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# ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO SECOND-ORDER DIFFERENTIAL EQUATIONS WITH FRACTIONAL DERIVATIVE PERTURBATIONS

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Abstract. In this article we study the asymptotic behavior of solutions to nonlinear second-order differential equations having perturbations that involve Caputo's derivatives of several fractional orders. We find sufficient conditions for all solutions to be asymptotic to a straight line.

## 1. Introduction

The aim of this article is to study the asymptotic properties of solutions to scalar second-order ordinary differential equations that are perturbed with a term involving fractional derivatives. In these equations, the fractional derivatives most frequently used are the Riemann-Liouville and the Caputo's fractional derivatives. For basic definitions of fractional calculus and fundamentals of the theory of fractional differential equations, we refer the reader to the monographs [28, 29].

Fractional derivatives play the role of a damping force in vibrating systems in viscous fluids; which is the case in the well known Bargley-Torvik equation,

$$u''(t) + A^c D^{3/2} u(t) = au(t) + \phi(t)\,. \tag{1.1}$$

This equation models the motion of a rigid plate immersing in a viscous liquid with the fractional damping term $A^c D^{3/2} u(t)$ which has Caputo's fractional derivative (see [36]). Solutions of the linear fractionally damped oscillator equation with the Caputo's derivative are analyzed in [26]. Existence results on boundary-value problems for the generalized Bagely-Torvik equation

$$u''(t) + A\,{}^c D^{\alpha} u(t) = f(t, u(t), {}^c D^{\beta} u(t), u'(t)) \tag{1.2}$$

and for some other fractional differential equations can be found in [2, 3, 34, 27]. An existence and uniqueness result for the multi-fractional initial-value problem

$$\begin{gathered} Au'' + \sum_{k=1}^{N} B_k\,{}^c D^{\alpha_k} u(t) = f(t, u), \\ u(0) = u_0, \quad u'(0) = c_1, \quad 0 < \alpha_k < 2,\ k = 1, 2, \dots, N \end{gathered} \tag{1.3}$$

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can be found in [33]. Caputo's fractional derivatives in equation (1.3) play the role of damping terms. Abstract evolution equations with the Caputo's fractional derivatives in the nonlinearities are studied in [13, 14] . Fractionally damped pendulums or oscillators are studied in [26, 33]. More articles devoted to this type of equations can be found in the list of references.

The following equation for a pendulum has the ordinary damping term $\lambda x'(t)$ and the fractional damping terms $\lambda_1\, {}^cD^{\beta_1}x(t), \dots, \lambda_m\, {}^cD^{\beta_m}x(t)$:

$$x''(t) + \lambda_1\, {}^cD^{\beta_1}x(t) + \dots \lambda_m\, {}^cD^{\beta_m}x(t) + \lambda x'(t) + \omega^2 x(t) = g(t, x(t), x'(t)),$$

where $t > 0$, $\beta_i \in (0,1)$, $i = 1, 2, \dots, m$.

In [26], the equation

$$x'' + \lambda_0^c D^\alpha x + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1, \quad \lambda > 0.$$

is analyzed by using the fractional version of the Laplace transformation. The Laplace image of $x(t)$ is

$$X(s) = \frac{sx_0 + x_1 + \lambda s^{\alpha-1}x_0}{s^2 + \lambda s^\alpha + \omega^2},$$

and the characteristic equation for the fractional differential equation is

$$s^2 + \lambda s^\alpha + \omega^2 = 0.$$

When $\alpha = p/q$ this characteristic equation is was analyzed in [28]. For the linear fractionally damped oscillator with $\alpha = 1/2$ the characteristic equation is

$$s^2 + \lambda s^{1/2} + \omega^2 = 0,$$

whose analysis is much more complicated than in the case of the harmonic oscillator with the classical damping term (see [26]). It is clear that the exact analysis of linear fractional systems is extraordinary difficult. Some analysis and simulations of fractional-order systems can be found in the book [28]. The form of the equation (1.3) enables us to avoid some difficulties in the study of the stability problem by using a desingularization method developed in [19, 20, 22].

