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UNIFORMLY FINALLY BOUNDED SOLUTIONS TO SYSTEMS OF DIFFERENTIAL EQUATIONS WITH VARIABLE STRUCTURE AND IMPULSES

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Abstract. We study nonlinear non-autonomous systems of ordinary differential equations with variable structure and impulses. The consecutive changes on right-hand sides of this system and the impulsive effects on the solution of the corresponding initial problem take place simultaneously at the moment when the solution cancels the switching functions. We find sufficient conditions for the uniform final boundedness of solutions. These results are obtained using a suitable variation of the Lyapunov second method.

1. Introduction

Gurgula and Perestyuk are the first who apply the Lyapunov second method to study the solutions of impulsive equations. In [13], they use the "classical" continuous Lyapunov functions to study the stability of the "zero solution" of such equations. Discontinuous Lyapunov functions were introduced by Bainov and Simeonov [4]. The solutions qualities of one special class differential equations with variable structure and impulses are studied for the first time in [17]. The equations, studied in this paper were introduced in [7, 8]. There are numerous applications of equations with impulsive effects. We will point the articles [1, 3, 5, 8, 9, 15, 16, 18, 20, 21, 22, 23]. The applications of differential equations with variable structure are mainly in the control theory and engineering practice: [2, 8, 11, 12, 14, 19]. The main object of this article is to study the following initial problem for nonlinear non-autonomous systems of ordinary differential equations with variable structure and impulses at non-fixed moments:

$$\frac{dx}{dt} = f_i(t,x), \quad \text{if } \varphi_i\big(x(t)\big) \neq 0,\ t_{i-1} < t < t_i; \tag{1.1}$$

$$\varphi_i\big(x(t_i)\big) = 0, \quad i = 1, 2, \dots; \tag{1.2}$$

$$x(t_i + 0) = x(t_i) + I_i\big(x(t_i)\big); \tag{1.3}$$

$$x(t_0) = x_0, \tag{1.4}$$

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where: $f_i : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$, $\varphi_i : \mathbb{R}^n \to \mathbb{R}$, $I_i : \mathbb{R}^n \to \mathbb{R}^n$, the initial point $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$.

Denote the solution of problem (1.1)–(1.4) by $x(t; t_0, x_0)$ which is left-continuous at each point in the domain. In general, this solution has a finite jump discontinuity on the right at the moments $t_1, t_2, \dots$. They are named moments of switching. The functions $I_1, I_2, \dots$ and $\varphi_1, \varphi_2, \dots$ are named impulsive and switching functions, respectively.

We use the following notation:

- $\Phi_i\big\{x \in \mathbb{R}^n : \varphi_i(x) = 0\big\}$, $i = 1, 2, \dots$, are called switching hypersurfaces;
- Id is an identity in $\mathbb{R}^n$;
- $\gamma(t_0, x_0) = \big\{x(t : t_0, x_0),\ t_0 \le t \le T\big\}$ is a trajectory of problem (1.1)–(1.4) for $t_0 \le t \le T$;
- $B_\delta(x_0) = \big\{x \in \mathbb{R}^n : \|x - x_0\| < \delta\big\}$, where $x_0 \in \mathbb{R}^n$ and $\delta = \text{const} > 0$;
- $B^c_\delta(x_0) = \big\{x \in \mathbb{R}^n : \|x - x_0\| \ge \delta\big\} = \mathbb{R}^n \backslash B_\delta(x_0)$.

Definition 1.1. We say that the solution of system of differential equations with variable structure and impulses (1.1)–(1.3) is:

- bounded, if $(\forall t_0 \in \mathbb{R}^+)(\forall \alpha = \mathit{const} > 0)\big(\exists \beta = \beta(t_0, \alpha) > 0\big)$ such that
$$\big(\forall x_0 \in B_\alpha(x_0)\big) \Rightarrow \big\|x(t; t_0, x_0)\big\| < \beta,\ t \ge t_0;$$
- uniformly bounded, if $(\forall t_0 \in \mathbb{R}^+)(\forall \alpha = \text{const} > 0)\big(\exists \beta = \beta(\alpha) > 0\big)$ such that
$$\big(\forall x_0 \in B_\alpha(x_0)\big) \Rightarrow \big\|x(t; t_0, x_0)\big\| < \beta,\ t \ge t_0;$$
- quasi-uniformly finally bounded, if $(\exists \beta = \mathit{const} > 0) : (\forall \alpha = \mathit{const} > 0)\big(\exists T = T(\alpha) > 0\big)$ such that
$$\big(\forall x_0 \in B_\alpha(x_0)\big) \Rightarrow \big\|x(t; t_0, x_0)\big\| < \beta,\ t \ge t_0 + T;$$
- uniformly finally bounded, if the solutions are uniformly bounded and quasi-uniformly finally bounded.

