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MULTIPLE SOLUTIONS TO ASYMMETRIC SEMILINEAR ELLIPTIC PROBLEMS VIA MORSE THEORY

LEANDRO RECOVA, ADOLFO RUMBOS

ABSTRACT. In this article we study the existence of solutions to the problem

$$
\begin{gathered}
-\Delta u = g(x,u) \quad \text{in } \Omega; \\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered}
$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 2$) and $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a differentiable function with $g(x,0) = 0$ for all $x \in \Omega$. By using minimax methods and Morse theory, we prove the existence of at least three nontrivial solutions for the case in which an asymmetric condition on the nonlinearity g is assumed. The first two nontrivial solutions are obtained by employing a cutoff technique used by Chang et al in [9]. For the existence of the third nontrivial solution, first we compute the critical group at infinity of the associated functional by using a technique used by Liu and Shaoping in [19]. The final result is obtained by using a standard argument involving the Morse relation.

1. Introduction

The goal of this article is to study the existence and multiplicity of solutions of the boundary-value problem

$$
\begin{gathered}
-\Delta u = g(x,u) \quad \text{in } \Omega; \\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.1}
$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded set with smooth boundary, $\partial\Omega$, and g is a differentiable function. By a solution of (1.1) we mean a weak solution, i.e., a function $u \in H_0^1(\Omega)$ satisfying

$$
\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega g(x,u) v dx, \tag{1.2}
$$

for any $v \in H_0^1(\Omega)$, where $H_0^1(\Omega)$ is the Sobolev space obtained through completion of $C_0^\infty(\Omega)$ with respect to the metric induced by the norm

$$
\|u\| = \Big(\int_\Omega |\nabla u|^2 dx \Big)^{1/2}, \quad \text{for all } u \in H_0^1(\Omega).
$$

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We will denote by $0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$ the distinct eigenvalues of the linear problem

$$\begin{gathered} -\Delta u = \lambda u \quad \text{in } \Omega; \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

The following conditions on g, and its primitive, $G(x,s) = \int_0^s g(x,\xi)d\xi$, for all $x \in \Omega$ and $s \in \mathbb{R}$, will be assumed throughout this article:

- (G1) $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is differentiable, $g(x,0) = 0$, and $g'(x,0) = \lambda_m$ with $m > 1$.
- (G2) There is $\lambda > 0$ with $\lambda \neq \lambda_1$, and $0 \le \alpha < 1$ such that
$$\lim_{s\to-\infty} \frac{g(x,s) - \lambda s}{|s|^\alpha} = 0.$$
- (G3) There are θ and s_0 with $0 < \theta < 1/2$ and $s_0 > 0$ such that
$$0 < G(x,s) \le \theta s g(x,s), \quad \text{for } s > s_0 \text{ and all } x \in \Omega.$$
- (G4) $\lim_{s\to\infty} g(x,s)/s^\sigma = 0$, where $\frac{1}{\theta} - 1 < \sigma \le \frac{N+2}{N-2}$, if $N \ge 3$, or $1 < \sigma < \infty$ if $N = 2$.
- (G5) $\sigma\theta < \min\big\{\frac{1}{1+\alpha}, \frac{N+2}{2N}\big\}$.
- (G6) There exists $s^- < 0$ such that
$$2G(x,s) - g(x,s)s \le 0, \text{ for all } s < s_-.$$

The main result of this article is the following.

Theorem 1.1. *Assume g satisfies* (G1)–(G6) *and there exists $t_0 > 0$ such that $g(x,t_0) = 0$. Then problem* (1.1) *has at least three nontrivial solutions.*

The work in this article was motivated by that of De Figueiredo's in [11]. In that paper, the author was interested in studying the solvability of the problem

$$\begin{gathered} -\Delta u = \lambda u + f(x,u) + t\varphi + h \quad \text{in } \Omega; \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

where φ is a positive eigenfunction associated with the the first eigenvalue λ_1 of $(-\Delta, H_0^1(\Omega))$, $t \in \mathbb{R}$ and $h \in C^\nu(\overline{\Omega})$, $0 < \nu \le 1$, $\int_\Omega h\varphi dx = 0$. In [11], the author assumed the following conditions on the nonlinearity f and its primitive F:

- (F1) $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a C^1 function.
- (F2) There exists $0 < \alpha < 1$ such that $\lim_{s\to-\infty} f(x,s)|s|^{-\alpha} = 0$.
- (F3) $\lim_{s\to-\infty} f'_s(x,s) = 0$.
- (F4) There are θ and s_0 with $0 < \theta < 1/2$ and $s_0 > 0$ such that $0 < F(x,s) \le \theta s f(x,s)$, for $s > s_0$ and all $x \in \Omega$.
- (F5) $\lim_{s\to+\infty} f(x,s)s^{-\sigma} = 0$, where $\sigma \le (N+2)/(N-2)$ if $N \ge 3$ or $1 < \sigma < \infty$ if $N = 2$.
- (F6) $f'_s(x,s) \ge -\mu$ where $\mu < \lambda - \lambda_k$.
- (F7) $\sigma\theta < \min\big\{\frac{1}{1+\alpha}, \frac{N+2}{N-2}\big\}$.

De Figueiredo proved that under the assumptions (F1)–(F7), there exists $\hat{t} > 0$ such that, for all $t \ge \hat{t}$, problem (1.3) has at least two solutions. De Figueiredo used a generalized version of the mountain pass theorem (see [22, Theorem 5.3]) which required the Palais-Smale (PS) condition to be verified. In [11], the author proved the (PS) condition for a general class of superlinear elliptic problems of the type (1.1) under the conditions (G1)–(G5), without the assumption that $g'(x,0) = \lambda_m$, with $m \neq 1$. In this articld, we will study the solvability of problem (1.1) under the conditions (G1)–(G6) and for the case in which 0 is a degenerate critical point of the associated functional of (1.1).

Many authors have studied problem (1.1) under different assumptions on g (See [4, 6, 8, 11, 19, 21, 22, 25]). Rabinowitz considered a similar problem in [22] where condition (G3) was valid for all $|s| > s_0$ and $g(x, s) = o(|s|)$ for small values of s. First, he proved the existence of a nontrivial solution by using the mountain pass theorem. Next, by assuming that g is Lipschitz continuous, Rabinowitz proved the existence of two nontrivial solutions u^-, u^+ such that $u^- < 0 < u^+$. Wang [25] also assumed condition (G3) for $|s| > s_0$, in addition to $g(0) = 0$ and $g'(0) = 0$. He proved the existence of three nontrivial solutions by using a Morse theory approach. In [21], Perera approached this problem by assuming that condition (G3) is valid for all $|s| > s_0$, and the existence of a constant $a > 0$ such that $g(0) = g(a) = 0$, and $g'(0) = \lambda$. Perera proved the existence of four nontrivial solutions for the cases where $\lambda \in (\lambda_j, \lambda_{j+1})$, $\lambda = \lambda_j < \lambda_{j+1}$, and $\lambda_j < \lambda = \lambda_{j+1}$, and $j \geq 3$. In this article, we are only assuming condition (G3) for large positive values of s. For large negative values of s we are assuming conditions (G2) and (G6). In this sense, g is said to be an asymmetric nonlinearity. We will show that problem (1.1) has at least three nontrivial solutions by using variational methods and Morse Theory.

Another work on asymmetric nonlinearities related to the work in this article is that of Liu and Shaoping [19]. In [19], the authors considered the model problem

$$\begin{gathered} -\Delta u = \lambda u + (u^+)^p \quad \text{in } \Omega; \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.4}$$

where $u^+ = \max\{0, u\}$, $1 < p < (N + 2)/(N - 2)$, and $\lambda \neq \lambda_1$. Liu and Shaoping proved that (1.4) has at least one nontrivial solution. They used Morse theory and computed the critical groups at infinity for the corresponding functional. The computation of the critical groups at infinity in [19] applies to the problem of this article because of conditions (G3) and (G6). We will use some of the techniques presented on [19] to obtain the existence of multiple solutions for our problem.

This article is organized as follows: Section 2 has some results in Morse Theory that will be used throughout the paper. In Section 3, we present some estimates for $g(x, s)$ and its primitive $G(x, s)$. In Section 4, we prove the Palais-Smale condition for the associated functional of problem (1.1). A local linking at the origin is proved in Section 5. In Section 6 we show the existence of two nontrivial solutions by employing the cutoff-technique used by Chang et al. in [9]. Finally, in Section 7, we prove the existence of at least three nontrivial solutions for problem (1.1) as stated in Theorem 1.1.

2. Preliminaries

We will denote by H the Sobolev space $H_0^1(\Omega)$ obtained by completion of $C_0^\infty(\Omega)$ with respect to the metric induced by the norm

$$\|u\| = \Big(\int_\Omega |\nabla u|^2 dx \Big)^{1/2}, \quad \text{for all } u \in H.$$

Let $J : H \to \mathbb{R}$ denote the functional associated with problem (1.1) given by

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega G(x, u) dx, \tag{2.1}$$

for $u \in H$. It is known that, by virtue of growth conditions on g imposed by the assumptions (G2) and (G4), $J \in C^2(H, \mathbb{R})$ with Fréchet derivatives given by

$$\langle J'(u), \varphi \rangle = \int_\Omega \nabla u \cdot \nabla \varphi dx - \int_\Omega g(x,u)\varphi dx, \quad \text{for } \varphi \in H, \tag{2.2}$$

and

$$\langle J''(u)v, \varphi \rangle = \int_\Omega \nabla v \cdot \nabla \varphi dx - \int_\Omega g'(x,u) v\varphi dx, \quad \text{for } u, v, \varphi \in H.$$

In view of (2.2) and (1.2), we see that critical points of J correspond to the weak solutions of problem (1.1).

Let $0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots$ denote the distinct eigenvalues of the linear problem

$$\begin{gathered} -\Delta u = \lambda u \quad \text{in } \Omega; \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

It is well–known that H can be decomposed as $H = H^- \oplus H^+$, where

$$H^- = \oplus_{j=1}^m \ker(-\Delta - \lambda_j I), \quad H^+ = (H^-)^\perp.$$

We will set $\dim H^- = d$.

To study the multiplicity of solutions of problem (1.1), it will be necessary to compute the critical groups of isolated critical points of the functional J defined in (2.1). Let X be a topological space. If $Y \subseteq X$ is a subset of X, we will say that (X, Y) is a topological pair. Denote by $H_q(X, Y)$ the q–singular relative homology group of the pair (X, Y) with coefficients in $\mathbb{Z}$. The critical groups basically describe the local behavior of the functional J near its critical points. For an isolated critical point u_0 of J, set $c = J(u_0)$ and put $J^c = \{u \in H | J(u) \le c\}$. The q-critical group of J at u_0 with coefficients in $\mathbb{Z}$ is defined by

$$C_q(J, u_0) = H_q(J^c \cap U_{u_0}, J^c \cap U_{u_0} \backslash \{u_0\}),$$

for all $q = 0, 1, 2, \dots$ (see Chang [6, Definition 4.1, page 32]), where U_{u_0} is an open neighborhood of u_0 such that u_0 is the unique critical point of J in U_{u_0}. According to the excision property in singular homology theory, the critical groups of isolated critical points are well–defined and they do not depend on a special choice of the neighborhood U_{u_0}. We will denote by $\widetilde{H}_q(X, Y)$ the q–singular reduced relative homology group of the pair (X, Y) with coefficients in $\mathbb{Z}$ (see Hatcher [13, page 110]).

Condition (G1) will allow us to compute the critical groups at the origin by using the decomposition $H = H^- \oplus H^+$. This is related to the concept of local linking at the origin introduced by Li and Liu [15], which we present next.