In the asymptotic theory of the $n$-th order nonlinear ordinary differential equations

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)}), \tag{1.4}$$

a classical problem is to establish some conditions for the existence of a solution which approaches a polynomial of degree $1 \le m \le n-1$ as $t \to \infty$. The first paper concerning this problem was published by Caligo [7] in 1941. He proved that if

$$|A(t)| < \frac{k}{t^{2+\rho}} \tag{1.5}$$

for all large $t$, where $k, \rho$ are given, then any solution $y(t)$ of the linear differential equation

$$y''(t) + A(t)y(t) = 0, \quad t > 0, \tag{1.6}$$

can be represented asymptotically as $y(t) = c_1 t + c_2 + o(1)$ when $t \to +\infty$, with $c_1, c_2 \in \mathbb{R}$ (see [1]). The first article on the nonlinear second-order differential equation

$$y''(t) + f(t, y(t)) = 0 \tag{1.7}$$

was published by Trench [37] in 1963. Then there are publications by Cohen [9], Trench [37], Kusano and Trench [15] and [16], Dannan [12], Constantin [10] and [11], Rogovchenko [31], Rogovchenko [32], Mustafa, Rogovchenko [25], Lipovan [17] and others. In the proofs of their results the key role is played by the Bihari inequality [6] which is a generalization of the Gronwall inequality. Some results on the existence of solutions of the $n$-th order differential equation approaching a polynomial function of the degree $m$ with $1 \le m \le n-1$ are proved by Philos, Purnaras and Tsamatos [30]. Their proofs are based on an application of the Schauder Fixed Point Theorem. The paper by Agarwal, Djebali, Moussaoui and Mustafa [1] surveys the literature concerning the topic in asymptotic integration theory of ordinary differential equations. Several conditions under which all solutions of the one-dimensional $p$-Laplacian equation

$$(|y'|^{p-1}y')' = f(t,y,y'), \quad p>1 \tag{1.8}$$

are asymptotic to $a+bt$ as $t\to\infty$ for some real numbers $a,b$ are proved in [24]. Some sufficient conditions for the existence of such solutions of the equation

$$(\Phi(y^{(n)})' = f(t,y), \quad n \ge 1, \tag{1.9}$$

where $\Phi:\mathbb{R}\to\mathbb{R}$ is an increasing homeomorphism with a locally Lipschitz inverse satisfying $\Phi(0)=0$ are given in the paper [21].

In the papers [22, 23] the fractional differential equation of the Caputo's type

$${}^cD_a^\alpha x(t) = f(t,x(t)), \quad a\ge 1,\ \alpha\in(1,2) \tag{1.10}$$

is studied. In [23] a higher order fractional differential equation is studied. In the both papers sufficient conditions under which all solutions of these equations are asymptotic to $at+b$, is proved. The problem of asymptotic integration of fractional differential equations of the Riemann-Liouville type is studied in [4, 5]. The obtained results are proved by an application of the fixed point method.

The aim of this paper is to give some conditions under which all solutions of a nonlinear second order differential equations perturbed by the Riemann-Liouville integral of a nonlinear function are asymptotic to $at+b$. The proof of this result is based on a desingularization method proposed by the author in the paper [19] (see also [20]).

## 2. Second-order ODEs perturbed with a fractional derivative

In this section we study the following fractional initial-value problem

$$u''(t) + f(t,u(t),u'(t)) + \sum_{i=1}^m r_i(t)\int_0^t (t-s)^{\alpha_i-1}h_i(\tau,u(\tau),u'(\tau))d\tau = 0, \tag{2.1}$$

$$u(1)=c_1 \quad u'(1)=c_2, \tag{2.2}$$

where $t>0$ and $0<\alpha<1$.

