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PRECISE ASYMPTOTIC BEHAVIOR OF STRONGLY DECREASING SOLUTIONS OF FIRST-ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study the asymptotic behavior of strongly decreasing solutions of the first-order nonlinear functional differential equation

$$x'(t) + p(t)|x(g(t))|^{\alpha-1}x(g(t)) = 0,$$

where α is a positive constant such that $0 < \alpha < 1$, $p(t)$ is a positive continuous function on $[a, \infty)$, $a > 0$ and $g(t)$ is a positive continuous function on $[a, +\infty)$ such that $\lim_{t\to\infty} g(t) = \infty$. Conditions which guarantee the existence of strongly decreasing solutions are established, and theorems are stated on the asymptotic behavior of such solutions, at infinity. The problem it is studied in the framework of regular variation, assuming that the coefficient $p(t)$ is a regularly varying function, and focusing on strongly decreasing solutions that are regularly varying. In addition, $g(t)$ is required to satisfy the condition

$$\lim_{t\to\infty} \frac{g(t)}{t} = 1.$$

Examples illustrating the results are also given.

1. Introduction

Consider the first-order nonlinear functional differential equation

$$x'(t) + p(t)|x(g(t))|^{\alpha-1}x(g(t)) = 0, \tag{1.1}$$

where α is a positive constant such that $0 < \alpha < 1$, $p(t)$ is a positive continuous function on $[a, \infty)$, $a > 0$ and $g(t)$ is a positive continuous function on $[a, +\infty)$ such that $\lim_{t\to\infty} g(t) = \infty$.

By a *solution* of the equation (1.1), we mean a function $x(t)$ which satisfies (1.1) for all $t \geq a$.

A solution $x(t)$ of the equation (1.1) is called *oscillatory*, if the terms $x(t)$ of the function are neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory*.

Assume that the solution $x(t)$ of (1.1) is nonoscillatory. Then it is either eventually positive or eventually negative. As $-x(t)$ is also a solution of (1.1), we may restrict ourselves to the case where $x(t) > 0$ for all large t.

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We are interested in the asymptotic behavior of positive solutions of (1.1) existing in some neighborhood of infinity and decreasing to zero as $t \to \infty$. Such solutions are often referred to as *strongly decreasing solutions* of (1.1).

An interesting question then arises whether (1.1) possess strongly decreasing solutions. If this is the case, is it possible to determine the asymptotic behavior at infinity of its strongly decreasing solutions precisely?

It seems to be difficult to answer the equation in general. So, we study the problem in the framework of regular variation, which means that the coefficient $p(t)$ is assumed to be a regularly varying function and our attention is focused only on strongly decreasing solutions which are regularly varying. Then, the question is answered in the affirmative in the case that the deviating argument $g(t)$ is required to satisfy the condition

$$\lim_{t\to\infty} \frac{g(t)}{t} = 1. \tag{1.2}$$

For the reader's convenience we recall here the definition of regularly varying functions, notations and some of basic properties including Karamata's integration theorem which will play an important role in establishing the main results of this paper.

Definition 1.1. A measurable function $f : [0,\infty) \to (0,\infty)$ is called *regularly varying of index* $\rho \in \mathbb{R}$ if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \text{for all } \lambda > 0.$$

The set of all regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. The symbol SV is often used to denote $\mathrm{RV}(0)$ in which case members of SV are called *slowly varying functions*. Since any function $f(t) \in\mathrm{RV}(\rho)$ is expressed as

$$f(t) = t^\rho g(t) \quad \text{with } g(t) \in \mathrm{SV},$$

the class SV of slowly varying functions is of fundamental importance in the theory of regular variation. Typical examples of slowly varying functions are all functions tending to positive constants as $t \to \infty$,

$$\prod_{n=1}^{N} (\log_n t)^{\alpha_n}, \quad \alpha_n \in \mathbb{R}, \quad \exp\Big\{\prod_{n=1}^{N} (\log_n t)^{\beta_n}\Big\}, \quad \beta_n \in (0,1),$$

where $\log_n t$ denotes the n-th iteration of $\log t$ and $\log t$ denotes the natural logarithm.

It is known that the function $2 + \sin(\log\log t)$ is regularly varying, whereas $2 + \sin(\log t)$ is not. The function

$$L(t) = \exp\big\{(\log t)^\theta \cos(\log t)^\theta\big\}, \quad \theta \in (0, \frac{1}{2}),$$

is a slowly varying function which is oscillating in the sense that

$$\limsup_{t\to\infty} L(t) = \infty \quad \text{and} \quad \liminf_{n\to\infty} L(t) = 0.$$

One of the most important properties of regularly varying functions is the following representation theorem.

Proposition 1.2. *A function $f(t) \in \mathrm{RV}(\rho)$ if and only if $f(t)$ is represented in the form*

$$f(t) = c(t) \exp\Big\{ \int_{t_0}^{t} \frac{\delta(s)}{s} ds \Big\}, \quad t \geq t_0, \tag{1.3}$$

for some $t_0 > 0$ and for some measurable functions $c(t)$ and $\delta(t)$ such that

$$\lim_{t\to\infty} c(t) = c_0 \in (0, \infty) \quad \text{and} \quad \lim_{t\to\infty} \delta(t) = \rho.$$

If $c(t) \equiv c_0$, then $f(t)$ is called a *normalized* regularly varying function of index ρ.

The following result illustrates operations which preserve slow variation.