Definition 2.1. Let J be a C^1 function defined on a Banach space H. We say that J has a local linking near the origin if H has a direct sum decomposition $H = H^- \oplus H^+$, with $\dim H^- < \infty$, $J(0) = 0$, and, for some $\delta > 0$,

$$\begin{gathered} J(u) \le 0, \quad \text{for } u \in H^-, \|u\| \le \delta; \\ J(u) > 0, \quad \text{for } u \in H^+, 0 < \|u\| \le \delta. \end{gathered} \tag{2.3}$$

Assume u is a critical point of J such that $J''(u)$ is a Fredholm operator. The Morse index of u, denoted by $\mu_0(u)$, is defined as the supremum of the dimensions of the vector subspaces of H on which $J''(u)$ is negative definite. The nullity of u, denoted by $\nu_0 = \nu_0(u)$, is defined as the dimension of the kernel of $J''(u)$.

We say that a functional J satisfies the Palais-Smale (PS) condition if any sequence $(u_n) \subset H$ for which $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence. We will say that $(u_n) \subset H$ is a (PS) sequence for J if

$$|J(u_n)| \leq C, \quad \text{for all } n \text{ and some constant } C,$$

and $J'(u_n) \to 0$ as $n \to \infty$.

Based on the notion of local linking at the origin, and assuming that J satisfies the (PS) condition, the critical groups $C_q(J, 0)$ can be calculated based on a result from Su [24].

Proposition 2.2 ([24, Proposition 2.3])**.** *Assume J satisfies the (PS) condition and that it has a local linking at 0 with respect to $H = H^- \oplus H^+$, where 0 has a Morse index μ_0 and nullity ν_0. Set $d = \dim H^-$. Then*

$$C_q(J, 0) = \begin{cases} \delta_{q,\mu_0}\mathbb{Z}, & \text{if } d = \mu_0; \\ \delta_{q,\mu_0+\nu_0}\mathbb{Z}, & \text{if } d = \mu_0 + \nu_0. \end{cases}$$

Thus, to compute the critical groups of J at the origin, we will show that the functional J satisfies the (PS) condition and that J satisfies a local linking condition at the origin with respect to the decomposition $H = H^- \oplus H^+$, where $H^- = \oplus_{j=1}^m \ker(-\Delta - \lambda_j I)$ and $H^+ = (H^-)^\perp$. This will be the content of Section 4.

Let $\mathcal{K} = \{u \in H : J'(u) = 0\}$ be the set of critical points of J and assume J satisfies the (PS) condition; then, $\mathcal{K}$ is a finite set. Set $a < \inf J(\mathcal{K})$. The critical groups of J at infinity are defined as in Bartsch and Li [4] by

$$C_q(J, \infty) = H_q(H, J^a), \quad q = 0, 1, 2, \dots. \tag{2.4}$$

Finally, we will need the Morse relation. Let $J : H \to \mathbb{R}$ be a functional that satisfies the (PS) condition. If the functional $J : H \to \mathbb{R}$ has a finite number of critical points, we can define the Morse–type number of the pair (H, J^a) by

$$M_q := M_q(H, J^a) = \sum_{u \in \mathcal{K}} \dim C_q(J, u), \quad q = 0, 1, 2, \dots. \tag{2.5}$$

Applying the infinite dimensional Morse Theory developed in [6, 20], we can derive the Morse relation

$$\sum_{q=0}^{\infty} M_q t^q = \sum_{q=0}^{\infty} \beta_q t^q + (1 + t) \sum_{q=0}^{\infty} a_q t^q, \tag{2.6}$$

where $\beta_q = \dim C_q(J, \infty)$, and a_q are non-negative numbers. The numbers β_q are also called the Betti numbers of the pair (H, J^a). As a consequence of equation (2.6), if $\beta_q \neq 0$ for some q, then J must have a critical point, say w, with $C_q(J, w) \ncong 0$. In fact, by expanding the equation (2.6), we have that

$$M_0 + M_1 t + \dots + M_q t^q + \dots = (\beta_0 + a_0) + (\beta_1 + a_1 + a_0)t + \dots + (\beta_q + a_q + a_{q-1})t^q + \dots$$

Observe that the term $\beta_q + a_q + a_{q-1} > 0$ since $\beta_q \neq 0$ and $a_q, a_{q-1} \geq 0$. Therefore, $M_q \neq 0$. This implies that there is at least one critical point $w \in \mathcal{K}$ such that $C_q(J, u) \ncong 0$.

3. Estimates on $G(x,s)$ and $g(x,s)$

In this section we establish some estimates for $g(x,s)$ and $G(x,s)$ that will be used throughout this work. First, from condition (G2), there exists $t_- < 0$ such that, for $s < t_-$, it follows that

$$|g(x,s) - \lambda s| < |s|^{\alpha},$$

so that

$$\lambda s - |s|^{\alpha} < g(x,s) < \lambda s + |s|^{\alpha}, \quad \text{for } s < t_-.$$

Then, there exists a constant $C_1 > 0$ such that

$$-C_1 + \lambda s - |s|^{\alpha} \le g(x,s) \le C_1 + \lambda s + |s|^{\alpha}, \tag{3.1}$$

for all $s \le 0$ and $x \in \Omega$. From (3.1), we have

$$|sg(x,s) - \lambda s^2| \le C_1|s| + |s|^{1+\alpha}, \quad \text{for } s \le 0. \tag{3.2}$$

Applying Young's Inequality,

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}, \quad \text{for } a, b \ge 0, \tag{3.3}$$

with $a = |s|$, $b = 1$, $p = 1 + \alpha$, and $q = (1+\alpha)/\alpha$, we can rewrite (3.2) as

$$|sg(x,s) - \lambda s^2| \le \frac{C_1\alpha}{1+\alpha} + \big(1 + \frac{C_1}{1+\alpha}\big)|s|^{1+\alpha}. \tag{3.4}$$

Setting $C_2 = \max\big(1 + \frac{C_1}{1+\alpha}, \frac{C_1\alpha}{1+\alpha}\big)$ in (3.4), we obtain

$$|g(x,s)s - \lambda s^2| \le C_2 + C_2|s|^{1+\alpha}, \quad \text{for } s \le 0, \text{ and } x \in \Omega. \tag{3.5}$$

By integrating the inequality in (3.1) and using the definition of G, we obtain

$$-C_1|s| + \frac{\lambda}{2}s^2 - \frac{1}{\alpha+1}|s|^{\alpha+1} \le G(x,s) \le C_1|s| + \frac{\lambda}{2}s^2 + \frac{1}{\alpha+1}|s|^{\alpha+1}, \tag{3.6}$$

for all $s \le 0$ and a.e $x \in \Omega$, or,

$$|G(x,s) - \frac{\lambda}{2}s^2| \le C_1|s| + \frac{1}{\alpha+1}|s|^{\alpha+1}, \tag{3.7}$$

for $s \le 0$ and $x \in \Omega$.

Next, we show that

$$|g(x,s)s - 2G(x,s)| \le C_4 + C_4|s|^{1+\alpha}, \tag{3.8}$$

for some constant $C_4 > 0$, $s \le 0$ and $x \in \Omega$. In fact, multiplying (3.1) by $s \le 0$, we obtain

$$C_1|s| + \lambda s^2 + |s|^{1+\alpha} \ge g(x,s)s \ge -C_1|s| + \lambda s^2 - |s|^{1+\alpha}. \tag{3.9}$$

Similarly, from (3.6), we have

$$2C_1 s - \lambda s^2 + \frac{2}{1+\alpha}|s|^{1+\alpha} \ge -2G(x,s) \ge -2C_1|s| - \lambda s^2 - \frac{2}{1+\alpha}|s|^{1+\alpha}. \tag{3.10}$$

Then, adding (3.9) and (3.10), we obtain

$$3C_1|s| + \big(1 + \frac{2}{1+\alpha}\big)|s|^{1+\alpha} \ge g(x,s)s - 2G(x,s) \ge -3C_1|s| - \big(1 + \frac{2}{1+\alpha}\big)|s|^{1+\alpha},$$

so that

$$|g(x,s)s - 2G(x,s)| \le 3C_1|s| + \big(1 + \frac{2}{1+\alpha}\big)|s|^{1+\alpha}, \quad \text{for } s \le 0.$$

Applying Young's Inequality (3.3) with $a = |s|$, $b = 1$, $p = 1+\alpha$, and $q = (1+\alpha)/\alpha$, we obtain

$$|g(x,s)s - 2G(x,s)| \le \frac{3C_1}{1+\alpha} + \big(\frac{2+3C_1\alpha}{1+\alpha} + 1\big)|s|^{1+\alpha}, \tag{3.11}$$

for $s \le 0$. Therefore, defining C_5 by

$$C_5 = \max\Big(\frac{3C_1}{1+\alpha}, \frac{2+3C_1\alpha}{1+\alpha} + 1\Big),$$

we obtain (3.8) from (3.11).

Combining (3.6) and condition (G4), we find a global estimate for $G(x,s)$ given by

$$G(x,s) \le C_6|s| + \frac{\lambda}{2}s^2 + \frac{1}{1+\alpha}|s|^{1+\alpha} + \frac{1}{\sigma+1}|s|^{\sigma+1}, \tag{3.12}$$

for all $s \in \mathbb{R}$ and $x \in \Omega$, and $C_6 = C_1 + C_2$.

Finally, from condition (G3), we can find $C_7, C_8 > 0$ such that

$$G(x,s) \ge C_7|s|^\mu - C_8, \tag{3.13}$$

for all $s \ge 0$, where $\mu = 1/\theta > 2$. In fact, from condition (G3) we have

$$0 \le \frac{\partial G}{\partial s}(x,s) - \frac{1}{s\theta}G(x,s), \tag{3.14}$$

for $s > s_0$. Multiplying (3.14) by the integrating factor $s^{-1/\theta}$ and integrating over the interval $[s_0, s]$, we obtain

$$0 \le -\frac{1}{s_0^{1/\theta}}G(x,s_0) + \frac{1}{s^{1/\theta}}G(x,s), \quad \text{for all } s > s_0.$$

Then, setting $C_7 = \frac{1}{s_0^{1/\theta}}G(x,s_0)$, we can find a constant $C_8 > 0$ such that

$$G(x,s) \ge C_7|s|^\mu - C_8,$$

for all $s > 0$ and $x \in \Omega$, which is (3.13).

The next lemma will be used in the proof of a local linking condition at the origin.

Lemma 3.1. *Assume that g satisfies condition* (G1) *and let $\varepsilon > 0$ be such that $\lambda_m + \varepsilon < \lambda_{m+1}$. Then, there exists $\delta_1 > 0$ such that*

$$|G(x,s)| \le \big(\frac{\lambda_m + \varepsilon}{2}\big)|s|^2, \tag{3.15}$$

for $|s| < \delta_1$ and $x \in \Omega$, where m is as given in (G1).

Proof. Since $g'_s(x,0) = \lambda_m$ for all $x \in \Omega$, there exists $\delta_1 > 0$ such that, for $|s| < \delta_1$,

$$|g(x,s) - \lambda_m s| \le \varepsilon|s|, \quad \text{for all } x \in \Omega;$$

then,

$$|g(x,s)| \le (\lambda_m + \varepsilon)|s|, \quad \text{for } |s| < \delta_1, \tag{3.16}$$

Therefore, we can show that

$$|G(x,s)| \le \big(\frac{\lambda_m + \varepsilon}{2}\big)|s|^2, \quad \text{for } |s| < \delta_1, \text{ and } x \in \Omega,$$

which is (3.15). $\square$

4. Palais-Smale condition

Assuming (G1)–(G5), we can show that the functional $J : H \to \mathbb{R}$ defined in (2.1) satisfies the Palais-Smale (PS) condition. The proof was done by De Figueiredo in [11] assuming that $\lambda \neq \lambda_j$ for all $j \in \mathbb{N}$. It turns out the result is true if we assume that $\lambda \neq \lambda_1$. We present the proof here for the reader's convenience.

Lemma 4.1 ([11, Lemma 1, page 291]). *If g and G satisfy* (G1)–(G5)*, then the functional $J : H \to \mathbb{R}$ defined by*

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega G(x,s)dx, \quad \textit{for } u \in H,$$

satisfies the Palais-Smale condition.

