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A FULLY NONLINEAR GENERALIZED MONGE-AMPÈRE PDE ON A TORUS

VAMSI P. PINGALI

Abstract. We prove an existence result for a "generalized" Monge-Ampère equation, introduced in [11], under some assumptions on a flat complex 3-torus. As an application we prove the existence of Chern connections on certain kinds of holomorphic vector bundles on complex 3-tori whose top Chern character forms are given representatives.

1. Introduction

The complex Monge-Ampère equation on a Kähler manifold was introduced by Calabi [4], and was solved by Aubin [1] and Yau [13]. Since then other such fully nonlinear equations were studied, namely, the Hessian and the inverse Hessian equations [7, 8, 9]. The inverse Hessian equations were introduced by Chen [5] in an attempt to find a lower bound on the Mabuchi energy. Actually, in [5] Chen conjectured that a fairly general fully nonlinear Monge-Ampère type PDE has a solution. Roughly speaking, instead of requiring the determinant of the complex Hessian of a function to be prescribed, it requires a combination of the symmetric polynomials of the Hessian to be given. A real version of such an equation was studied by Krylov [10] and a general existence result was proven by reducing it to a Bellman equation. In view of these developments a "generalized Monge-Ampère" equation was introduced in [11] and a few local "toy models" were studied. As expected, the equation is quite challenging. The main problem is to find techniques to prove *a priori* estimates in order to use the method of continuity to solve the equation. In this paper we study this equation on a flat complex torus wherein curvature issues do not play a role. The aim of this basic example is to give insight into studying this equation in a more general setting. We prove an existence result (theorem 2.1) in this paper.

A small geometric application of this result is also provided - Given a (k, k) form η representing the kth Chern character class $[\mathrm{tr}((\Theta)^k)]$ of a vector bundle on a compact complex manifold, it is very natural to ask whether there is a metric whose induced Chern connection realises $\mathrm{tr}((\Theta)^k) = \eta$. As phrased this question seems almost intractable. It is not even obvious as to whether there is *any* connection satisfying this requirement, leave aside a Chern connection. Work along these lines

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was done by Datta in [6] using the h-principle. Therefore, it is more reasonable to ask whether equality can be realised for the top Chern character form. To restrict ourselves further we ask whether any given metric h_0 may be conformally deformed to $h_0 e^{-\phi}$ so as to satisfy a fully nonlinear PDE of the type treated in [11]. Admittedly the result we have in this direction (theorem 2.3) imposes quite a few restrictive assumptions on the type of vector bundles involved. However, the goal is to simply introduce the problem and solve it in a basic case to highlight the difficulties involved.

2. Summary of results

We prove an existence and uniqueness theorem for a "generalized" Monge-Ampère type equation [11] on a flat, complex 3-Torus. In whatever follows $dd^c = \sqrt{-1}\partial\bar{\partial}$ and $\omega_f = \omega + dd^c f$.

Theorem 2.1. *Let $(X, \omega = \sqrt{-1}\omega_{i\bar{j}} dz^i \wedge d\bar{z}^j)$ be a flat, Kähler complex 3-torus (i.e. the $\omega_{i\bar{j}}$ are constants) $\frac{\mathbb{C}^3}{\Lambda}$ and $\alpha \geq \tilde{\epsilon}\omega \wedge \omega$ ($\tilde{\epsilon} > 0$) be a smooth harmonic (i.e. constant coefficient) $(2,2)$ form on X satisfying $\omega^3 - \alpha \wedge \omega > 0$. The following equation has a unique smooth solution ϕ satisfying $3(\omega + dd^c\phi)^2 - \alpha > 0$ and $\int_X \phi = 0$:*

$$T(\phi) = \omega_\phi^3 - \alpha \wedge \omega_\phi = \eta = e^F(\omega^3 - \omega \wedge \alpha) > 0, \tag{2.1}$$

where $\int_X \eta = \int_X (\omega^3 - \alpha \wedge \omega)$ and by $\alpha \geq \tilde{\epsilon}\omega \wedge \omega$ we mean that $(\alpha - \tilde{\epsilon}\omega \wedge \omega) = (\sqrt{-1})^2 \sum_i f_i \phi_i \wedge \bar{\phi}_i \wedge \Phi_i \wedge \bar{\Phi}_i$ for smooth functions $\tilde{\epsilon} > 0$, $f_i \geq 0$, and $(1,0)$-forms ϕ_i, Φ_i.

