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GENERAL BOUNDARY STABILIZATION RESULT OF MEMORY-TYPE THERMOELASTICITY WITH SECOND SOUND

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Abstract. In this article we consider an n-dimensional system of visco-thermoelasticity with second sound, where a viscoelastic dissipation is acting on a part of the boundary. We prove an explicit general decay rate result without imposing $u_0 = 0$ as in [17]. This allows a larger class of relaxation functions and initial data, hence, generalizes some previous results existing in the literature.

1. Introduction

The Classical Fourier law of heat conduction expresses that the heat flux within a medium is proportional to the local temperature gradient in the system. A well known consequence of this law is that heat perturbations propagates with an infinite speed. Experiments showed that heat conduction in some dielectric crystals at low temperatures is free of this paradox and disturbances, which are almost entirely thermal, propagate in a finite speed. This phenomenon in dielectric crystals is called second sound (see [6]). To overcome this physical paradox, Maxwell [7], Cattaneo [3] adopted a non-classical heat flux Maxwell-Cattaneo law to get rid of this unphysical results. This Maxwell-Cattaneo relation contains an extra inertial term with respect to the Fourier law

$$\tau_0 q_t + q + \kappa \nabla\theta = 0,$$

where $q = q(x,t) \in \mathbb{R}^n$ is the heat flux vector, τ_0 is the relaxation time and κ is the heat conductivity. The conservation of energy equation introduces the hyperbolic equation, which describes heat propagation with finite speed.

Result concerning existence, blow up, and asymptotic behavior of smooth, as well as weak solutions in thermoelasticity with second sound have been established over the past two decades by many mathematicians. Tarabek [20] treated problems related to

$$\begin{gathered} u_{tt} - a(u_x,\theta,q)u_{xx} + b(u_x,\theta,q)\theta_x = \alpha_1(u_x,\theta)qq_x \\ \theta_t + g(u_x,\theta,q)q_x + d(u_x,\theta,q)u_{tx} = \alpha_2(u_x,\theta)qq_t \\ \tau(u_x,\theta)q_t + q + k(u_x,\theta)\theta_x = 0, \end{gathered} \tag{1.1}$$

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in both bounded and unbounded situations and established global existence results for small initial data. He also showed that these “ classical” solutions tend to equilibrium as t tends to infinity; however, no rate of decay has been discussed. In his work, Tarabek used the usual energy argument and exploited some relations from the second law of thermodynamics to overcome the difficulty arising from the lack of Poincaré’s inequality in the unbounded domains. Racke [18] discussed (1.1) and established exponential decay results for several linear and nonlinear initial boundary value problems. In particular he studied (1.1), with $\alpha_1 = \alpha_2 = 0$, and for a rigidly clamped medium with temperature hold constant on the boundary. i.e

$$u(t,0) = u(t,1) = 0, \quad \theta(t,0) = \theta(t,1) = \bar{\theta}, \quad t \geq 0,$$

and showed that, for small enough initial data and classical solutions decay exponentially to the equilibrium state. Messaoudi and Said-Houari [9] extended the decay result of [18] to the case when $\alpha_1 \neq 0$, $\alpha_2 \neq 0$.

For the multi-dimensional case ($n = 2, 3$), Racke [19] established an existence result for the following n-dimensional problem

$$\begin{gathered} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div} u) + \beta\nabla\theta = 0 \quad \text{in } \Omega\times(0,+\infty) \\ \theta_t + \gamma \operatorname{div} q + \delta \operatorname{div} u_t = 0 \quad \text{in } \Omega\times(0,+\infty) \\ \tau q_t + q + \kappa\nabla\theta = 0 \quad \text{in } \Omega\times(0,+\infty) \\ u(.,0) = u_0, \quad u_t(.,0) = u_1, \quad \theta(.,0) = \theta_0, \quad q(.,0) = q_0 \quad \text{in } \Omega \\ u = \theta = 0 \quad \text{on } \Gamma\times[0,+\infty), \end{gathered} \tag{1.2}$$

where Ω is a bounded domain of $\mathbb{R}^n$, with a smooth boundary Γ, $u = u(x,t)$, $q = q(x,t) \in \mathbb{R}^n$, and $\mu, \lambda, \beta, \gamma, \delta, \tau, \kappa$, are positive constants, where μ, λ are Lame moduli and τ is the relaxation time, a small parameter compared to the others. In particular if $\tau = 0$, (1.2) reduces to the system of classical thermoelasticity, in which the heat flux is given by Fourier’s law instead of Cattaneo’s law. He also proved, under the conditions $rotu = rotq = 0$, an exponential decay result for (1.2). This result applies automatically to the radially symmetric solution, since it is only a special case.

Messaoudi [8] considered (1.2), in the presence of a source term, and proved a blow up result for solutions with negative initial energy. This result was extended later to certain solutions with positive energy by Messaoudi and Said-Houari [10].

In this article, we are concerned with the system

$$\begin{gathered} u_{tt} - \mu\Delta u - (\mu+\lambda)\nabla(\operatorname{div} u) + \beta\nabla\theta = 0 \quad \text{in } \Omega\times(0,+\infty) \\ c\theta_t + \kappa \operatorname{div} q + \beta \operatorname{div} u_t = 0 \quad \text{in } \Omega\times(0,+\infty) \\ \tau_0 q_t + q + \kappa\nabla\theta = 0 \quad \text{in } \Omega\times(0,+\infty) \\ u(.,0) = u_0, \quad u_t(.,0) = u_1, \quad \theta(.,0) = \theta_0, \quad q(.,0) = q_0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \Gamma_0\times[0,+\infty) \\ u(x,t) = -\int_0^t g(t-s)(\mu\frac{\partial u}{\partial\nu} + (\mu+\lambda)(\operatorname{div} u)\nu)(s)ds \quad \text{on } \Gamma_1\times[0,+\infty) \\ \theta = 0 \quad \text{on } \Gamma\times[0,+\infty), \end{gathered} \tag{1.3}$$

which models the transverse vibration of a thin elastic body, taking in account the heat conduction given by Cattaneo’s law. Here, Ω is a bounded domain of $\mathbb{R}^n$ ($n \geq 2$) with a smooth boundary Γ, such that $\{\Gamma_0, \Gamma_1\}$ is a partition of Γ, ν is the outward normal to Γ, $u = u(x,t) \in \mathbb{R}^n$ is the displacement vector, $q = q(x,t) \in \mathbb{R}^n$ is the heat flux vector, $\theta = \theta(x,t)$ is the difference temperature and the relaxation function g is a positive differentiable function. The coefficients $c, \kappa, \beta, \mu, \lambda$, τ_0 are positive constants, where μ, λ are Lame moduli and τ_0 is the relaxation time, a small parameter compared to the others. The boundary condition on Γ_1 is the nonlocal boundary condition responsible for the memory effect.

Messaoudi and Al-Shehri treated system (1.3) of thermoelasticity with second sound in [13] subject to boundary condition of memory type. If k is the resolvent kernel of $-g'/g(0)$, they showed in [12] that the energy decays at the same rate as of $(-k')$, while in [13], when $(-k')$ decays exponentially, the energy decays at a polynomial rate. Recently, Mustafa [17] treated system (1.3) for k satisfying

$$k(0) > 0, \quad \lim_{t\to\infty} k(t) = 0, \quad k'(t) \le 0, \tag{1.4}$$

$$k''(t) \ge H(-k'(t)), \quad \forall t > 0, \tag{1.5}$$

where H is a positive function, which is linear or strictly increasing, strictly convex of class C^2 on $(0, r]$, $r < 1$, and $H(0) = 0$ and proved for $u_0 = 0$ on Γ_1, an explicit energy decay formula which is not necessarily of exponential or polynomial-type decay.