Proof. In what follows, we use the same symbol C to denote all constants that come up in the estimates. Let (u_n) be a (PS) sequence for J in $H = H_0^1(\Omega)$; that is, (u_n) satisfies

$$|J(u_n)| = \Big|\frac{1}{2}\int_\Omega |\nabla u_n|^2 dx - \int_\Omega G(x,u_n)dx\Big| \leq C, \quad \text{for all } n, \tag{4.1}$$

and some constant $C > 0$, and

$$|\langle J'(u_n), v\rangle| = \Big|\int_\Omega \nabla u_n \cdot \nabla v dx - \int_\Omega g(x,u_n)v dx\Big| \leq \varepsilon_n \|v\|, \quad \text{for all } n, \tag{4.2}$$

where $\varepsilon_n \to 0$ as $n \to \infty$ and $v \in H$. By virtue of the subcritical growth condition in (G4), it suffices to prove that $(\|u_n\|)$ is bounded ([23, Proposition 2.2, p.73]). First, notice that

$$\begin{aligned}
&\int_\Omega [g(x,u_n)u_n - 2G(x,u_n)]dx \\
&= \int_\Omega \big[g(x,u_n)u_n - |\nabla u_n|^2 + |\nabla u_n|^2 - 2G(x,u_n)\big]dx \\
&\leq \Big|\int_\Omega [|\nabla u_n|^2 - g(x,u_n)u_n dx]\Big| + 2|J(u_n)|.
\end{aligned}$$

Thus, setting $v = u_n$ in (4.2) and using (4.1) we obtain

$$\int_\Omega [g(x,u_n)u_n - 2G(x,u_n)]\, dx \leq \varepsilon_n \|u_n\| + C, \quad \text{for all } n. \tag{4.3}$$

The integral on the left side of (4.3) can be split in three parts,

$$\int_\Omega [g(x,u_n)u_n - 2G(x,u_n)]dx = \Big[\int_{\Omega_n^-} + \int_{\Omega_n^0} + \int_{\Omega_n^+}\Big][g(x,u_n)u_n - 2G(x,u_n)]dx,$$

where $\Omega_n^- = \{x \in \Omega : u_n \leq 0\}$, $\Omega_n^0 = \{x \in \Omega : 0 \leq u_n \leq s_0\}$, and $\Omega_n^+ = \{x \in \Omega : u_n > s_0\}$. The first integral is estimated using (3.8) as follows,

$$\int_{\Omega_n^-} [g(x,u_n)u_n - 2G(x,u_n)]dx \leq C + C\int |u_n^-|^{1+\alpha}dx, \quad \text{for all } n, \tag{4.4}$$

where $u^- = \max\{0, -u\}$. The second integral taken over Ω_n^0 is bounded uniformly with respect to n. The third integral can be estimated using (G3) as follows:

$$\int_{\Omega_n^+} [g(x,u_n)u_n - 2G(x,u_n)]dx \geq \big(\frac{1}{\theta} - 2\big)\int_{\Omega_n^+} G(x,u_n)dx, \quad \text{for all } n. \tag{4.5}$$

Thus, combining (4.5) with (4.3) and (4.4), we obtain

$$\int_{\Omega_n^+} G(x,u_n)dx \le C + \varepsilon_n \|u_n\| + C\|u_n^-\|_{L^{1+\alpha}}^{1+\alpha}, \quad \text{for all } n, \tag{4.6}$$

and some constant $C > 0$. Set $v = u_n^-$ in (4.2). Then, it follows that

$$\Big|\int_\Omega |\nabla u_n^-|^2 dx - \int_{u_n<0} g(x,u_n)u_n dx\Big| \le \varepsilon_n \|u_n^-\|, \quad \text{for all } n. \tag{4.7}$$

Next, compute

$$\begin{aligned}
&\Big|\int |\nabla u_n^-|^2 dx - 2\int_{u_n<0} G(x,u_n)dx\Big| \\
&= \Big|\int |\nabla u_n^-|^2 dx - \int_{u_n<0} g(x,u_n)u_n dx + \int_{u_n<0} [g(x,u_n)u_n dx - 2G(x,u_n)]dx\Big|,
\end{aligned}$$

and use (3.8) and (4.7) to obtain

$$\Big|\int |\nabla u_n^-|^2 dx - 2\int_{u_n<s_0} G(x,u_n)dx\Big| \le C + \varepsilon_n \|u_n^-\| + C\|u_n^-\|_{L^{1+\alpha}}^{1+\alpha}, \tag{4.8}$$

for all n. Similarly, using (3.5), we obtain

$$\Big|\int |\nabla u_n^-|^2 dx - \lambda \int |u_n^-|^2 dx\Big| \le C + \varepsilon_n \|u_n^-\| + C\|u_n^-\|_{L^{1+\alpha}}^{1+\alpha}, \quad \text{for all } n. \tag{4.9}$$

There are two cases to consider: (i) ($\|u_n^-\|$) is bounded, and (ii) $\|u_n^-\| \to \infty$, passing to a subsequence, if necessary. If case (i) holds, then the estimate (4.8) implies that

$$\int_{u_n<0} G(x,u_n)dx \le C, \quad \text{for all } n. \tag{4.10}$$

Indeed, using (4.8) we obtain

$$\begin{aligned}
\Big|2\int_{u_n<0} G(x,u_n)dx\Big| &\le \Big|2\int_{u_n<0} G(x,u_n)dx - \int_\Omega |\nabla u_n^-|^2 dx\Big| + \Big|\int_\Omega |\nabla u_n^-|^2 dx\Big| \\
&\le C + \varepsilon_n \|u_n^-\| + C\|u_n^-\|_{L^{1+\alpha}}^{1+\alpha} + \|u_n^-\|^2.
\end{aligned}$$

Thus, by the Sobolev inequality,

$$\Big|2\int_{u_n<0} G(x,u_n)dx\Big| \le C + \varepsilon_n \|u_n^-\| + C\|u_n^-\|^{1+\alpha} + \|u_n^-\|^2.$$

Hence, since we are assuming ($\|u_n^-\|$) is bounded, it follows that

$$\Big|2\int_{u_n<0} G(x,u_n)dx\Big| \le C, \quad \text{for all } n,$$

which shows (4.10).

Next, notice that

$$\begin{aligned}
\frac{1}{2}\|u_n^+\|^2 &= J(u_n) + \int_\Omega G(x,u_n)dx - \frac{1}{2}\|u_n^-\|^2 \\
&= J(u_n) + \int_{u_n\ge 0} G(x,u_n)dx + \int_{u_n<0} G(x,u_n)dx - \frac{1}{2}\|u_n^-\|^2.
\end{aligned}$$

Thus, using (4.1), (4.6) and (4.10), we have

$$\|u_n^+\|^2 \le C + 2\varepsilon_n \|u_n^+\|, \tag{4.11}$$

since we are assuming that $(\|u_n^-\|)$ is bounded. It follows from (4.11) that $\|u_n^+\|$ is bounded, since $\varepsilon_n \to 0$ as $n \to \infty$. It then follows that $(\|u_n\|)$ is bounded.

Next, consider the case (ii) in which $\|u_n^-\| \to \infty$, and let us show that this cannot hold, completing in this way the proof of the lemma.

Using the fact that $u_n^+ = u_n + u_n^-$, we obtain

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla u_n^+|^2 dx &= \frac{1}{2}\int_\Omega |\nabla u_n|^2 dx - \frac{1}{2}\int_\Omega |\nabla u_n^-|^2 dx \\ &= \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\|u_n^-\|^2 + \int_\Omega G(x,u_n)dx - \int_\Omega G(x,u_n)dx,\end{aligned}$$

which we can write as

$$\frac{1}{2}\int_\Omega |\nabla u_n^+|^2 dx = J(u_n) + \frac{1}{2}\int_\Omega \left[2G(x,u_n) - |\nabla u_n^-|^2\right] dx,$$

so that, by (4.1), (4.6) and (4.8),

$$\frac{1}{2}\int_\Omega |\nabla u_n^+|^2 dx \le C + \varepsilon_n \|u_n^-\| + C\|u_n^-\|_{L^{1+\alpha}}^{1+\alpha}, \quad \text{for all } n. \tag{4.12}$$

Next, use the fact that $u_n^- = u_n^+ - u_n$ to estimate

$$\begin{aligned}\Big|\int \nabla u_n^- \cdot \nabla v dx - \lambda \int u_n^- v\Big| \le \Big|\int \nabla u_n \cdot \nabla v dx - \int g(x,u_n) v dx\Big| \\ + \int |\nabla u_n^+ \cdot \nabla v| dx + \int_{u_n>0} |g(x,u_n)v| dx \\ + \int_{u_n<0} |g(x,u_n)v - \lambda u_n^- v| dx;\end{aligned}$$

so that, by (4.2),

$$\Big|\int \nabla u_n^- \cdot \nabla v dx - \lambda \int u_n^- v\Big| \le \varepsilon_n \|v\| + I_1 + I_2 + I_3, \quad \text{for all } n, \tag{4.13}$$

where

$$I_1 = \int |\nabla u_n^+ \cdot \nabla v| dx, \quad I_2 = \int_{u_n \ge 0} |g(x,u_n)v| dx,$$
$$I_3 = \int_{u_n<0} |g(x,u_n)v - \lambda u_n^- v| dx.$$

We will estimate I_1, I_2, and I_3 separately. To estimate I_1, use Hölder's inequality to get

$$I_1 = \int |\nabla u_n^+ \cdot \nabla v| dx \le \|u_n^+\| \|v\|. \tag{4.14}$$

To estimate I_2, apply Hölder's inequality with

$$p = \frac{2N}{N+2} \tag{4.15}$$

and $q = 2N/(N-2)$ for $N \ge 3$. If $N = 2$, take $1 \le p \le 1/(\sigma\theta)$ which can be done since (G_5) implies $\sigma\theta < 1$. Then,

$$\begin{aligned}I_2 = \int_{u_n>0} |g(x,u_n)v| dx &\le \Big(\int |g(x,u_n)|^p\Big)^{1/p} \Big(\int |v|^q\Big)^{1/q} \\ &\le \Big(\int |C + C|u_n^+|^\sigma|^p\Big)^{1/p} \|v\|_{L^q},\end{aligned}$$

so that,

$$I_2 \le \big(C + C\|u_n^+\|_{L^{p\sigma}}^{\sigma}\big)\,\|v\|_{L^q}. \tag{4.16}$$

Finally, use (3.1) to obtain the following estimate for I_3:

$$I_3 = \int_{u_n<0} |g(x,u_n)v - \lambda u_n^- v dx| dx \le (C + C\|u_n^-\|^{\alpha})\|v\|. \tag{4.17}$$

Combining (4.14), (4.16), and (4.17) into (4.13), we have the estimate

$$\Big|\int \nabla u_n^- \cdot \nabla v dx - \lambda \int u_n^- v\Big| \le (C + \|u_n^+\| + C\|u_n^+\|_{L^{p\sigma}}^{\sigma} + C\|u_n^-\|^{\alpha})\|v\|, \tag{4.18}$$

for all n. Set

$$K_n = C + \|u_n^+\| + C\|u_n^+\|_{L^{p\sigma}}^{\sigma} + C\|u_n^-\|^{\alpha}, \quad \text{for all } n; \tag{4.19}$$

then (4.18) can be written as

$$\Big|\int \nabla u_n^- \cdot \nabla v dx - \lambda \int u_n^- v\Big| \le K_n\|v\|, \quad \text{for all } n. \tag{4.20}$$

The goal next is to show that

$$\frac{K_n}{\|u_n^-\|} \to 0 \quad \text{as } n \to \infty. \tag{4.21}$$

where K_n is as given by (4.19). First, from (4.12), and the fact that $\alpha < 1$, it follows that

$$\frac{\|u_n^+\|}{\|u_n^-\|} \to 0, \quad \text{as } n \to \infty. \tag{4.22}$$

Secondly, we claim that

$$\frac{\|u_n^+\|_{L^{p\sigma}}^{\sigma}}{\|u_n^-\|} \to 0 \quad \text{as } n \to \infty. \tag{4.23}$$