**Definition 2.1.** A function $u:[0,T)\to\mathbb{R}$, $0<T\le\infty$, is called a solution of (2.1) if $u\in C^2$ on the interval $(0,T)$, $\lim_{\tau\to 0^+}u(t)$ exists and $u(t)$ satisfies (2.1) on the interval $(0,T)$. This solution is called global if it exists for all $t\in[0,\infty)$.

We assume the following hypotheses:

- (H1) Every solution of the equation (2.1) is global;
- (H2) The functions $f(t,u,v), h_i(t,u,v)$, $i=1,2,\dots,m$ are continuous on $D=\{(t,u,v): t\in[0,\infty),\ u,v\in\mathbb{R}\}$ and the functions $r_i(t)$, $i=1,2,\dots,m$ are continuous on the interval $[0,\infty)$;

(H3) There exist continuous, nonnegative functions $h_i : [0,\infty) \to \mathbb{R}$, $i = 1,2,3$ and continuous, positive and nondecreasing functions $g_j : [0,\infty) \to \mathbb{R}$ such that
$$|f(t,u,v)| \le Se^{-\gamma t}\Big(h_1(t)g_1\big(\frac{|u|}{t}\big) + h_2(t)g_2(|v|) + h_3(t)\Big), \quad t > 0,$$
where $S, \gamma > 0$;

(H4) There exist continuous, nonnegative functions $h_{ij} : [0,\infty) \to \mathbb{R}$, $i = 1,2,\dots,m$; $j = 1,2,3$ and continuous positive, nondecreasing functions $G_{ij} : [0,\infty) \to \mathbb{R}$, $i = 1,2,\dots,m$; $j = 1,2,3$ such that
$$|f_i(t,u,v)| \le h_{1i}(t)G_{ij}\big(\frac{|u|}{t}\big) + h_{2i}(t)G_{2i}(|v|) + h_{3i}(t), \quad t > 0;$$
for all $(t,u,v) \in D$, $i = 1,2,\dots,m$;

(H5) $|r_i(t)| \le S_i e^{-\omega_i t}$, $t \ge 0$, where $S_i > 0$, $\omega_i > 1$, $i = 1,2,\dots,m$;

(H6) There exist numbers $p_i > 1$, $i = 1,2,\dots,m$ such that $p_i(\alpha_i - 1) + 1 > 0$ with
$$\int_0^\infty h_i(s)^q < \infty, \int_0^\infty h_{ij}(s)^q < \infty, \quad i = 1,2,\dots,m;\ j = 1,2,3,$$
where $q = q_1 q_2 \dots q_m$, $q_i = p_i/(p_i - 1)$, $i = 1,2,\dots,m$;

(H7)
$$\int_0^\infty \frac{\tau^{q-1} d\tau}{\omega(\tau)} = \infty,$$
where
$$\omega(w) = g_1(w)^q + g_2(w)^q + \sum_{i=1}^m \sum_{j=1}^2 G_{ij}(w)^q.$$

**Theorem 2.2.** *If the conditions* (H1)–(H7) *are satisfied then for every global solution $u(t)$ of* (2.1) *there exist real numbers $a, b$ such that $u(t) = at + b + o(t)$ as $t \to \infty$.*

For the proof of this theorem we use the following lemma, proved in [19].

**Lemma 2.3.** *Let $p_j, \alpha_j$, $j = 1,2,\dots,m$ satisfy* (H4). *Then*
$$\int_0^t (t-s)^{p_j(\alpha_j - 1)} e^{p_j s} ds \le Q_j e^{p_j t}, \quad t \ge 0,\ j = 1,2,\dots,m,$$
*where*
$$Q_j = \frac{\Gamma(1 + p_j(\alpha_j - 1))}{p^{1 + p_j(\alpha_j - 1)}},$$
*and*
$$\Gamma(x) = \int_0^\infty s^{x-1} e^{-s} ds, \quad x > 0$$
*which is the Euler gamma function.*