Proposition 1.3. *Let $L(t)$, $L_1(t)$, $L_2(t)$ be slowly varying. Then, $L(t)^\alpha$ for any $\alpha \in \mathbb{R}$, $L_1(t) + L_2(t)$, $L_1(t)L_2(t)$ and $L_1(L_2(t))$ (if $L_2(t) \to \infty$) are slowly varying.*

A slowly varying function may grow to infinity or decay to zero as $t \to \infty$. But its order of growth or decay is several limited as in shown in the following.

Proposition 1.4. *Let $f(t) \in \mathrm{SV}$. Then, for any $\varepsilon > 0$,*

$$\lim_{t\to\infty} t^{\varepsilon} f(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} t^{-\varepsilon} f(t) = 0.$$

A simple criterion for determining the regularly of differentiable positive functions follows (see [6]).

Proposition 1.5. *A differentiable positive function $f(t)$ is a normalized regularly varying function of index ρ if and only if*

$$\lim_{t\to\infty} t \frac{f'(t)}{f(t)} = \rho.$$

The following proposition, known as Karamata's integration theorem [19, 20], is of highest importance in handling slowly and regularly varying functions analytically. Here and throughout the symbol $\sim$ is used to denote the asymptotic equivalence of two positive functions, that is

$$f(t) \sim g(t),\ t \to \infty \iff \lim_{t\to\infty} \frac{g(t)}{f(t)} = 1.$$

Proposition 1.6. *Let $L(t) \in \mathrm{SV}$. Then*

(i) *if $\alpha > -1$,*
$$\int_a^t s^\alpha L(s) ds \sim \frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(ii) *if $\alpha < -1$,*
$$\int_t^\infty s^\alpha L(s) ds \sim -\frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(iii) *if $\alpha = -1$,*
$$l(t) = \int_a^t \frac{L(s)}{s} ds \in \mathrm{SV}, \quad \text{and} \quad \lim_{t\to\infty} \frac{L(t)}{l(t)} = 0,$$
$$m(t) = \int_t^\infty \frac{L(s)}{s} ds \in SV, \quad \text{and} \quad \lim_{t\to\infty} \frac{L(t)}{m(t)} = 0.$$

Here in defining $m(t)$ it is assumed that $L(t)/t$ is integrable near the infinity.

We now define the class of nearly regularly varying functions. To this end it is convenient to introduce the following notation.

Notation. Let $f(t)$ and $g(t)$ be two positive continuous functions defined in a neighborhood of infinity, say for $t \geq T$. We use the notation $f(t) \asymp g(t)$, $t \to \infty$, to denote that there exist positive constants k and K such that

$$kg(t) \leq f(t) \leq Kg(t) \quad \text{for } t \geq T.$$

Clearly, $f(t) \sim g(t)$, $t \to \infty$, implies $f(t) \asymp g(t)$, $t \to \infty$, but not conversely. It is easy to see that if $f(t) \asymp g(t)$, $t \to \infty$ and if $\lim_{t\to\infty} g(t) = 0$, then $\lim_{t\to\infty} f(t) = 0$.

Definition 1.7. If $f(t)$ satisfies $f(t) \asymp g(t)$, $t \to \infty$, for some $g(t)$ which is regularly varying of index ρ, then $f(t)$ is called a *nearly regularly varying function of index* ρ.

For example, the function $2 + \sin(\log t)$ is nearly slowly varying because $2 + \sin(\log t) \asymp 2 + \sin(\log\log t)$, $t \to \infty$. It follows that, for any $\rho \in \mathbb{R}$, $t^\rho(2 + \sin(\log t))$ is nearly regularly varying, but not regularly varying, of index ρ.

For a complete exposition of theory of regular variation and its applications we refer the reader to the book by Bingham, Goldie and Teugels [1]. See also Seneta [21], Geluk and Haan [4]. A comprehensive survey of results up to 2000 on the asymptotic analysis of second order ordinary differential equations by means of regular variation can be found in the monograph of Marić [16]. Since the publication of [16] there has been an increasing interest in the analysis of ordinary differential equations by means of regularly varying functions, and thus theory of regular variation has proved to be a powerful tool of determining the accurate asymptotic behavior of positive solutions for a variety of nonlinear differential equations of Emden-Fowler and Thomas-Fermi types. See, for example, the papers [3, 6, 7, 9, 10, 11, 12, 13, 14, 15, 17, 18].

2. Main results

In this section we establish conditions under which (1.1) possess strongly decreasing solutions, and study the asymptotic behavior of such solutions. To this end, the following lemmas provide useful tools.

Lemma 2.1. *Assume that* (1.2) *is satisfied. Then, for any regularly varying function $f(t)$, it holds that*

$$f(g(t)) \sim f(t) \quad \text{as } t \to \infty.$$

Proof. Suppose that $f \in \mathrm{RV}(\sigma)$. By Proposition 1.2, $f(t)$ is represented as

$$f(t) = c(t) \exp\Big\{\int_{t_0}^{t} \frac{\delta(s)}{s} ds\Big\}, \quad t \geq t_0,$$

for some $t_0 > 0$ and for some measurable functions $c(t)$ and $\delta(t)$ such that

$$\lim_{t\to\infty} c(t) = c_0 \in (0,\infty) \quad \text{and} \quad \lim_{t\to\infty} \delta(t) = \sigma.$$

We may assume that $c(g(t))/c(t) \leq 2$ and $|\delta(t)| \leq 2|\sigma|$ for $t \geq t_0$. Then,

$$\begin{aligned}\frac{f(g(t))}{f(t)} &= \frac{c(g(t))}{c(t)} \exp\Big\{\int_t^{g(t)} \frac{\delta(s)}{s} ds\Big\} \leq 2\exp\Big\{2|\sigma|\Big|\int_t^{g(t)} \frac{ds}{s}\Big|\Big\} \\ &= 2\exp\Big\{2|\sigma|\,\Big|\log\Big(\frac{g(t)}{t}\Big)\Big|\Big\} \to 1, \quad t \to \infty,\end{aligned}$$

which means that $f(g(t)) \sim f(t)$, $t \to \infty$. The proof is complete. $\square$

Next we have a generalized L' Hospital's rule.