Definition 1.2. We say that a sequence of scalar piecewise continuous Lyapunov functions
$$\big\{V_i,\ V_i : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+,\ i = 1, 2, \dots\big\},$$
corresponds to the system of differential equations with variable structure and impulses (1.1)–(1.3) if:

(1) $V_i \in C[\mathbb{R}^+ \times \mathbb{R}^n \backslash \Phi_i, \mathbb{R}^+]$, $i = 1, 2, \dots$;
(2) $V_i(t, 0) = 0,\ t \in \mathbb{R}^+,\ i = 1, 2, \dots$;
(3) For every point $(t, x_{\Phi_i}) \in \mathbb{R}^+ \times \Phi_i$ and for each $i = 1, 2, \dots$, there exist the limits:
$$\lim_{x \to x_{\Phi_i}, \Phi_i(x) < 0,} V_i(t, x) = V_i(t, x_{\Phi_i} - 0) = V_i(t, x_{\Phi_i}),$$
$$\lim_{x \to x_{\Phi_i}, \Phi_i(x) > 0,} V_i(t, x) = V_i(t, x_{\Phi_i} + 0).$$

We note that in general,
$$V_i(t, x_{\Phi_i}) = V_i(t, x_{\Phi_i} - 0) \ne V_i(t, x_{\Phi_i} + 0).$$

Definition 1.3. Let $\{V_i : \ i = 1, 2, \dots\}$ be a sequence of scalar piecewise continuous Lyapunov functions. Then for every point $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n \backslash \Phi_i$ and for each $i = 1, 2, \dots$, we define the derivative of V_i at point (t, x) with respect to system (1.1)–(1.3), as follows:

$$\dot{V}_i(t, x) = \dot{V}_{i,(1.1)-(1.3)}(t, x) = \lim_{h \to +0} \frac{1}{h}\big(V_i(t + h, x + hf_i(t, x)) - V_i(t, x)\big).$$

Remark 1.4. It can be shown that for $i = 1, 2, \dots$ and for every point $(t, x) = \big(t, x(t; t_0, x_0)\big) \in [t_0, \infty) \times (\mathbb{R}^n \backslash \Phi_i)$, it holds

$$\begin{aligned}\dot{V}(t, x) &= D^+_{(1.1)-(1.3)} V_i\big(t, x(t; t_0, x_0)\big) \\ &= \limsup_{h \to +0} \frac{1}{h}\Big(V_i\big(t + h, x(t + h; t_0, x_0)\big) - V_i\big(t, x(t; t_0, x_0)\big)\Big).\end{aligned}$$

In other words, the derivative of each Lyapunov function $V_i, \ i = 1, 2, \dots$, at every point $(t, x) = \big(t, x(t; t_0, x_0)\big)$ with respect to the system of differential equations (1.1)–(1.3) coincides with the upper right Dini derivative at the same point with respect to the solution of system under consideration.

We shall use the following class of scalar functions

$$K = \big\{a \in C[\mathbb{R}^+, \mathbb{R}^+], \ a \uparrow\uparrow, \ a(0) = 0\big\},$$

i.e. a is a strictly monotonically increasing function and $a(0) = 0$. We use the following conditions:

(H1) The functions f_i belong to $C[\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n]$, $i = 1, 2, \dots$;

(H2) There exist the constants $C_{f_i} > 0$ such that

$$\big(\forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n\big) \Rightarrow \|f_i(t, x)\| \le C_{f_i}, \ i = 1, 2, \dots;$$

(H3) The functions φ_i belong to $C^1[\mathbb{R}^n, \mathbb{R}]$, $i = 1, 2, \dots$;

(H4) There exist constants $C_{\operatorname{grad} \varphi_i} > 0$ such that

$$(\forall x \in \mathbb{R}^n) \Rightarrow \big\| \operatorname{grad} \varphi_i(x) \big\| \le C_{\operatorname{grad} \varphi_i}, \quad i = 1, 2, \dots;$$