*Proof of Theorem 2.2.* Let $u(t)$ be a solution of (2.1) corresponding to the initial conditions (2.2). Then
$$\begin{aligned} u'(t) &= c_2 - \int_1^t f(s,u(s),u'(s)) ds \\ &\quad - \sum_{i=1}^m \int_1^t r_i(s) \int_0^s (s-\tau)^{\alpha_i - 1} f_i(\tau, u(\tau), u'(\tau)) d\tau ds, \end{aligned} \tag{2.3}$$

$$
\begin{aligned}
u(t) = c_1 + c_2(t-1) - \int_1^t (t-s) f(s, u(s), u'(s)) ds \\
- \sum_{i=1}^m \int_1^t (t-s) r_i(s) \Big( \int_0^s (s-\tau)^{\alpha_i - 1} f_i(\tau, u(\tau), u'(\tau)) d\tau \Big) ds.
\end{aligned} \tag{2.4}
$$

From conditions (H3)–(H5) it follows that for $t \geq 1$,

$$
\begin{aligned}
|u'(t)| \leq |c_2| + \int_1^t [h_1(s) g_1(\frac{|u(s)|}{s}) + h_2(s) g_2(|u'(s)|) + h_3(s)] ds \\
+ \sum_{i=1}^m \int_1^t |r_i(s)| \int_0^s (s-\tau)^{\alpha_i - 1} \Big[ h_{1i}(\tau) G_{1i}(\frac{|u(\tau)|}{\tau}) \\
+ h_{2i}(\tau) G_{2i}(|u'(\tau)|) + h_{3i}(\tau) \Big] d\tau ds
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{|u(t)|}{t} \leq C + \int_1^t [h_1(s) g_1(\frac{|u(s)|}{s}) + h_2(s) g_2(|u'(s)|) + h_3(s)] ds \\
+ \sum_{i=1}^m \int_1^t |r_i(s)| \int_0^s (s-\tau)^{\alpha_i - 1} \Big[ h_{1i}(\tau) G_{1i}(\frac{|u(\tau)|}{\tau}) \\
+ h_{2i}(\tau) G_{2i}(|u'(\tau)|) + h_{3i}(\tau) \Big] d\tau ds,
\end{aligned}
$$

where $C = |c_1| + |c_2|$. If $q_i = p_i/(p_i - 1)$ then using Lemma 2.3 and the Hölder inequality we estimate

$$
\begin{aligned}
&\int_0^s (s-\tau)^{\alpha_i - 1} k_{1i}(\tau) G_{1i}(\frac{|u(\tau)|}{\tau}) d\tau \\
&\leq \Big( \int_0^s (s-\tau)^{p_i(\alpha_i - 1)} e^{p_i \tau} d\tau \Big)^{1/p_i} \Big( \int_0^s e^{-q_i \tau} h_{1i}(\tau)^{q_i} G_{1i}(\frac{|u(\tau)|}{\tau})^{q_i} d\tau \Big)^{1/q_i} \\
&\leq Q_i e^s \Big( \int_0^s e^{-q_i \tau} h_{1i}(\tau)^{q_i} G_{1i}(\frac{|u(\tau)|}{\tau})^{q_i} d\tau \Big)^{1/q_i},
\end{aligned}
$$

$$
\int_0^s (s-\tau)^{\alpha_i - 1} h_{2i}(\tau) G_{2i}(|u'(\tau)|) d\tau \leq Q_i e^s \Big( \int_0^s e^{-q_i \tau} h_{2i}(\tau)^{q_i} G_{2i}(|u'(\tau)|)^{q_i} d\tau \Big)^{1/q_i},
$$

$$
\int_0^s (s-\tau)^{\alpha_i - 1} h_{3i}(\tau) d\tau \leq Q_i e^s \Big( \int_0^s e^{-q_i \tau} h_{3i}(\tau)^{q_i} d\tau \Big)^{1/q_i}.
$$