Lemma 2.2 ([5]). *Let $f(t), g(t) \in \mathrm{C}^1[T,\infty)$ and suppose that*

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = \infty \quad \textit{and} \quad g'(t) > 0 \quad \textit{for all large } t,$$

or

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = 0 \quad \textit{and} \quad g'(t) < 0 \quad \textit{for all large } t.$$

Then

$$\liminf_{t\to\infty} \frac{f'(t)}{g'(t)} \le \liminf_{t\to\infty} \frac{f(t)}{g(t)}, \quad \limsup_{t\to\infty} \frac{f(t)}{g(t)} \le \limsup_{t\to\infty} \frac{f'(t)}{g'(t)}.$$

The main results of this article are described in the following two theorems.

Theorem 2.3. *Suppose that $p(t)$ is regularly varying and $g(t)$ satisfies* (1.2). *Then,* (1.1) *possesses strongly decreasing slowly varying solutions if and only if*

$$p \in \mathrm{RV}(-1) \quad \textit{and} \quad \int_a^\infty p(t)dt < \infty, \tag{2.1}$$

in which case any such solution $x(t)$ obeys the unique decay law

$$x(t) \sim \Big((1-\alpha)\int_t^\infty p(s)ds\Big)^{\frac{1}{1-\alpha}}, \quad t \to \infty. \tag{2.2}$$

Theorem 2.4. *Suppose that $p(t)$ is regularly varying and $g(t)$ satisfies* (1.2). *Then,* (1.1) *possesses strongly decreasing regularly varying solutions of negative index ρ if and only if*

$$p \in \mathrm{RV}(\lambda) \quad \textit{with} \quad \lambda < -1, \tag{2.3}$$

in which case ρ is given by

$$\rho = \frac{\lambda+1}{1-\alpha}, \tag{2.4}$$

and any such solution $x(t)$ obeys the unique decay law

$$x(t) \sim (\frac{tp(t)}{-\rho})^{\frac{1}{1-\alpha}}, \quad t \to \infty. \tag{2.5}$$

Proof. We give simultaneous proof of both Theorems 2.3 and 2.4. We assume that $p \in \mathrm{RV}(\lambda)$ is represented in the form

$$p(t) = t^\lambda l(t), \quad l \in \mathrm{SV}, \tag{2.6}$$

and seek strongly decreasing regularly varying solutions $x(t)$ of (1.1) expressed as

$$x(t) = t^\rho \xi(t), \quad \xi \in \mathrm{SV}, \quad \rho \le 0. \tag{2.7}$$

Our aim is to solve the integral equation

$$x(t) = \int_t^\infty p(s)x(g(s))^\alpha ds, \tag{2.8}$$

in some neighborhood of infinity in the class of regularly varying functions.

Suppose that (1.1) has a strongly decreasing solution $x \in \mathrm{RV}(\rho)$. Using (2.6), (2.7), (1.2) and Lemma 2.1, we have

$$\int_t^\infty p(s)x(g(s))^\alpha ds = \int_t^\infty s^\lambda g(s)^{\alpha\rho} l(s)\xi(g(s))^\alpha ds \sim \int_t^\infty s^{\lambda+\alpha\rho} l(s)\xi(s)^\alpha ds, \tag{2.9}$$

as $t \to \infty$. The convergence of the last integral means that $\lambda + \alpha\rho \le -1$.

First consider the case that $\lambda + \alpha\rho = -1$. Then, from (2.8) and (2.9) it follows that

$$x(t) \sim \int_t^\infty s^{-1} l(s)\xi(s)^\alpha ds \in \mathrm{SV}, \quad t \to \infty, \tag{2.10}$$

which implies that $x \in \mathrm{SV}$; that is, $\rho = 0$ $(x(t) = \xi(t))$, and so we see that $\lambda = -1$. Now let $\eta(t)$ denote the right-hand side of (2.10). Then,

$$-\eta'^{-1} l(t)\xi(t)^\alpha \sim t^{-1} l(t)\eta(t)^\alpha,$$

which can be rewritten as

$$-\eta(t)^{-\alpha}\eta'^{-1} l(t) = p(t), \quad \text{or} \quad \Big(\frac{\eta(t)^{1-\alpha}}{1-\alpha}\Big)' \sim p(t), \tag{2.11}$$

as $t \to \infty$. Since $\eta(t) \to 0$, $t \to \infty$, (2.11) implies that $p(t)$ is integrable on $[a, \infty)$, and integrating (2.11) from $t \to \infty$, we easily find that

$$x(t) \sim \Big((1-\alpha)\int_t^\infty p(s)ds\Big)^{\frac{1}{1-\alpha}}, \quad t \to \infty.$$

Consider next the case that $\lambda + \alpha\rho < -1$. Then, applying Part (ii) of Proposition 1.6 to (2.9), we obtain

$$x(t) \sim \frac{t^{\lambda+\alpha\rho+1} l(t)\xi(t)^\alpha}{-(\lambda + \alpha\rho + 1)}, \quad t \to \infty, \tag{2.12}$$