In fact, using the estimates in (3.13) and (4.6), we obtain the estimate

$$\int (u_n^+)^{1/\theta} dx \le C + \varepsilon_n\|u_n\| + C\|u_n^-\|_{L^{1+\alpha}}^{1+\alpha}, \text{ for all } n. \tag{4.24}$$

Now, by the Sobolev inequality, we obtain from (4.24) that

$$\begin{aligned}\Big(\int (u_n^+)^{1/\theta} dx\Big)^{\theta} &\le \big(C + \varepsilon_n\|u_n\| + C\|u_n^-\|^{1+\alpha}\big)^{\theta}\\ &\le C + C\varepsilon_n\|u_n\|^{\theta} + C\|u_n^-\|^{\theta(1+\alpha)}.\end{aligned} \tag{4.25}$$

Choose $\alpha' \in (\alpha, 1)$, and divide both sides of (4.25) by $\|u_n^-\|^{\theta(1+\alpha')}$, to obtain

$$\frac{\|u_n^+\|_{L^{1/\theta}}}{\|u_n^-\|^{\theta(1+\alpha')}} \le \frac{C}{\|u_n^-\|^{\theta(1+\alpha')}} + C\frac{\varepsilon_n\|u_n\|^{\theta}}{\|u_n^-\|^{\theta(1+\alpha')}} + \frac{C}{\|u_n^-\|^{\theta(\alpha'-\alpha)}}, \quad \text{for all } n,$$

so that, in view of (4.22),

$$\frac{\|u_n^+\|_{L^{1/\theta}}}{\|u_n^-\|^{\theta(1+\alpha')}} \to 0, \quad \text{as } n \to \infty. \tag{4.26}$$

We claim that

$$\frac{u_n^+}{\|u_n^-\|^{1/\sigma}} \to 0 \quad \text{as } n \to \infty \text{ in the } L^{p\sigma} \text{ norm}; \tag{4.27}$$

this will establish (4.23). To show (4.27), set $p_1 = 1/(p\sigma\theta)$, where p is as given in (4.15), and note that $p_1 > 1$ by (G_5). Next, use Hölder's inequality to obtain

$$
\begin{aligned}
\int_\Omega \Big(\frac{|u_n^+|}{\|u_n^-\|^{\frac{1}{\sigma}}}\Big)^{p\sigma} dx &\leq C\Big(\int_\Omega \frac{|u_n^+|^{1/\theta}}{\|u_n^-\|^{\frac{1}{\sigma\theta}}} dx\Big)^{p\sigma\theta} \\
&\leq \frac{C}{\|u_n^-\|^p}\Big(\int_\Omega \frac{|u_n^+|^{1/\theta}}{\|u_n^-\|^{1+\alpha'}} \|u_n^-\|^{1+\alpha'} dx\Big)^{p\sigma\theta}
\end{aligned}
$$

so that

$$
\int_\Omega \Big(\frac{|u_n^+|}{\|u_n^-\|^{\frac{1}{\sigma}}}\Big)^{p\sigma} dx \leq C\|u_n^-\|^{p\sigma\theta(1+\alpha')-p}\Big(\int_\Omega \frac{|u_n^+|^{1/\theta}}{\|u_n^-\|^{1+\alpha'}} dx\Big)^{p\sigma\theta}. \tag{4.28}
$$

By using (4.26), we see that the term in parenthesis in (4.28) approaches zero as $n \to \infty$.

Next, notice that

$$
\|u_n^-\|^{p\sigma\theta(1+\alpha')-p} \to 0, \quad \text{as } n \to \infty,
$$

by condition (G5), since we are assuming $\|u_n^-\| \to \infty$ as $n \to \infty$. Hence,

$$
\int_\Omega \Big(\frac{|u_n^+|}{\|u_n^-\|^{\frac{1}{\sigma}}}\Big)^{p\sigma} dx \to 0, \quad \text{as } n \to \infty,
$$

which is (4.27). We have therefore established (4.23).

Use (4.19) to obtain

$$
\frac{K_n}{\|u_n^-\|} = \frac{C}{\|u_n^-\|} + \frac{\|u_n^+\|}{\|u_n^-\|} + C\frac{\|u_n^+\|_{L^{p\sigma}}^\sigma}{\|u_n^-\|} + C\frac{1}{\|u_n^-\|^{1-\alpha}}, \tag{4.29}
$$

where

$$
\frac{\|u_n^+\|}{\|u_n^-\|} \to 0 \quad \text{as } n \to \infty,
$$

by (4.22), and

$$
\frac{\|u_n^+\|_{L^{p\sigma}}^\sigma}{\|u_n^-\|} \to 0 \quad \text{as } n \to \infty
$$

by (4.23). Hence, since $\alpha < 1$ and $\|u_n^-\| \to \infty$ as $n \to \infty$, we obtain from (4.29) that

$$
\frac{K_n}{\|u_n^-\|} \to 0 \quad \text{as } n \to \infty,
$$

which is (4.21).

Next, combine (4.20) and (4.21) to obtain

$$
\lim_{n\to\infty} \Big(\int_\Omega \frac{\nabla u_n^-}{\|u_n^-\|} \cdot \nabla v dx - \lambda \int_\Omega \frac{u_n^-}{\|u_n^-\|} v\Big) = 0, \quad \text{for all } v \in H. \tag{4.30}
$$

Set $w_n = u_n^-/\|u_n^-\|$ for all n. Since $\|w_n\| = 1$, for all n, passing to a subsequence if necessary, we may assume that there exists $w_0 \in H$ such that $w_n \rightharpoonup w_0$ (weakly) in $H_0^1(\Omega)$ and $w_n \to w_0$ strongly in $L^2(\Omega)$. We may also assume that $w_n(x) \to w(x)$ for a.e $x \in \Omega$. It follows from (4.30) with $v = w_n$ that

$$
\int w_0^2 = \lambda^{-1},
$$

since we are assuming that $\lambda > 0$. It then follows from (4.30) that

$$\int_\Omega \nabla w_0 \cdot \nabla v dx - \lambda \int_\Omega w_0 v dx = 0, \text{ for all } v \in H_0^1(\Omega);$$

that is, w_0 is a nontrivial weak solution of the problem

$$\begin{aligned} -\Delta w_0 &= \lambda w_0 \text{in } \Omega; \\ w_0 &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4.31}$$

By the maximum principle, $w_0 < 0$ in Ω. Thus, w_0 is an eigenfunction of (4.31) that does not change sign in Ω. Hence, $\lambda = \lambda_1$, the first eigenvalue of (4.31), which is the case we are excluding. Therefore, we obtain a contradiction. The proof of Lemma 4.1 is now completed. □

5. Local linking at the origin

The notion of local linking at the origin was introduced by Li and Liu in [15] and [16]. We present the definition given in Li and Willem [14].

Definition 5.1 ([14, Section 0]). *Let J be a C^1 function defined on a Banach space H. We say that J has a local linking near the origin if H has a direct sum decomposition $H = H^- \oplus H^+$ with $\dim H^- < \infty$, $J(0) = 0$, and, for some $\delta > 0$,*

$$\begin{aligned} J(u) &\leq 0, \quad \text{for } u \in H^-, \ \|u\| \leq \delta; \\ J(u) &> 0, \quad \text{for } u \in H^+, \ 0 < \|u\| \leq \delta. \end{aligned} \tag{5.1}$$

Lemma 5.2. *Assume* (G1)–(G5) *hold. Then, J has a local linking at* 0 *with respect to the decomposition $H = H^- \oplus H^+$, where $H^- = \oplus_{j \leq m} \ker(-\Delta - \lambda_j I)$, and $H^+ = (H^-)^\perp$.*

Proof. The proof is based on arguments presented in papers by Li and Willem in [14, Theorem 4], and by Li and Liu in [17, Theorem 3.1].

First, let us show that there exists $\delta > 0$ such that $J(u) \leq 0$ for $u \in H^-$ if $\|u\| < \delta$. In fact, by the definition of $H^- = \oplus_{j \leq m} \ker(-\Delta - \lambda_j I)$, we have

$$\int_\Omega |\nabla u|^2 dx \leq \lambda_m \int_\Omega u^2 dx, \quad \text{for } u \in H^-. \tag{5.2}$$

Since H^- is finite–dimensional, there exists $C > 0$ such that

$$\|u\|_\infty \leq C\|u\|, \quad \text{for } u \in H^-, \tag{5.3}$$

where $\|u\|_\infty = \sup\{|u(x)| : x \in \Omega\}$. Select $u \in H^-$ such that $\|u\| \leq \frac{\delta_1}{C}$, so that $|u(x)| \leq \delta_1$, for a.e $x \in \Omega$, where δ_1 is given in Lemma 3.1. Then, from Lemma 3.1 it follows that

$$-\big(\frac{\lambda_m + \varepsilon}{2}\big)|u(x)|^2 \leq G(x,u) \leq \big(\frac{\lambda_m + \varepsilon}{2}\big)|u(x)|^2, \quad \text{for } |u(x)| < \delta_1. \tag{5.4}$$

It follows from (5.2) that

$$J(u) \leq \int_\Omega [\frac{\lambda_m}{2} u^2 - G(x, u(x))] dx, \quad \text{for } u \in H^-, \|u\| \leq \frac{\delta_1}{C}. \tag{5.5}$$

Hence, using the estimate in (5.4), we obtain from (5.5) that

$$J(u) \leq -\frac{\varepsilon}{2} \int_\Omega u^2 dx \leq 0, \quad \text{for } u \in H^- \text{ and } \|u\| \leq \frac{\delta_1}{C}. \tag{5.6}$$

Next, we need to show that $J(u) > 0$ for $0 < \|u\| < \delta$, for $u \in H^+$, where δ will be chosen shortly. Since we already have the estimate (3.16) in Lemma 3.1 for $|s| \le \delta_1$, we need an estimate for $|s| > \delta_1$. In fact, using the estimate (3.12), for $\frac{|s|}{\delta_1} > 1$, and the assumption that $0 \le \alpha < 1$ in (G_1), we have that

$$\begin{aligned}
G(x,s) &\le C|s| + \frac{\lambda}{2}s^2 + \frac{1}{1+\alpha}|s|^{1+\alpha} + \frac{1}{\sigma+1}|s|^{\sigma+1}, \\
&= C\delta_1\frac{|s|}{\delta_1} + \frac{\delta_1^2\lambda}{2}\big(\frac{s}{\delta_1}\big)^2 + \frac{\delta_1^{1+\alpha}}{1+\alpha}\big(\frac{|s|}{\delta_1}\big)^{1+\alpha} + \frac{\delta_1^{\sigma+1}}{\sigma+1}\big(\frac{|s|}{\delta_1}\big)^{\sigma+1} \\
&\le \frac{1}{\delta_1^{\sigma+1}}\Big(C\delta_1 + \frac{\delta_1^2\lambda}{2} + \frac{\delta_1^{1+\alpha}}{1+\alpha} + \frac{\delta_1^{\sigma+1}}{\sigma+1}\Big)|s|^{\sigma+1},
\end{aligned}$$

so that

$$G(x,s) \le C_\varepsilon|s|^{\sigma+1}, \quad \text{for all } |s| > \delta_1, \tag{5.7}$$

where C_ε is given by

$$C_\varepsilon = \frac{1}{\delta_1^{\sigma+1}}\Big(C\delta_1 + \frac{\delta_1^2\lambda}{2} + \frac{\delta_1^{1+\alpha}}{1+\alpha} + \frac{\delta_1^{\sigma+1}}{\sigma+1}\Big).$$

Combining (3.15) and (5.7), it follows that

$$G(x,s) \le \big(\frac{\lambda_m+\varepsilon}{2}\big)|s|^2 + C_\varepsilon|s|^{\sigma+1}, \tag{5.8}$$

for all $s \in \mathbb{R}$ and $x \in \Omega$. Then, for $u \in H^+$ and using (5.8), we have

$$\begin{aligned}
J(u) &= \frac{1}{2}\|u\|^2 - \int_\Omega G(x,u)dx \\
&\ge \frac{1}{2}\|u\|^2 - \big(\frac{\lambda_m+\varepsilon}{2}\big)\int_\Omega u^2dx - C_\varepsilon\int_\Omega |u|^{\sigma+1}dx.
\end{aligned}$$