(H5) The functions I_i belong to $C[\mathbb{R}^n, \mathbb{R}^n]$, $i = 1, 2, \dots$;

(H6) There exist the constants $C_{\varphi_{i+1}(Id+I_i)} > 0$ such that

$$(\forall x \in \Phi_i) \Rightarrow \big|\varphi_{i+1}\big((Id + I_i)(x)\big)\big| = \big|\varphi_{i+1}\big(x + I_i(x)\big)\big| \ge C_{\varphi_{i+1}(Id+I_i)}, \ i = 1, 2, \dots;$$

(H7) The next inequalities are valid

$$\varphi_i\big((Id + I_{i-1})(x)\big) \cdot \big\langle \operatorname{grad} \varphi_i(x), f_i(t, x)\big\rangle < 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \ i = 1, 2, \dots;$$

(H8) The series

$$\sum_{i=1}^{\infty} \frac{C_{\varphi_i(Id+I_{i-1})}}{C_{\operatorname{grad} \varphi_i} \cdot C_{f_i}}$$

diverges;

(H9) There exist the constants $C_{\langle \operatorname{grad} \varphi_i, f_i\rangle} > 0$ such that

$$\big(\forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n\big) \Rightarrow \big|\big\langle \operatorname{grad} \varphi_i(x), f_i(t, x)\big\rangle\big| \ge C_{\langle \operatorname{grad} \varphi_i, f_i\rangle}, \ i = 1, 2, \dots;$$

(H10) For every point $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$ and for each $i = 1, 2, \dots$, the solution of initial problem

$$\frac{dx}{dt} = f_i(t, x), \quad x(t_0) = x_0$$

exists and it is unique for $t \ge t_0$;

(H11) There exists constant $C_I > 0$ such that
$$(\forall x \in \mathbb{R}^n) \Rightarrow \big\| I_i(x) \big\| \le C_I, \quad i = 1, 2, \dots .$$

2. Preliminary results

Theorem 2.1 ([10]). *Assume conditions* (H1)–(H7). *Then*

(1) *If the trajectory $\gamma(t_0, x_0)$ of problem* (1.1)–(1.4) *meets consecutively the switching hypersurfaces Φ_i and Φ_{i+1}, then the following estimate is valid for the switching moments t_i and t_{i+1}:*
$$t_{i+1} - t_i \ge \frac{C_{\varphi_{i+1}(Id+I_i)}}{C_{\operatorname{grad}\varphi_{i+1}} C_{f_{i+1}}}, \quad i = 1, 2, \dots ;$$
(2) *If the trajectory $\gamma(t_0, x_0)$ meets all the switching hypersurfaces Φ_i, $i = 1, 2, \dots$, and condition (H8) is satisfied, then the switching moments increase indefinitely; i.e. $\lim_{i\to\infty} t_i = \infty$ is satisfied.*

Remark 2.2. If the following inequalities are satisfied
$$0 < C_{f_1} = C_{f_2} = \dots ;$$
$$0 < C_{\operatorname{grad}\varphi_1} = C_{\operatorname{grad}\varphi_2} = \dots ;$$
$$0 < C_{\varphi_2(Id+I_1)} = C_{\varphi_3(Id+I_2)} = \dots ,$$
then it is easy to establish that
$$\sum_{j=1}^{\infty} \frac{C_{\varphi_j(Id+I_{j-1})}}{C_{\operatorname{grad}\varphi_j} C_{f_j}} = \infty.$$
Therefore, condition (H8) follows by the equalities above. If these equalities and the conditions (H1)–(H7) are satisfied, and using the previous theorem, we deduce that the switching moments increase indefinitely.

Theorem 2.3 ([10]). *Assume conditions* (H1), (H3), (H5), (H7), (H9), (H10). *Then the trajectory of problem* (1.1)–(1.4) *meets every one of the hypersurfaces Φ_i, $i = 1, 2, \dots$.*

Using Theorem 2.1 and condition (H10), we obtain the next theorem.