This shows that $\rho = \lambda + \alpha\rho + 1$, or

$$\rho = \frac{\lambda+1}{1-\alpha}.$$

Since $\rho < 0$ in this case, we must have $\lambda < -1$. Noting that (2.12) is rewritten as

$$x(t) \sim \frac{tp(t)x(t)^\alpha}{-\rho}, \quad t \to \infty,$$

we immediately obtain the asymptotic relation for $x(t)$:

$$x(t) \sim \Big(\frac{tp(t)}{-\rho}\Big)^{\frac{1}{1-\alpha}}, \quad t \to \infty.$$

The above observations can be summarized as follows. A strongly decreasing regularly varying solution $x(t)$ of (1.1) is either slowly varying ($\rho = 0$) in which case $p(t)$ satisfies (2.1), or regularly varying of negative index ρ in which case $p(t)$ satisfies (2.3) and ρ is given by (2.4). Furthermore, the asymptotic behavior of $x(t)$ is governed by the unique formula (2.2) and (2.5) according as $\rho = 0$ and $\rho < 0$. This completes the proof of the "only if" parts of Theorems 2.3 and 2.4.

The proof of the "if" parts of the theorems proceeds as follows. Assume that $p(t)$ satisfies either (2.1) or (2.3). Let

$$X(t) = \begin{cases} \big((1-\alpha)\int_t^\infty p(s)ds\big)^{\frac{1}{1-\alpha}} & \text{if } \lambda = -1, \\ \big(\frac{tp(t)}{-\rho}\big)^{\frac{1}{1-\alpha}} & \text{if } \lambda < -1, \text{ where } \rho = \frac{\lambda+1}{1-\alpha}. \end{cases} \tag{2.13}$$

It can be shown that $X(t)$ satisfies the integral asymptotic relation

$$\int_t^\infty p(s)X(g(s))^\alpha ds \sim X(t), \quad t \to \infty. \tag{2.14}$$

In fact, if $\lambda = -1$ (and $p(t)$ is integrable on $[a, \infty)$), then

$$\begin{aligned}\int_t^\infty p(s)X(g(s))^\alpha ds &\sim \int_t^\infty p(s)X(s)^\alpha ds = \int_t^\infty p(s)\Big((1-\alpha)\int_s^\infty p(r)dr\Big)^{\frac{\alpha}{1-\alpha}} ds \\ &= \Big((1-\alpha)\int_t^\infty p(s)ds\Big)^{\frac{1}{1-\alpha}} = X(t), \quad t \to \infty,\end{aligned}$$

and if $\lambda < -1$, then using the expression for $X(t) = t^\rho (l(t)/(-\rho))^{1/(1-\alpha)}$, we find that

$$\begin{aligned}\int_t^\infty p(s)X(g(s))^\alpha ds &\sim \int_t^\infty p(s)X(s)^\alpha ds = \int_t^\infty s^{\lambda+\alpha\rho} l(s)(\frac{l(s)}{-\rho})^{\frac{\alpha}{1-\alpha}} ds \\ &= \int_t^\infty s^{\rho-1} l(s)(\frac{l(s)}{-\rho})^{\frac{\alpha}{1-\alpha}} ds \sim t^\rho \frac{l(t)}{-\rho}(\frac{l(t)}{-\rho})^{\frac{\alpha}{1-\alpha}} \\ &= t^\rho(\frac{l(t)}{-\rho})^{\frac{1}{1-\alpha}} = X(t), \quad \text{as } t \to \infty.\end{aligned}$$

In view of (2.14) there exists $T > a$ such that $T_0 := \inf_{t \ge T} g(t) \ge a$ and

$$\frac{1}{2}X(t) \le \int_t^\infty p(s)X(g(s))^\alpha ds \le 2X(t), \quad t \ge T. \tag{2.15}$$

Notice that $X(t)$ is decreasing in case $\lambda = -1$. We may assume that $X(t)$ is also decreasing on $[T_0, \infty)$ in case $\lambda < -1$. This follows from [1, Theorem 1.5.3] which asserts that any function $f \in \mathrm{RV}(\rho)$ with nonzero index ρ is asymptotic to a monotone function.

Choose positive constants k and K such that

$$k \le 2^{-\frac{1}{1-\alpha}}, \quad K \ge 2^{\frac{1}{1-\alpha}}, \tag{2.16}$$

and define the set $\mathcal{X}$ of continuous functions by

$$\mathcal{X} = \big\{x \in C[T_0, \infty) : kX(t) \le x(t) \le KX(t),\ t \ge T_0\big\}. \tag{2.17}$$

Finally consider the mapping $F : \mathcal{X} \to C[T_0, \infty)$ defined by

$$Fx(t) = \begin{cases} \int_t^\infty p(s)x(g(s))^\alpha ds & \text{for } t \ge T, \\ \frac{Fx(T)}{X(T)}X(t) & \text{for } T_0 \le t \le T. \end{cases} \tag{2.18}$$

We will show that the Schauder-Tychonoff fixed point theorem (see e.g., [2, Chapter I]) is applicable to F acting on $\mathcal{X}$. Let $x \in \mathcal{X}$. Using (2.15)–(2.18), we see that

$$\begin{aligned}Fx(t) &\le \int_t^\infty p(s)(KX(g(s)))^\alpha ds \le 2K^\alpha X(t) \le KX(t), \\ Fx(t) &\ge \int_t^\infty p(s)(kX(g(s)))^\alpha ds \ge \frac{1}{2}k^\alpha X(t) \ge kX(t),\end{aligned}$$

so that $kX(t) \le Fx(t) \le KX(t)$ for $t \ge T$. The last inequality clearly holds for $T_0 \le t \le T$ since $k \le Fx(T)/X(T) \le K$. Thus, F maps X into itself.