Thus, applying the Sobolev inequality, and the fact that $\|u\|^2 \ge \lambda_{m+1}\|u\|_{L^2}^2$ for $u \in H^+$, we obtain

$$J(u) \ge \frac{1}{2}\big[1 - \big(\frac{\lambda_m+\varepsilon}{\lambda_{m+1}}\big) - \tilde{C}_\varepsilon\|u\|^{\sigma-1}\big]\|u\|^2 \quad \text{for } u \in H^+. \tag{5.9}$$

Next, choose $\rho > 0$ such that

$$\rho < \Big[\frac{1}{2\tilde{C}_\varepsilon}\Big(1 - \big(\frac{\lambda_m+\varepsilon}{\lambda_{m+1}}\big)\Big)\Big]^{\frac{1}{\sigma-1}}.$$

Then, for $u \in H^+$ such that $\|u\| < \delta$, where $\delta = \min\{\frac{\delta_1}{C}, \rho\}$, we obtain from (5.9) that

$$J(u) > 0, \quad \text{for } u \in H^+, 0 < \|u\| < \delta,$$

and the lemma is proved. □

By Lemma 5.2, J satisfies a local linking condition at the origin with respect to the decomposition $H = H^- \oplus H^+$. In this case, 0 has Morse Index μ_0 and nullity ν_0 given by

$$\mu_0 = \sum_{j=1}^{m-1} \dim\ker(-\Delta - \lambda_j I), \tag{5.10}$$

$$\nu_0 = \dim\ker(-\Delta - \lambda_m I), \tag{5.11}$$

respectively, where we are assuming that $m > 1$ by (G1). Therefore, using Proposition 2.2, (5.10), (5.11), and since $\dim H^- = \mu_0 + \nu_0 = d$, we obtain

$$C_q(J, 0) = \delta_{q,d}\mathbb{Z}. \tag{5.12}$$

6. Existence of two nontrivial solutions

In this section we prove the existence of two nontrivial solutions of problem (1.1) under the assumptions (G1)–(G5) and $g(x, t_0) = 0$ for all $x \in \overline{\Omega}$ and some $t_0 > 0$. We will employ the cutoff technique used by Chang, Li and Liu in [9, Theorem B].

Proposition 6.1. *Assume g satisfies* (G1)–(G5) *and suppose there exists $t_0 > 0$ such that $g(x, t_0) = 0$ for all $x \in \overline{\Omega}$. Then, problem* (1.1) *has a nontrivial solution, u_0, such that*

$$C_q(J, u_0) = \delta_{q,0}\mathbb{Z}. \tag{6.1}$$

Proof. Define $\overline{g} : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$\overline{g}(x, s) = \begin{cases} g(x, s), & \text{if } s \in [0, t_0]; \\ 0, & \text{if } s \notin [0, t_0]. \end{cases}$$

Define the functional $\overline{J} : H \to \mathbb{R}$ by

$$\overline{J}(u) = \frac{1}{2}\|u\|^2 - \int_\Omega \overline{G}(x, u)dx, \quad \text{for } u \in H, \tag{6.2}$$

where $\overline{G}(x, s) = \int_0^s \overline{g}(x, \xi)d\xi$, for $x \in \Omega$, $s \in \mathbb{R}$. In order to show the existence of a nontrivial solution for problem (1.1), we will first show that $\overline{J}$ has a minimizer.

Let $M = \sup_{x\in\overline{\Omega}, s\in[0,t_0]} |\overline{g}(x, s)|$; then, using Hölder and Poincaré's inequalities we have

$$\begin{aligned} \overline{J}(u) &\geq \frac{1}{2}\|u\|^2 - M\int_\Omega |u|dx \\ &\geq \frac{1}{2}\|u\|^2 - M|\Omega|^{1/2}\|u\|_{L^2(\Omega)} \\ &\geq \frac{1}{2}\|u\|^2 - c\|u\|, \quad \text{for all } u \in H, \end{aligned}$$

which shows that $\overline{J}$ is coercive and bounded below. Also, $\overline{J}$ is weakly lower semicontinuous. Thus, there exists a global minimizer u_0 of $\overline{J}$ such that

$$\overline{J}(u_0) = \inf_{u\in H} \overline{J}(u).$$

(See Evans [10, Page 488]). The function $\overline{g}$ is locally Lipschitz continuous; thus, it follows that u_0 is a classical solution of the problem

$$\begin{aligned} -\Delta u &= \overline{g}(x, u) \quad \text{in } \Omega; \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{6.3}$$

(See Agmon [1]). Let $\Omega_- = \{x \in \Omega : u_0(x) < 0\}$. Then, by the definition of $\overline{g}$, u solves the BVP,

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega_-; \\ u &= 0 \quad \text{on } \partial\Omega_-, \end{aligned} \tag{6.4}$$

which has only the trivial solution $u \equiv 0$. It then follows that $\Omega_- = \emptyset$. Similarly, if we consider the set $\Omega_{t_0} = \{x \in \Omega : u_0(x) > t_0\}$, it can be shown that $\Omega_{t_0} = \emptyset$.

Therefore, we have $0 \leq u_0 \leq t_0$ in Ω. Using the strong maximum principle, we can show that

$$0 < u_0(x) < t_0, \quad \text{for all } x \in \Omega, \tag{6.5}$$

$$\frac{\partial u_0}{\partial \nu}(x) < 0, \quad \text{on } \partial\Omega, \tag{6.6}$$

where ν is the outward unit normal vector on $\partial\Omega$.

We claim that u_0 is also a local minimizer for J. It follows from (6.5) and (6.6) that there exists $\delta > 0$ such that $u \in C_0^1(\Omega)$ and $\|u - u_0\|_{C^1} < \delta$ imply that $0 < u(x) < t_0$. Thus, there is a C^1 neighborhood of u_0 on which $J(u) \geq J(u_0)$; so that u_0 is a C^1 local minimizer of J. Then, using a result due to Brézis and Nirenberg [5], we conclude that u_0 is also a minimizer in the H_0^1 topology.

Finally, using Chang [6, Example 1, page 33], we see that

$$C_q(J, u_0) = \delta_{q,0}\mathbb{Z}. \tag{6.7}$$

Notice that this implies that $u_0 \neq 0$ by comparison with (5.12), since $d \geq 1$ by virtue of (G_1). □

Before we prove the next theorem, we will need the following variant of the Mountain Pass Lemma in Chang [7].

Proposition 6.2 ([7, Corollary 1.2])**.** *Suppose that $J \in C^{2-0}(H, \mathbb{R})$ satisfies the (PS) condition, with u_0 a local minimum. If there exists $v_0 \in H$ such that $v_0 \neq u_0$ and $J(v_0) = J(u_0)$, then J has at least a nontrivial critical point.*

Next, we show that there is an additional critical point of J of mountain pass type.

Theorem 6.3. *Assume g satisfies the hypotheses of Proposition 6.1. Then, problem* (1.1) *has two nontrivial solutions u_0 and u_1 such that $0 < u_0 < u_1$, where u_0 is given by Proposition* (6.1)*. Moreover, if the critical points at level $c_1 = J(u_1)$ are isolated, there exists a critical point $\widetilde{u}_1$ with*

$$C_q(J, \widetilde{u}_1) = \delta_{q,1}\mathbb{Z}, \quad \textit{for } q = 1, 2, 3, \dots. \tag{6.8}$$

Proof. Let u_0 be the local minimizer of the functional J defined in (2.1) that is given by Proposition 6.1. Assume that u_0 is isolated. It follows from the result of Proposition 6.1 that u_0 is a C^2 solution of the boundary-value problem in (1.1) satisfying

$$0 < u_0(x) < t_0, \quad \text{for all } x \in \Omega. \tag{6.9}$$

We will prove the existence of a mountain pass critical point, u_1, of the functional J with the property that

$$u_0(x) < u_1(x), \quad \text{for all } x \in \Omega. \tag{6.10}$$

Consider the modified functional $\widetilde{J} : H \to \mathbb{R}$ given by

$$\widetilde{J}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega [G(x, v + u_0) - G(x, u_0) - g(x, u_0)v] dx, \tag{6.11}$$

for all $v \in H$. This functional was obtained by setting

$$\widetilde{J}(v) = J(u_0 + v) - J(u_0), \quad \text{for all } v \in H, \tag{6.12}$$

and observing that the fact that u_0 is a critical point of J implies that

$$\int_\Omega \nabla u_0 \cdot \nabla v = \int_\Omega g(x, u_0(x)) v(x)\, dx, \quad \text{for all } v \in H.$$

It follows from (6.12) and the assumption that J has an isolated local minimum at u_0 that the functional $\widetilde{J}$ defined by (6.11) has an isolated local minimum at 0.

Put

$$\widetilde{g}(x,s) = g(x, u_0(x) + s) - g(x, u_0(x)), \quad x \in \Omega,\ s \in \mathbb{R}, \tag{6.13}$$

and set $\widetilde{G}(x,s) = \int_0^s \widetilde{g}(x,\xi) d\xi$, so that

$$\widetilde{G}(x,s) = G(x, s + u_0(x)) - G(x, u_0(x)) - s g(x, u_0(x)), \quad x \in \Omega,\ s \in \mathbb{R}. \tag{6.14}$$

In view of (6.11) and (6.14), we see that

$$\widetilde{J}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega \widetilde{G}(x, v(x))\, dx, \quad \text{for } v \in H. \tag{6.15}$$

Next, define the truncated versions of $\widetilde{g}$ and $\widetilde{G}$ in (6.13) and (6.14), respectively:

$$\widetilde{g}_+(x,s) = \begin{cases} \widetilde{g}(x,s), & x \in \Omega,\ s \geq 0; \\ 0, & x \in \Omega,\ s < 0; \end{cases} \tag{6.16}$$

and

$$\widetilde{G}_+(x,s) = \begin{cases} \widetilde{G}(x,s), & x \in \Omega,\ s \geq 0; \\ 0, & x \in \Omega,\ s < 0. \end{cases} \tag{6.17}$$

We can then define the truncated version of $\widetilde{J}$ as follows

$$\widetilde{J}_+(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega \widetilde{G}_+(x, v(x))\, dx, \quad \text{for } v \in H. \tag{6.18}$$

We note that the truncated functional in (6.18) can be written in terms of $\widetilde{J}$ as follows:

$$\widetilde{J}_+(v) = \widetilde{J}(v^+) + \frac{1}{2}\|v^-\|^2 \quad \text{for } v \in H, \tag{6.19}$$

where $v^+(x) = \max\{v(x), 0\}$, for $x \in \Omega$, is the positive part of v in Ω, and $v^- = (-v)^+$ the negative part.

It follows from (6.19) and the assumption that that $\widetilde{J}$ has an isolated local minimum at 0 that the functional $\widetilde{J}_+$ defined by (6.18) and (6.17) has an isolated local minimum at 0. We will next show that $\widetilde{J}_+$ satisfies the (PS) condition and the assumptions of Proposition 6.2 (which is [7, Corollary 1.2]).