Remark 2.2. Let χ be a harmonic (with respect to ω) Kähler form. Define $\tilde{\omega}$ as $\tilde{\omega} = \omega + \frac{\chi}{3}$ and assume that $\tilde{\omega}^3 - \tilde{\omega}^2 \wedge \chi > \frac{-2\chi^3}{27}$. As an interesting consequence one can see that the equation

$$\tilde{\omega}_\phi^3 = \chi \wedge \tilde{\omega}_\phi^2 \tag{2.2}$$

has a unique solution satisfying $\tilde{\omega}_\phi > 0$ and $3\tilde{\omega}_\phi^2 > 2\chi \wedge \tilde{\omega}_\phi$ if we also assume that χ satisfies $\int_X \omega^3 = \int_X \chi \wedge \omega^2$. Indeed, equation 2.2 maybe rewritten as

$$0 = \tilde{\omega}_\phi^3 - \chi \wedge \tilde{\omega}_\phi^2 = \omega_\phi^3 - \omega_\phi \wedge \frac{\chi^2}{3} - \frac{2\chi^3}{27}.$$

Thus we recover existence for an inverse Hessian equation in this very special case by taking $\alpha = \frac{\chi^2}{3}$ and $\eta = \frac{2\chi^3}{27}$. (It is easy to verify the remaining conditions of theorem 2.1.) This also shows that solving the equation in general would give an alternate proof of existence for some inverse Hessian equations, i.e., some of the results in [8].

A consequence of theorem 2.1 and the Calabi conjecture is the following theorem that deals with the existence of a Chern connection with a prescribed top Chern form.

Theorem 2.3. *Let X be a compact complex manifold of dimension n and (V, h_0) be a rank k hermitian holomorphic vector bundle over X. We denote the (normalised) curvature matrix of the Chern connection ∇_0 associated to h_0 as $\Theta_0 = \frac{\sqrt{-1}}{2\pi} F_0$ where F_0 is the curvature matrix of ∇_0. In the following two cases, given an (n,n)*

form η representing the top Chern character class of V, there exists a smooth metric $h = h_0 e^{-2\pi\phi}$ such that its top Chern-Weil form of the Chern character class is η.

1. *X is a surface, i.e., $n = 2$, $\operatorname{tr}(\Theta_0) > 0$, and $(\operatorname{tr}(\Theta_0))^2 + k(\eta - \operatorname{tr}(\Theta_0^2)) > 0$.*
2. *X is a complex 3-torus, $k\omega = \operatorname{tr}(\Theta_0)$ is a harmonic positive form, $\alpha = \frac{3(\operatorname{tr}(\Theta_0))^2}{k^2} - \frac{3\operatorname{tr}(\Theta_0^2)}{k} > 0$ is harmonic, $-2(\operatorname{tr}(\Theta_0))^3 + 3k\operatorname{tr}(\Theta_0) \wedge \operatorname{tr}(\Theta_0^2) > 0$, and $k^2(\eta - \operatorname{tr}(\Theta_0^3) - 2(\operatorname{tr}(\Theta_0))^3 + 3k\operatorname{tr}(\Theta_0) \wedge \operatorname{tr}(\Theta_0^2) > 0$.*

Remark 2.4. We recall that the curvature of a connection $\nabla = d + A$ on a rank-k vector bundle is defined locally as a $k \times k$-matrix of 2-forms $F = dA + A \wedge A$ where the connection A is locally a $k \times k$-matrix of 1-forms. The trace alluded to in theorem 2.3 is the trace of the matrix F giving rise to a single 2-form (as opposed to the traces of 2-forms (giving rise to single functions) that occur later on in this paper).