Since $kX(t) \le Fx(t) \le KX(t)$ for $t \ge T_0$, the set $F(\mathcal{X})$ is uniformly bounded on $[T_0, \infty)$. Since

$$0 \ge (Fx)'^{\alpha} p(t)X(g(t))^\alpha, \quad t \ge T,$$

for all $x \in \mathcal{X}$. $F(\mathcal{X})$ is equicontinuous on $[T, \infty)$, and hence on $[T_0, \infty)$. Then, the relative compactness of $F(\mathcal{X})$ follows from the Ascoli's theorem.

Finally, let $\{x_n(t)\}$ be any sequence in $\mathcal{X}$ converging, as $n \to \infty$, to $x(t)$ in $\mathcal{X}$ uniformly on compact subintervals of $[T_0, \infty)$. Then, by (2.18) we have

$$|Fx_n(t) - Fx(t)| \le \int_t^\infty p(s)|x_n(g(s))^\alpha - x(g(s))^\alpha| ds, \quad t \ge T, \tag{2.19}$$

and, for $T_0 \le t \le T$,

$$|Fx_n(t) - Fx(t)| = \frac{|Fx_n(T) - Fx(T)|}{X(T)} X(t) \le |Fx_n(T) - Fx(T)|. \tag{2.20}$$

Application of the Lebesgue dominated convergence theorem to the right-hand side of (2.19) ensures that, as $n \to \infty$, $Fx_n(t)$ converges to $Fx(t)$ uniformly on $[T, \infty)$, and using this fact in (2.20) we conclude that the convergence $Fx_n(t) \to Fx(t)$ is uniform on the entire interval $[T_0, \infty)$.

Thus all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled, and hence there exists $x \in \mathcal{X}$ such that $x = Fx$, which implies in particular that $x(t)$ satisfies the integral equation (2.8) on $[T, \infty)$, that is, $x(t)$ is a strongly decreasing solution of equation (1.1). The membership $x \in \mathcal{X}$ implies that $x(t)$ is a nearly regularly varying of the same index as $X(t)$. It remains to verify that $x(t)$ is certainly a regularly varying with the help of the generalized L'Hospital's rule (Lemma 2.2).

Let $x(t)$ be the strongly decreasing solution of (1.1) obtained above as a solution of the integral equation (2.8). Define the function $u(t)$ by

$$u(t) = \int_t^\infty p(s)X(g(s))^\alpha ds, \tag{2.21}$$

and put

$$m = \liminf_{t\to\infty} \frac{x(t)}{u(t)}, \quad M = \limsup_{t\to\infty} \frac{x(t)}{u(t)}. \tag{2.22}$$

Since $x(t) \asymp X(t)$, $t \to \infty$, it is clear that $0 < m \le M < \infty$. We now apply Lemma 2.2 to M, obtaining

$$\begin{aligned} M &\le \limsup_{t\to\infty} \frac{x'(t)}{u'(t)} = \limsup_{t\to\infty} \frac{p(t)x(g(t))^\alpha}{p(t)X(g(t))^\alpha} \\ &= \Big(\limsup_{t\to\infty} \frac{x(g(t))}{X(g(t))}\Big)^\alpha = \Big(\limsup_{t\to\infty} \frac{x(t)}{X(t)}\Big)^\alpha \\ &= \Big(\limsup_{t\to\infty} \frac{x(t)}{u(t)}\Big)^\alpha = M^\alpha, \end{aligned}$$

where the relation $u(t) \sim X(t)$, $t \to \infty$, (cf. (2.14)) has been used in the last step. Thus, $M \le M^\alpha$, which implies $M \le 1$ because of $\alpha < 1$. Likewise, application of Lemma 2.2 to m leads to $m \ge 1$. It follows therefore that $m = M = 1$; that is,

$$\lim_{t\to\infty} \frac{x(t)}{u(t)} = 1;$$

i.e.,

$$x(t) \sim u(t) \sim X(t), \quad t \to \infty,$$

which shows that $x(t)$ is a regularly varying function of index $\rho = 0$ or of index $\rho = (\lambda + 1)/(1 - \alpha) < 0$ according as $\lambda = -1$ or $\lambda < -1$. This completes the proof of Theorems 2.3 and 2.4. $\square$

3. Perturbations of equation (1.1)

Consider the following perturbation of equation (1.1),

$$x'(t) + p(t)|x(g(t))|^{\alpha-1}x(g(t)) + q(t)|x(h(t))|^{\beta-1}x(h(t)) = 0, \tag{3.1}$$

where α is a positive constant such that $0 < \alpha < 1$, $p(t)$ is a positive continuous function on $[a,\infty)$, $a > 0$ and $g(t)$ is a positive continuous function on $[a,+\infty)$ such that $\lim_{t\to\infty} g(t) = \infty$, β is a positive constant, $q(t)$ is a positive continuous function on $[a,\infty)$ and $h(t)$ is a continuous deviating argument on $[a,\infty)$ such that $\lim_{t\to\infty} h(t) = \infty$.

Our purpose here is to show that the structure of strongly decreasing solutions of equation (1.1) remains essentially unchanged provided the perturbation is sufficiently small in a definite sense. The main result is described in the following theorem.

