T_1}}(\exp C'T_1-1),
\end{aligned} \tag{4.10}$$

where $T_1\in[0,T_0]$, and the constant $C'=C'(\|u_0\|_{B^{\alpha}_{1,\infty}})$. Thus, if $C'(\exp T_1-1)<\frac{1}{2}$, we can deduce that $u^{(n)}$ converges to $u\in L^\infty([0,T_1];B^{\alpha+2s-2}_{1,\infty})$ in Y_{T_1}. By the well-known interpolation inequality in the Besov spaces we have $u^n\to u$ in $L^\infty([0,T_1];B^{\beta}_{1,\infty})$ for all $\beta\in[\alpha+2s-2,\alpha]$. Moreover, the estimate (4.8) allows us to conclude that $u\in C^1([0,T_1];B^{\alpha+2s-2}_{1,\infty})$. Letting $\epsilon\to 0, n\to\infty$, we find that u is a solution to (1.1) in $B^{\beta}_{1,\infty}$.

Next we prove that the solution is unique. Suppose that u, v in $L^\infty([0,T_1];B^{\beta}_{1,\infty})$ are two solutions of (1.1) associated with the initial condition u_0, v_0. Then $u-v$ satisfies the equation

$$\begin{gathered}
u-v=\nabla\cdot[(u-v)\nabla(-\Delta)^{-s}u]+\nabla\cdot[v\nabla(-\Delta)^{-s}(u-v)],\\
(u-v)(x,0)=u_0-v_0.
\end{gathered}$$

Working similarly with (4.10), we can obtain

$$\begin{aligned}
\|(u-v)(t)\|_{B^{\beta}_{1,\infty}}&\le M2^{j(\beta-\alpha)}\|(u-v)(x,0)\|_{B^{\beta}_{1,\infty}}\\
&\quad+M\|u-v\|_{C([0,T_1];B^{\beta}_{1,\infty})}(\exp C''T_1-1),
\end{aligned}$$

for $M=M(\|(u_0\|_{B^{\beta}_{1,\infty}},\|v_0\|_{B^{\beta}_{1,\infty}})$. If $M(\exp C''T_1-1)<1$, then we get the uniqueness of solutions in $C([0,T_1];B^{\beta}_{1,\infty})$. The proof is complete.

5. Proof of Theorem 1.2

When $d\ge 3$, we know $(-\Delta)^{-1}u=c(n,s)|x|^{-d+2}*u$. Let $P_\epsilon u=c(n,s)(|x|^{-d+2}*\sigma_\epsilon)*u=(-\Delta)^{-1}(\sigma_\epsilon*u)=(-\Delta)^{-1}u_\epsilon$, where σ is defined in the beginning of Section 4. Consider the successive approximation sequence $\{u^{(n)}\}$ satisfying

$$\begin{gathered}
u^{(1)}=S_2(u_0)\\
\partial_t u^{(n+1)}=\nabla\cdot(u^{(n+1)}\nabla(P_\epsilon u^{(n)}_\epsilon))=\nabla u^{(n+1)}\cdot\nabla(-\Delta)^{-1}u^{(n)}_\epsilon-u^{(n+1)}u^{(n)}_\epsilon\\
u^{(n+1)}(x,0)=u_0^{n+1}=S_{n+2}u_0.
\end{gathered} \tag{5.1}$$

Appling Δ_j on (5.1),

$$\begin{aligned}
\partial_t\Delta_j u^{(n+1)}&=\sum[\Delta_j,\partial_i(-\Delta)^{-s}u^{(n)}_\epsilon]\partial_i u^{(n+1)}\\
&\quad+\nabla(-\Delta)^{-s}u^{(n)}_\epsilon\nabla\Delta_j(u^{(n+1)})-\Delta_j(u^{(n+1)}u^{(n)}_\epsilon).
\end{aligned} \tag{5.2}$$