The hypotheses of theorem 2.3 require some discussion. As a warm-up example, let us consider the question for a line bundle; i.e., given a metric h_0 on a hermitian holomorphic line bundle L on a complex n-fold with the curvature form denoted as Θ_{h_0}, can we find a new metric $h = e^{-\phi}h_0$ such that the top Chern character form $(\frac{\sqrt{-1}}{2\pi}(\Theta_{h_0} + dd^c\phi))^n = \eta$ where $[\eta] = \operatorname{ch}_n(L)$? This is just the "usual" Monge-Ampère equation. To prove existence, the commonly made assumption is $\Theta_{h_0} > 0$. So it is not at all surprising (and almost inevitable) that a "generalized" version of such an equation would warrant more positivity assumptions, some of which might seem a little less geometric than desired.

Nevertheless, here are a few examples (certainly not exhaustive) that satisfy the hypotheses:

1. X is any compact complex surface, (V, h_0) is any rank-k hermitian holomorphic vector bundle over X such that $\operatorname{tr}(\Theta_0) > 0$ and $\eta = (\operatorname{tr}(\Theta_0))^2 + \epsilon g \operatorname{tr}(\Theta_0)$ where $\int_X g \operatorname{tr}(\Theta_0) = 0$ and $\epsilon \ll 1$.
2. X is the complex 3-torus with the standard lattice $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Choose three line bundles $(L_1, h_1), (L_2, h_2), (L_3, h_3)$ so that their Chern forms are $\omega_1 = \sqrt{-1}\sum dz^i \wedge d\bar{z}^i$, $\omega_2 = \sqrt{-1}(3dz^2 \wedge d\bar{z}^2 + dz^3 \wedge d\bar{z}^3)$, $\omega_3 = 2dz^3 \wedge d\bar{z}^3$. Take (V, h_0) to be their direct sum and $\eta = \operatorname{tr}(\Theta_0^3) + \epsilon g$ where $\epsilon << 1$ and $\int g = 0$.

3. Proofs of main theorems

Unless specified otherwise, for the remainder of the paper we denote all the constants (independent of the relevant quantities) appearing in the estimates by C by default. We first prove the following useful lemma.

Lemma 3.1. *Let X be a Kähler 3-manifold. If γ is a non-negative real $(1,1)$ form and β be a strongly strictly positive real $(2,2)$ form (hence $*\beta > 0$ for the Hodge star of any Kähler metric) such that $\gamma^3 - \beta \wedge \gamma > 0$ then $3\gamma^2 - \beta > 0$ and $\gamma > 0$.*

Proof. Since $\gamma^3 > 0$ it is clear that $\gamma > 0$. Let $*$ denote the Hodge star with respect to γ. Notice that $\beta \wedge \gamma = *\beta \wedge \frac{\gamma^2}{2}$. Since we are dealing with top forms, we may divide by $(*\beta)^3$ to get $\frac{\gamma^3}{(*\beta)^3} - \frac{*\beta\wedge\gamma^2}{2(*\beta)^3} > 0$. At a point p, choose coordinates so that the strictly positive form $*\beta$ is $\sqrt{-1}\sum dz^i \wedge d\bar{z}^i$ and γ is diagonal with eigenvalues λ_i. Then at p, $6\lambda_1\lambda_2\lambda_3 - (\sum_{i<j}\lambda_i\lambda_j) > 0$ thus implying that $6\lambda_i > 1$. This means $6\gamma - *\beta > 0$. Applying $*$ we see that $3\gamma^2 - \beta > 0$. ∎

We need another lemma.

Lemma 3.2. *Let X be a Kähler 3-manifold. If γ is a positive real $(1,1)$ form, $\eta > 0$ is a $(3,3)$ form, and β be a strongly strictly positive real $(2,2)$ form, then the functions $\mathcal{F} : \gamma \to \frac{\beta\wedge\gamma}{\gamma^3}$ and $\mathcal{G} : \gamma \to \frac{\eta}{\gamma^3}$ are convex.*

Proof. Fix a Kähler form ω for X and let $*$ be its Hodge star. Choose coordinates so that $\omega = \sqrt{-1}\sum dz^i \wedge d\bar{z}^i$ at a point p. By a linear change of coordinates $*\beta$ may be diagonalised at p. Hence $\beta = -b_3 dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 - b_2 dz^3 \wedge d\bar{z}^3 \wedge dz^1 \wedge d\bar{z}^1 - b_1 dz^2 \wedge d\bar{z}^2 \wedge dz^3 \wedge d\bar{z}^3$ at p. By scaling z_i appropriately we may assume that $b_i = 1$. At p the function $\mathcal{F}$ is $A \to \frac{\mathrm{tr}(A)}{6\det(A)}$ where A is a positive hermitian matrix. The fact that this and $G(A) = \frac{1}{\det(A)}$ are convex is proven in [10]. (Notice that $\mathcal{G}(A) = KG(A)$ for some positive constant K.) $\square$

It is easy to see that the set $\mathcal{S}$ of $\gamma > 0$ in lemma 3.1 satisfying $\gamma^3 - \beta\wedge\gamma > 0$ is a convex open set. In fact a stronger statement holds.