Theorem 3.1. *Assume that $p \in \mathrm{RV}(\lambda)$ satisfies (2.1) or (2.3). Let $X(t)$ denote the function defined by (2.13). Suppose moreover that*

$$\lim_{t\to\infty} \frac{q(t)X(h(t))^\beta}{p(t)X(g(t))^\alpha} = 0. \tag{3.2}$$

(i) Let (2.1) hold. Then, equation (3.1) possesses strongly decreasing slowly varying solutions all of which enjoy one and the same asymptotic behavior

$$x(t) \sim \Big((1-\alpha)\int_t^\infty p(s)ds\Big)^{\frac{1}{1-\alpha}}, \quad t\to\infty. \tag{3.3}$$

(ii) Let (2.3) hold. Then, equation (3.1) possesses strongly decreasing regularly varying of the unique negative index $\rho = \frac{\lambda+1}{1-\alpha}$ all of which enjoy one and the same asymptotic behavior

$$x(t) \sim \Big(\frac{tp(t)}{-\rho}\Big)^{\frac{1}{1-\alpha}}, \quad t\to\infty. \tag{3.4}$$

Proof. Choose positive constants k and K satisfying

$$k \le 2^{-\frac{1}{1-\alpha}}, \quad K \ge 4^{\frac{1}{1-\alpha}}, \tag{3.5}$$

Since $X(t)$ satisfies (2.14), there exists $T > a$ such that $T_0 := \inf_{t\ge T} g(t) \ge a$ and

$$\frac{1}{2}X(t) \le \int_t^\infty p(s)X(g(s))^\alpha ds \le 2X(t), \quad t \ge T. \tag{3.6}$$

In view of (3.2) we may assume that T is chosen so that

$$\frac{q(t)X(h(t))^\beta}{p(t)X(g(t))^\alpha} \le \frac{k^\alpha}{K^\beta} \quad \text{for } t\ge T. \tag{3.7}$$

Let

$$\mathcal{X} = \big\{x \in C[T_0,\infty) : kX(t) \le x(t) \le KX(t),\ t\ge T_0\big\} \tag{3.8}$$

and consider the mapping $G : \mathcal{X} \to C[T_0,\infty)$ defined by

$$Gx(t) = \begin{cases} \int_t^\infty \big(p(s)x(g(s))^\alpha + q(s)x(h(s))^\beta\big)ds & \text{for } t\ge T,\\ \frac{Gx(T)}{X(T)}X(t) & \text{for } T_0 \le t \le T. \end{cases} \tag{3.9}$$

One can prove that (i) G maps $\mathcal{X}$ into itself, (ii) $G(\mathcal{X})$ is relatively compact in $C[T_0,\infty)$ and (iii) G is a continuous mapping.

(i) $G(\mathcal{X}) \subset \mathcal{X}$. Let $x \in \mathcal{X}$. Then, since (3.7) implies

$$\begin{aligned} p(t)x(g(t))^\alpha + q(t)x(h(t))^\beta &= p(t)x(g(t))^\alpha\Big(1 + \frac{q(t)x(h(t))^\beta}{p(t)x(g(t))^\alpha}\Big) \\ &\le p(t)x(g(t))^\alpha\Big(1 + \frac{K^\beta q(t)X(h(t))^\beta}{k^\alpha p(t)X(g(t))^\alpha}\Big) \\ &\le 2p(t)x(g(t))^\alpha, \end{aligned}$$

for $t \ge T$, using (3.6) and (3.5), we see that

$$Gx(t) \le 2\int_t^\infty p(s)x(g(s))^\alpha ds \le 2K^\alpha \int_t^\infty p(s)X(g(s))^\alpha ds \le 4K^\alpha X(t) \le KX(t),$$

for $t \ge T$. Since

$$Gx(t) \ge \int_t^\infty p(s)x(g(s))^\alpha ds \ge k^\alpha \int_t^\infty p(s)X(g(s))^\alpha ds \ge \frac{1}{2}k^\alpha X(t) \ge kX(t),$$

for $t \ge T$, we see that $kX(t) \le Gx(t) \le KX(t)$ for $t \ge T$. It is clear that this inequality holds also for $T_0 \le t \le T$. This shows that G is a self-map on $\mathcal{X}$.

(ii) $G(\mathcal{X})$ is relatively compact. It is clear that $G(\mathcal{X})$ is uniformly bounded on $[T_0, \infty)$. $G(\mathcal{X})$ is equicontinuous on $[T, \infty)$ since it holds that

$$0 \ge (Gx)'^\alpha p(t)X(g(t))^\alpha + K^\beta q(t)X(h(t))^\beta), \quad t \ge T,$$

for all $x \in \mathcal{X}$. The equicontinuity on $[T_0, T]$ is evident.

(iii) G is continuous. Let $\{x_n(t)\}$ be a sequence in $\mathcal{X}$ converging, as $n \to \infty$, to $x(t)$ in $\mathcal{X}$ uniformly on any compact subinterval of $[T_0, \infty)$. We then have

$$|Gx_n(t) - Gx(t)| \le \int_t^\infty \Big(p(s)|x_n(g(s))^\alpha - x(g(s))^\alpha| + q(s)|x_n(h(s))^\beta - x(h(s))^\beta|\Big)ds \tag{3.10}$$

for $t \ge T$, and

$$|Gx_n(t) - Gx(t)| \le |Gx_n(T) - Gx(T)| \quad \text{for } T_0 \le t \le T,$$

from which the uniform convergence of $Gx_n(t) \to Gx(t)$ on $[T_0, \infty)$ follows as a consequence of application of the Lebesgue dominated convergence theorem to the right-hand side of (3.10).

Therefore, by the Schauder-Tychonoff fixed point theorem there exists a fixed point $x \in \mathcal{X}$ of G, which satisfies the integral equation

$$x(t) = \int_t^\infty \big(p(s)x(g(s))^\alpha + q(s)x(h(s))^\beta\big)ds \tag{3.11}$$

for $t \ge T$. Hence $x(t)$ is a strongly decreasing solution of (3.1) on $[T, \infty)$ which is nearly regularly varying. That $x(t)$ is certainly regularly varying can be proved with the help of Lemma 2.2.