First, notice that $\widetilde{J}_+ \in C^{2-0}(H, \mathbb{R})$. Next, we will see that $\widetilde{J}_+$ satisfies the (PS) condition. Thus, let (v_n) be a (PS) sequence for $\widetilde{J}_+$ in H. To show that (v_n) has a convergent subsequence in H, it is sufficient to show that (v_n) is a bounded sequence (see [23, Chapter 2, Proposition 2.2]). We have that

$$|\widetilde{J}_+(v_n)| \leq C, \quad \text{for all } n, \tag{6.20}$$

$$\widetilde{J}'_+(v_n) \to 0, \quad \text{as } n \to \infty. \tag{6.21}$$

It follows from (6.21) that

$$|\langle \widetilde{J}'_+(v_n), v\rangle| = \Big|\int_\Omega (\nabla v_n \cdot \nabla v - \widetilde{g}_+(x, v_n) v) dx\Big| \leq \varepsilon_n \|v\|, \quad \text{for all } n, \tag{6.22}$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Let $n_1 \in \mathbb{N}$ be such that $\varepsilon_n \leq 1$ for all $n \geq n_1$. Set $v = v_n$ in (6.22) to get

$$\|v_n\| \geq \Big| \int_\Omega |\nabla v_n|^2 dx - \int_\Omega \widetilde{g}_+(x, v_n) v_n dx \Big|, \quad \text{for } n \geq n_1. \tag{6.23}$$

Therefore, using (6.20) and (6.23) we obtain, for $n \geq n_1$,

$$\begin{aligned}
&C + \mu^{-1}\|v_n\| \\
&\geq \widetilde{J}_+(v_n) - \mu^{-1}\langle \widetilde{J}'_+(v_n), v_n\rangle \\
&\geq \widetilde{J}_+(v_n) - \mu^{-1}\Big(\int_\Omega |\nabla v_n|^2 dx - \int_\Omega \widetilde{g}_+(x, v_n) v_n dx\Big), \\
&\geq \int_\Omega \Big(\frac{1}{2}|\nabla v_n|^2 - \widetilde{G}_+(x, v_n)\Big) dx - \mu^{-1}\Big(\int_\Omega [|\nabla v_n|^2 - \widetilde{g}_+(x, v_n) v_n] dx\Big), \\
&\geq \big(\frac{1}{2} - \frac{1}{\mu}\big)\|v_n\|^2 + \int_\Omega T_n dx.
\end{aligned}$$

where $T_n = \mu^{-1}\widetilde{g}_+(x, v_n)v_n - \widetilde{G}_+(x, v_n)$. Note that

$$s\widetilde{g}_+(x, s) - \widetilde{G}_+(x, s) = 0, \quad \text{for all } s \leq 0. \tag{6.24}$$

Let $\Omega = \Omega_{1,n} \cup \Omega_{2,n}$, where

$$\Omega_{1,n} = \{x \in \Omega : v_n(x) \leq s_0\}, \quad \Omega_{2,n} = \{x \in \Omega : v_n(x) > s_0\},$$

for all n, where s_0 is given by (G3). Then,

$$C + \mu^{-1}\|v_n\| \geq \big(\frac{1}{2} - \frac{1}{\mu}\big)\|v_n\|^2 + \int_{\Omega_{1,n}} T_n dx + \int_{\Omega_{2,n}} T_n \, dx. \tag{6.25}$$

Note that we have also used (6.24). By (G3), $(2^{-1} - \mu^{-1}) > 0$, and the second integral in (6.25) is nonnegative. Define

$$K_2 = \max_{x\in\overline{\Omega}, s\leq s_0} |\mu^{-1}\widetilde{g}_+(x, s) - \widetilde{G}_+(x, s)|.$$

Then,

$$\Big| \int_{\Omega_{1,n}} T_n dx\Big| \leq K_2|\Omega| \quad \text{for all } n.$$

Thus, (6.25) becomes

$$C + \frac{1}{\mu}\|v_n\| \geq \big(\frac{1}{2} - \frac{1}{\mu}\big)\|v_n\|^2 - K_2|\Omega|, \quad \text{for } n \geq n_1. \tag{6.26}$$

Therefore, it follows from (6.26) that (v_n) is bounded. Hence, $\widetilde{J}_+$ satisfies the (PS) condition.

Before we proceed with the proof, we will derive an estimate for $\widetilde{G}(x, s)$ for positive values of s.

Apply (3.13) to (6.14), using the estimate in (6.9), to get that

$$\widetilde{G}(x, s) \geq C_5|s + u_0(x)|^\mu - C_6 - |G(x, u_0(x))| - |s||g(x, u_0(x))|,$$

for $s \geq 0$ and $x \in \Omega$. Thus, there exists a constant $C_9 > 0$ such that

$$\widetilde{G}(x, s) \geq C_9|s|^\mu - C_9|u_0(x)|^\mu - C_6 - C_{10} - C_{11}|s|, \tag{6.27}$$

for $s \geq 0$ and $x \in \Omega$, where we have set

$$C_{10} = \max_{x\in\overline{\Omega}, 0\leq\xi\leq t_o} |G(x,\xi)|, \quad C_{11} = \max_{x\in\overline{\Omega}, 0\leq\xi\leq t_o} |g(x,\xi)|.$$

Thus, setting

$$C_{12} = C_9 t_o^{\mu} + C_6 + C_{10}$$

we obtain from (6.27) that

$$\widetilde{G}(x,s) \geq C_9|s|^{\mu} - C_{11}|s| - C_{12} \tag{6.28}$$

Hence, using the assumption that $\mu > 2$, we deduce from (6.28) the existence of positive constants C_{13} and C_{14} such that

$$\widetilde{G}(x,s) \geq C_{13}|s|^{\mu} - C_{14}, \quad \text{for } s \geq 0 \text{ and } x \in \Omega. \tag{6.29}$$

Next, we show that

$$\lim_{t\to\infty} \widetilde{J}_+(t\varphi_1) = -\infty. \tag{6.30}$$

In fact, use the estimate on (6.29) to obtain

$$\begin{aligned}\widetilde{J}_+(t\varphi_1) &= \frac{t^2}{2}\|\varphi_1\|^2 - \int_\Omega \widetilde{G}_+(x,t\varphi_1)dx \\ &\leq \frac{t^2}{2}\|\varphi_1\|^2 - C_{13}\frac{t^{\mu}}{2}\int_\Omega |\varphi_1|^{\mu}dx + C_{14}|\Omega|,\end{aligned}$$

so that

$$\lim_{t\to\infty} \widetilde{J}_+(t\varphi_1) = -\infty,$$

since $\mu > 2$, which is (6.30).

We have already noted that, since we are assuming u_0 is a strict local minimizer of J, it follows that 0 is a strict local minimizer of $\widetilde{J}_+$. It then follows from (6.30) and the intermediate value theorem that there exists $v_0 \in H$ such that $v_0 \neq 0$ and $\widetilde{J}_+(v_0) = 0$. Then, by the variant of the Mountain Pass Lemma in Chang [6] (See Proposition 6.2), $\widetilde{J}_+$ has a nontrivial critical point v_1 of mountain–pass type. We note that v_1 is a solution to the boundary-value problem

$$\begin{gathered}-\Delta v = \widetilde{g}_+(x,v(x)), \quad \text{for } x \in \Omega; \\ v = 0, \quad \text{on } \partial\Omega.\end{gathered}$$

It then follows from the definition of $\overline{g}_+$ in (6.16), elliptic regularity theory, and the maximum principle that $v_1(x) > 0$ for all $x \in \Omega$. Consequently, v_1 solves the boundary-value problem

$$\begin{gathered}-\Delta v = \widetilde{g}(x,v(x)), \quad \text{for } x \in \Omega; \\ v = 0, \quad \text{on } \partial\Omega.\end{gathered}$$

Hence, in view of the definition of $\widetilde{g}$ in (6.13), the function $u_1 = u_0 + v_1$ is the critical point of J of mountain-pass type satisfying

$$u_0(x) < u_1(x), \quad \text{for all } x \in \Omega.$$

Moreover, if the critical points of the level set K_{c_1}, with $c_1 = J(u_1)$, are isolated, then, using [20, Corollary 8.5], there exists $\widetilde{u}_1 \in K_{c_1}$ such that

$$C_q(J,\widetilde{u}_1) \cong \delta_{q,1}\mathbb{Z}. \tag{6.31}$$

□

7. Critical groups $C_q(J,\infty)$

In this section we compute the critical groups $C_q(J,\infty)$, for $q = 1, 2, \dots,$ as defined in (2.4). We will assume that conditions (G1)–(G6) are satisfied. We will use the technique outlined by Liu and Shaoping in [19, Proposition 3.1].

Let $a = \inf J(\mathcal{K})$, where $\mathcal{K} = \{u \in H : J'(u) = 0\}$ is the critical set of J. First, we will show that any compact set $A \subset J^{-M}$ is contractible in J^{-M}, for some constant $M > -a$. This will imply that $\widetilde{H}_q(J^{-M}) = 0$ for all $q \in \mathbb{Z}$. Then, by using the exact homology sequence of the pair (H, J^{-M}), we will show that $\widetilde{H}_q(H, J^{-M}) = 0$ for all $q \in \mathbb{Z}$.

First, note that, by combining the conditions (G3) and (G6), we can find a constant $K_1 > 0$ such that

$$2G(x,s) - sg(x,s) \le K_1, \quad \text{for all } s \in \mathbb{R} \text{ and } x \in \Omega. \tag{7.1}$$

The following proposition is based on a result from Liu and Shaoping in [19, Proposition 3.1].

Proposition 7.1. *Under the conditions* (G1)–(G6)*, any compact subset A of the sublevel set $J^{-M} = \{u \in H : J(u) \le -M\}$ is contractible in J^{-M} for*

$$M \ge \max\{K_1|\Omega|, -a\},$$

where K_1 is given in (7.1)*.*

Proof. **Step 1:** Let A be a compact subset of J^{-M}, where $M > \max\{K_1|\Omega|, -a\}$. First, we show how to deform A to a subset $A_1 \subset J^{-2M}$ in J^{-M}. Compute

$$J(tu) = \frac{t^2}{2}\|u\|^2 - \int_\Omega G(x,tu)dx, \quad \text{for } t \in \mathbb{R}, \tag{7.2}$$

and $u \in A$. Then, using (7.2), we can show that

$$\frac{d}{dt}[J(tu)] = \frac{1}{t}\Big[2J(tu) + \int_\Omega \big(2G(x,tu) - g(x,tu)tu\big)dx\Big], \tag{7.3}$$

for $t > 0$. Using (7.1) and (7.2) we obtain from (7.3) that

$$\frac{d}{dt}[J(tu)] - \frac{2}{t}J(tu) \le \frac{K_1|\Omega|}{t} \tag{7.4}$$

for all $t \ge 1$. Multiply (7.4) by the integrating factor $1/t^2$ and integrate from 1 to $t > 1$ to obtain

$$J(tu) \le t^2 J(u) + \frac{t^2}{2}K_1|\Omega| - \frac{1}{2}K_1|\Omega|, \tag{7.5}$$

for all $t \ge 1$. Define a map η_1 on $[0,1] \times A$ by

$$\eta_1(t,u) = (1+t)u, \quad \text{for } u \in A. \tag{7.6}$$

Then, η_1 is continuous. Also, $\eta_1(t,u) \in J^{-M}$ for all $t \in [0,1]$. In fact, from (7.5), we have

$$J(\eta_1(t,u)) \le (1+t)^2 J(u) + \frac{K_1|\Omega|}{2}[(1+t)^2 - 1]. \tag{7.7}$$

Since $K_1|\Omega| < -M$, and $J(u) \le -M$, we obtain from (7.7) that

$$J(\eta_1(t,u)) \le (1+t)^2 J(u) + M(1+t)^2 - M \le -M$$

Thus,

$$J((1+t)u) \le -M, \text{ for all } 0 \le t \le 1.$$

Therefore, η_1 defines a continuous map from $[0,1]\times A$ to J^{-M}. Set $A_1=\eta_1(1,A)$. Then, A_1 is a compact set. We claim that $A_1\subset J^{-2M}$. In fact, setting $t=1$ in (7.5), and using the assumption that $K_1|\Omega|\le M$, we obtain $J(2u)\le -2M$. Therefore, $A_1\subset J^{-2M}$. Thus, η_1 defines a deformation from A to A_1 in J^{-M}.

In what follows, we will use the fact that, if $u\in J^{-M}$, then

$$J(tu)\le -M,\quad \text{for } t\ge 1; \tag{7.8}$$

this is a consequence of (7.5).