Lemma 3.3. *Let γ_1, γ_2 lie in $\mathcal{S}$ and $\gamma_t = t\gamma_1 + (1-t)\gamma_2$. Then $3\gamma_t^2 - \beta > Ct\gamma_1^2$ where C depends only on γ_1 and β.*

Proof. Notice that

$$\begin{aligned} \gamma_t^3 - \beta\wedge\gamma_t &= \gamma_t^3\big(1 - \frac{\beta\wedge\gamma_t}{\gamma_t^3}\big) \\ &\geq \gamma_t^3\Big(1 - t\frac{\beta\wedge\gamma_1}{\gamma_1^3} - (1-t)\frac{\beta\wedge\gamma_2}{\gamma_2^3}\Big)\,, \end{aligned} \tag{3.1}$$

where the last inequality follows from lemma 3.2. Since γ_1 and γ_2 lie in $\mathcal{S}$,

$$\gamma_t^3\Big(1 - t\frac{\beta\wedge\gamma_1}{\gamma_1^3} - (1-t)\frac{\beta\wedge\gamma_2}{\gamma_2^3}\Big) \geq t\gamma_t^3\Big(1 - \frac{\beta\wedge\gamma_1}{\gamma_1^3}\Big) > \tilde{C}t\gamma_t^3\,, \tag{3.2}$$

where $\tilde{C}$ is a small positive constant depending only on γ_1 and β. Putting 3.1 and 3.2 together we have

$$\gamma_t^3 - \frac{\beta}{1-\tilde{C}t}\wedge\gamma_t > 0\,.$$

This implies (by using lemma 3.1) that

$$\begin{gathered} 3\gamma_t^2 - \frac{\beta}{1-\tilde{C}t} > 0 \\ \Rightarrow 3\gamma_t^2 - \beta > \frac{\tilde{C}t}{1-\tilde{C}t}\beta > Ct\gamma_1\,. \end{gathered}$$

$\square$

Proof of Theorem 2.1. We use the method of continuity. Consider the family of equations for t in $[0,1]$

$$(\omega + dd^c\phi_t)^3 - \alpha\wedge(\omega + dd^c\phi_t) = \frac{e^{tF}\int_X(\omega^3 - \alpha\wedge\omega)}{\int_X e^{tF}(\omega^3 - \alpha\wedge\omega)}(\omega^3 - \alpha\wedge\omega)\,. \tag{3.3}$$

At $t = 0$, $\phi = 0$ is a solution. By lemma 3.1 ellipticity is preserved along the path. We verify that [11, theorem 2.1] applies here. Indeed, we notice that $T(\phi) - T(0) = \int_0^1 \frac{dT(t\phi)}{dt}dt = dd^c\phi\wedge\int_0^1(3\omega_{t\phi}^2 - \alpha)dt$ and that lemma 3.3 (along with the substitution $\tilde{t} = 1 - t$ in the integral) implies that the conditions of theorem 2.1 are satisfied.

This proves that the set of t for which solutions exist is open, solutions are unique and have an *a priori* C^0 bound. To prove that it is closed we need $C^{2,\beta}$ *a priori* estimates (by Schauder theory this is enough to bootstrap the regularity). We proceed to find such estimates for $(\omega + dd^c\phi)^3 - \alpha \wedge (\omega + dd^c\phi) = f\omega^3$. Locally $\omega = \sqrt{-1}\sum dz^i \wedge d\bar{z}^i$, $u = \sum |z|^2 + \phi$, and

$$\det(dd^c u) - \operatorname{tr}(A dd^c u) = f \tag{3.4}$$

for some hermitian positive matrix A. If α is diagonalised such that $\alpha = dz^1 \wedge d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^2 + \dots$ then $A = \frac{1}{6}\mathrm{Id}$.