Let

$$u(t) = \int_t^\infty \big(p(s)X(g(s))^\alpha + q(s)X(h(s))^\beta\big)ds \tag{3.12}$$

and consider the inferior and superior limits of $x(t)/u(t)$:

$$m = \liminf_{t\to\infty} \frac{x(t)}{u(t)}, \quad M = \limsup_{t\to\infty} \frac{x(t)}{u(t)}, \tag{3.13}$$

The fact that $x(t) \asymp X(t)$, $t \to \infty$, guarantees that $0 < m \le M < \infty$. We notice that

$$p(t)X(g(t))^\alpha + q(t)X(h(t))^\beta \sim p(t)X(g(t))^\alpha, \quad t \to \infty, \tag{3.14}$$

(cf. (3.2)) which implies taht

$$p(t)x(g(t))^\alpha + q(t)x(h(t))^\beta \sim p(t)x(g(t))^\alpha, \quad t \to \infty. \tag{3.15}$$

We now apply Lemma 2.2 to m and M. Using (3.14), (3.15) and the relation $u(t) \sim X(t)$, $t \to \infty$, which follows from (3.14), we obtain

$$\begin{aligned} M &\le \limsup_{t\to\infty} \frac{x'(t)}{u'(t)} = \limsup_{t\to\infty} \frac{p(t)x(g(t))^\alpha + q(t)x(h(t))^\beta}{p(t)X(g(t))^\alpha + q(t)X(h(t))^\beta} \\ &= \limsup_{t\to\infty} \frac{p(t)x(g(t))^\alpha}{p(t)X(g(t))^\alpha} = \Big(\limsup_{t\to\infty} \frac{x(g(t))}{X(g(t))}\Big)^\alpha \\ &= \Big(\limsup_{t\to\infty} \frac{x(t)}{X(t)}\Big)^\alpha = \Big(\limsup_{t\to\infty} \frac{x(t)}{u(t)}\Big)^\alpha = M^\alpha. \end{aligned}$$

Thus, we have $M \le M^\alpha$, which implies $M \le 1$ because $\alpha < 1$. Similarly, Lemma 2.2 applied to m leads to $m \ge m^\alpha$ which gives $m \ge 1$. It follows that $m = M = 1$; that is,

$$\lim_{t\to\infty} \frac{x(t)}{u(t)} = 1 \implies x(t) \sim u(t) \sim X(t), \quad t \to \infty.$$

We conclude therefore that $x(t)$ is slowly varying if $\lambda = -1$ and regularly varying of negative index $\rho = \frac{\lambda+1}{1-\alpha}$ if $\lambda < -1$. The proof is complete. $\square$

Remark 3.2. It is worth noticing that in Theorem 3.1 the exponent β may be any constant (larger or smaller than 1), the coefficient $q(t)$ may not be regularly varying, and the only requirement for the deviating argument $h(t)$ is that $\lim_{t\to\infty} h(t) = \infty$.

Remark 3.3. In equation (3.1) suppose that

$$q \in \mathrm{RV}(\mu) \quad \text{and} \quad h \in \mathrm{RV}(\nu), \quad \nu \ge 0. \tag{3.16}$$

Then,

$$p(t)X(g(t))^\alpha \in \mathrm{RV}(\lambda + \alpha\rho), \quad q(t)X(h(t))^\beta \in \mathrm{RV}(\mu + \beta\rho\nu),$$

and so condition (3.2) is satisfied if

$$\mu + \beta\rho\nu < \lambda + \alpha\rho, \tag{3.17}$$

which gives, via Theorem 3.1, a practical criterion for the existence of strongly decreasing solutions for equation (3.1) with regularly varying $q(t)$ and $h(t)$. Note that if $\rho = 0$ ($\lambda = -1$), then (3.17) reduces to $\mu < -1$.

Corollary 3.4. Assume that $p(t)$ satisfies (2.1) or (2.3), and that $g(t)$ satisfies (1.2). Suppose moreover that $q(t)$ and $h(t)$ satisfy (3.16).

(i) Let $\lambda = -1$. If $\mu < -1$, then (3.1) possesses strongly decreasing slowly varying solutions $x(t)$ all of which enjoy the unique asymptotic behavior

$$x(t) \sim \Big((1-\alpha)\int_t^\infty p(s)ds\Big)^{\frac{1}{1-\alpha}}, \quad t \to \infty.$$

(ii) Let $\lambda < -1$. If (3.17) holds, then (3.1) possesses strongly decreasing regularly varying solutions $x(t)$ of negative index ρ all of which enjoy the unique asymptotic behavior

$$x(t) \sim \Big(\frac{tp(t)}{-\rho}\Big)^{\frac{1}{1-\alpha}}, \quad t \to \infty.$$

4. Examples

In this section we give four examples illustrating the main results of this article.