The remainder of the argument follows the same steps as in Liu and Shaoping in [19, Proposition 2.1].

Step 2: In this step, we show how to deform the set A_1 obtained in Step 1 to a subset of smooth functions.

Since the functional $J:H\to\mathbb{R}$ is continuous on H and A_1 is compact, there exists $\varepsilon>0$ such that, for all $u\in A_1$,

$$\|v-u\|<\varepsilon \Rightarrow |J(v)-J(u)|<\frac{M}{2}. \tag{7.9}$$

On the other hand, since the set $C_0^1(\Omega)$ is dense in H, for each $u\in A_1$, there exists $u^\varepsilon\in C_0^1(\Omega)$ such that

$$\|u-u^\varepsilon\|<\varepsilon. \tag{7.10}$$

Note that $\{B_\varepsilon(u^\varepsilon)\}_{u\in A_1}$ is an open cover for A_1. Thus, since A_1 is compact, there exist smooth functions $u_1^\varepsilon,u_2^\varepsilon,\dots,u_n^\varepsilon$ such that

$$A_1\subset \cup_{i=1}^n B_\varepsilon(u_i^\varepsilon).$$

Let $\{\beta_i\}_{i=1}^n$ be a partition of unity subordinate to the cover $\{B_\varepsilon(u_i^\varepsilon)\}_{i=1}^n$, where the functions $\{\beta_i\}_{i=1}^n$ are Lipschitz continuous. Then, for any $u\in A_1$,

$$\|\sum_{i=1}^n\beta_i(u)u_i^\varepsilon-u\|=\|\sum_{i=1}^n\beta_i(u)u_i^\varepsilon-\sum_{i=1}^n\beta_i(u)u\|\le\|u_i^\varepsilon-u\|,\quad\text{for some } j.$$

Hence,

$$\|\sum_{i=1}^n\beta_i(u)u_i^\varepsilon-u\|\le\varepsilon, \tag{7.11}$$

where we used (7.10) and the fact $\sum_{i=1}^n\beta_i(u)=1$. Let $u^*(u)=\sum_i\beta_i(u)u_i^\varepsilon$, for all $u\in A_1$. Then, u^* is continuous. Let η_2 be a map defined on $[0,1]\times A_1$ by

$$\eta_2(t,u)=(1-t)u+tu^*(u),\quad\text{for } t\in[0,1] \text{ and } u\in A_1. \tag{7.12}$$

Note that η_2 is continuous. Next, we show that $\eta_2(t,u)\in J^{-\frac{3}{2}M}$ for all $t\in[0,1]$ and $u\in A_1$. Indeed, setting $v=(1-t)u+tu^*(u)$ and using (7.11) and (7.12), we obtain

$$\|v-u\|=t\|u^*-u\|<\varepsilon. \tag{7.13}$$

Then, using (7.9) we obtain

$$|J(v)-J(u)|<\frac{M}{2}, \tag{7.14}$$

in view of (7.13). Since $J(u)\le -2M$, we obtain from (7.14) that

$$J(v)<-\frac{3M}{2},\quad \text{for all } t\in[0,1], \text{ and } u\in A_1.$$

Define $A_2 = \eta_2(A_1, 1)$. We have $A_2 \subset J^{-\frac{3}{2}M} \cap C_0^1(\Omega)$. Therefore, we have deformed the set A_1 into a compact subset A_2 of $J^{-\frac{3}{2}M} \cap C_0^1(\Omega)$.

Note that there exists a constant $\overline{M} > 0$ such that

$$|\nabla u(x)| \leq \overline{M}, \quad \text{for all } u \in A_2. \tag{7.15}$$

In fact,

$$\overline{M} = \max_{1\leq i\leq n} \max_{x\in\overline{\Omega}} |\nabla u_i^{\varepsilon}(x)|.$$

Step 3: In this step, we will deform the subset A_2 from Step 2 to a subset of functions with nonzero positive part. First, note that, since $J : H \to \mathbb{R}$ is continuous and A_2 is compact, there exists $\varepsilon_1 > 0$ such that, for all $u \in A_3$,

$$\|v - u\| < \varepsilon_1 \Rightarrow |J(v) - J(u)| < \frac{M}{2}. \tag{7.16}$$

Let $d(x) = \operatorname{dist}(x, \partial\Omega)$, for $x \in \Omega$. By [12, Lemma 14.16], there exists $\nu > 0$ such that d is smooth in the set $\Gamma_\nu = \{x \in \overline{\Omega} : d(x) < \nu\}$. Define $\varphi_\varepsilon : \overline{\Omega} \to \mathbb{R}$ by

$$\varphi_\varepsilon(x) = \begin{cases} 2\overline{M}d(x), & \text{if } x \in \Gamma_\varepsilon; \\ 2\overline{M}\varepsilon, & \text{if } x \in \overline{\Omega}\backslash\Gamma_\varepsilon, \end{cases}$$

where $\varepsilon > 0$ is such that $\varepsilon < \nu$ and

$$\int_{\Gamma_\varepsilon} |\nabla d|^2 dx < \frac{\varepsilon_1^2}{4\overline{M}^2}. \tag{7.17}$$

It follows from (7.17) that

$$\|\varphi_\varepsilon\| < \varepsilon_1. \tag{7.18}$$

Furthermore, for every $u \in A_2$, we have

$$u(x) + \varphi_\varepsilon(x) > 0, \quad \text{for } x \text{ near } \partial\Omega. \tag{7.19}$$

In fact, if $u(x) > 0$ for x near $\partial\Omega$, the statement in (7.19) is true. If not, there exists $x_0 \in \partial\Omega$ such that $u(x) < 0$ for $x \in B_{\delta_0}(x_0) \cap \Omega$ for some $\delta_0 > 0$. Let $\vec{n}$ be a unit normal vector to $\partial\Omega$ that points towards Ω. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = u(x_0 + t\vec{n}), \quad \text{for all } t \in \mathbb{R}.$$

By the intermediate value theorem, there exists $\xi \in (0, t)$ such that $f(t) = f'(\xi)t$, for $t > 0$ in some neighborhood of 0; then,

$$u(x_0 + t\vec{n}) = (\nabla u(x_0 + \xi\vec{n}) \cdot \vec{n})t, \quad \text{for } t > 0 \text{ small enough.}$$

Since $u(x_0 + t\vec{n}) < 0$, $|u(x_0 + t\vec{n})| = -u(x_0 + t\vec{n})$, for $t > 0$ small enough. Then, using (7.15), we obtain

$$-u(x_0 + t\vec{n}) = |\nabla u(x_0 + \xi\vec{n})|t \leq \overline{M}t, \quad \text{for } t > 0 \text{ small enough.}$$

So that,

$$-u(x_0 + t\vec{n}) < 2\overline{M}t, \quad \text{for } t > 0 \text{ small enough.} \tag{7.20}$$

Observe that, for $t > 0$ small enough, $d(x_0 + t\vec{n}) = t$. We can therefore rewrite (7.20) as

$$-u(x_0 + t\vec{n}) < 2\overline{M}d(x_0 + t\vec{n}), \quad \text{for } t > 0 \text{ small enough;}$$

so that

$$-u(x_0 + t\vec{n}) < \varphi_\varepsilon(x_0 + t\vec{n}), \quad \text{for } t > 0 \text{ small enough.}$$

Therefore, $v(x_0 + t\vec{n}) > 0$ for $t > 0$ small enough. Thus, $v = u + \varphi_\varepsilon$ has a positive part, v^+.

Define a map η_3 on $[0,1] \times A_2$ by

$$\eta_3(t,u) = u + t\varphi_\varepsilon, \quad \text{for all } u \in A_2, t \in [0,1].$$

Then, η_3 is continuous. We claim that $\eta_3(t,u) \in J^{-M}$, for $u \in A_2$ and $t \in [0,1]$. Indeed, for $v = u + t\varphi_\varepsilon$, and $0 \le t \le 1$, using (7.18), we obtain

$$\|v - u\| = t\|\varphi_\varepsilon\| \le \|\varphi_\varepsilon\| \le \varepsilon_1.$$

Then, it follows from (7.16) that

$$|J(v) - J(u)| < \frac{M}{2}.$$

Since $J(u) \le -\frac{3}{2}M$, we obtain

$$J(v) < \frac{M}{2} + J(u) < \frac{M}{2} - \frac{3M}{2} = -M.$$

Thus $\eta_3 : [0,1] \times A_2 \to J^{-M}$ is a continuous map. Put $A_3 = \eta_3(1, A_2)$. Therefore, A_3 is a compact subset of the level set J^{-M} whose elements have nonzero positive part. This concludes the proof of Step 3.

Step 4: In this step, our goal is to deform the set A_3 into a set of functions u, for which $J(u^+) < 0$. For each element $u \in A_3$, we have

$$J(tu^+) = \frac{t^2}{2}\|u^+\|^2 - \int_\Omega G(x, tu^+)dx. \tag{7.21}$$

Noting that A_3 is compact, we set

$$M_1 = \max_{\xi \in A_3} \|\xi^+\|^2. \tag{7.22}$$

Similarly, $\int_\Omega G(x,u^+)dx$ attains a minimum m_1 in A_3 given by

$$m_1 = \inf_{\xi^+ \in A_3} \int_\Omega G(x, \xi^+)dx. \tag{7.23}$$

It follows from (7.22), (7.23) and (7.21) that

$$J(tu^+) \le t^2[\frac{M_1}{2} - \frac{m_1}{t^2}], \quad \text{for } u \in A_3, \text{ and } t > 0. \tag{7.24}$$

Next, choose $T_1 > 0$ such that

$$\beta = \frac{m_1}{T_1^2} - \frac{M_1}{2} > 0. \tag{7.25}$$

Then, by virtue of (7.24) and (7.25),

$$J(tu^+) \le -\beta t^2, \quad \text{for } u \in A_3, \text{ and } t \ge T_1.$$

Since we want $J(tu^+) \le -M$, we can choose t such that

$$t \ge \big(\frac{M}{\beta}\big)^{1/2}.$$

Put

$$T_4 = \max\big\{T_1, \big(\frac{M}{\beta}\big)^{1/2}\big\}. \tag{7.26}$$

Now, define a map η_4 on $[0,1] \times A_3$ by

$$\eta_4(t,u) = [(1-t) + tT_4]\, u, \quad \text{for } t \in [0,1], \text{ and } u \in A_3.$$

Then, η_4 is continuous, and $\eta_4(t,u) \in J^{-M}$ by (7.8). Thus, η_4 defines a continuous map from $[0,1] \times A_3$ to J^{-M}. Put $A_4 = \eta_4(1, A_3)$. Then, A_4 is a compact set. Also, $A_4 \subset J^{-M}$, and $J(u^+) \leq -M$, for all $u \in A_4$. This concludes the proof of Step 4.

Step 5: In this step, the goal is to deform the set A_4 to a set of functions, u, in J^{-M} such that $J(u^+)$ is negatively large enough. First, notice that, for $0 \leq s \leq 1$,

$$
\begin{aligned}
J(-su^-) &= \frac{s^2}{2} \int_\Omega [|\nabla u^-|^2 - G(x, -su^-)] dx \\
&\leq \frac{1}{2} \max_{\xi \in A_4, 0 \leq s \leq 1} \Big| \int_\Omega [|\nabla \xi^-|^2 - G(x, -s\xi^-)] dx \Big|.
\end{aligned}
$$

Set

$$
C_{11} = \frac{1}{2} \max_{\xi \in A_4, 0 \leq s \leq 1} \Big| \int_\Omega [|\nabla \xi^-|^2 - G(x, -s\xi^-)] dx \Big|.
$$

Then,

$$
J(-su^-) \leq C_{11}, \quad \text{for } s \in [0,1], \text{ and } u \in A_4. \tag{7.27}
$$

This estimate will also be used in Step 6.