C^1 *estimate:* For this we shall not make the assumption that α is harmonic. This assumption will be used only in the higher order estimates. Following [3] let O be a point where $\beta = \ln(|\nabla\phi|^2) - \gamma(\phi)$ achieves its maximum. (If we prove that β is bounded, then so is the first derivative. So assume that $|\nabla\phi| > 1$ without loss of generality. β is Blocki's function. γ will be chosen later.) Differentiating once we see that $\det(dd^c u)\operatorname{tr}((dd^c u)^{-1}(dd^c u_k)) - \operatorname{tr}(A_{,k} dd^c u) - \operatorname{tr}(A dd^c u_k) = f_k$ (and similarly for $\bar{k}$). Let L be the matrix $\det(dd^c u)(dd^c u)^{-1} - A > 0$. Hence

$$\operatorname{tr}(L dd^c u_i) = f_i + \operatorname{tr}(A_{,i} dd^c u)\,. \tag{3.5}$$

At O we may assume that $\phi_{i\bar{j}}$ is diagonal. Besides, $\beta_k = 0$ there and $\operatorname{tr}(L\beta_{k\bar{l}}) \le 0$ at O. The first condition implies that

$$\begin{aligned} &\frac{1}{|\nabla\phi|^2}\Big(\sum \phi_{ik}\phi_{\bar{i}} + \phi_i\phi_{\bar{i}k}\Big) - \gamma'\phi_k = 0 \\ &\Rightarrow \frac{1}{|\nabla\phi|^2}(\phi_{kk}\phi_{\bar{k}} + \phi_k\phi_{\bar{k}k}) = \gamma'\phi_k \end{aligned} \tag{3.6}$$

at O. Moreover,

$$\begin{aligned} \beta_{k\bar{l}} = &-\frac{1}{|\nabla\phi|^4}\Big(\sum \phi_{ik}\phi_{\bar{i}} + \phi_i\phi_{\bar{i}k}\Big)\Big(\sum \phi_{j\bar{l}}\phi_{\bar{j}} + \phi_j\phi_{\bar{j}\bar{l}}\Big) \\ &+ \frac{1}{|\nabla\phi|^2}\Big(\sum \phi_{ik\bar{l}}\phi_{\bar{i}} + \phi_{ik}\phi_{\bar{i}\bar{l}} + \phi_{i\bar{l}}\phi_{\bar{i}k} + \phi_i\phi_{\bar{i}k\bar{l}}\Big) - \gamma''\phi_{\bar{l}}\phi_k - \gamma'\phi_{k\bar{l}}\,. \end{aligned} \tag{3.7}$$

Noticing that $dd^c u_i = dd^c\phi_i$, and using 3.5 and 3.6 we get (at O)

$$\begin{aligned} 0 &\ge \operatorname{tr}(L\beta_{k\bar{l}}) \\ &= -((\gamma')^2 + \gamma'')\operatorname{tr}(L\phi_k\phi_{\bar{l}}) - \gamma'\operatorname{tr}(L dd^c u) + \gamma'\operatorname{tr}(L) \\ &\quad + \frac{1}{|\nabla\phi|^2}\Big(\sum \phi_{\bar{i}}(f_i + \operatorname{tr}(A_{,i} dd^c u)) \\ &\quad + \phi_i(f_{\bar{i}} + \operatorname{tr}(A_{,\bar{i}} dd^c u)) + \operatorname{tr}(L\phi_{ik}\phi_{\bar{i}\bar{l}}) + \operatorname{tr}(L\phi_{i\bar{l}}\phi_{\bar{i}k})\Big) \\ &\ge -((\gamma')^2 + \gamma'')\operatorname{tr}(L\phi_k\phi_{\bar{l}}) - \gamma'[3\det(dd^c u) - \operatorname{tr}(A dd^c u)] \\ &\quad + \gamma'\Big(\det(dd^c u)\sum_{i=1}^{3}\frac{1}{u_{i\bar{i}}} - \operatorname{tr}(A)\Big) - 2\frac{|\nabla f|}{|\nabla\phi|} - \frac{C\operatorname{tr}(A dd^c u)}{|\nabla\phi|} \\ &\quad + \frac{1}{|\nabla\phi|^2}\Big(\sum \operatorname{tr}(L\phi_{ik}\phi_{\bar{i}\bar{l}}) + \operatorname{tr}(L\phi_{i\bar{l}}\phi_{\bar{i}k})\Big) \\ &\ge -((\gamma')^2 + \gamma'')\operatorname{tr}(L\phi_k\phi_{\bar{l}}) - \gamma'[3f + 2\operatorname{tr}(A dd^c u)] + \gamma'\Big(\det(dd^c u)\sum_{i=1}^{3}\frac{1}{u_{i\bar{i}}}\Big) \end{aligned}$$