Theorem 2.4. *Assume conditions* (H1)–(H8), (H10). *Then the solution of problem* (1.1)–(1.4) *exists and it is unique for $t \ge t_0$.*

We introduce a piecewise differentiable function $V : [t_0, \infty) \to \mathbb{R}^+$ with points of discontinuity, which coincide with the moments of switching $t_1, t_2, \dots$, by
$$\begin{aligned} V(t) &= V\big(t, x(t; t_0, x_0)\big) \\ &= \begin{cases} V_1\big(t, x(t; t_0, x_0)\big), \quad \text{if } t_0 \le t \le t_1; \\ V_{i+1}\big(t, x(t; t_0, x_0)\big) \\ = V_{i+1}\Big(t, x(t_i + 0; t_0, x_0) + \int_{t_i}^{t} f_{i+1}\big(\tau, x(\tau; t_0, x_0)\big) d\tau\Big) \\ = V_{i+1}\Big(t, x(t_i; t_0, x_0) + I_i\big(x(t_i; t_0, x_0)\big) + \int_{t_i}^{t} f_{i+1}\big(\tau, x(\tau; t_0, x_0)\big) d\tau\Big), \\ \quad \text{if } t_i < t \le t_{i+1},\ i = 1, 2, \dots . \end{cases} \end{aligned} \tag{2.1}$$
It is obvious that the function is continuous from the left in its domain.

The next theorem contains sufficient conditions under which function, introduced above is monotonically decreasing.

Theorem 2.5. *Assume that:*

(1) *Conditions (H1)–(H10) hold.*

(2) *There exists a sequence of scalar piecewise continuous Lyapunov functions*

$$\big\{V_i; V_i : \mathbb{R}^+ \times D \to \mathbb{R}^+,\ i = 1, 2, \dots\big\},$$

corresponding to the impulsive system of differential equations (1.1)–(1.3)*, and constant* $\rho > 0$ *such that*

(2.1) *It is satisfied that*

$$V_{i+1}\big(t + 0, x + I_i(x)\big) = V_{i+1}\big(t, x + I_i(x)\big) \le V_i(t, x),\ (t, x) \in \mathbb{R}^+ \times \big(B^c_\rho(0) \cap \Phi_i\big),$$

for $i = 1, 2, \dots$*;*

(2.2) *It is satisfied that*

$$\dot{V}_i(t, x) \le 0,\ (t, x) \in \mathbb{R}^+ \times \big(B^c_\rho(0) \cap \Phi_i\big), \quad i = 1, 2, \dots .$$

Then function $V(t) = V\big(t, x(t; t_0, x_0)\big)$*, defined in* (2.1)*, is monotonically decreasing in every interval* (t_*, t^*)*, for which:* $t_0 < t_*$ *and*

$$\|x(t; t_0, x_0)\| \ge \rho, \quad t_* < t < t^*.$$

Proof. According to Theorem 2.1, $\lim_{i\to\infty} t_i = \infty$. This implies that there are numbers k and p such that

$$t_k \le t_* < t_{k+1} < \dots < t_{k+p} \le t^* < t_{k+p+1}.$$

Within each of the open intervals

$$(t_*, t_{k+1});\ (t_{k+1}, t_{k+2}); \dots ; (t_{k+p-1}, t_{k+p});\ (t_{k+p}, t^*) \tag{2.2}$$

in accordance with condition (2.2) of the theorem, it is satisfied that

$$\frac{d}{dt}V(t) = D^+_{(1.1)-(1.3)}V_i\big(t, x(t; t_0, x_0)\big) = \dot{V}_{(1.1)-(1.3)}(t, x) = \dot{V}(t, x) \le 0.$$

So, we conclude that the function $V = V(t)$ is monotonically decreasing in each one of these intervals.

By condition (2.1) for $i = k + 1, k + 2, \dots , k + p$, we have

$$\begin{aligned} V(t_i + 0) - V(t_i) &= V_{i+1}\big(t_i + 0, x(t_i + 0; t_0, x_0)\big) - V\big(t_i, x(t_i; t_0, x_0)\big) \\ &= V_{i+1}\big(t_i, x(t_i; t_0, x_0) + I_i\big(x(t_i; t_0, x_0)\big)\big) - V_i\big(t_i, x(t_i; t_0, x_0)\big) \\ &\le 0. \end{aligned}$$

In this way, we obtain that function V is monotonically decreasing in the union of intervals (2.2), i.e. interval (t_*, t^*). The proof is complete. □

The next theorem contains sufficient conditions for the function V, introduced above, to be bounded.