When $p\ge 2$, multiplying both sides of (5.2) by $p\Delta_j u^{(n+1)}|\Delta_j u^{(n+1)}|^{p-2}$ and integrating over $\mathbb{R}^d$, we obtain

$$\frac{d}{dt}\|\Delta_j u^{(n+1)}\|^p_{L^p}=\sum\int p\Delta_j u^{(n+1)}|\Delta_j u^{(n+1)}|^{p-2}[\Delta_j,\partial_i(-\Delta)^{-1}u^{(n)}_\epsilon]\partial_i u^{(n+1)}$$

$$\begin{aligned}&+\int p\Delta_j u^{(n+1)}|\Delta_j u^{(n+1)}|^{p-2}\nabla(-\Delta)^{-1}u_\epsilon^{(n)}\nabla\Delta_j(u^{(n+1)})\\&-\int p\Delta_j u^{(n+1)}|\Delta_j u^{(n+1)}|^{p-2}\Delta_j(u^{(n+1)}u_\varepsilon^{(n)}) = J_1''+J_2''+J_3''.\end{aligned}$$

By Hölder's inequality,

$$J_1'' \le C\|\Delta_j u^{(n+1)}\|_{L^p}^{p-1}\|[\Delta_j,\partial_i(-\Delta)^{-1}u_\epsilon^{(n)}]\partial_i u^{(n+1)}\|_{L^p}.$$

From the proof of Proposition 2.3 (iv),

$$J_3'' \le \|\Delta_j u^{(n+1)}\|_{L^p}^{p-1}[\|u^{(n)}\|_{L^\infty}\|\Delta_j u^{(n+1)}\|_{L^p}+\|u^{(n+1)}\|_{L^\infty}\|\Delta_j u^{(n)}\|_{L^p}].$$

Also we obtain

$$\begin{aligned}J_2'' &= \int \nabla(-\Delta)^{-1}u_\epsilon^{(n)}\nabla(|\Delta_j(u^{(n+1)})|^p)\\&= \int u_\epsilon^{(n)}|\Delta_j u^{(n+1)}|^p dx \le \|u^{(n)}\|_{L^\infty}\|\Delta_j u^{(n+1)}\|_{L^p}^p.\end{aligned}$$

Now we obtain

$$\begin{aligned}\frac{d}{dt}\|\Delta_j u^{(n+1)}\|_{L^p} &\le C\|[\Delta_j,\partial_i(-\Delta)^{-1}u^{(n)}]\partial_i u^{(n+1)}\|_{L^p}\\&\quad + C\|u^{(n)}\|_{L^\infty}\|\Delta_j u^{(n+1)}\|_{L^p}+C\|u^{(n+1)}\|_{L^\infty}\|\Delta_j u^{(n)}\|_{L^p}.\end{aligned}$$

When, $\alpha > d/p$,

$$\frac{d}{dt}\|u^{(n+1)}\|_{\dot B^\alpha_{p,\infty}} \le C\|u^{(n+1)}\|_{B^\alpha_{p,\infty}}\|u^{(n)}\|_{B^\alpha_{p,\infty}}.$$

Letting $p\to\infty$,

$$\begin{aligned}\frac{d}{dt}\|\Delta_j u^{(n+1)}\|_{L^\infty} &\le C\|[\Delta_j,\partial_i(-\Delta)^{-1}u^{(n)}]\partial_i u^{(n+1)}\|_{L^\infty}\\&\quad + C\|u^{(n)}\|_{L^\infty}\|\Delta_j u^{(n+1)}\|_{L^\infty}+C\|\Delta_j(u^{(n+1)}u^{(n)})\|_{L^\infty}.\end{aligned}$$

So for any $\alpha>0$,

$$\frac{d}{dt}\|u^{(n+1)}\|_{\dot B^\alpha_{\infty,\infty}} \le C\|u^{(n+1)}\|_{B^\alpha_{\infty,\infty}}\|u^{(n)}\|_{B^\alpha_{\infty,\infty}}.$$