Next, using the estimate (3.13), we obtain

$$
J(tu^+) \leq \frac{t^2}{2} \|u^+\|^2 - \frac{C_7 t^\mu}{2} \|u^+\|^\mu_{L^\mu} - C_8 |\Omega|.
$$

So that

$$
J(tu^+) \to -\infty, \quad \text{as } t \to \infty,
$$

since $\mu > 2$. Thus, we can choose T_5 large enough such that

$$
J(T_5 u^+) \leq -M - C_{11}, \quad \text{for all } u \in A_4.
$$

Define a map η_5 on $[0,1] \times A_4$ by

$$
\eta_5(t,u) = [(1-t) + tT_5]\, u^+ - u^-, \quad \text{for } u \in A_4, \text{ and } t \in [0,1].
$$

Then, η_5 is continuous and $\eta_5(t,u) \in J^{-M}$ for all $t \in [0,1]$ and $u \in A_4$ by (7.8). Thus, η_5 defines a map from $[0,1] \times A_4$ to J^{-M}. Put $A_5 = \eta_5(1, A_4)$. Thus, A_5 is a compact set and

$$
A_5 \subset J^{-M} \cap \{u \in A_4 | J(u^+) \leq -M - C_{11}\}, \tag{7.28}
$$

and η_5 defines a deformation from A_4 to A_5. This concludes the proof of Step 5.

Step 6: In this step, we will deform the set A_5 obtained in Step 5 to a subset of nonnegative functions in J^{-M}. For each element $u \in A_5$, we have

$$
J(u^+ - su^-) = J(u^+) + J(-su^-) \quad \text{for all } u \in A_5.
$$

So that, using (7.27) and (7.28), we obtain

$$
J(u^+ - su^-) \leq -M, \quad \text{for all } s \in [0,1], \text{ and } u \in A_5. \tag{7.29}
$$

Define a map η_6 on $[0,1] \times A_5$ by

$$
\eta_6(t,u) = u^+ - (1-t)u^-.
$$

Then η_6 is continuous and $\eta_6(t,u) \in J^{-M}$ for all $t \in [0,1]$ and $u \in A_5$, by virtue of (7.29). Thus, η_6 defines a map from $[0,1] \times A_5$ to J^{-M}. Put $A_6 = \eta_6(1, A_5)$. Then, A_6 is a compact set and its elements are nonnegative functions. This concludes the proof of Step 6.

Step 7: Define $B_1^+ = \{u \in H | \|u\| = 1 \text{ and } u \geq 0\}$. We saw in Step 6 that A_6 is compact and

$$A_6 \subset J^{-M} \cap \{u \in H : u \geq 0\}.$$

We show that $J^{-M} \cap \{u \in H : u \geq 0\}$ and B_1^+ are homotopic.

Observe that for every $u \in H\backslash\{0\}$ such that $u \geq 0$ on Ω, there exists a unique $t^*(u) > 0$ such that

$$J(t^*(u)u) = -M. \tag{7.30}$$

Furthermore, the map $u \mapsto t^*(u)$ is continuous for $u \in \{u \in H : u \geq 0\}$. In fact, it follows from (3.13) that

$$J(tu) \leq \frac{t^2}{2}\|u\|^2 + C_8|\Omega| - C_7 t^\mu \int_\Omega |u|^\mu dx.$$

So that, since $\mu > 2$, we have that

$$\lim_{t\to\infty} J(tu) = -\infty, \quad \text{for } u \in H\backslash\{0\},\ u \geq 0 \text{ in } \Omega. \tag{7.31}$$

Also, $J(0) = 0$. It then follows by the intermediate value theorem and (7.31) that, for each $u \in H\backslash\{0\}$ with $u \geq 0$, there exists $t^* > 0$ such that,

$$J(t^*u) = -M.$$

So that, using (7.4),

$$\frac{d}{dt}J(tu)\big|_{t=t^*} \leq -\frac{1}{t^*}M < 0.$$

Hence, by the implicit function theorem, t^* is unique and is a continuous function of u for $u \in H\backslash\{0\}$, $u \geq 0$ in Ω, which proves (7.30). Furthermore, $J(tu) \leq -M$ for all $t \geq t^*(u)$.

Next, set

$$B = \{tv : v \in B_1^+ \text{ and } t \geq t^*(v)\}. \tag{7.32}$$

We show that

$$B = J^{-M} \cap \{u \in H : u \geq 0 \text{ in } \Omega\}. \tag{7.33}$$

To see why (7.33) is true, take $u \in H$ with $u \geq 0$ in Ω, and $J(u) \leq -M$; so that

$$u = \|u\|u_1, \quad \text{where } u_1 = \frac{1}{\|u\|}u \in B_1,$$

and $\|u\| \geq t^*(u_1)$, since $J(\|u\|u_1) \leq -M$. Hence,

$$J^{-M} \cap \{u \in H : u \geq 0 \text{ in } \Omega\} \subseteq B. \tag{7.34}$$

Next, let $u \in B$. Then, there exists $v \in B_1^+$ and $t \geq t^*(v)$ such that $u = tv$, where $v \in B_1^+$ and $t \geq t^*(v)$. Then, by the definition of $t^*(v)$ it follows that

$$J(tv) \leq -M,$$

which shows that $u \in J^{-M}$. Hence, $u \in J^{-M} \cap \{u \in H : u \geq 0 \text{ in } \Omega\}$. Thus

$$B \subseteq J^{-M} \cap \{u \in H : u \geq 0, \text{ in } \Omega\}. \tag{7.35}$$

The inclusion (7.34) and (7.35) establish (7.33).

Next, we show that B and B_1^+ are homotopic. This will imply that

$$J^{-M} \cap \{u \in H : u \geq 0 \text{ in } \Omega\} \cong B_1^+.$$

Define $f : B \to B_1^+$ as follows: For each $u \in B$, $u \in J^{-M}$ and $u \geq 0$ in Ω, so that $u \neq 0$; thus, we can define

$$f(u) = \frac{1}{\|u\|}u, \quad \text{for all } u \in B.$$

Define $g : B_1^+ \to B$ by $g(u) = t^*(u)u$ for all $u \in B_1^+$. Then,

$$f \circ g(u) = \frac{1}{t^*(u)\|u\|}(t^*(u)u) = u,$$

for all $u \in B_1^+$. So, $f \circ g = id_{B_1^+}$. On the other hand, for $u \in B$,

$$g \circ f(u) = t^*\big(\frac{u}{\|u\|}\big)\frac{u}{\|u\|}. \tag{7.36}$$

We claim that

$$t^*(u) = \frac{1}{\|u\|}t^*\big(\frac{u}{\|u\|}\big). \tag{7.37}$$

By the definition of t^*, we have $J(t^*(u)u) = -M$. Similarly,

$$J\Big(t^*\big(\frac{u}{\|u\|}\big)\frac{u}{\|u\|}\Big) = -M.$$

Then, by the uniqueness of t^* we obtain (7.37). Therefore, we can rewrite (7.36) as

$$g \circ f(u) = t^*(u)u. \tag{7.38}$$

Next, we build a homotopy from g to id_B by $H : [0,1] \times B \to B$ given by

$$H(s,u) = [st^*(u) + (1-s)]u. \tag{7.39}$$

Then, $H(0,u) = u$ and $H(1,u) = t^*(u)u = g \circ f(u)$. Note that

$$t^*(u) \leq st^*(u) + (1-s) \leq 1, \quad \text{for all } s \in [0,1],$$

since $t^*(u) \leq 1$ for $u \in B$, by virtue of (7.33) Hence, $J(H(s,u)) \leq -M$ for all $s \in [0,1]$. It follows that B and B_1^+ are homotopic. Therefore, since

$$B = J^{-M} \cap \{u \in H : u \geq 0\},$$

we obtain that $J^{-M} \cap \{u \in H : u \geq 0\}$ and B_1^+ are homotopic.

Step 8: In this step, we show that B_1^+ is contractible. Let u_0 be any element in B_1^+, so that $\|u_0\| = 1$ and $u_0 \geq 0$. Define $H : [0,1] \times B_1^+ \to B_1^+$ by

$$H(t,u) = \frac{tu_0 + (1-t)u}{\|tu_0 + (1-t)u\|}, \quad t \in [0,1], u \in B_1^+.$$

Note that, for any $t \in [0,1]$ and $u \in B_1^+$, $tu_0 + (1-t)u \geq 0$. Furthermore,

$$\|tu_0 + (1-t)u\| \neq 0, \quad \text{for all } t \in [0,1], \text{ and } u \in B_1^+.$$

Otherwise there would exist $t_1 \in [0,1]$ and $u_1 \in B_1^+$ such that

$$\|t_1u_0 + (1-t_1)u_1\| = 0.$$

Then, $t_1u_0 + (1-t_1)u_1 = 0$. So that

$$t_1u_0 = -(1-t_1)u_1,$$

where $t_1u_0 \geq 0$ and $u_1 \geq 0$, so that $t_1u_0 \leq 0$. Thus, $t_1 = 0$, so that $u_1 = 0$, which is impossible. Thus, $H : [0,1] \times B_1^+ \to B_1^+$ defines a homotopy with $H(0,u) = u$, that is, $H(0,.) = id_{B_1^+}$, and $H(1,u) = u_0$ for all $u \in B_1^+$.

Step 9: By Step 6 we have that

$$A_6 \subset J^{-M} \cap \{u \in H | u \geq 0\}.$$

In Step 7, we showed that $J^{-M} \cap \{u \in H | u \geq 0\}$ is homotopic to B_1^+. In Step 8, we showed that B_1^+ is contractible in J^{-M}. Therefore, A_6 is contractible in J^{-M}. This concludes the proof of Proposition 7.1. □

The previous proposition implies that $\widetilde{H}_q(J^{-M}) = 0$, for all $q \in \mathbb{Z}$. Then, it follows from the exactness of the homology sequence of the pair (H, J^{-M}),

$$\begin{aligned} \dots \xrightarrow{i_*} \widetilde{H}_{q+1}(H) \cong \{0\} \xrightarrow{j_*} H_{q+1}(H, J^{-M}) \xrightarrow{\partial_*} \widetilde{H}_q(J^{-M}) \\ \cong \{0\} \xrightarrow{i_*} \widetilde{H}_q(H) \cong \{0\} \xrightarrow{\partial_*} \dots, \end{aligned}$$

that ∂_* is an isomorphism. Therefore,

$$C_q(J, \infty) = H_q(H, J^{-M}) = 0, \tag{7.40}$$

for all $q \in \mathbb{Z}$.

Now we present the proof of the main result.

Proof of Theorem 1.1. Let u_0 be as given in Theorem 6.1 and u_1 as given in Theorem (6.3). Assume by way of contradiction that $0, u_0$, and u_1 are the only critical points of J. Then, $\mathcal{K} = \{0, u_0, u_1\}$. Using the Morse relation (2.6) with $t = -1$, we obtain

$$\sum_{q=0}^{\infty} M_q (-1)^q = \sum_{q=0}^{\infty} \beta_q (-1)^q, \tag{7.41}$$

where M_q are the Morse type numbers defined in (2.5) and $\beta_q = \dim C_q(J, \infty)$ are the Betti numbers for $q = 0, 1, 2, \dots$. First, the left side of (7.41) is given by

$$\sum_{q=0}^{\infty} M_q (-1)^q = M_0 - M_1 + (-1)^d M_d,$$

where

$$\begin{gathered} M_d = \dim C_d(J, 0) = 1, \quad M_0 = \dim C_0(J, u_0) = 1, \\ M_1 = \dim C_1(J, u_1) = 1, \end{gathered} \tag{7.42}$$

where we have used (5.12), (6.7), and (6.1), respectively.

The Betti numbers are given by $\beta_q = \dim C_q(J, \infty)$, where the critical groups at infinity were computed in (7.40), so that

$$C_q(J, \infty) = 0, \quad \text{for } q = 0, 1, 2, \dots.$$

Then, $\beta_q = 0$ for all $q = 0, 1, 2, \dots$. Hence, substituting (7.42) in (7.41), we obtain

$$(-1)^d = 0,$$

which is a contradiction. Therefore, J must have at least four critical points; that is, problem (1.1) must have at least three nontrivial weak solutions. □

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