Our aim in this work is to investigate (1.3) for resolvent kernels satisfying (1.4) and (1.5), when $u_0 \neq 0$ on Γ_1 is taken into account. The proof is based on the multiplier method and makes use of some estimates of [17] with the necessary modification needed to obtain our result. The paper is organized as follows. In section 2, we present some notations and material needed for our work. In section 3, we establish some technical lemmas and state our main theorem, while the proof of our main result will be given in section 4.

2. Notation and transformation

In this section we introduce some notation and prove some lemmas. To establish our result, we shall make the following assumption:

(A1) The partition Γ_0 and Γ_1 are closed, disjoint, with $\operatorname{meas}(\Gamma_0) > 0$ and satisfying

$$\Gamma_1 = \{x \in \Gamma : m(x).\nu \ge \delta > 0\}, \quad \Gamma_0 = \{x \in \Gamma : m(x).\nu \le 0\},$$

where $m(x) = x - x_0$, for some $x_0 \in \mathbb{R}^n$.

Similarly to [11, 12, 14], applying Volterra's inverse operator, the boundary condition

$$u(x,t) = -\int_0^t g(t-s)\Big(\mu\frac{\partial u}{\partial \nu} + (\mu+\lambda)(\operatorname{div} u)\nu\Big)(s)ds, \quad \text{on } \Gamma_1 \times [0,+\infty), \tag{2.1}$$

can be transformed into

$$\mu\frac{\partial u}{\partial \nu} + (\mu+\lambda)(\operatorname{div} u)\nu = -\frac{1}{g(0)}(u_t + k * u_t), \quad \text{on } \Gamma_1 \times [0,+\infty),$$

where * denotes the convolution product

$$(\varphi * \psi)(t) = \int_0^t \varphi(t-s)\psi(s)\,ds,$$

and k is the resolvent kernel of $(-g'/g(0))$ which satisfies

$$k + \frac{1}{g(0)}(g' * k) = -\frac{1}{g(0)} g'.$$

Taking $\eta = 1/g(0)$, we arrive at

$$\mu \frac{\partial u}{\partial \nu} + (\mu+\lambda)(\operatorname{div} u)\nu = -\eta(u_t + k(0)u - k(t)u_0 + k' * u), \quad \text{on } \Gamma_1 \times [0,+\infty). \tag{2.2}$$

Then, we will use the boundary relation (2.2) instead of the fourth equation in (1.3). Let us define

$$(\varphi \circ \psi)(t) = \int_0^t \varphi(t-s)|\psi(t) - \psi(s)|^2 ds,$$
$$(\varphi \Diamond \psi)(t) = \int_0^t \varphi(t-s)(\psi(t) - \psi(s)) ds.$$

By using Hôlder's inequality, we have

$$|(\varphi \Diamond \psi)(t)|^2 \le \Big(\int_0^t |\varphi(s)| ds \Big) (|\varphi| \circ \psi)(t). \tag{2.3}$$

Lemma 2.1 ([15])**.** *If $\varphi, \psi \in C^1(\mathbb{R}^+)$, then*

$$(\varphi * \psi)\psi_t = -\frac{1}{2}\varphi(t)|\psi(t)|^2 + \frac{1}{2}\varphi' \circ \psi - \frac{1}{2}\frac{d}{dt}\Big(\varphi \circ \psi - \Big(\int_0^t \varphi(s)ds\Big)|\psi(t)|^2\Big). \tag{2.4}$$

Let us define

$$V = \{v \in (H^1(\Omega)) : v = 0 \quad \text{on } \Gamma_0\}.$$

The well-posedness of system (1.3) is presented in the following theorem, which can be proved, using the Galerkin method as in [1, 4] and the reference therein.

Theorem 2.2. *Let $k \in W^{2,1}(\mathbb{R}^+) \cap W^{1,\infty}(\mathbb{R}^+)$, $u_0 \in ((H^2(\Omega)) \cap V)^n$, $\theta_0 \in H_0^1(\Omega)$, $q_0 \in (H^1(\Omega))^n$, and $u_1 \in V^n$, with*

$$\frac{\partial u_0}{\partial \nu} + \eta u_0 = 0 \quad \text{on } \Gamma_1. \tag{2.5}$$

Then there exists a unique strong solution u of system (1.3)*, such that*

$$u \in C(\mathbb{R}^+; (H^2(\Omega) \cap V)^n) \cap C^1(\mathbb{R}^+; V^n),$$
$$\theta \in C(\mathbb{R}^+; H_0^1(\Omega)) \cap C^1(\mathbb{R}^+; L^2(\Omega)),$$
$$q \in C(\mathbb{R}^+; (H^1(\Omega))^n) \cap C^1(\mathbb{R}^+; (L^2(\Omega))^n).$$

3. Decay of solutions

In this section we discuss the asymptotic behavior of the solutions of system (1.3) when the resolvent kernel k satisfies the assumption

(B1) $k : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^2 function such that

$$k(0) > 0, \quad \lim_{t\to\infty} k(t) = 0, \quad k'(t) \le 0,$$

and there exists a positive function $H \in C^1(\mathbb{R}^+)$, with $H(0) = 0$, and H is linear or strictly increasing and strictly convex C^2 function on $(0, r]$, $r < 1$, such that

$$k''(t) \ge H(-k'(t)), \quad \forall t > 0.$$

It is a routine procedure to define the first-order energy of system (1.3) by (see Lemma 3.2 below).

$$
\begin{aligned}
E_1(t) = &\frac{1}{2}\int_\Omega \big[|u_t|^2 + \mu|\nabla u|^2 + (\mu+\lambda)(\operatorname{div} u)^2 + c\theta^2 + \tau_0 q^2\big]dx \\
&- \frac{\eta}{2}\int_{\Gamma_1}(k'\circ u)(t)d\Gamma_1 + \frac{\eta}{2}\int_{\Gamma_1} k(t)|u|^2 d\Gamma_1.
\end{aligned}
\tag{3.1}
$$

Now, we differentiate (1.3), with respect to t, to obtain

$$
\begin{gathered}
u_{ttt} - \mu\Delta u_t - (\mu+\lambda)\nabla(\operatorname{div} u_t) + \beta\nabla\theta_t = 0 \quad \text{in } \Omega\times(0,+\infty) \\
c\theta_{tt} + \kappa \operatorname{div} q_t + \beta \operatorname{div} u_{tt} = 0 \quad \text{in } \Omega\times(0,+\infty) \\
\tau_0 q_{tt} + q_t + \kappa\nabla\theta_t = 0 \quad \text{in } \Omega\times(0,+\infty)
\end{gathered}
\tag{3.2}
$$

and the boundary condition (2.2) to obtain

$$
\mu\frac{\partial u_t}{\partial\nu} + (\mu+\lambda)(\operatorname{div} u_t)\nu = -\eta(u_{tt} + k(0)u_t + k' * u_t), \quad \text{on } \Gamma_1\times\mathbb{R}^+. \tag{3.3}
$$

Consequently, similar computations yield the second-order energy of system (1.3):

$$
\begin{aligned}
E_2(t) = &\frac{1}{2}\int_\Omega \big[|u_{tt}|^2 + \mu|\nabla u_t|^2 + (\mu+\lambda)(\operatorname{div} u_t)^2 + c\theta_t^2 + \tau_0 q_t^2\big]dx \\
&- \frac{\eta}{2}\int_{\Gamma_1}(k'\circ u_t)(t)d\Gamma_1 + \frac{\eta}{2}\int_{\Gamma_1} k(t)|u_t|^2 d\Gamma_1.
\end{aligned}
$$

Theorem 3.1. *Given $(u_0, u_1, \theta_0, q_0) \in (H^2(\Omega)\cap V)^n \times V^n \times H_0^1(\Omega) \times (H^1(\Omega))^n$, we assume that* (A1) *and* (B1) *hold. Then there exist positive constants c_1, c_2, k_1, k_2, k_3, ε_0 and t_1 such that:*

(I) In the special case $H(t) = ct^p$, where $1 \le p < 3/2$, the solution of (1.3) *satisfies*

$$
E_1(t) \le \Big(\frac{c_1 + c_2\int_{t_1}^t [k(s)\int_{\Gamma_1}|u_0|^2 d\Gamma_1]^{2p-1}ds}{t}\Big)^{\frac{1}{2p-1}} - \frac{\eta}{2}\Big(\int_{\Gamma_1}|u_0|^2 d\Gamma_1\Big)\int_t^\infty k^2(s)ds
\tag{3.4}
$$

for all $t \ge t_1$.