$$
\begin{aligned}
&- C - \frac{C\,\mathrm{tr}(Add^cu)}{|\nabla\phi|} + \frac{1}{|\nabla\phi|^2}\Big(\sum \mathrm{tr}(L\phi_{ik}\phi_{\bar{i}\bar{l}}) + \mathrm{tr}(L\phi_{i\bar{l}}\phi_{\bar{i}k})\Big)\\
&\geq -((\gamma')^2+\gamma'')\,\mathrm{tr}(L\phi_k\phi_{\bar{l}}) - 2\gamma'\,\mathrm{tr}(Add^cu)\\
&\quad + \gamma'[f+\mathrm{tr}(Add^cu)]\sum\frac{1}{u_{i\bar{i}}} - C - \frac{C}{|\nabla\phi|}\,\mathrm{tr}(Add^cu)\,.
\end{aligned}
$$

Note that C can potentially depend on γ and hence on $\|\phi\|_{C^0}$. If we choose γ so that $\gamma' > E > 0$, and $-((\gamma')^2+\gamma'') > Q > 0$ (where E and Q are arbitrary positive constants), then this forces $(dd^cu)^{-1}(O)$ to be bounded. For instance γ can be chosen [3] to be $\gamma(x) = \frac{1}{2}\ln(2x+1)$. Assume that $|\nabla\phi| \to \infty$. If $\sum\frac{1}{u_{i\bar{i}}} > 2+\epsilon$ uniformly then surely $\Delta u(O)$ is bounded. This observation actually implies that $\Delta u(O)$ is bounded.

Lemma 3.4. *At any point Q if $\Delta u \to \infty$, then $\sum\frac{1}{u_{i\bar{i}}} > 2+\epsilon$ for some uniform ϵ.*

Proof. Choose normal coordinates for ω around Q so that dd^cu is diagonal at Q. Recall that $\omega^3 - \alpha\wedge\omega > \tilde{\epsilon}\omega^3$ forces $A_{ii} < 1-\tilde{\epsilon}$. Let $u_{i\bar{i}}(Q) = \lambda_i$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq C > 0$. (If λ_3 gets arbitrarily close to 0, then the lemma is obviously true.) If $\lambda_1 \to \infty$ it is clear from the equation $\lambda_1\lambda_2\lambda_3 = f + \sum A_{ii}\lambda_i$ that λ_3 should be bounded. Solving for λ_1, one can see that $\lambda_2 \to \frac{A_{11}}{\lambda_3}$. This means that $\sum\frac{1}{\lambda_i}$ goes to $\frac{1}{\lambda_3}+\frac{\lambda_3}{A_{11}} \geq 2(1/A_{11})^{1/2} > 2+\epsilon$. □

Lemma 3.4 implies that L is bounded below and above at O. This means that $\nabla\phi$ is bounded at O.

$C^{1,1}$ *estimate:* Define $g = \frac{\alpha\wedge\omega_\phi}{\omega^3} - \phi$. Locally $g = \mathrm{tr}(Add^cu) - \phi$. If g is bounded, then thanks to the previous C^0 estimate on ϕ, so is $\mathrm{tr}(Add^cu)$. This will give us the desired bound on $\Delta\phi$ and hence on $dd^c\phi$, i.e. the $C^{1,1}$ estimate.