These inequalities yield

$$
\begin{aligned}
\frac{|u(t)|}{t} &\leq C + S \int_1^t e^{-\gamma s} \Big( h_1(s) g_1\Big(\frac{|u(s)|}{s}\Big) + h_2(s) g_2(|u'(s)|) + h_3(s) \Big) ds \\
&+ \sum_{i=1}^m S_i Q_i \int_1^t e^{-(\omega_i - 1)s} \Big\{ \Big( \int_0^s e^{-q_i \tau} h_{1i}(\tau)^{q_i} G_{1i}(\frac{|u(\tau)|}{\tau})^{q_i} d\tau \Big)^{1/q_i} \\
&+ \Big( \int_0^s e^{-q_i \tau} h_{2i}(\tau)^{q_i} G_{2i}(|u'(\tau)|)^{q_i} d\tau \Big)^{1/q_i} + \Big( \int_0^s e^{-q_i \tau} h_{3i}(\tau)^{q_i} d\tau \Big)^{1/q_i} \Big\} ds
\end{aligned}
$$

Since $\omega_i > 1$ and $\gamma > 0$, we have the estimate

$$
\frac{|u(t)|}{t} \leq C + S \int_0^t e^{-\gamma s} \Big( h_1(s) g_1 \frac{|u(s)|}{s} ) + h_2(s) g_2(|u'(s)|) + h_3(s) \Big) ds
$$

$$
\begin{aligned}
&+\sum_{i=1}^{m} S_i\frac{Q_i}{\omega_i-1}\Big\{\Big(\int_0^t e^{-q_i\tau}h_{1i}(\tau)^{q_i}G_{1i}\big(\frac{|u(\tau)|}{\tau}\big)^{q_i}d\tau\Big)^{1/q_i}\\
&+\Big(\int_0^t e^{-q_i\tau}h_{2i}(\tau)^{q_i}G_{2i}(|u'(\tau)|)^{q_i}d\tau\Big)^{1/q_i}+\Big(\int_0^t e^{-q_i\tau}h_{3i}(\tau)^{q_i}d\tau\Big)^{1/q_i}d\tau\Big\}.
\end{aligned}
$$

Denoting by $z(t)$ the right-hand side of this inequality, we have

$$
\frac{|u(t)|}{t}\le z(t),\quad |u'(t)|\le z(t),\quad t\ge 0.
$$

Since $g_1, g_2, G_{1i}, G_{2i}, G_{3i}$ are nondecreasing functions these inequalities yield

$$
\begin{aligned}
z(t)\le{}& C+S\int_0^t e^{-\gamma s}\Big(h_1(s)g_1(z(s))+h_2(s)g_2(z(s))+h_3(s)\Big)ds\\
&+\sum_{i=1}^{m} S_i\frac{Q_i}{\omega_i-1}\Big\{\Big(\int_0^t e^{-q_i\tau}h_{1i}(\tau)^{q_i}G_{1i}(z(\tau))^{q_i}d\tau\Big)^{1/q_i}\\
&+\Big(\int_0^t e^{-q_i\tau}h_{2i}(\tau)^{q_i}G_{2i}(z(\tau))^{q_i}d\tau\Big)^{1/q_i}+\Big(\int_0^t e^{-q_i\tau}h_{3i}(\tau)^{q_i}d\tau\Big)^{1/q_i}d\tau\Big\}.
\end{aligned}
$$

Let $Q=\max\{\frac{S_iQ_i}{\omega_i-1}, i=1,2,\dots,m\}$ and $q=q_1q_2\dots q_m$. Then using the inequality $(\sum_{i=1}^{3m+2}a_i)^q\le(3m+2)^{q-1}(\sum_{i=1}^{3m+2}a_i^q)$ for any nonnegative numbers $a_i$, $i=1,2,\dots,3m+2$, we obtain the estimate