(II) In the general case, the solution of (1.3) *satisfies*

$$
\begin{aligned}
E_1(t) \le\ &k_1 H_1^{-1}\Big(\frac{k_2 + k_3(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_{t_1}^t H_0(k(s))ds}{t}\Big) \\
&- \frac{\eta}{2}\Big(\int_{\Gamma_1}|u_0|^2 d\Gamma_1\Big)\int_t^\infty k^2(s)ds \quad \text{for all } t \ge t_1,
\end{aligned}
\tag{3.5}
$$

where

$$
H_1(t) = tH_0'(\varepsilon_0 t), \quad H_0(t) = H(D(t)),
$$

provided that D is a positive C^1 function, with $D(0) = 0$, for which H_0 is strictly increasing and strictly convex C^2 function on $(0, r]$ and

$$
\int_0^{+\infty} \frac{-k'(s)}{H_0^{-1}(k''(s))}ds < +\infty. \tag{3.6}
$$

Remarks. 1. If $u_0 = 0$ on Γ_1, we then obtain the results in Mustafa [17].

2. If $\int_0^\infty H_0(k(s))ds < +\infty$, then (3.5) reduces to

$$E_1(t) \le k_1 H_1^{-1}(\frac{c}{t}) - \frac{\eta}{2}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) \int_t^\infty k^2(s)ds,$$

which clearly shows that $\lim_{t\to\infty} E_1(t) = 0$, in this case.

3. The usual decay rate estimate, already proved for k satisfying $k'' \ge d(-k')^p$, $1 \le p < 3/2$, is a special case of our result. We will provide a "simpler" proof for this special case.

4. The condition $k'' \ge d(-k')^p$, $1 \le p < 3/2$ assumes $-k'(t) \le \omega e^{-dt}$ when $p = 1$ and $-k'(t) \le \omega / t^{\frac{1}{p-1}}$ when $1 < p < 3/2$. Our result allows resolvent kernels whose derivatives are not necessarily of exponential or polynomial decay. For instance, if

$$k'(t) = -\exp(-\sqrt{t}),$$

then $k''(t) = H(-k'(t))$ where, for $t \in (0, r]$, $r < 1$,

$$H(t) = \frac{t}{2\ln(1/t)}\,.$$

Since

$$H'(t) = \frac{1 + \ln(1/t)}{2[\ln(1/t)]^2} \quad \text{and} \quad H''(t) = \frac{\ln(1/t) + 2}{2t[\ln(1/t)]^3},$$

the function H satisfies hypothesis (B1) on the interval $(0, r]$ for any $0 < r < 1$. Also, by taking $D(t) = t^\alpha$, (3.6) is satisfied for any $\alpha > 1$. Therefore, an explicit rate of decay can be obtained by Theorem 3.1. The function $H_0(t) = H(t^\alpha)$ has derivative

$$H_0'(t) = \frac{\alpha t^{\alpha-1}[1 + \ln(1/t^\alpha)]}{2[\ln(1/t^\alpha)]^2}.$$

Then, we do some direct calculations and use (3.5) to deduce that

$$\begin{aligned} E_1(t) \le k_1 &\Big(\frac{k_2 + k_3\big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\big) \int_{t_1}^t H_0(k(s))ds}{t}\Big)^{1/(2\alpha)} \\ &- \frac{\eta}{2}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) \int_t^\infty k^2(s)ds, \quad \forall t \ge t_1, \end{aligned}$$

for any $\alpha > 1$, where $H_0(k(s)) = \frac{(k(s))^\alpha}{2\ln(1/(k(s))^\alpha)}$. Therefore, taking $\alpha \to 1$, the energy decays at the following rate

$$\begin{aligned} E_1(t) \le k_1 &\Big(\frac{k_2 + k_3(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_{t_1}^t H(k(s))ds}{t}\Big)^{1/2} \\ &- \frac{\eta}{2}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) \int_t^\infty k^2(s)ds, \quad \forall t \ge t_1. \end{aligned} \tag{3.7}$$

If $\int_0^\infty H(k(s))ds < +\infty$, then equation (3.7) reduces to

$$E_1(t) \le \frac{C}{t^{\frac{1}{2}}} - \frac{\eta}{2}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) \int_t^\infty k^2(s)ds, \quad \forall t \ge t_1.$$

5. The well-known Jensen's inequality will be of essential use in establishing our main result. If F is a convex function on $[a, b]$, $f : \Omega \to [a, b]$ and h are integrable functions on $\Omega, h(x) \geq 0$, and $\int_\Omega h(x)dx = k > 0$, then Jensen's inequality states that

$$F\Big[\frac{1}{k}\int_\Omega f(x)h(x)dx\Big] \leq \frac{1}{k}\int_\Omega F[f(x)]h(x)dx.$$

6. As in [17] and since $\lim_{t\to\infty} k(t) = 0$, $\lim_{t\to+\infty}(-k'(t))$ cannot be equal to a positive number, and so it is natural to assume that $\lim_{t\to+\infty}(-k'(t)) = 0$, in the same way, we deduce that $\lim_{t\to+\infty} k''(t) = 0$. Hence there is $t_1 > 0$ large enough such that $k'(t_1) < 0$ and

$$\max\{k(t), -k'(t), k''(t)\} < \min\{r, H(r), H_0(r)\}, \quad \forall t \geq t_1. \tag{3.8}$$

As k' is nondecreasing, $k'(0) < 0$ and $k'(t_1) < 0$, then $k'(t) < 0$ for any $t \in [0, t_1]$ and

$$0 < -k'(t_1) \leq -k'(t) \leq -k'(0), \quad \forall t \in [0, t_1].$$

Therefore, since H is a positive continuous function, we have

$$a \leq H(-k'(t)) \leq b, \quad \forall t \in [0, t_1],$$

for some positive constants a and b. Consequently, for all $t \in [0, t_1]$,

$$k''(t) \geq H(-k'(t)) \geq a = \frac{a}{k'(0)}k'(0) \geq \frac{a}{k'(0)}k'(t)$$

which gives

$$k''(t) \geq d(-k'(t)), \quad \forall t \in [0, t_1], \tag{3.9}$$

for some positive constant d.