Differentiating equation 3.4 we see that

$$
\begin{aligned}
&\mathrm{tr}\left((\det(dd^cu)(dd^cu)^{-1} - A)dd^cu_k\right) = f_k\\
&\Rightarrow \mathrm{tr}(Ldd^cu_k) = f_k\,,
\end{aligned}
$$

where the matrix $L = \det(dd^cu)(dd^cu)^{-1} - A > 0$ is defined as before. In whatever follows, upper indices do not denote the inverse matrix. They just denote the original matrix itself and are used to make the Einstein summation convention work nicely.

Differentiating again and taking the trace after multiplication with A we see that

$$
\begin{aligned}
&A^{k\bar{l}}\,\mathrm{tr}\left((\det(dd^cu)(dd^cu)^{-1} - A)dd^cu_{k\bar{l}}\right)\\
&= A^{k\bar{l}}f_{k\bar{l}} + \det(dd^cu)A^{k\bar{l}}\,\mathrm{tr}\left((dd^cu)^{-1}dd^cu_{\bar{l}}(dd^cu)^{-1}dd^cu_k\right)\\
&\quad - \det(dd^cu)A^{k\bar{l}}\,\mathrm{tr}\left((dd^cu)^{-1}dd^cu_{\bar{l}}\right)\mathrm{tr}\left((dd^cu)^{-1}dd^cu_k\right)
\end{aligned}
$$

which implies

$$
\begin{aligned}
A^{k\bar{l}}\,\mathrm{tr}\left(Ldd^cu_{k\bar{l}}\right) &= A^{k\bar{l}}f_{k\bar{l}} + \det(dd^cu)A^{k\bar{l}}\,\mathrm{tr}\left((dd^cu)^{-1}dd^cu_{\bar{l}}(dd^cu)^{-1}dd^cu_k\right)\\
&\quad - \det(dd^cu)A^{k\bar{l}}\,\mathrm{tr}\left((dd^cu)^{-1}dd^cu_{\bar{l}}\right)\mathrm{tr}\left((dd^cu)^{-1}dd^cu_k\right).
\end{aligned} \tag{3.8}
$$

Upon differentiating g we see that

$$
\begin{aligned}
g_k &= \mathrm{tr}(Add^cu_k) - \phi_k\,,\\
g_{k\bar{l}} &= \mathrm{tr}(Add^cu_{k\bar{l}}) - \phi_{k\bar{l}}\,.
\end{aligned} \tag{3.9}
$$

Let us assume that g attains its maximum at a point P. At P, $g_k = 0$, $u_k = \phi_k$, $u_{k\bar{l}} = \phi_{k\bar{l}} + \delta_{k\bar{l}}$, and $\operatorname{tr}(L[g_{k\bar{l}}]) = L^{k\bar{l}} g_{k\bar{l}} \leq 0$. Choose normal coordinates for ω around P so that $dd^c u$ is diagonal at P. Putting these observations, and equations 3.1, 3.8 and 3.9 together we see that at P (all the arbitrary constants that occur below are positive by convention)

$$
\begin{aligned}
0 &\geq -L^{k\bar{l}}\phi_{k\bar{l}} + A^{k\bar{l}} f_{k\bar{l}} + \det(dd^c u) A^{k\bar{l}} \operatorname{tr}\big((dd^c u)^{-1} dd^c u_{\bar{l}} (dd^c u)^{-1} dd^c u_k\big) \\
&\quad - \det(dd^c u) A^{k\bar{l}} \operatorname{tr}\big((dd^c u)^{-1} dd^c u_{\bar{l}}\big) \operatorname{tr}\big((dd^c u)^{-1} dd^c u_k\big) \\
&\geq -L^{k\bar{l}} u_{k\bar{l}} + \operatorname{tr}(L) + A^{k\bar{l}} f_{k\bar{l}} \\
&\quad - \det(dd^c u) A^{k\bar{l}} \operatorname{tr}\big((dd^c u)^{-1} dd^c u_{\bar{l}}\big) \operatorname{tr}\big((dd^c u)^{-1} dd^c u_k\big) \\
&\geq -3\det(dd^c u) + \operatorname{tr}(L) + A^{k\bar{l}} u_{k\bar{l}} - C - A^{k\bar{l}} \frac{(f_k + \operatorname{tr}(A dd^c u_k))(f_l + \operatorname{tr}(A dd^c u_l))}{\det(dd^c u)} \\
&\geq -2A^{k\bar{l}} u_{k\bar{l}} + \operatorname{tr}(L) - C - A^{k\bar{l}} \frac{(f_k + u_k)(f_l + u_l)}{f + \operatorname{tr}(A dd^c u)} \\
&\geq -2A^{k\bar{l}} u_{k\bar{l}} + \det(dd^c u) \operatorname{tr}((dd^c u)^{-1}) - C_1 - \frac{C_2}{f + \operatorname{tr}(A dd^c u)} \\
&= -2A^{k\bar{l}} u_{k\bar{l}} + (f + \operatorname{tr}(A dd^c u)) \operatorname{tr}((dd^c u)^{-1}) - C_1 - \frac{C_2}{f + \operatorname{tr}(A dd^c u)}\,.
\end{aligned} \tag{3.10}
$$