Example 4.1. Consider the equation (1.1) with $p(t)$ satisfying

$$p(t) \sim \frac{\exp(-\sqrt{\log t})}{t\sqrt{\log t}}, \quad t \to \infty, \tag{4.1}$$

and call it equation (E1). Obviously $p \in \mathrm{RV}(-1)$ and $p(t)$ is integrable near the infinity, and so by Theorem 2.3 equation (E1), for any $g(t)$ satisfying (1.2), has strongly decreasing slowly varying solutions $x(t)$ all of which enjoy the unique asymptotic behavior

$$x(t) \sim \Big((1-\alpha)\int_t^\infty p(s)ds\Big)^{\frac{1}{1-\alpha}} \sim (2(1-\alpha))^{\frac{1}{1-\alpha}} \exp\big(-\sqrt{\log t}\big), \quad t \to \infty. \tag{4.2}$$

If in particular

$$p(t) = \frac{\exp(-\sqrt{\log t})}{t\sqrt{\log t}} \exp\Big(\frac{\alpha}{1-\alpha}(\sqrt{\log g(t)} - \sqrt{\log t})\Big),$$

then $p(t)$ satisfies (4.1) and equation (E1) possesses an exact slowly varying solution

$$x_0(t) = (2(1-\alpha))^{\frac{1}{1-\alpha}} \exp\big(-\sqrt{\log t}\big).$$

Example 4.2. Consider the equation (1.1) with $p(t)$ satisfying

$$p(t) \sim t^{-\alpha-1}L(t), \quad t \to \infty, \tag{4.3}$$

where $L(t)$ is any continuous slowly varying function, and call it equation (E2). Since $\lambda = -\alpha - 1 < -1$, from Theorem 2.4 it follows that equation (E2) possesses strongly decreasing solutions belonging to the class $\mathrm{RV}(-\frac{\alpha}{1-\alpha})$ and that any such solution $x(t)$ enjoys the asymptotic behavior

$$x(t) \sim \big(\frac{1-\alpha}{\alpha}\big)^{\frac{1}{1-\alpha}} t^{-\frac{\alpha}{1-\alpha}} L(t)^{\frac{1}{1-\alpha}}, \quad t \to \infty. \tag{4.4}$$

If in particular

$$p(t) = t^{-\alpha-1}L(t)\Big(\frac{g(t)}{t}\Big)^{\frac{\alpha^2}{1-\alpha}} \Big(\frac{L(t)}{L(g(t))}\Big)^{\frac{\alpha}{1-\alpha}} \Big(1 - \frac{tL'(t)}{\alpha L(t)}\Big),$$

where $L(t)$ is a continuously differentiable slowly varying function, then $p(t)$ satisfies (4.3) (use Lemma 2.1 and Proposition 1.5) and equation (E2) has an exact strongly decreasing regularly varying solution

$$x_0(t) = \Big(\frac{1-\alpha}{\alpha}\Big)^{\frac{1}{1-\alpha}} t^{-\frac{\alpha}{1-\alpha}} L(t)^{\frac{1}{1-\alpha}},$$

for any deviating argument $g(t)$ satisfying (1.2).

Example 4.3. Consider equation (3.1) in which $\alpha < 1$, $p(t)$ satisfies (4.1), $\beta > 0$ is a constant and $q(t)$ and $h(t)$ satisfy (3.16). This equation is referred to as equation (EP1). By (i) of Corollary 3.4 one concludes that (EP1) possesses strongly decreasing slowly varying solutions all of which enjoy the asymptotic behavior (4.2). It is to be noted that the perturbed term may be superlinear ($\beta > 1$) or sublinear ($\beta < 1$), and that any deviating argument, retarded, advanced or otherwise, is admitted as $h(t)$ as long as it is regularly varying of nonnegative index. For instance, $h(t)$ may be any one of the following:

$$t \pm \tau, \quad t \pm \sqrt{t}, t \pm \log t, \quad ct, \quad t^{\theta}, \quad \log t,$$

where τ, c and θ are positive constants.

For example, if $\alpha < 1$, $\mu < -1$ and $g(t) \sim t$, $t \to \infty$, then the equation (EP1)

$$x'(t) + \frac{\exp(-\sqrt{\log t})}{t\sqrt{\log t}} |x(g(t))|^{\alpha-1} x(g(t)) + t^{\mu} L(t) |x(h(t))|^{\beta-1} x(h(t)) = 0, \tag{4.5}$$

always possesses strongly decreasing slowly varying solutions $x(t)$ all of which behave like

$$x(t) \sim (2(1-\alpha))^{\frac{1}{1-\alpha}} \exp\big(-\sqrt{\log t}\big), \quad t \to \infty,$$

for any constant $\beta > 0$, any $L \in \mathrm{SV}$ and any $h \in \mathrm{RV}(\nu)$, $\nu \geq 0$, such that $\lim_{t\to\infty} h(t) = \infty$.

Example 4.4. Consider equation (3.1) in which $\alpha < 1$, $p(t)$ satisfies (4.3), and β, $q(t)$ and $h(t)$ are as in the above equation (EP1). We call this equation (EP2). Since $\lambda = -\alpha - 1$ and $\rho = -\frac{\alpha}{1-\alpha}$, condition (3.17) is reduced to

$$\mu < \frac{\alpha\beta\nu - 1}{1-\alpha}, \tag{4.6}$$

which ensures the existence of strongly decreasing regularly varying solutions $x(t)$ of negative index for equation (EP2) having the asymptotic behavior

$$x(t) \sim \big(\frac{1-\alpha}{\alpha}\big)^{\frac{1}{1-\alpha}} t^{-\frac{\alpha}{1-\alpha}} L(t)^{\frac{1}{1-\alpha}}, \quad t \to \infty. \tag{4.7}$$

In particular, if $\mu < -\frac{1}{1-\alpha}$, then the equation

$$x'^{-\alpha-1} L(t) |x(g(t))|^{\alpha-1} x(t + \sin t) + t^{\mu} M(t) |x(\log t)|^{\beta-1} x(\log t) = 0$$

has strongly decreasing solutions $x(t)$ satisfying (4.7) for any positive constant β and for any continuous slowly varying functions $L(t)$ and $M(t)$.

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