Theorem 2.6. *Assume that:*

(1) *Conditions (H1)–(H11) hold;*

(2) *There exists a sequence of scalar piecewise continuous Lyapunov functions*

$$\big\{V_i; V_i : \mathbb{R}^+ \times D \to \mathbb{R}^+,\ i = 1, 2, \dots\big\},$$

corresponding to the impulsive system of differential equations (1.1)–(1.3)*, and constant* $\rho > 0$ *such that*

(2.1) *The functions $a, b \in K$ and constant $\rho > 0$ correspond on the upper sequence of Lyapunov functions satisfy*
(2.1.1) $a(\|x\|) \le V_i(t,x) \le b(\|x\|)$, $(t,x) \in \mathbb{R}^+ \times B^c_\rho(0)$, $i = 1, 2, \dots$,
(2.1.2) $\lim_{u\to\infty} a(u) = \infty$;
(2.2) *It is satisfied that*

$$V_{i+1}\big(t+0, x+I_i(x)\big) = V_{i+1}\big(t, x+I_i(x)\big) \\ \le V_{i+1}(t,x), \quad (t,x) \in \mathbb{R}^+ \times \big(B^c_\rho(0) \cap \Phi_i\big), \quad i = 1, 2, \dots;$$

(2.3) *It is satisfied that*

$$\dot V_i(t,x) \le 0, \ (t,x) \in \mathbb{R}^+ \times \big(B^c_\rho(0) \backslash \Phi_i\big), \quad i = 1, 2, \dots .$$

Then

$$(\forall \alpha = const \ge \rho + C_i > 0)(\forall x_0 \in B_\alpha(0)) \Rightarrow V(t) = V\big(t, x(t;t_0,x_0)\big) < \alpha(\beta), \ t \ge t_0,$$

where $\beta = \beta(\alpha) > a^{-1}\big(b(\alpha + C_I)\big)$.

Proof. Let α be an arbitrary positive constant, satisfying the inequality $\alpha \ge \rho + C_I$. From the definition of constant β, it follows:

$$\alpha(\beta) > b(a + C_I) > a(\alpha + C_I) \ and \ \beta > \alpha + C_I.$$

Let $(t_0, x_0) \in \mathbb{R}^+ \times B_\alpha(0)$. Assume that

$$(\exists t^* > t_0): \ V(t^*) = V\big(t^*, x(t^*;t_0,x_0)\big) \ge \alpha(\beta).$$

We will note that, it is possible for point t^* to be a switching moment, i.e. to exist a number $k \in N$ such that $t^* = t_k$. From the assumptions made and the conditions of theorem, we obtain

$$b\big(\|x(t^*;t_0,x_0)\|\big) \ge V(t^*) \ge \alpha(\beta) > b(\alpha),$$

i.e. $\|x(t^*;t_0,x_0)\| > \alpha$. From the inequality above and having in mind that the solution of problem is continuous on the left (including the switching points), we conclude that there is a point t_* such that

$$t_0 < t_* < t^*; \tag{2.3}$$

$$\rho < \alpha - C_I \le \|x(t_*;t_0,x_0)\| \le \alpha; \tag{2.4}$$

$$\alpha \le \|x(t;t_0,x_0)\|, \ t_* < t \le t^*. \tag{2.5}$$

Using successively condition (2.2) of this Theorem, inequalities (2.3) and (2.4), Theorem 2.5, and finally the assumptions made, we conclude that

$$b\big(\|x(t_*;t_0,x_0)\|\big) \ge V\big(t_*, x(t_*;t_0,x_0)\big) = V(t_*) \ge V(t^*) > b(\alpha),$$

from there we have $\|x(t_*;t_0,x_0)\| > \alpha$, which contradicts (2.4). The proof is complete. □

3. Main results

The main objective of this section is finding sufficient conditions for the uniform final boundedness of the solutions of system (1.1)–(1.3). The conditions are obtained by using the sequences of peacewise continuous scalar Lyapunov functions.

Theorem 3.1. *Assume that* (H1)–(H11) *and conditions (2.1) and (2.2) of Theorem 2.6 are satisfied. Also assume that*

$$\dot{V}_i(t,x) \le -c(\|x\|), \quad (t,x) \in \mathbb{R}^+ \times \big(B_\rho^c(0)\backslash\Phi_i\big),\ i = 1,2,\dots,$$

where the function $c \in K$. Then the solutions of system of differential equations with variable structure and impulses (1.1)–(1.3) *are uniformly finally bounded.*