$$
\begin{aligned}
&z(t)^q\\
&\le(3m+2)^{q-1}\Big(C^q+S^q\int_1^t e^{-\gamma s}\Big(\int_1^t(h_1(s)g_1(z(s))+h_2(s)g_2(z(s))+h_3(s))ds\Big)^q\\
&+Q^q\sum_{i=1}^{m}\Big\{\Big(\int_0^t e^{-q_i\tau}h_{1i}(\tau)^{q_i}G_{1i}(z(\tau))^{q_i}d\tau\Big)^{\hat q_i}\\
&+\Big(\int_0^t e^{-q_i\tau}h_{2i}(\tau)^{q_i}G_{2i}(z(\tau))^{q_i}d\tau\Big)^{\hat q_i}+\Big(\int_0^t e^{-q_i\tau}h_{3i}(\tau)^{q_i}d\tau\Big)^{\hat q_i}d\tau\Big\},
\end{aligned}
$$

where $\hat q_i=q_1q_2\dots q_{i-1}q_{i+1}\dots q_m$. If $\hat p_i=\frac{\hat q_i}{\hat q_i-1}$ and $p=\frac{q}{q-1}$, then using the Hölder inequality we obtain the following inequalities

$$
\begin{aligned}
&\int_0^t e^{-\gamma s}\Big\{\int_1^s\Big(h_1(\tau)g_1(z(\tau))+h_2(\tau)g_2(z(\tau))+h_3(\tau)\Big)d\tau\Big\}^q ds\\
&\le\big(\frac{1}{p\gamma}\big)^{1/p}\int_0^t\Big(h_1(s)g_1(z(s))+h_2(s)g_2(z(s))+h_3(s)\Big)^q ds\\
&\le 3^{q-1}\big(\frac{1}{p\gamma}\big)^{1/p}\int_0^t\Big(h_1(s)^qg_1(z(s))^q+h_2(s)^qg_2(z(s))^q+h_3(s)^q\Big)ds,
\end{aligned}
$$

$$
\begin{aligned}
&\Big(\int_0^t e^{-q_i\tau}h_{1i}(\tau)^{q_i}G_{1i}(z(\tau))^{q_i}d\tau\Big)^{\hat q_i}\\
&\le\Big(\int_0^t e^{-\hat p_iq_is}ds\Big)^{\frac{1}{\hat p_i}}\Big(\int_0^t h_{1i}(s)^qG_{1i}(z(s))^q ds\Big)\\
&\le\frac{1}{(\hat p_iq_i-1)^{1/\hat p_i}}\int_0^t h_{1i}(s)^qG_{1i}(z(s))^q ds,
\end{aligned}
$$

$$\Big(\int_0^t e^{-q_i\tau} h_{2i}(\tau)^{q_i} G_{2i}(z(\tau))^{q_i} d\tau\Big)^{\hat{q}_i} \le \frac{1}{(\hat{p}_i q_i - 1)^{1/\hat{p}_i}} \int_0^t h_{2i}(s)^q G_{2i}(z(s))^q ds,$$
$$\int_0^t e^{-q_i s} h_{3i}(s)^{q_i} ds \le \frac{1}{(\hat{p}_i q_i - 1)^{1/\hat{p}_i}} \int_0^t h_{3i}(s)^q ds.$$

From these inequalities and (H6) it follows that there exist a constant $A > 0$ such that

$$\begin{aligned} z(t)^q &\le A + A\int_0^t [h_1(s)^q g_1(z(s))^q + h_2(s)^q g_2(z(s)) + h_3(s)^q] ds \\ &\quad + A\sum_{i=1}^m \int_0^t h_{1i}(s)^q G_{1i}(z(s))^q ds + A\sum_{i=1}^m \int_0^t h_{2i}(s)^q G_{2i}(z(s))^q ds. \end{aligned}$$

This inequality implies that the function $v(t) = z(t)^q$ satisfy the inequality

$$v(t) \le A + \int_0^t F(s)\omega(v(s)^{\frac{1}{q}}) ds, \quad t \ge 0,$$

where

$$\begin{gathered} \omega(z) = g_1(z)^q + g_2(z)^q + \sum_{i=1}^m [G_{1i}(z)^q + G_{2i}(z)^q], \\ F(t) = A\Big(h_1(t)^q + h_2(t)^q + \sum_{i=1}^m [h_{1i}(t)^q + h_{2i}(t)^q]\Big). \end{gathered}$$