Lemma 3.2. *Under the assumptions of Theorem 3.1, the energies of the solution of* (1.3) *satisfy*

$$\begin{aligned} E_1'(t) \leq &-\int_\Omega |q|^2 dx - \frac{\eta}{2}\int_{\Gamma_1} |u_t|^2 d\Gamma_1 + \frac{\eta}{2}k'(t)\int_{\Gamma_1} |u|^2 d\Gamma_1 \\ &- \frac{\eta}{2}\int_{\Gamma_1} (k'' \circ u)(t) d\Gamma_1 + \frac{\eta}{2}k^2(t)\int_{\Gamma_1} |u_0|^2 d\Gamma_1. \end{aligned} \tag{3.10}$$

$$E_2'(t) \leq -\int_\Omega |q_t|^2 dx \leq 0. \tag{3.11}$$

Proof. Multiplying $(1.3)_1$ by u_t, $(1.3)_2$ by θ, and $(1.3)_3$ by q and integrating over Ω, using integration by parts, the boundary conditions (2.2) and (2.4), one can easily find that

$$\begin{aligned} E_1'(t) = &-\int_\Omega |q|^2 dx - \eta\int_{\Gamma_1} |u_t|^2 d\Gamma_1 + \frac{\eta}{2}k'(t)\int_{\Gamma_1} |u|^2 d\Gamma_1 \\ &- \frac{\eta}{2}\int_{\Gamma_1} (k'' \circ u)(t) d\Gamma_1 + \eta\int_{\Gamma_1} k(t)(u_0.u_t) d\Gamma_1 \end{aligned}$$

Young's inequality then yields

$$\int_{\Gamma_1} k(t)(u_0.u_t) d\Gamma_1 \leq \frac{1}{2}\int_{\Gamma_1} |u_t|^2 d\Gamma_1 + \frac{1}{2}k^2(t)\int_{\Gamma_1} |u_0|^2 d\Gamma_1,$$

and consequently, we obtain (3.10) for strong solutions. Estimate (3.11) is established in a similar way using (3.2) and (3.3). $\square$

Remark 3.3. If $u_0 = 0$ on Γ_1, then $E_1'(t) \le 0$, hence $E_1(t) \le E_1(0)$. If $u_0 \ne 0$ on Γ_1, then

$$E_1(t) \le E_1(0) + \frac{\eta}{2}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) \int_0^t k^2(s)ds \le A, \tag{3.12}$$

for some $A > 0$.

Lemma 3.4. *Under the assumptions* (A1) *and* (B1)*, the solution of* (1.3) *satisfies: for any* $\epsilon > 0$,

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u_t.[M + (n-1)u]dx \\ &\le -\int_\Omega |u_t|^2 dx - \mu \int_\Omega |\nabla u|^2 dx - \frac{\mu+\lambda}{2}\int_\Omega (\operatorname{div} u)^2 dx + C\int_\Omega |\nabla\theta|^2 dx \\ &\quad - \frac{\mu\delta}{2}\int_{\Gamma_1} |\nabla u|^2 d\Gamma_1 - (\mu+\lambda)\delta \int_{\Gamma_1} (\operatorname{div} u)^2 d\Gamma_1 + C(1+\frac{1}{\epsilon})\int_{\Gamma_1} |u_t|^2 d\Gamma_1 \\ &\quad + Ck^2(t)\int_{\Gamma_1} |u|^2 d\Gamma_1 - \frac{C}{\epsilon}\int_{\Gamma_1} (k' \circ u)(t) d\Gamma_1 \\ &\quad + \epsilon \int_{\Gamma_1} |u|^2 d\Gamma_1 + C(1+\frac{1}{\epsilon})k^2(t)\int_{\Gamma_1} |u_0|^2 d\Gamma_1, \end{aligned} \tag{3.13}$$

where

$$M = (M_1, M_2, \dots, M_n)^T, \quad \textit{such that } M_i = 2m.\nabla u^i,$$

and C *is a "generic" positive constant independent of* ϵ.

For a proof of the above lemma, see [13, 17].

4. Proof of main results

In this section we prove our main result.

Proof of Theorem 3.1. Taking $E(t) = E_1(t) + E_2(t)$, we define

$$L(t) = NE(t) + \int_\Omega u_t.[M + (n-1)u]dx. \tag{4.1}$$

From (3.10), (3.11), and (3.13), we obtain

$$\begin{aligned} L'(t) &\le -N\int_\Omega |q|^2 dx - N\int_\Omega |q_t|^2 dx - \frac{N\eta}{2}\int_{\Gamma_1} |u_t|^2 d\Gamma_1 - \frac{N}{2}\eta\int_{\Gamma_1} (k'' \circ u)(t) d\Gamma_1 \\ &\quad - \int_\Omega |u_t|^2 dx - \mu\int_\Omega |\nabla u|^2 dx - \frac{\mu+\lambda}{2}\int_\Omega (\operatorname{div} u)^2 dx \\ &\quad - \frac{\mu\delta}{2}\int_{\Gamma_1} |\nabla u|^2 d\Gamma_1 - (\mu+\lambda)\delta\int_{\Gamma_1} (\operatorname{div} u)^2 d\Gamma_1 + C\int_{\Gamma_1} |u_t|^2 d\Gamma_1 \\ &\quad + \frac{C}{\epsilon}k^2(t)\int_{\Gamma_1} |u|^2 d\Gamma_1 - \frac{C}{\epsilon}\int_{\Gamma_1} (k' \circ u)(t) d\Gamma_1 + \epsilon\int_{\Gamma_1} |u|^2 d\Gamma_1 + C\int_\Omega |\nabla\theta|^2 dx \\ &\quad + C(1+\frac{1}{\epsilon})k^2(t)\int_{\Gamma_1} |u_0|^2 d\Gamma_1 + N\frac{\eta}{2}k^2(t)\int_{\Gamma_1} |u_0|^2 d\Gamma_1. \end{aligned}$$

By using $(1.3)_3$ and

$$\int_{\Gamma_1} |u|^2 d\Gamma_1 \le c_0 \int_\Omega |\nabla u|^2 dx \tag{4.2}$$

and $\int_\Omega |\theta|^2 dx \le c_p \int_\Omega |\nabla\theta|^2 dx$, for some positive constants c_0 and c_p, we arrive at

$$\begin{aligned} L'(t) \le & -(N-C_1)\int_\Omega |q|^2 dx - (N-C_1)\int_\Omega |q_t|^2 dx - \int_\Omega |u_t|^2 dx \\ & - \int_{\Gamma_1} k(t)|u|^2 d\Gamma_1 - \Big(\mu - \epsilon c_0 - \frac{C}{\epsilon}k^2(t) - c_0 k(t)\Big)\int_\Omega |\nabla u|^2 dx \\ & - \frac{\mu+\lambda}{2}\int_\Omega (\operatorname{div} u)^2 dx - \big(\frac{N}{2}\eta - C\big)\int_{\Gamma_1} |u_t|^2 d\Gamma_1 \\ & - C\int_{\Gamma_1} (k'\circ u)(t) d\Gamma_1 - \int_\Omega |\theta|^2 dx \\ & + \big[\frac{N}{2}\eta + C(1+\frac{1}{\epsilon})\big]k^2(t)\int_{\Gamma_1} |u_0|^2 d\Gamma_1. \end{aligned} \tag{4.3}$$

At this point, we choose our constants carefully. We first, fix ϵ so small that $\epsilon c_0 = \frac{1}{2}\mu$ and pick N large enough so that $L \sim E$,

$$a_1 = \frac{N}{2}\eta - C \ge 0 \quad \text{and} \quad a_2 = N - C_1 > 0.$$

Thus, (4.3) simplifies to

$$\begin{aligned} L'(t) \le & -\int_\Omega |u_t|^2 dx - \Big(\frac{\mu}{2} - \frac{C}{\epsilon}k^2(t) - c_0 k(t)\Big)\int_\Omega |\nabla u|^2 dx \\ & - \frac{\mu+\lambda}{2}\int_\Omega (\operatorname{div} u)^2 dx - \int_{\Gamma_1} k(t)|u|^2 d\Gamma_1 - a_2\int_\Omega |q|^2 dx - \int_\Omega |\theta|^2 dx \\ & - C\int_{\Gamma_1} (k'\circ u)(t) d\Gamma_1 + Ck^2(t)\int_{\Gamma_1} |u_0|^2 d\Gamma_1. \end{aligned}$$

Using the fact that $\lim_{t\to+\infty} k(t) = 0$, we obtain

$$L'(t) \le -mE_1(t) + Ck^2(t)\int_{\Gamma_1} |u_0|^2 d\Gamma_1 - c\int_{\Gamma_1} (k'\circ u)(t) d\Gamma_1, \quad \forall t \ge t_1, \tag{4.4}$$

for some t_1 large enough and some positive constants m and c.