It is easy to prove that when $p=1$,

$$\begin{aligned}\frac{d}{dt}\int |u^{(n+1)}|dx &= \int \frac{u^{(n+1)}}{|u^{(n+1)}|}u_t^{(n+1)}dx\\&= \int \nabla\cdot(|u^{(n+1)}|(|x|^{-d+2}*\sigma_\epsilon)*u^{(n)}) = 0.\end{aligned}$$

Noting $u^{(n+1)}(x,0)=u_0^{n+1}=S_{n+2}u_0$, we know $\|u^{(n+1)}\|_{L^1}\le C$.

When $p\ge 2$,

$$\begin{aligned}\frac{d}{dt}\int |u^{(n+1)}|^p dx &= p\int |u^{(n+1)}|^{p-2}u^{(n+1)}u_t^{(n+1)}dx\\&= p\int |u^{(n+1)}|^{p-2}u^{(n+1)}\nabla\cdot(u^{(n+1)}\nabla(-\Delta)^{-1}u_\epsilon^{(n)})\\&= -(p-1)\int \nabla|u^{(n+1)}|^p\cdot(\nabla(-\Delta)^{-1}u_\epsilon^{(n)})\\&= (p-1)\int |u^{(n+1)}|^p u_\epsilon^{(n)} \le C\|u^{(n+1)}\|_{L^p}^p\|u^{(n)}\|_{L^\infty}.\end{aligned}$$

It means

$$\frac{d}{dt}\|u^{(n+1)}\|_{L^p} \le \|u^{(n+1)}\|_{L^p}\|u^{(n)}\|_{L^\infty} \le C\|u^{(n+1)}\|_{B^\alpha_{p,\infty}}\|u^{(n)}\|_{B^\alpha_{p,\infty}}.$$

Again letting $p \to \infty$, we obtain

$$\frac{d}{dt}\|u^{(n+1)}\|_{L^\infty} \le C\|u^{(n+1)}\|_{B^\alpha_{\infty,\infty}}\|u^{(n)}\|_{B^\alpha_{\infty,\infty}}.$$

Collecting the estimates above, we now obtain

$$\begin{aligned}\frac{d}{dt}\|u^{(n+1)}\|_{B^\alpha_{p,\infty}} &\le C\|u^{(n+1)}\|_{B^\alpha_{p,\infty}}\|u^{(n)}\|_{B^\alpha_{p,\infty}}, \quad p<\infty,\ \alpha>\frac{d}{p},\\ \frac{d}{dt}\|u^{(n+1)}\|_{B^\alpha_{\infty,\infty}} &\le C\|u^{(n+1)}\|_{B^\alpha_{\infty,\infty}}\|u^{(n)}\|_{B^\alpha_{\infty,\infty}}, \quad p=\infty,\ \alpha>0.\end{aligned}$$

For any $\beta \in (d/p, \alpha)$, $p < \infty$, or $\beta \in (0, \alpha)$, $p = \infty$ we have

$$\frac{d}{dt}\|u^{(n+1)}\|_{B^\beta_{p,\infty}} \le \|u^{(n+1)}\|_{B^\beta_{p,\infty}}\|u^{(n)}\|_{B^\beta_{p,\infty}},$$

and

$$\begin{aligned}\frac{d}{dt}\|u^{(n+1)} - u^{(n)}\|_{B^\beta_{p,\infty}} &\le \|u^{(n+1)} - u^{(n)}\|_{B^\beta_{p,\infty}}\|u^{(n)}\|_{B^\beta_{p,\infty}}\\ &\quad + \|u^{(n)} - u^{(n-1)}\|_{B^\beta_{p,\infty}}\|u^{(n)}\|_{B^\beta_{p,\infty}}.\end{aligned}$$

The rest of the proof is similar to the proof of Theorem 1.1, we omit it.

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