From (H6) it follows that $\int_0^\infty F(s)ds < \infty$, and from the Bihari inequality we obtain

$$v(t) \le K_0 = \Omega^{-1}[\Omega(A) + \int_0^\infty F(s)ds] < \infty, \quad t \ge 0,$$

where

$$\Omega(u) = \int_{v_0}^v \frac{\sigma}{\omega(\sigma)}\,.$$

Note that $\Omega(A) + \int_0^\infty F(s)ds$ is always in the range of $\Omega^{-1}$, as $\omega(\infty) = \infty$ by (H7). This implies that there is a constant $K > 0$ such that

$$|u'(t)| \le z(t) \le K, \quad \frac{|u(t)|}{t} \le z(t) \le K, \quad t \ge 0.$$

In conclusion, we obtain the existence of the limit

$$\lim_{t\to\infty} \frac{|u(t)|}{t} = c,$$

which completes the proof. $\square$

## 3. Example

The following example is a fractional modification of the Caligo's example mentioned in the introduction.

$$
\begin{aligned}
&u''(t) + Se^{-\gamma t}\Big\{\omega^2 \frac{1}{(t+1)^{1+\frac{1}{q}}}\big(\frac{u(t)}{t}\big) + k_1 \frac{1}{(t+1)^{1+\frac{1}{q}}} u'(t) + k_2 \frac{1}{t^{1+\frac{1}{q}}}\Big\} \\
&+ \sum_{i=1}^{m} S_i e^{-\omega_i t} \int_0^t (t-s)^{\alpha_i - 1}\Big\{\frac{\eta_{1i}}{(s+1)^{1+\frac{1}{q_i}}} \ln\big[\big(\frac{u(s)}{s}\big)^{q_i} + 2\big]^{1/q_i} \\
&+ \frac{\eta_{2i}}{(s+1)^{1+\frac{1}{q_i}}}\big(\ln\big[u'(s)\big]^{q_i} + 2\big)^{1/q_i} + \frac{\eta_{3i}}{(s+1)^{1+\frac{1}{q_i}}}\Big\} ds = 0,
\end{aligned} \tag{3.1}
$$

where $S$, $\gamma$, $\omega$, $k_1$, $k_2$, $\eta_{1i}$, $\eta_{2i}$, $\eta_{3i}$, $i = 1, 2, \dots, m$ are positive numbers and $\gamma$, $\omega_i$, $q$, $q_i$, $\alpha_i$ satisfy the conditions in Theorem 2.2. Here

$$
h_i(t) = \frac{k_i}{(t+1)^{1+\frac{1}{q}}}, \quad h_{ji}(t) = \frac{\eta_{ji}}{(t+1)^{1+\frac{1}{q_i}}},
$$

$i = 1, 2, \dots, m$, $j = 1, 2, 3$, $g_1(u) = g_1(u) = [\ln(u^q + 2)]^{\frac{1}{q}}$, $g_{1i}(u) = g_{2i}(u) = [\ln(u^{q_i} + 2)]^{1/q_i}$. Since

$$
\int_0^\infty h_i(s)^q ds = \int_0^\infty \frac{1}{(s+1)^{1+q}} ds = \frac{1}{q}
$$

and

$$
\int_0^\infty \frac{\sigma^{q-1} d\sigma}{g_1(\sigma)^q} = \int_0^\infty \frac{\sigma^{q-1} d\sigma}{[\ln(\sigma^q + 2)]} = \frac{1}{q} \int_0^\infty \frac{d\tau}{\ln(\tau + 2)} = \infty,
$$

all conditions of Theorem 2.2 are satisfied and therefore for any solution of (3.1) there exist constants $a, b \in \mathbb{R}$ such that $u(t) = at + b + o(t)$ as $t \to \infty$.

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