Now we use (3.9), and (3.10) to conclude that, for any $t \ge t_1$,

$$\begin{aligned} & -\int_0^{t_1} k'(s)\int_{\Gamma_1} |u(t)-u(t-s)|^2 d\Gamma_1 ds \\ & \le \frac{1}{d}\int_0^{t_1} k''(s)\int_{\Gamma_1} |u(t)-u(t-s)|^2 d\Gamma_1 ds \\ & \le -c[E_1'(t) - \frac{\eta}{2}k^2(t)\int_{\Gamma_1} |u_0|^2 d\Gamma_1]. \end{aligned} \tag{4.5}$$

Next we take $F(t) = L(t) + cE_1(t)$, which is clearly equivalent to $E(t)$, and use (4.4) and (4.5), to obtain: for all $t \ge t_1$ with some new positive constant $C > 0$,

$$F'(t) \le -mE_1(t) + Ck^2(t)\int_{\Gamma_1} |u_0|^2 d\Gamma_1 - c\int_{t_1}^t k'(s)\int_{\Gamma_1} |u(t)-u(t-s)|^2 d\Gamma_1 ds. \tag{4.6}$$

Similarly to [17] we consider two cases:

(I) $H(t) = ct^p$ and $1 \le p < 3/2$: If $1 < p < 3/2$, one can easily show that $\int_0^{+\infty} [-k'(s)]^{1-\delta_0} ds < +\infty$ for any $\delta_0 < 2-p$. Using this fact, (3.10), (3.12) and

(4.2) and choosing t_1 even larger if needed, we deduce that, for all $t \geq t_1$

$$\begin{aligned} \eta(t) &= \int_{t_1}^{t} [-k'(s)]^{1-\delta_0} \int_{\Gamma_1} |u(t)-u(t-s)|^2 \, d\Gamma_1 ds \\ &\leq 2\int_{t_1}^{t} [-k'(s)]^{1-\delta_0} \int_{\Gamma_1} (|u(t)|^2 + |u(t-s)|^2) \, d\Gamma_1 ds \\ &\leq cA \int_0^{+\infty} [-k'(s)]^{1-\delta_0} ds < 1. \end{aligned} \tag{4.7}$$

Then, Jensen's inequality, (3.10), hypothesis (B1), and (4.7) lead to

$$\begin{aligned} & -\int_{t_1}^{t} k'(s) \int_{\Gamma_1} |u(t)-u(t-s)|^2 \, d\Gamma_1 ds \\ &= \int_{t_1}^{t} [-k'(s)]^{\delta_0} [-k'(s)]^{1-\delta_0} \int_{\Gamma_1} |u(t)-u(t-s)|^2 \, d\Gamma_1 ds \\ &= \int_{t_1}^{t} [-k'(s)]^{p-1+\delta_0(\frac{\delta_0}{p-1+\delta_0})} [-k'(s)]^{1-\delta_0} \int_{\Gamma_1} |u(t)-u(t-s)|^2 \, d\Gamma_1 ds \\ &\leq \eta(t) \Big[\frac{1}{\eta(t)} \int_{t_1}^{t} [-k'(s)]^{(p-1+\delta_0)} [-k'(s)]^{1-\delta_0} \int_{\Gamma_1} |u(t)-u(t-s)|^2 \, d\Gamma_1 ds \Big]^{\frac{\delta_0}{p-1+\delta_0}} \\ &\leq \Big[\int_{t_1}^{t} [-k'(s)]^{p} \int_{\Gamma_1} |u(t)-u(t-s)|^2 \, d\Gamma_1 ds \Big]^{\frac{\delta_0}{p-1+\delta_0}} \\ &\leq c\Big[\int_{t_1}^{t} k''(s) \int_{\Gamma_1} |u(t)-u(t-s)|^2 \, d\Gamma_1 ds \Big]^{\frac{\delta_0}{p-1+\delta_0}} \\ &\leq c\Big[-E_1'(t) + \frac{\eta}{2} k^2(t) \Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1 \Big) \Big]^{\frac{\delta_0}{p-1+\delta_0}}. \end{aligned}$$

Then, particularly for $\delta_0 = 1/2$, we find that (4.6) becomes

$$F'(t) \leq -mE_1(t) + c\Big[-E_1'(t) + \frac{\eta}{2} k^2(t) (\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \Big]^{\frac{1}{2p-1}} + Ck^2(t) \int_{\Gamma_1} |u_0|^2 d\Gamma_1,$$

for all $t \geq t_1$. However,

$$\begin{aligned} & \Big(F(t) + \frac{\eta}{2} \Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1 \Big) \int_t^{\infty} k^2(s) ds \Big)' \\ &\leq F'(t) \\ &\leq -mE_1(t) + c\Big[-E_1'(t) + \frac{\eta}{2} k^2(t) \Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1 \Big) \Big]^{\frac{1}{2p-1}} + Ck^2(t) \int_{\Gamma_1} |u_0|^2 d\Gamma_1, \end{aligned}$$

hence, for all $t \geq t_1$, we have

$$\begin{aligned} & \Big(F(t) + \frac{\eta}{2} \Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1 \Big) \int_t^{\infty} k^2(s) ds \Big)' \\ &\leq -m\Big[E_1(t) + \frac{\eta}{2} \Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1 \Big) \int_t^{\infty} k^2(s) ds \Big] \\ &\quad + c\Big[-E_1'(t) + \frac{\eta}{2} k^2(t) \Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1 \Big) \Big]^{\frac{1}{2p-1}} + Ck^2(t) \int_{\Gamma_1} |u_0|^2 d\Gamma_1 \end{aligned}$$

$$+ m\frac{\eta}{2}\Big(\int_{\Gamma_1}|u_0|^2d\Gamma_1\Big)\int_t^\infty k^2(s)ds. \tag{4.8}$$

Using hypothesis (B1), for all $t \geq t_1$, we have

$$\int_t^\infty k^2(s)ds \leq k(t)\int_t^\infty k(s)ds \leq k(t)\int_0^\infty k(s)ds, \text{ and } k^2(t) \leq c'k(t). \tag{4.9}$$

Then (4.8) becomes, for some positive constant C

$$\begin{aligned}
&\Big(F(t) + \frac{\eta}{2}\Big(\int_{\Gamma_1}|u_0|^2d\Gamma_1\Big)\int_t^\infty k^2(s)ds\Big)' \\
&\leq -m\Big[E_1(t) + \frac{\eta}{2}\Big(\int_{\Gamma_1}|u_0|^2d\Gamma_1\Big)\int_t^\infty k^2(s)ds\Big] \\
&\quad + c\Big[-E_1'(t) + \frac{\eta}{2}k^2(t)\Big(\int_{\Gamma_1}|u_0|^2d\Gamma_1\Big)\Big]^{\frac{1}{2p-1}} + Ck(t)\int_{\Gamma_1}|u_0|^2d\Gamma_1.
\end{aligned}$$