Proof. According to Theorem 2.6, the solution of system (1.1)–(1.4) is uniformly bounded. More precisely, let:

$$\alpha \ge \rho; \tag{3.1}$$

$$\beta = \beta(\alpha) > \max\big\{\alpha + C_I, a^{-1}\big(b(\alpha)\big)\big\}; \tag{3.2}$$

$$(t_0, x_0) \in \mathbb{R}^+ \times B_\alpha(0). \tag{3.3}$$

Then

$$\big\|x(t;t_0,x_0)\big\| < \beta, \quad t \ge t_0. \tag{3.4}$$

The statement, formulated above will be valid if we replace the inequalities (3.1) and (3.2) with

$$\alpha \ge \rho + C_I; \tag{3.5}$$

$$\beta = \beta(\alpha) > a^{-1}\big(b(\alpha + C_I)\big). \tag{3.6}$$

Indeed, inequality (3.1) obviously follows from (3.5). Using (3.6), we obtain

$$\beta > a^{-1}\big(b(\alpha + C_I)\big) > a^{-1}\big(b(\alpha)\big); \tag{3.7}$$

$$a(\beta) > b(\alpha + C_I) > a(\alpha + C_I) \Leftrightarrow \beta > \alpha + C_I. \tag{3.8}$$

From these two inequalities, we obtain (3.2). Finally, by (3.3), (3.5) and (3.6) follows (3.4).

Let $B = \rho + C_I$. We shall show that

$$\begin{aligned}&(\forall\alpha = const > \rho + C_I)\big(\exists T = T(\alpha) > 0\big) : \big(\forall x_0 \in B_\alpha(0)\big)\\ &\Rightarrow \|x(t;t_0,x_0)\| < \beta,\ t \ge t_0 + T,\end{aligned}$$

whence, it follows that the solutions of the system are quasi-uniformly finally bounded.

There is $\beta > B$ and therefore $a(\beta) > a(B)$. Let ν be a natural number such that

$$\nu - 1 \le \frac{a(\beta) - a(B)}{a(B)} < \nu.$$

Denote

$$\theta_k = t_0 + k\frac{a(B)}{c(\rho)}, \quad k = 0,1,\dots,\nu.$$

We will show that regardless of the the choice of initial point $x_0 \in B_\alpha(0)$, the next estimates are valid:

$$V(t) = V\big(t, x(t;t_0,x_0)\big) < (\nu + 1 - k)a(B), \quad t \ge \theta_k,\ k = 0,1,\dots,\nu. \tag{3.9}$$

We shall prove the statement by induction. For $k = 0$, i.e. for $t \ge \theta_0 = t_0$, using Theorem 2.6 we have

$$V(t) = V\big(t, x(t;t_0,x_0)\big) < a(\beta) < (\nu + 1 - 0)a(B), \quad t \ge t_0 = \theta_0.$$

Assume that

$$V(t) < (\nu + 1 - k)a(B), \quad t \ge \theta_k. \tag{3.10}$$

We shall show that

$$V(t) < (\nu - k)a(B),\ t \geq \theta_{k+1}.$$

If the opposite is true, i.e.

$$(\exists t^* \geq \theta_{k+1}) : V(t^*) \geq (\nu - k)a(B),$$

then by (3.10) for $t = \theta_k$, the inequality above and monotony of function V (see Theorem 2.5), we obtain

$$V(\theta_k) - V(\theta_{k+1}) \leq V(\theta_k) - V(t^*) \leq a(B). \tag{3.11}$$

On the other hand, using the fact that function V is a piecewise differentiable, and also the inequality of condition (2) of this theorem, we arrive at the estimate

$$\begin{aligned} V(\theta_{k+1}) &\leq V(\theta_k) + \int_{\theta_k}^{\theta_{k+1}} \frac{d}{dt} V(\tau) d\tau \\ &< V(\theta_k) - c(\rho)(\theta_{k+1} - \theta_k) \\ &= V(\theta_k) - a(B). \end{aligned}$$

which contradicts (3.11). We substitute

$$T = T(\alpha) = \frac{a(B)}{c(\rho)} \nu.$$

Then

$$t \geq t_0 + T = t_0 + \frac{a(B)}{c(\rho)} \nu = \theta_\nu.$$

Then by (3.9) and condition (2.1.1) of Theorem 2.6, finally we obtain

$$a\big(\|x(t; t_0, x_0)\|\big) \leq V(t) = V\big(t, x(t; t_0, x_0)\big) < a(B),$$

i.e. $\|x(t; t_0, x_0)\| < \beta$. The proof is complete. □

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