Let $u_{l\bar{l}}$ at P be λ_l. Thus at P,

$$
\begin{aligned}
0 &\geq -2\sum_{l=1}^{3} A_{l\bar{l}}\lambda_l + \sum_{l=1}^{3} A_{l\bar{l}}\lambda_l \sum_{k=1}^{3} \frac{1}{\lambda_k} - C_1 - \frac{C_2}{f + \operatorname{tr}(A dd^c u)} \\
&= \Big(\sum_{k=1}^{3} \frac{1}{\lambda_k} - 2\Big) \sum_{l=1}^{3} A_{l\bar{l}}\lambda_l - C_1 - \frac{C_2}{f + \operatorname{tr}(A dd^c u)}\,.
\end{aligned} \tag{3.11}
$$

Using lemma 3.4 we see that if $\Delta u \to \infty$ at P, then

$$
0 \geq \epsilon \sum_{l=1}^{3} A_{l\bar{l}}\lambda_l - C_1 - \frac{C_2}{f + \operatorname{tr}(A dd^c u)}\ . \tag{3.12}
$$

It is clear from equation 3.12 that $\operatorname{tr}(A dd^c u)$ is bounded at P and hence so is g. As mentioned earlier this implies the desired $C^{1,1}$ estimate.

$C^{2,\beta}$ *estimate:* Rewriting the equation (just as in [11]) $-1 = -\frac{\eta}{(\omega+dd^c\phi)^3} - \frac{\alpha\wedge(\omega+dd^c\phi)}{(\omega+dd^c\phi)^3}$ and using lemma 3.2 we see that the (complex version [2][12] of) Evans-Krylov theory applies to it. This proves the desired estimate.

Proof of Theorem 2.3. The curvature $\Theta(h) = \Theta_0 + dd^c\phi$. Hence $\operatorname{tr}((\Theta_0 + dd^c\phi)^n) = \eta$. This equation reduces in the two cases of the theorem to

$$
\Big(dd^c\phi + \frac{\operatorname{tr}(\Theta_0)}{k}\Big)^2 = \frac{\eta - \operatorname{tr}((\Theta_0)^2)}{k} + \frac{(\operatorname{tr}(\Theta_0))^2}{k^2}
$$

and

$$
\begin{aligned}
&\Big(dd^c\phi + \frac{\operatorname{tr}(\Theta_0)}{k}\Big)^3 - \Big(dd^c\phi + \frac{\operatorname{tr}(\Theta_0)}{k}\Big) \wedge \Big(\frac{-3\operatorname{tr}(\Theta_0^2)}{k} + 3\frac{(\operatorname{tr}(\Theta_0))^2}{k^2}\Big) \\
&= \frac{\eta - \operatorname{tr}(\Theta_0^3)}{k} - \frac{2(\operatorname{tr}(\Theta_0))^3 - 3k\operatorname{tr}(\Theta_0)\wedge\operatorname{tr}(\Theta_0^2)}{k^3}
\end{aligned}
$$

respectively. The first equation may be solved under the given hypotheses using Aubin-Yau's solution [13][1] of the Calabi conjecture [4]. The second one is solved using theorem 2.1.

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Vamsi P. Pingali
Department of Mathematics, 412 Krieger Hall, Johns Hopkins University, Baltimore, MD 21218, USA
E-mail address: vpingali@math.jhu.edu