Now we multiply by $\big[E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2d\Gamma_1)\int_t^\infty k^2(s)ds\big]^{2p-2}$, using that $E_1'(t) \leq \frac{\eta}{2}k^2(t)(\int_{\Gamma_1}|u_0|^2d\Gamma_1)$ we obtain

$$\begin{aligned}
&\Big(\Big[F(t) + \frac{\eta}{2}\Big(\int_{\Gamma_1}|u_0|^2d\Gamma_1\Big)\int_t^\infty k^2(s)ds\Big]\Big[E_1(t) \\
&+ \frac{\eta}{2}\Big(\int_{\Gamma_1}|u_0|^2d\Gamma_1\Big)\int_t^\infty k^2(s)ds\Big]^{2p-2}\Big)' \\
&\leq \Big(F(t) + \frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2d\Gamma_1)\int_t^\infty k^2(s)ds\Big)' \\
&\quad \times \Big[E_1(t) + \frac{\eta}{2}\Big(\int_{\Gamma_1}|u_0|^2d\Gamma_1\Big)\int_t^\infty k^2(s)ds\Big]^{2p-2} \\
&\leq -m\Big[E_1(t) + \frac{\eta}{2}\Big(\int_{\Gamma_1}|u_0|^2d\Gamma_1\Big)\int_t^\infty k^2(s)ds\Big]^{2p-1} \\
&\quad + c\Big[E_1(t) + \frac{\eta}{2}\Big(\int_{\Gamma_1}|u_0|^2d\Gamma_1\Big)\int_t^\infty k^2(s)ds\Big]^{2p-2} \\
&\quad \times \Big[-E_1'(t) + \frac{\eta}{2}k^2(t)(\int_{\Gamma_1}|u_0|^2d\Gamma_1)\Big]^{\frac{1}{2p-1}} \\
&\quad + C\Big[k(t)\int_{\Gamma_1}|u_0|^2d\Gamma_1\Big]\Big[E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2d\Gamma_1)\int_t^\infty k^2(s)ds\Big]^{2p-2}.
\end{aligned}$$

Then, applying Young's inequality, with $\sigma = 2p-1$, and $\sigma' = \frac{2p-1}{2p-2}$, for

$$\Big[-E_1'(t) + \frac{\eta}{2}k^2(t)\Big(\int_{\Gamma_1}|u_0|^2d\Gamma_1\Big)\Big]^{\frac{1}{2p-1}}\Big[E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2d\Gamma_1)\int_t^\infty k^2(s)ds\Big]^{2p-2}$$

and

$$\Big[k(t)\int_{\Gamma_1}|u_0|^2d\Gamma_1\Big]\Big[E_1(t) + \frac{\eta}{2}\Big(\int_{\Gamma_1}|u_0|^2d\Gamma_1\Big)\int_t^\infty k^2(s)ds\Big]^{2p-2}$$

we obtain, for $C > 0$,

$$\Big(\Big[F(t) + \frac{\eta}{2}\Big(\int_{\Gamma_1}|u_0|^2d\Gamma_1\Big)\int_t^\infty k^2(s)ds\Big]$$

$$
\begin{aligned}
&\times \Big[E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_t^{\infty} k^2(s)ds\Big]^{2p-2}\Big)' \\
&\le -m\Big[E_1(t) + \frac{\eta}{2}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) \int_t^{\infty} k^2(s)ds\Big]^{2p-1} \\
&\quad + 2\varepsilon\Big[E_1\Big(t\Big) + \frac{\eta}{2}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) \int_t^{\infty} k^2(s)ds\Big]^{2p-1} \\
&\quad + C_\varepsilon\Big[-E_1'(t) + \frac{\eta}{2}k^2(t)\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big)\Big] + C[k(t)\int_{\Gamma_1} |u_0|^2 d\Gamma_1]^{2p-1}.
\end{aligned}
$$

Consequently, for $2\varepsilon < m$, we obtain

$$
F_0'(t) \le -m'\Big[E_1(t)+\frac{\eta}{2}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) \int_t^{\infty} k^2(s)ds\Big]^{2p-1} + C\Big[k(t)\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big]^{2p-1},
$$

where m' is some positive constant and

$$
\begin{aligned}
F_0(t) &= \Big[F(t) + \frac{\eta}{2}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) \int_t^{\infty} k^2(s)ds\Big] \\
&\quad \times \Big[E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_t^{\infty} k^2(s)ds\Big]^{2p-2} \\
&\quad + C_\varepsilon\Big[E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_t^{\infty} k^2(s)ds\Big].
\end{aligned}
$$

Also, it is easy to show that this inequality is true for $p = 1$. Once again, we use the fact that $E_1'(t) \le \frac{\eta}{2}k^2(t)(\int_{\Gamma_1} |u_0|^2 d\Gamma_1)$ to deduce that

$$
\begin{aligned}
&\Big(t\Big[E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_t^{\infty} k^2(s)ds\Big]^{2p-1}\Big)' \\
&\le \Big[E_1(t) + \frac{\eta}{2}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) \int_t^{\infty} k^2(s)ds\Big]^{2p-1} \\
&\le -\frac{1}{m'}F_0'(t) + \frac{C}{m'}\Big[k(t)\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big]^{2p-1},
\end{aligned}
$$

for all $t \ge t_1$. A simple integration over (t_1, t) yields

$$
\begin{aligned}
&t\Big[E_1(t) + \frac{\eta}{2}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) \int_t^{\infty} k^2(s)ds\Big]^{2p-1} \\
&\le \frac{1}{m'}F_0(t_1) + \frac{C}{m'}\int_{t_1}^{t} \Big[k(s)\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big]^{2p-1} ds \\
&\quad + t_1\Big[E_1(t_1) + \frac{\eta}{2}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) \int_{t_1}^{\infty} k^2(s)ds\Big]^{2p-1} \\
&\le c_1 + \frac{C}{m'}\int_{t_1}^{t} \Big[k(s)\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big]^{2p-1},
\end{aligned}
$$

hence

$$
\begin{aligned}
&\Big[E_1(t) + \frac{\eta}{2}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) \int_t^{\infty} k^2(s)ds\Big]^{2p-1} \\
&\le \frac{c_1}{t} + \frac{C}{m't}\int_{t_1}^{t} \Big[k(s)\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big]^{2p-1} ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
E_1(t) \le & \Big(\frac{c_1 + c_2 \int_{t_1}^t [k(s) \int_{\Gamma_1} |u_0|^2 d\Gamma_1]^{2p-1} ds}{t}\Big)^{\frac{1}{2p-1}} \\
& - \frac{\eta}{2}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) \int_t^\infty k^2(s) ds.
\end{aligned}
$$

(II) The general case: As in [17], we define

$$
I(t) = \int_{t_1}^t \frac{-k'(s)}{H_0^{-1}(k''(s))} \int_{\Gamma_1} |u(t) - u(t-s)|^2 d\Gamma_1 ds
$$

where H_0 is such that (3.6) is satisfied. As in (4.7), we find that for all $t \ge t_1$, $I(t)$ satisfies

$$
I(t) < 1. \tag{4.10}
$$

We also assume, without loss of generality that $I(t) \ge \beta_0 > 0$, for all $t \ge t_1$; otherwise (4.6) yields an explicit decay. In addition, we define $\lambda(t)$ by

$$
\lambda(t) = \int_{t_1}^t k''(s) \frac{-k'(s)}{H_0^{-1}(k''(s))} \int_{\Gamma_1} |u(t) - u(t-s)|^2 d\Gamma_1 ds,
$$

and infer from (B1) and the properties of H_0 and D that

$$
\frac{-k'(s)}{H_0^{-1}(k''(s))} \le \frac{-k'(s)}{H_0^{-1}(H(-k'(s)))} = \frac{-k'(s)}{D^{-1}(-k'(s))} \le k_0,
$$

for some positive constantk_0. Then, using (3.8), (3.10), and (3.12), one can easily see that for all $t \ge t_1$, $\lambda(t)$ satisfies

$$
\begin{aligned}
\lambda(t) &\le k_0 \int_{t_1}^t k''(s) \int_{\Gamma_1} |u(t) - u(t-s)|^2 d\Gamma_1 ds \\
&\le cA \int_{t_1}^t k''(s) ds \le cA(-k'(t_1)) \\
&< \min\{r, H(r), H_0(r)\},
\end{aligned} \tag{4.11}
$$

for t_1 even larger (if needed). In addition, we can easily see that

$$
\lambda(t) \le -c\Big[E_1'(t) - \frac{\eta}{2} k^2(t) \int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big], \quad \forall t \ge t_1. \tag{4.12}
$$

Since H_0 is strictly convex on $(0, r]$, and $H_0(0) = 0$, we have

$$
H_0(\theta x) \le \theta H_0(x),
$$

provided $0 \le \theta \le 1$, and $x \in (0, r]$. Then using this fact, (4.10) ,(4.11), and Jensen's inequality leads to

$$
\begin{aligned}
\lambda(t) &= \frac{1}{I(t)} \int_{t_1}^t I(t) H_0[H_0^{-1}(k''(s))] \frac{-k'(s)}{H_0^{-1}(k''(s))} \int_{\Gamma_1} |u(t) - u(t-s)|^2 d\Gamma_1 ds \\
&\ge \frac{1}{I(t)} \int_{t_1}^t H_0[I(t) H_0^{-1}(k''(s))] \frac{-k'(s)}{H_0^{-1}(k''(s))} \int_{\Gamma_1} |u(t) - u(t-s)|^2 d\Gamma_1 ds \\
&\ge H_0\Big(\frac{1}{I(t)} \int_{t_1}^t I(t) H_0^{-1}(k''(s)) \frac{-k'(s)}{H_0^{-1}(k''(s))} \int_{\Gamma_1} |u(t) - u(t-s)|^2 d\Gamma_1 ds\Big) \\
&= H_0\Big(-\int_{t_1}^t k'(s) \int_{\Gamma_1} |u(t) - u(t-s)|^2 d\Gamma_1 ds\Big), \quad \forall t \ge t_1.
\end{aligned}
$$

This implies

$$-\int_{t_1}^{t} k'(s)\int_{\Gamma_1} |u(t)-u(t-s)|^2 d\Gamma_1 ds \le H_0^{-1}(\lambda(t)), \quad \forall t \ge t_1,$$

and (4.6) becomes

$$F'(t) \le -mE_1(t) + Ck^2(t)\int_{\Gamma_1} |u_0|^2 d\Gamma_1 + cH_0^{-1}(\lambda(t)), \quad \forall t \ge t_1. \tag{4.13}$$

Now, using that $H_0' > 0$, $H_0'' > 0$, $\varepsilon_0 < r$, $c_0 > 0$, and

$$E_1'(t) \le \frac{\eta}{2}k^2(t)(\int_{\Gamma_1} |u_0|^2 d\Gamma_1),$$

we find that the functional

$$\begin{aligned} F_1(t) = H_0'\Big(\varepsilon_0 \frac{E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1)\int_t^\infty k^2(s)ds}{E_1(0) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1)\int_0^\infty k^2(s)ds}\Big)F(t) \\ + c_0\Big(E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1)\int_t^\infty k^2(s)ds\Big) \end{aligned}$$

satisfies

$$\begin{aligned} F_1'(t) &= \Big(\varepsilon_0 \frac{E_1'(t) - \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1)k^2(t)}{E_1(0) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1)\int_0^\infty k^2(s)ds}\Big) \\ &\quad \times H_0''\Big(\varepsilon_0 \frac{E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1)\int_t^\infty k^2(s)ds}{E_1(0) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1)\int_0^\infty k^2(s)ds}\Big)F(t) \\ &\quad + H_0'\Big(\varepsilon_0 \frac{E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1)\int_t^\infty k^2(s)ds}{E_1(0) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1)\int_0^\infty k^2(s)ds}\Big)F'(t) \\ &\quad + c_0\Big[E_1'(t) - \frac{\eta}{2}k^2(t)(\int_{\Gamma_1} |u_0|^2 d\Gamma_1)\Big] \\ &\le H_0'\Big(\varepsilon_0 \frac{E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1)\int_t^\infty k^2(s)ds}{E_1(0) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1)\int_0^\infty k^2(s)ds}\Big)F'(t) \\ &\quad + c_0\Big[E_1'(t) - \frac{\eta}{2}k^2(t)(\int_{\Gamma_1} |u_0|^2 d\Gamma_1)\Big], \quad \forall t \ge t_1. \end{aligned}$$

Hence, using (4.13), for all $t \ge t_1$, we obtain

$$\begin{aligned} &F_1'(t) \\ &\le -mE_1(t)H_0'\Big(\varepsilon_0 \frac{E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1)\int_t^\infty k^2(s)ds}{E_1(0) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1)\int_0^\infty k^2(s)ds}\Big) \\ &\quad + C\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big)k^2(t)H_0'\Big(\varepsilon_0 \frac{E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1)\int_t^\infty k^2(s)ds}{E_1(0) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1)\int_0^\infty k^2(s)ds}\Big) \\ &\quad + cH_0^{-1}(\lambda(t))H_0'\Big(\varepsilon_0 \frac{E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1)\int_t^\infty k^2(s)ds}{E_1(0) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1)\int_0^\infty k^2(s)ds}\Big) \\ &\quad + c_0\Big[E_1'(t) - \frac{\eta}{2}k^2(t)(\int_{\Gamma_1} |u_0|^2 d\Gamma_1)\Big]. \end{aligned} \tag{4.14}$$

Let H_0^* be the convex conjugate of H_0 in the sense of Young (see [2, p. 61-64]), then

$$H_0^*(s) = s(H_0')^{-1}(s) - H_0[(H_0')^{-1}(s)], \quad \text{if } s \in (0, H_0'(r)], \tag{4.15}$$

and H_0^* satisfies the Young inequality

$$AB \le H_0^*(A) + H_0(B), \quad \text{if } A \in (0, H_0'(r)],\ B \in (0, r], \tag{4.16}$$

with

$$A = H_0'\Big(\varepsilon_0 \frac{E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_t^\infty k^2(s)ds}{E_1(0) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_0^\infty k^2(s)ds}\Big), \quad B = H_0^{-1}(\lambda(t)).$$

Using (4.11), (4.14), and (4.16), we arrive at

$$\begin{aligned}
F_1'(t) &\le -mE_1(t)H_0'\Big(\varepsilon_0 \frac{E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_t^\infty k^2(s)ds}{E_1(0) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_0^\infty k^2(s)ds}\Big) \\
&\quad + C(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) k^2(t) H_0'\Big(\varepsilon_0 \frac{E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_t^\infty k^2(s)ds}{E_1(0) + \frac{\eta}{2}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) \int_0^\infty k^2(s)ds}\Big) \\
&\quad + c\lambda(t) + cH_0^*\Big(H_0'\Big(\varepsilon_0 \frac{E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_t^\infty k^2(s)ds}{E_1(0) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_0^\infty k^2(s)ds}\Big)\Big) \\
&\quad + c_0\Big[E_1'(t) - \frac{\eta}{2} k^2(t)\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big)\Big].
\end{aligned}$$

Then using (4.12) and (4.15), we obtain that for all $t \ge t_1$,

$$\begin{aligned}
F_1'(t) &\le -mE_1(t)H_0'\Big(\varepsilon_0 \frac{E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_t^\infty k^2(s)ds}{E_1(0) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_0^\infty k^2(s)ds}\Big) \\
&\quad + Ck^2(t)\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) H_0'\Big(\varepsilon_0 \frac{E_1(t) + \frac{\eta}{2}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) \int_t^\infty k^2(s)ds}{E_1(0) + \frac{\eta}{2}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) \int_0^\infty k^2(s)ds}\Big) \\
&\quad - c\Big[E_1'(t) - \frac{\eta}{2} k^2(t) \int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big] \\
&\quad + c(\varepsilon_0 \frac{E_1(t) + \frac{\eta}{2}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) \int_t^\infty k^2(s)ds}{E_1(0) + \frac{\eta}{2}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) \int_0^\infty k^2(s)ds}) \\
&\quad \times H_0'\Big(\varepsilon_0 \frac{E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_t^\infty k^2(s)ds}{E_1(0) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_0^\infty k^2(s)ds}\Big) \\
&\quad + c_0\Big[E_1'(t) - \frac{\eta}{2} k^2(t)(\int_{\Gamma_1} |u_0|^2 d\Gamma_1)\Big].
\end{aligned}$$

Consequently, with a suitable choice of c_0, we obtain, for all $t \ge t_1$,

$$\begin{aligned}
F_1'(t) &\le -m\Big(\frac{E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_t^\infty k^2(s)ds}{E_1(0) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_0^\infty k^2(s)ds}\Big) \\
&\quad \times H_0'\Big(\varepsilon_0 \frac{E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_t^\infty k^2(s)ds}{E_1(0) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_0^\infty k^2(s)ds}\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ m\Big(\frac{\eta}{2}\Big(\int_{\Gamma_1}|u_0|^2 d\Gamma_1\Big)\int_t^\infty k^2(s)ds\Big)\\
&\times H_0'\Big(\varepsilon_0\frac{E_1(t)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_t^\infty k^2(s)ds}{E_1(0)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_0^\infty k^2(s)ds}\Big)\\
&+ C(k^2(t)\int_{\Gamma_1}|u_0|^2 d\Gamma_1)H_0'(\varepsilon_0\frac{E_1(t)+\frac{\eta}{2}\Big(\int_{\Gamma_1}|u_0|^2 d\Gamma_1\Big)\int_t^\infty k^2(s)ds}{E_1(0)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_0^\infty k^2(s)ds})\\
&+ c\Big(\varepsilon_0\frac{E_1(t)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_t^\infty k^2(s)ds}{E_1(0)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_0^\infty k^2(s)ds}\Big)\\
&\times H_0'\Big(\varepsilon_0\frac{E_1(t)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_t^\infty k^2(s)ds}{E_1(0)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_0^\infty k^2(s)ds}\Big).
\end{aligned}
\tag{4.17}
$$

Using (4.9), for some positive constant C, (4.17) becomes

$$
\begin{aligned}
F_1'(t) \le &-m\Big(\frac{E_1(t)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_t^\infty k^2(s)ds}{E_1(0)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_0^\infty k^2(s)ds}\Big)\\
&\times H_0'\Big(\varepsilon_0\frac{E_1(t)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_t^\infty k^2(s)ds}{E_1(0)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_0^\infty k^2(s)ds}\Big)\\
&+ C\Big(\int_{\Gamma_1}|u_0|^2 d\Gamma_1\Big)k(t)H_0'\Big(\varepsilon_0\frac{E_1(t)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_t^\infty k^2(s)ds}{E_1(0)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_0^\infty k^2(s)ds}\Big)\\
&+ c\Big(\varepsilon_0\frac{E_1(t)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_t^\infty k^2(s)ds}{E_1(0)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_0^\infty k^2(s)ds}\Big)\\
&\times H_0'\Big(\varepsilon_0\frac{E_1(t)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_t^\infty k^2(s)ds}{E_1(0)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_0^\infty k^2(s)ds}\Big).
\end{aligned}
\tag{4.18}
$$

In a similar way, using (4.15) and (4.16), we find that, for t_1 even larger (if needed),

$$
\begin{aligned}
&k(t)\Big[H_0'(\varepsilon_0\frac{E_1(t)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_t^\infty k^2(s)ds}{E_1(0)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_0^\infty k^2(s)ds})\Big]\\
&\le \Big(\varepsilon_0\frac{E_1(t)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_t^\infty k^2(s)ds}{E_1(0)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_0^\infty k^2(s)ds}\Big)\\
&\quad\times H_0'\Big(\varepsilon_0\frac{E_1(t)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_t^\infty k^2(s)ds}{E_1(0)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_0^\infty k^2(s)ds}\Big)\\
&\quad+ H_0(k(t)), \quad \forall t \ge t_1.
\end{aligned}
$$

So, with a suitable choice of ε_0, (4.18) becomes

$$
\begin{aligned}
F_1'(t) \le &-\ell\Big(\frac{E_1(t)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_t^\infty k^2(s)ds}{E_1(0)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_0^\infty k^2(s)ds}\Big)\\
&\times H_0'\Big(\varepsilon_0\frac{E_1(t)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_t^\infty k^2(s)ds}{E_1(0)+\frac{\eta}{2}(\int_{\Gamma_1}|u_0|^2 d\Gamma_1)\int_0^\infty k^2(s)ds}\Big)
\end{aligned}
$$

$$+ C\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) H_0(k(t)),$$

for all $t \geq t_1$.

Hence,

$$\begin{aligned} F_1'(t) &\leq -\ell H_1\Big(\frac{E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_t^\infty k^2(s) ds}{E_1(0) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_0^\infty k^2(s) ds}\Big) \\ &\quad + C\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) H_0(k(t)), \quad \forall t \geq t_1, \end{aligned} \tag{4.19}$$

where $H_1(t) = tH_0'(\varepsilon_0 t)$.

Since $H_1'(t) = H_0'(\varepsilon_0 t) + \varepsilon_0 t H_0''(\varepsilon_0 t)$, then using the strict convexity of H_0 on $(0, r]$, we find that $H_1'(t)$, $H_1(t) > 0$, on $(0, 1]$. Thus, taking in account that $E_1'(t) \leq \frac{\eta}{2} k^2(t) \big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\big)$ and (4.19), for all $t \geq t_1$, we have

$$\begin{aligned} &\Big[tH_1\Big(\frac{E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_t^\infty k^2(s) ds}{E_1(0) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_0^\infty k^2(s) ds}\Big)\Big]' \\ &\leq H_1\Big(\frac{E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_t^\infty k^2(s) ds}{E_1(0) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_0^\infty k^2(s) ds}\Big) \\ &\leq -\frac{1}{\ell} F_1'(t) + \frac{C}{\ell}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) H_0(k(t)). \end{aligned}$$

A simple integration over (t_1, t) yields

$$\begin{aligned} &tH_1\Big(\frac{E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_t^\infty k^2(s) ds}{E_1(0) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_0^\infty k^2(s) ds}\Big) \\ &\leq t_1 H_1\Big(\frac{E_1(t_1) + \frac{\eta}{2}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) \int_{t_1}^\infty k^2(s) ds}{E_1(0) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_0^\infty k^2(s) ds}\Big) \\ &\quad + \frac{1}{\ell} F_1(t_1) + \frac{C}{\ell}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) \int_{t_1}^t H_0(k(s)) ds. \end{aligned}$$

This gives, for some positive constant k_2 and for all $t \geq t_1$,

$$H_1\Big(\frac{E_1(t) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_t^\infty k^2(s) ds}{E_1(0) + \frac{\eta}{2}(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_0^\infty k^2(s) ds}\Big) \leq \frac{k_2}{t} + \frac{C}{\ell t}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) \int_{t_1}^t H_0(k(s)) ds.$$

Therefore, for some positive constants k_1 and k_3, we obtain

$$\begin{aligned} E_1(t) &\leq k_1 H_1^{-1}\Big(\frac{k_2 + k_3(\int_{\Gamma_1} |u_0|^2 d\Gamma_1) \int_{t_1}^t H_0(k(s)) ds}{t}\Big) \\ &\quad - \frac{\eta}{2}\Big(\int_{\Gamma_1} |u_0|^2 d\Gamma_1\Big) \int_t^\infty k^2(s) ds. \end{aligned}$$

This completes the proof. □

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